Introduction to MATHEMATICAL PHYSICS

Lectures Notes 2024/2025

Degree in Physics, 2nd year, 2nd term

A. Ponno
Department of Mathematics “T. Levi-Civita”
University of Padua, Italy
ponno@math.unipd.it

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"If a leaden ball, projected from the top of a mountain by the force of gunpowder with a given velocity, and in a direction parallel to the horizon, is carried in a curve line to the distance of two miles before it falls to the ground; the same, if the resistance of the air were taken away, with a double or decuple velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line, which it might describe, till at last it should fall at the distance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or lastly, so that it might never fall to the earth, but go forward into the celestial spaces, and proceed in its motion in infinitum*. And after the same manner that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the whole earth, the moon also, either by the force of gravity, if it is endued with gravity, or by any other force, that impels it towards the earth, may be perpetually drawn aside towards the earth, out of the rectilinear way, which by its innate force it would pursue; and would be made to revolve in the orbit which it now describes; nor could the moon without some such force, be retained in its orbit. If this force was too small, it would not sufficiently turn the moon out of a rectilinear course: if it was too great, it would turn it too much, and draw down the moon from its orbit towards the earth. It is necessary, that the force be of a just quantity, and it belongs to the mathematicians to find the force, that may serve exactly to retain a body in a given orbit, with a given velocity; and vice versa, to determine the curvilinear way, into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear way, by means of a given force."*

I. Newton, Book I of the *Philosophiae Naturalis Principia Mathematica*, 1687 [34].

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Preface

The present notes deal with classical mechanics and its mathematical methods, which means making use of advanced tools of analysis, geometry and algebra in order to solve classes of problems relevant to physics first, and, in so doing, to get a more abstract, higher level formulation of the physical laws of motion which is relevant far beyond mechanics (what is usually referred to as "formalism").

The notes are meant to be an integration of, or complement to the class lectures, where most of the problems are worked out in a complete way. A complete preparation to the exam is based on the use of both of them. I intentionally avoid to write down the detailed solutions of exercises and problems in the notes, but for examples, with the aim to stress the importance, for the students, to attend the class lectures and realize which kind of ideas and technical efforts are required, and how long it takes to solve specific problems up to the end, in a complete way. However, almost all exercises and problems are formulated in a step by step guided form. In the present notes I distinguish exercises, which are of a purely, maybe minor technical content, from problems, that have instead more physical relevance, require something new to be tackled, or constitute the starting point for the introduction of advanced tools. The topics in the notes are reported according to the sequence of those treated in the class lectures, although sometimes the correspondence can be a bit different, for the obvious reasons concerning the difference in between giving lectures at the blackboard and writing lecture notes.

The approach to the subject, in my course, is partially founded on *problem solving*. In tackling an interesting problem, one is required to understand completely what has to be done, to thus plan a strategy of solution, and to finally perform detailed computations; a complete check of the procedure and of the final formulas is required at the end. The method of solution of a specific problem is not unique, and the choice is a matter of convenience, time, and taste. In order to simply get the final answer one may certainly choose the shortest path. However, one can consider to follow also longer, alternative ways to the solution, where the use of, and the need to develop higher mathematics is required. This is the point where building up a general theory and its mathematical apparatus emerges, and this is exactly the way it works in research (with a nontrivial additional ingredient: also the problem is new there, and one does not even know whether it is solvable in advance). Although mathematics plays a dominant role in this course, I recommend the students to always make use of physical intuition during the preliminary analysis of the problem at hand, and in the final stage, when a result in closed mathematical form has been obtained. The dimensional correctness and the physical meaning of a formula, or a theorem, must always be considered.

Concerning the role and effectiveness of mathematics in physics (see e.g. [43]), and viceversa, a remark is in order. The *truth value* in mathematics is given by the proof: a statement within a certain axiomatic framework is true if it is proven. On the other hand, the truth value in physics is given by the experimental evidence. Every experiment requires a reference, theoretical model defining the quantities to be measured and related to each other, and any theory is built up on, and must refer to experimental results, tests and facts. The relevance of any physical theory, whatever be the level of generality, rigour and difficulty involved, is based on its ability to explain experimental facts and to predict new ones. The history of theoretical and mathematical physics works this way. The mathematical apparatus of classical mechanics built up first by Newton, was founded on the experiments and data of Brahe, Kepler, Galilei, Boyle, Hooke and others, and was then extended by Euler, Laplace, Lagrange, Hamilton, Poincaré, Einstein (just to mention a few of them) to the impressive actual level leading to space explorations, to the prediction and measurement of gravitational waves, and to the evidence of dark matter. The overall process took, if we just start from Renaissance, five centuries, but almost four millennia if we start from the birth of astronomy within the Sumerian and Babylonian civilizations [28].

`Padova, April 2, 2025`

`Antonio Ponno`

Chapter 1

Newtonian mechanics

Part of the material contained in this chapter maight be known to the reader, at the level of a general physics course; a good reference is [25]. Newtonian mechanics is not only necessary as a basis for its subsequent developments, namely the Lagrangian and the Hamiltonian formulation, but also useful to solve fundamental problem of physics.

1.1 The Newton equation

The extreme synthesis of the monumental work of Newton [34] is encoded in the equation ruling the motion of a point mass, or particle, subject to a given force, i.e. $ma = f$, where m is the mass of the particle, a its acceleration, and f the force acting on it. The Newton equation, written in more explicit form, reads

$$\boxed{m\ddot{x}(t) = f(x(t), \dot{x}(t), t)} \ , \tag{1.1}$$

where over-dots denote total derivatives with respect to time t, one per dot. Here the curve $\mathbb{R} \ni t \mapsto x(t) \in \mathbb{R}^d$ is referred to as the trajectory, or motion of the particle, which gives the position of the latter in the physical space $\mathbb{R}^d$ ($d = 1, 2, 3$) at time t. The components of the vector position $x(t) = (x_1(t), \ldots, x_d(t))$, are determined with respect to a given reference frame (a basis in $\mathbb{R}^d$), and each of them is supposed to be smooth enough in t (class C^2, at least). We recall that the instantaneous velocity of the particle is the local tangent vector to the curve $t \mapsto x(t)$, namely

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \dot{x}(t) := \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \ ,$$

whereas the instantaneous acceleration is the local tangent vector to the velocity curve $t \mapsto v(t)$, namely

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \dot{v}(t) = \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = \ddot{x}(t) \ .$$

The force acting on the particle, in the Newtonian scheme, is the *known*, vector valued function $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ appearing on the right hand side of (1.1). Once the form of the force is specified, the Newton equation (1.1) becomes a vector valued, ordinary differential equation (ODE) of second order in normal form: the derivative of highest order - the second one - is given by a function of the derivatives of lower order and of time. Once specified the force, the unknown of the Newton equation is the motion $t \mapsto x(t)$, which is uniquely determined, locally in time, by assigning the initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$, i.e. the position and the velocity of the particle at some initial instant t_0, when f is smooth enough (e.g. C^1 in all arguments).

Remark 1.1. *The fact that the force f depends on the position and the velocity of the particle is an assumption, which allows to solve the problem of predicting the motion and turns out to be in agreement with a wide experimental evidence. However, there are situations where such an assumption does not work. An example is found in classical electrodynamics of point charges: the force exerted on a charged particle by the electromagnetic field generated by the particle itself is proportional to the derivative of the acceleration. Starting with a finite size particle, a non trivial limit procedure, which also requires a mass renormalization, leads to the Abraham-Lorentz radiation reaction force [24, 36]*

$$f_{rad} = \frac{2q^2}{3c^3}\,\dot{a} = \frac{2q^2}{3c^3}\,\dddot{x}\ , \tag{1.2}$$

where q is the charge of the particle and c is the speed of light. Which is the general solution of the non Newtonian equation $ma = f_{rad}$ for the free charge? This is known as the runaway problem. Hint: look for solutions of the form $x(t) = e^{\lambda t}\xi$, where ξ is a constant vector.

►**Problem 1.1.** *Consider equation (1.1) in dimension $d = 1$, with initial conditions $x(t_0) := x_0$ and $\dot{x}(t_0) := v_0$. Show that $x(t)$ is uniquely determined in a suitable neighbourhood of t_0. Hint: assume $f(x, v, t)$ smooth enough and Taylor expand $x(t_0 + h)$ in h, to third order included (with a remainder of order h^4). Result:*

$$\begin{aligned} x(t_0+h) &= x_0 + v_0 h + \frac{1}{m} f(x_0, v_0, t_0)\frac{h^2}{2} + \frac{1}{m}\left[\frac{\partial f}{\partial x}(x_0, v_0, t_0) v_0 + \right. \\ &+ \left. \frac{\partial f}{\partial v}(x_0, v_0, t_0)\frac{1}{m} f(x_0, v_0, t_0) + \frac{\partial f}{\partial t}(x_0, v_0, t_0)\right]\frac{h^3}{6} + O(h^4)\ . \end{aligned} \tag{1.3}$$

Hint: start with the identity

$$x(t_0 + h) = x(t_0) + \int_{t_0}^{t_0+h} \dot{x}(s_1)\ \mathrm{d}s_1\ ;$$

Go on by writing $\dot{x}(s_1) = \dot{x}(t_0) + \int_{t_0}^{s_1} \ddot{x}(s_2)\mathrm{d}s_2$. You need other two steps. Can you write down explicitly the remainder $O(h^4)$?

The Newton equation (1.1) can be rewritten in the equivalent first order form

$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m} f(x, v, t) \end{cases} . \tag{1.4}$$

The mathematical problem of solving (1.1), or (1.4), once the initial conditions $x(t_0)$ and $\dot{x}(t_0)$ are specified, thus determining the motion of the particle, is named Cauchy problem, or initial value problem. The first order form (1.4) of the Newton equation clarifies that the state of a particle is determined by its position and its velocity at the same time: $(x(t), v(t))$. The space of states is thus $\mathbb{R}^d \times \mathbb{R}^d$, i.e. the space of ordered pairs position-velocity. The space of states is also called, in jargon, the *phase space*, and the curve $t \mapsto (x(t), v(t))$ is called *phase curve*, to be distinguished from the trajectory, or orbit, i.e. its projection $t \mapsto x(t)$ onto the physical space of positions. The right hand side of (1.4) is called the *vector field* of the system, which is the tangent vector to the phase curve passing through (x, v) at time t. It may be useful to rewrite system (1.4) in the differential form

$$\begin{cases} \mathrm{d}x = v \ \mathrm{d}t \\ \mathrm{d}v = \frac{1}{m} f(x, v, t) \ \mathrm{d}t \ . \end{cases} \tag{1.5}$$

The meaning of (1.5) is the following: in a time $\mathrm{d}t$ the state (x, v) of the particle evolves to $(x', v') = (x + \mathrm{d}x, v + \mathrm{d}v) = (x + v\mathrm{d}t, v + (f/m)\mathrm{d}t)$.

Remark 1.2. *By* $\mathrm{d}x(t) = v(t)\mathrm{d}t$ *we mean here a shorthand notation for the standard Taylor expansion* $\Delta x(t) := x(t+\Delta t) - x(t) = v(t)\Delta t + o(\Delta t)$ *as* $\Delta t \to 0$*; the same for* $\mathrm{d}v(t)$*. Writing the Newton equation in the differential form (1.5) is not only convenient, but becomes mandatory when, for example, the force contains stochastic terms, as in the theory of Brownian motion, where derivatives become rigorously meaningless. If one replaces the differential symbol* d *with the finite difference symbol* Δ *in (1.5), which is equivalent to neglect the* $o(\Delta t)$ *remainder in the first order Taylor expansion, one gets a (first, rough) numerical algorithm to be implemented on a computer in order to numerically solve the initial value problem.*

►**Problem 1.2.** *Consider, for* $d = 1$*, the finite difference approximation of system (1.5) by replacing* $\mathrm{d}x$ *and* $\mathrm{d}t$ *with* Δx *and* $h := \Delta t$*, respectively. You get*

$$\begin{cases} x(t+h) = x(t) + v(t)h \\ v(t+h) = v(t) + \frac{1}{m} f(x(t), v(t), t)h \end{cases} ; \quad t = 0, h, \ldots, nh \ . \tag{1.6}$$

This is the so called Euler algorithm, the simplest one to numerically integrate the Newton dynamics on a computer. The initial conditions are $x_0 := x(0)$ *and* $v_0 := v(0)$*. Setting* $t = 0$ *in (1.6), you can determine* $x_1 := x(h)$ *and* $v_1 := v(h)$*. More in general, substituting into (1.6)* $t = (k-1)h$*, you can rewrite it as follows:*

$$\begin{cases} x_k = x_{k-1} + v_{k-1}h \\ v_k = v_{k-1} + \frac{1}{m} f_{k-1}h \end{cases} ; \quad k = 1, \ldots, n \ , \tag{1.7}$$

where $x_k := x(kh)$*,* $v_k := v(kh)$ *and* $f_{k-1} := f(x_{k-1}, v_{k-1}, (k-1)h)$*. Now, system (1.7) is in the form of a recurrent sequence, or iterated map, namely* $(x_k, v_k) = M(x_{k-1}, v_{k-1})$*, which is* *what a computer is able to solve: compute the right hand side, get the new value, reinsert it on the right and side and so on for a finite number n of steps. Now, fix the time-step h and the number of steps n such that $nh = T$, a prefixed time.*

1. *Solve analytically system (1.7) for the three cases: $f = 0$, $f = -mg$, $f = -kx$, i.e. get an explicit formula for x_k and v_k.*

2. *You could make use of a computer to solve (1.7). Play the game choosing h small enough and n large enough. Plot the sequence $\{x_k, v_k\}_{k=1,\dots,n}$ in the phase plane (x, v), for the three cases. What about the third one?*

3. *Solve the "real" Newtonian system (1.4) in the three cases, getting explicit formulas for $x_{real}(t)$ and $v_{real}(t)$ in terms of the initial conditions x_0 and v_0.*

4. *Compute the exact errors $E_x = |x_{real}(T) - x_n|$ and $E_v = |v_{real} - v_n|$ for the three cases. What general conclusion can be drawn? What about the limit of the errors as $n \to \infty$, or $h \to 0$ at fixed $T = nh$?*

Hint: in the third case, $f = -kx$, you might consider to write the recurrent map for the complex variable $z_k := (\omega x_k + \imath v_k)/\sqrt{2}$, where $\omega := \sqrt{k/m}$ and $\imath$ is the imaginary unit.

Excellent references on ODEs and much of the mathematics around them are [2] and [23]. Perhaps the best reference on numerical algorithms for ODEs is [22].

1.2 Force models and their properties

Solving "explicitly" the Newton equation (1.1) is certainly not possible in the general case. However, in certain special cases where the force displays a particular dependence on its arguments. As an example, if the force f does not depend explicitly on time, i.e. $\partial f/\partial t = 0$, the Newton equation (1.1), or (1.4), is said to be *autonomous*, as opposed to the general, non-autonomous case. Such a uniformity of the force with respect to time may help solving some problems in some particular cases, when the force is also independent of the position x, or when the force is independent of the velocity, or when the dimension of the problem is $d = 1$. In such cases, also depending on the space dimension d of the problem, one can determine, completely or in part, the main features of the motions. Some relevant cases are analyzed below.

1.2.1 Spatially uniform forces

This is the case of forces dependent only on the velocity (and time). In such a case, the second of Newton equations becomes a vector valued, first order ODE for the velocity: $m\dot{v} = f(v, t)$. If the latter equation can be solved for $v(t)$, the motion is then determined by an integration with respect to time: $x(t) = x(t_0) + \int_{t_0}^{t} v(s)\mathrm{d}s$. Such a procedure works, for example, when $f(v, t)$ is a linear (non homogeneous, in general) function of the velocity. An interesting solvable sub-case is the autonomous one-dimensional case: $f(v)$, $d = 1$. In the latter case the velocity equation can be solved by separation of variables:

$$\dot{v} = \frac{1}{m} f(v) := g(v) \; ; \quad \Leftrightarrow \quad \frac{\mathrm{d}v}{g(v)} = \mathrm{d}t \; ; \quad \Leftrightarrow \quad F(v) = t + c \; ,$$

the last step being obtained by indefinite integration of both sides, with F a primitive of $1/g$. In those intervals where F is monotonic, one finally gets $v(t) = F^{-1}(t+c)$. However, in general one will not be able to find the function F or, even if able, will not be able to invert it. Moreover, also supposing that a closed formula in terms of elementary functions has been obtained, this will be unreadable but for a few trivial cases, and extracting simple informations concerning the behaviour of the solution may turn out to be very difficult.

Remark 1.3. *In fact, a much more effective technique works here. Indeed, it is sufficient to draw the graph of $g(v) = f(v)/m$. Then, the equation $\dot{v} = f(v)/m$ tells us that $v(t)$ is a monotonically increasing (decreasing) function of time where $f(v) > 0$ ($f(v) < 0$, respectively), and that the zeroes of f are constant velocity motions. In particular, if v_* is a zero of f such that $f(v) > 0$ for $v < v_*$, and $f(v) < 0$ for $v > v_*$, then v_* is locally attracting, i.e. $\lim_{t\to+\infty} v(t) = v_*$ for all initial conditions $v(0)$ close enough to v_*.*

1.2.2 Positional, potential and conservative forces

Positional forces are independent of the velocity. In such a case, the Newton equation reads $m\ddot{x} = f(x,t)$. Here too, if $f(x,t)$ is a linear (non homogenous) function of x, for example, the problem can be solved. Relevant sub-cases are the following.

- *Potential forces* are those positional forces that are minus the gradient of a given scalar function $U(x,t)$:

  $$f(x,t) = -\nabla_x U(x,t) = -\frac{\partial U}{\partial x}(x,t) \; . \tag{1.8}$$

  The scalar function U is named *potential energy.* Such a condition is not sufficient, in general, to solve the Newton equation. However, large part of physics rests on such an assumption, which is the standard one in both classical and quantum mechanics.

- *Conservative forces* are potential forces independent of time:

  $$f(x) = -\nabla_x U(x) \; . \tag{1.9}$$

  The jargon is due to the fact that in such a case there exists a conservation law, namely the *energy conservation law.* Indeed, the following important theorem holds.

Proposition 1.1. *The* total energy function

$$H(x,\dot{x}) := K + U := \frac{m|\dot{x}|^2}{2} + U(x) \ , \tag{1.10}$$

sum of the kinetic energy $K := m|v|^2/2$ and of the potential energy $U(x)$, is constant along the solutions of the Newton equation $m\ddot{x} = -\nabla_x U(x)$. The same holds for $m\ddot{x} = f$, for any force of the form $f = -\nabla_x U(x) + \dot{x} \times b$, where $b(x,\dot{x},t)$ is any vector valued function of its arguments. The constant energy value E is determined by the initial data: $E = H(x(0),\dot{x}(0))$.

Proof. The time derivative of $H(x(t),\dot{x}(t))$, by the chain rule and making use of the Newton equation $m\ddot{x} = f$, reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(x(t),\dot{x}(t)) &= \frac{\partial H}{\partial x}\cdot\dot{x} + \frac{\partial H}{\partial \dot{x}}\cdot\ddot{x} = \frac{\partial U}{\partial x}\cdot\dot{x} + m\dot{x}\cdot\frac{1}{m}f = \\ &= \dot{x}\cdot\left(f + \frac{\partial U}{\partial x}\right) \equiv 0 \end{aligned} \tag{1.11}$$

under the hypotheses made. Thus $H(x(t),\dot{x}(t)) = H(x(0),\dot{x}(0)) := E$. □

The geometric consequence of this theorem is that the phase curve of a particle subject to a conservative force lies on the sub-manifod of the phase space determined by the equation $H(x,v) = E$, i.e. the set $H^{-1}(E) = \{(x,v) : H(x,v) = E\}$, the value E of the energy being determined by the initial condition: $E = H(x(0),\dot{x}(0))$. Such a conclusion can be also understood by checking that the gradient of H is orthogonal to the vector field of the Newton system, namely

$$\begin{pmatrix} \nabla_x U \\ mv \end{pmatrix} \cdot \begin{pmatrix} v \\ -\frac{1}{m}\nabla_x U + \frac{1}{m} v \times b \end{pmatrix} = 0$$

The constant energy manifold, embedded in the $2d$-dimensional phase space, has dimension $2d-1$, or co-dimesdion 1.

Remark 1.4. *In particular, if $d = 1$, the constant energy manifold $H^{-1}(E)$ consists of (the union of) curves, which implies that motions can be determined geometrically.*

Force terms orthogonal to the velocity of the particle are called gyroscopic. One example is the magnetic part of the Lorentz force, i.e. $(q/c)v \times B$, acting on a charged particle (charge q) moving in a given magnetic field B. Another example is the Coriolis "apparent" force, i.e. $2mv \times \omega$, acting on a particle moving in a non inertial frame rotating with instantaneous angular velocity ω.

1.2.3 Central forces: angular momentum and areal velocity

Central forces are those forces parallel to the vector position x when this is referred to a particular origin O, called the center of the force, namely:

$$f(x, v, t) = \psi(x, v, t)x = \varphi(x, v, t)\hat{x} \ , \tag{1.12}$$

where $\psi, \varphi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are given scalar functions, $\varphi(x, v, t) = \psi(x, v, t)|x|$ and $\hat{x} = x/|x|$. In such a case there exists another conservation law.

Proposition 1.2. *The vector valued function* angular momentum

$$\ell(x, \dot{x}) := x \times m\dot{x} \tag{1.13}$$

is constant along the solutions of the Newton equation $m\ddot{x} = \varphi\hat{x}$. In such a case the trajectory of the particle in the physical space $\mathbb{R}^d$ lies on the plane orthogonal to $\ell = x(0) \times m\dot{x}(0)$ if the latter is different from zero. The invariant plane of motion has equation $\ell \cdot x = 0$. If $x(0)$ is parallel to $\dot{x}(0)$, then $\ell = 0$ for all times, and $x(t)$ is parallel to $\dot{x}(t)$ for any t, the motion taking place on a line.

Proof. Taking the time derivative of the angular momentum (1.13) along the solutions of the Newton equation $m\ddot{x} = f$ (any f for the moment), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell(x(t), \dot{x}(t)) = \dot{x} \times m\dot{x} + x \times m\ddot{x} = x \times f \ .$$

The latter quantity, called the moment of the force, is zero if (and actually only if) f is a central force, i.e. $f = \varphi\hat{x}$. Since $x(t) \cdot \ell(0) = x(t) \cdot \ell(t) \equiv 0$, the orbit lies on the plane of equation $\ell \cdot x = 0$, orthogonal to ℓ and passing through the center of force, if $\ell \neq 0$. If $\ell = 0$, then $x(t)$ is parallel to $\dot{x}(t)$ for all times, which means that the motion takes plane on the line connecting the particle with the center of force at the initial time. □

A significant property of central forces is that the vector position $x(t)$ sweeps equal areas in equal times, or, which is the same, that the area swept by $x(t)$ increases linearly with time. When referred to planetary motions, this is called the second Kepler law, but is actually a property of all central motions, i.e. motions of particles subject to a central force, and is a consequence of the law of conservation of angular momentum.

►**Problem 1.3.** *Show that the area $A(t)$ swept by the vector position satisfies*

$$\dot{A}(t) = \frac{|\ell(t)|}{2m} \ , \tag{1.14}$$

whatever the motion of the particle, i.e. whatever be the force f. Then, in particular, for central forces $\ell(t) = \ell(0)$ and the areal velocity $\dot{A}$ is constant, i.e. $A(t) = A(0) + \frac{|\ell|}{2m}t$. Hint: start by observing that, by the definition of cross product, the area swept by the vector position in a time $\mathrm{d}t$ *is* $\mathrm{d}A = |x \times \mathrm{d}x|/2$.

1.2.4 Central, positional, autonomous conservative forces

Among the central forces, a special role is played by those forces of the form

$$f(x) = \varphi(|x|)\hat{x} = -\nabla U(|x|) \; ; \;\; U(|x|) = -\int \varphi(s)\, \mathrm{d}s\Big|_{s=|x|} \; . \tag{1.15}$$

Forces of this kind are, among others, the Newton-Hooke gravitational force $\varphi(r) = -GMm/r^2$, where M is the mass of the star placed at the center of force, m the mass of the planet, and G the Newton gravitational constant; the Coulomb electrostatic force $\varphi(r) = qQ/r^2$, where q is the charge of the particle and Q is the other charge placed at the center of force; the Hooke elastic force $\varphi(r) = -kr$, k being the elastic constant. The spherical symmetry of the force (1.15) implies that the Newton equation $m\ddot{x} = f(x)$ is invariant under any rotation R: if $x = R\xi$, then $f(R\xi) = Rf(\xi)$ (why?), and $\ddot{\xi} = f(\xi)$. Recall that $R^T R = \mathbf{1}_d$, where the superscript T denotes transposition, whereas $\mathbf{1}_d$ is the $d \times d$ identity.

Exercise 1.1. *Prove the right hand side of (1.15), i.e. that if $f(x) = \varphi(|x|)\hat{x}$ then there exists $U(|x|)$ such that $f = -\nabla U$. Hint: show first that $\nabla U(|x|) = U'(|x|)\hat{x}$; recall that $|x|^2 = x_1^2 + \cdots + x_d^2$.*

Exercise 1.2. *Show that the hypothesis of spherical symmetry can be removed: a central (positional, autonomous) conservative force is necessarily spherically symmetric. In other words, you have to prove that $\varphi(x)\hat{x} = -\nabla U$ implies that φ, and thus U, are functions of $|x|$ only. Hint: ($d = 3$) pass to spherical polar coordinates (r, θ, ϕ) and show, making use of the chain rule, that $\partial U/\partial\theta = \partial U/\partial\phi = 0$. What about $d = 2$?*

Central, positional, autonomous, conservative (spherically symmetric) forces are extremely interesting because in this case both energy and angular momentum are constants of motion, and this in turn allows to reduce the study of the motion to a one-dimensional Newtonian problem (that of the radial motion). Thus, such motions are “integrable”, i.e. solvable (in a geometric sense, in general, i.e. without explicitly solving the Newton equation, in general).

1.3 Problems of single particle motion

In this Section some physically relevant Newtonian problems are presented, grouped together according to the classification made above.

1.3.1 The 3D harmonic oscillator

The first important example of central conservative motion is that of a particle of mass m attracted by the center of force by the Hooke elastic force, i.e. connected to the origin by an ideal spring of zero rest length. The Newton equation is

$$m\ddot{x} = -kx \; , \quad x \in \mathbb{R}^3 \; , \tag{1.16}$$

where $k > 0$ is the elastic constant (we will also comment about the case $k < 0$; see below). In the notations introduced above, the force is of the form $f = \varphi(|x|)\hat{x}$, where $\varphi(|x|) = -k|x|$ (recall that $x = |x|\hat{x}$). By defining $\omega = \sqrt{k/m}$ one can rewrite (1.16) in the form $\ddot{x} = -\omega^2 x$, i.e. the 3D version of the harmonic oscillator equation. The problem we want to solve here, is to determine the form of the orbit, i.e. the form of the curve $t \mapsto x(t)$.

Due to the conservation of the angular momentum $\ell = x \times m\dot{x}$, the motion takes place on the plane of equation $\ell \cdot x = 0$. We can then rotate the axes of the reference frame in such a way to bring the unit vector $\hat{e}_3$ to coincide with $\hat{\ell}$. Thus, x and $\dot{x}$ have two components only (with respect to an arbitrary basis $\hat{e}_1, \hat{e}_2$ on the plane of motion), and (1.16), by components, reads

$$\begin{cases} \ddot{x}_1 = -\omega^2 x_1 \\ \ddot{x}_2 = -\omega^2 x_2 \end{cases} ; \quad \omega = \sqrt{k/m} \ . \tag{1.17}$$

These are two equations of 1D harmonic oscillator, and their general solutions read

$$\begin{cases} x_1(t) = a\cos(\omega t + \varphi) \\ x_2(t) = b\cos(\omega t + \psi) \end{cases} , \tag{1.18}$$

where a, b, φ, ψ are arbitrary constants. With no loss of generality we can choose the amplitudes $a, b > 0$ (one can always fix a sign playing with the phases φ and ψ. Now, defining $\delta := \psi - \varphi$, and $\theta := \omega t + \varphi$, we rewrite (1.18) as

$$\begin{cases} x_1(t) = a\cos(\theta) \\ x_2(t) = b\cos(\theta + \delta) \end{cases} . \tag{1.19}$$

This is clearly the equation of a 2π-periodic plane curve in parametric form (the parameter is the angle θ). We are going to show that such a curve is precisely an ellipse whose axes are rotated, in general, with respect to the canonical ones.

As a preliminary remark, we observe that when $\delta = 0$, (1.18) describes a segment on the line of Cartesian equation $x_2 = (b/a)x_1$, the segment being determined by the condition $|x_1 \leq a$, which implies $|x_2| \leq b$; both of them being due to $|\cos\theta| \leq 1$. Thus the motion corresponding to $\delta = 0$ is linear. The same holds for $\delta = \pi$, where the line equation becomes $x_2 = -(b/a)x_1$. The cases of linear motion $\delta = 0, \pi$ (and thus $\delta = n\pi$, for any integer n) must corresponds to $\ell = 0$, when the 3D equation (1.16) degenerates into an identical 1D equation. In the sequel we will suppose $\ell \neq 0$.

By eliminating $\cos\theta$ and $\sin\theta$ in (1.19) we get

$$\frac{x_1^2}{a^2\sin^2\delta} + \frac{x_2^2}{b^2\sin^2\delta} - \frac{2x_1x_2\cos\delta}{ab\sin^2\delta} = 1 \ , \tag{1.20}$$

that can be rewritten in the form $x \cdot \mathsf{M}x = 1$; explicitly

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} \frac{1}{a^2\sin^2\delta} & \frac{-\cos\delta}{ab\sin^2\delta} \\ \frac{-\cos\delta}{ab\sin^2\delta} & \frac{1}{b^2\sin^2\delta} \end{pmatrix}}_{\mathsf{M}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \ . \tag{1.21}$$

The matrix M defining the homogeneous quadratic form on the right hand side is real symmetric, and by the spectral theorem [15] is orthogonally similar to a real diagonal one: there exists a rotation matrix R, $\mathsf{R}^T\mathsf{R} = \mathbb{1}_2$, such that $\mathsf{R}^T\mathsf{M}\mathsf{R} = \mathrm{diag}(\lambda_-, \lambda_+)$, where λ_- and λ_+ are the two real eigenvalues of M, whereas the two columns of R are the corresponding, mutually orthogonal, unit eigenvectors of u^- and u^+ (in this order). As a consequence, the change of coordinates $y = \mathsf{R}^T x$ diagonalizes the quadratic form $x \cdot \mathsf{M}x$. Indeed, upon substituting $x = \mathsf{R}y$ in the latter expression, we get

$$(\mathsf{R}y) \cdot \mathsf{M}\mathsf{R}y = y \cdot (\mathsf{R}^T\mathsf{M}\mathsf{R})y = y \cdot \mathrm{diag}(\lambda_-, \lambda_+)y \ ,$$

i.e. we map equation (1.21) into

$$\lambda_- y_1^2 + \lambda_+ y_2^2 = 1 \ . \tag{1.22}$$

This is the Cartesian canonical equation of a conic section. The possible cases are: an ellipse, if λ_- and λ_+ are both positive, or a hyperbola, if one is positive and the other one is negative (both negative is impossible). The latter case is not expected on physical grounds: the force is attractive and grows with the distance from the center, which excludes unbounded motions. In fact, by a quick inspection of the matrix M, one gets $\mathrm{tr}(\mathsf{M}) = \lambda_- + \lambda_+ > 0$, $\det(\mathsf{M})_= \lambda_-\lambda_+ > 0$, which imply $0 < \lambda_- < \lambda_+$, so that (1.22) is the equation of an ellipse whose lengths of the major and minor semi-axes are $1/\sqrt{\lambda_-}$ and $1/\sqrt{\lambda_+}$, respectively. The explicit computation of the eigenvalues yields

$$\lambda_\pm = \frac{a^2 + b^2 \pm \sqrt{(a^2+b^2)^2 - 4a^2b^2\sin^2\delta}}{2a^2b^2\sin^2\delta} \ . \tag{1.23}$$

Exercise 1.3. *Check that $\lambda_+ > \lambda_- > 0$.*

Remark 1.5. *Concerning the form of the matrix R whose (ordered) columns are the eigenvectors u^- and u^+, one can obviously compute it. On the other hand, the unit vector $u^- \in \mathbb{R}^2$ can always be written in the form $u^- = (\cos\phi, \sin\phi)^T$. The only choices for the unit vector u^+, orthogonal to u^-, are $u^+ = \pm(-\sin\phi, \cos\phi)^T$ (check it). The two choices of sign yield*

$$\mathsf{R}_+(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \quad , \quad \mathsf{R}_-(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \ .$$

One has $\det(\mathsf{R}_+) = +1$ and, in particular, $\mathsf{R}_+(0) = \mathbb{1}_2 = \mathrm{diag}(1,1)$. On the other hand, $\det(\mathsf{R}_-) = -1$ and $\mathsf{R}_-(\phi)$ does not coincide with the identity for any value of ϕ; in particular, $\mathsf{R}_-(0) = \mathrm{diag}(1,-1)$. Recall that the relation $\mathsf{R}^T\mathsf{R} = \mathbb{1}_2$ implies $\det(\mathsf{R}) = \pm 1$. Thus, proper rotations are orthogonal matrices with determinant $+1$: they rotate the Cartesian axes preserving their mutual initial orientation. Orthogonal matrices with determinant -1 correspond to the initial inversion of an axis (the second one), followed by a proper rotation: $\mathsf{R}_-(\phi) = \mathsf{R}_+(\phi)\mathrm{diag}(1,-1)$ (check it).

►**Problem 1.4.** *Prove that unbounded solutions of (1.16) are forbidden, a priori, by the energy conservation law (reduction to plane motion can be used, but is not necessary).*

►**Problem 1.5.** *Suppose to reverse the sign of the elastic constant k in (1.16). This is the equation of the 3D hyperbolic oscillator (but nothing oscillates), or harmonic repulsor (but there is nothing harmonic here). Set $k = -|k|$ and call $\omega = \sqrt{|k|/m}$. You arrive at (1.17) with a plus sign on the right hand side. The solution of the two equations can be written in the form $x_i(t) = a_i e^{\omega t} + b_i e^{-\omega t}$, $i = 1, 2$ (check it). Eliminate the exponentials and get an equation of the form $x \cdot \mathsf{M}x = 1$. Prove that this can be diagonalized to the canonical equation of a hyperbola. Hint: $e^{\omega t}e^{-\omega t} = 1$. What about the energy conservation law in this case?*

The conclusion of this section is the following: *the orbits of problem (1.16) are ellipses if $k > 0$ and hyperbolas if $k < 0$.*

Exercise 1.4. *Start from the geometric definition of ellipse as the locus of points of the plane such that the sum of their distances d_1 and d_2 from two fixed points F_1 and F_2, called foci, is constant: $d_1 + d_2 = 2a$, with the focal distance $\overline{F_1F_2} = 2f$. Set the foci F_2 and F_1 on the x axis of the Cartesian plane at $(f, 0)$ and $(-f, 0)$, respectively. First, check that when the point is on the y axis its distance b from the origin is given by $b^2 = a^2 - f^2$. Now, check that the distances of a point $P = (x, y)$ on the locus are given by $d_1^2 = (x - f)^2 + y^2$, $d_2^2 = (x + f)^2 + y^2$, respectively. Now obtain the Cartesian canonical equation of the ellipse, namely*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ , \tag{1.24}$$

where a and b are the lengths of the major and minor semi-axes, respectively. Hint: square $d_1 + d_2 = 2a$, isolate $2d_1d_2$ and square again.

Exercise 1.5. *Set $x = f + r\cos\theta$ and $y = r\sin\theta$ into (1.24), so that r is the distance from the focus F_1 and θ the inclination angle of F_1P with respect to the x axis (counterclockwise). Now obtain the polar equation of the ellipse, namely*

$$r = \frac{p}{1 + \varepsilon\cos\theta} \ , \tag{1.25}$$

where the quantities

$$\varepsilon := \frac{f}{a} \quad ; \quad p := \frac{a^2 - f^2}{a} \tag{1.26}$$

are called eccentricity (ε) and parameter (p) of the ellipse, respectively. Hint: substitute $x = f + r\cos\theta, y = r\sin\theta$ in (1.24) written in the form $(a^2 - f^2)x^2 + a^2y^2 = a^2(a^2 - f^2)$; solve for r; recall that $r > 0$.

Remark 1.6. *Equation (1.34) defines all the conic sections. When $\varepsilon = 0$, (1.34) is the equation of a circle: $r(\theta) = p$. For proper ellipses, $0 < \varepsilon = f/a < 1$. When $\varepsilon = 1$, (1.34) is the polar equation of a parabola. Finally, if $\varepsilon > 1$ (1.34) defines hyperbolas. Observe that when $\varepsilon \geq 1$ the denominator on the right hand side of (1.34) vanishes at at two finite angles: $\pm\pi$ at $\varepsilon = 1$ and $\pm\hat{\theta}(\varepsilon)$ at $\varepsilon > 1$. In the latter case the angle $\pm\hat{\theta}$ are the inclination angles of the asymptotes of the hyperbola (one branch). See [14] for further details.*

For a complete introduction to the geometry and physical relevance of the conic sections (and much more) see [4].

1.3.2 The Kepler-Newton problem

The so called Kepler-Newton problem consists in the study of the motion of a particle of mass m subject to a conservative central force inversely proportional to the square of the distance from the center, namely

$$m\ddot{x} = -\frac{k}{|x|^2}\,\hat{x} = -\frac{k}{|x|^3}\,x \ , \quad x \in \mathbb{R}^3 \ , \tag{1.27}$$

where $k \neq 0$ is a constant. The physically relevant cases of attractive force, with $k > 0$, are the gravitational one, where $k = GMm$, G being the gravitational Newton constant, m the mass of a planet and M the mass of a star (at rest in the center), or the Coulomb one, where, for example, $k = e^2$, e being the common charge of the electron and the proton, which is the case of the Hydrogen atom (with the proton at rest in the center). A repulsive force, with $k < 0$, rules the dynamics of a particle of given charge q repelled by a center of equal charge q; here $k = -q^2$.

The (invention and) study of problem (1.27) was motivated by the three empirical laws induced by Kepler on the basis of the analysis of the data collected by Brahe. The three Kepler laws are the following.

1. Planets move on elliptical orbits, with the Sun in one of the foci.

2. The vector Sun-planet sweeps equal areas in equal times (constant areal velocity).

3. If, for any planet, T and a denote the revolution period and the length of the orbital major semi-axis, then the ratio T^2/a^3 is a constant, independent of the planet.

Modern data relative to the orbits in the solar system are reported in the Table below.

Planet	a [A.U.]	T (years)	ε	α	m_P/m_S
Mercury	0.39	0.24	0.205	7^o00	$1.6\ 10^{-7}$
Venus	0.72	0.61	0.006	$3^o23'$	$2.4\ 10^{-6}$
Earth	1	1	0.016	$0^o00'$	$3.0\ 10^{-6}$
Mars	1.52	1.88	0.093	$1^o51'$	$3.2\ 10^{-7}$
Jupiter	5.20	11.83	0.048	$1^o18'$	$0.9\ 10^{-3}$
Saturn	9.55	29.43	0.055	$2^o29'$	$2.8\ 10^{-4}$
Uranus	19.22	84.18	0.046	$0^o46'$	$4.4\ 10^{-5}$
Neptune	30.11	164.56	0.008	$1^o46'$	$0.5\ 10^{-4}$
Pluto	39.60	247.47	0.246	$17^o07'$	$1.7\ 10^{-6}$

Table 1.1: In the present table, for each planet, a is the major semi-axis length, in astronomic units (A.U.), i.e. referred to the value of a for the Earth; T is the revolution period, in years (again referred to the value of T for the Earth); ε is the eccentricity, α the inclination angle of the orbital plane with respect to that of the Earth; m_P/m_S is the ratio mass of the planet to mass of the Sun.

Exercise 1.6. *Plot T versus a in log-log scale. Check whether the data are compatible with a fitting line of slope* $3/2$.

In what follows, we study equation (1.27) with $k > 0$ under the hypothesis $\ell \neq 0$, with the aim of deducing (and extend) the three Kepler laws. As in the case of the 3D harmonic oscillator, we rotate the reference system in such a way that $\hat{e}_3 = \hat{\ell}$. On the plane of motion, orthogonal to ℓ, we could rewrite equation (1.27) in Cartesian (two) components. However, to our purposes here, this is not the convenient choice. Instead, we pass to plane polar coordinates. We attach to each point in the plane a local, orthonormal polar basis, defined as follows. Consider the transformation to plane polar coordinates defined by

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases} . \tag{1.28}$$

Then the unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ tangent to the coordinate lines $\theta = \textit{constant}$ and $r = \textit{constant}$, respectively, are given by

$$\hat{e}_r := \frac{\partial x}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \; ; \; \hat{e}_\theta = \frac{1}{r}\frac{\partial x}{\partial \theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} . \tag{1.29}$$

Observe that the vectors $\hat{e}_r$ and $\hat{e}_\theta$ depend only on the angle θ; moreover, the the polar basis $\hat{e}_r, \hat{e}_\theta$ preserves the orientation of the plane: $\hat{e}_r(0) = \hat{e}_1 = (1,0)^T$, $\hat{e}_\theta(0) = \hat{e}_2 = (0,1)^T$. One easily checks the fundamental relations:

$$\frac{\mathrm{d}\hat{e}_r}{\mathrm{d}\theta} = \hat{e}_\theta \; ; \; \frac{\mathrm{d}\hat{e}_\theta}{\mathrm{d}\theta} = -\hat{e}_r . \tag{1.30}$$

We now start from the obvious relation $x = r\hat{e}_r(\theta)$ (we are just rewriting (1.28) making use of the expression (1.29) of $\hat{e}_r$). Along the motions, i.e. the solutions of (1.27), r and θ depend on time. Thus, starting from $x(t) = r(t)\hat{e}_r(\theta(t))$, one easily finds

$$\dot{x} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \; ; \tag{1.31}$$

$$\ddot{x} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta . \tag{1.32}$$

Such a computation makes use of the chain rule together with repeated use of (1.30). Upon substitution of (1.32) into the left hand side of (1.27), now thought of in the plane ($x \in \mathbb{R}^2$), one gets

$$m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta = -\frac{k}{r^2}\,\hat{e}_r ,$$

i.e., since $\hat{e}_r$ and $\hat{e}_\theta$ are linearly independent,

$$\begin{cases} m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2} \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \end{cases} . \tag{1.33}$$

This is equation (1.27) written on the plane of motion in plane polar coordinates. The second equation, i.e. the θ component, expresses nothing but the conservation of the angular momentum. Indeed, multiplying it by r, one gets

$$0 = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) .$$

On the other hand, the angular momentum in polar coordinates reads

$$\ell = x \times m\dot{x} = r\hat{e}_r \times m(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = mr^2\dot{\theta}\hat{e}^3 \ , \tag{1.34}$$

where $\hat{e}_3 = \hat{\ell} = \hat{e}_r \times \hat{e}_\theta$. Thus, $mr^2\dot{\theta} = \ell_3$, and system (1.33) can be rewritten in the form

$$\begin{cases} m\ddot{r} = \frac{\ell_3^2}{mr^3} - \frac{k}{r^2} \\ \dot{\theta} = \frac{\ell_3}{mr^2} \end{cases} . \tag{1.35}$$

Remark 1.7. *In what did up to now we have never made use of the specific form of the force $\varphi(r)$. Convince yourself that, for a central force of the form $f = \varphi(r)\hat{e}_r$ (in the plane of motion), one gets, in place of (1.35), the system*

$$\begin{cases} m\ddot{r} = \frac{\ell_3^2}{mr^3} + \varphi(r) \\ \dot{\theta} = \frac{\ell_3}{mr^2} \end{cases} . \tag{1.36}$$

Now the radial equation (the first one) in (1.35) involves only the variable $r(t)$, so that the problem has been reduced to study the one-dimensional radial motion (we will com back below to such an analysis). In principle, solving for $r(t)$, one then determines $\theta(t)$ from the second of equations (1.35). Such a procedure is possible but does not lead to a solution expressed in simple closed form. However, the form of the orbit can be explicitly determined in the polar form $\theta \mapsto r(\theta)$ since, if $\ell_3 > 0$, for example, then $\dot{\theta} > 0$ and the function $t \mapsto \theta(t)$ is invertible, so that $t(\theta)$ exists for all $\theta \in \mathbb{R}$. By the chain rule, and identifying $r(\theta) = r(t(\theta))$, we get

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{mr^2}{\ell_3}\dot{r} \quad \Leftrightarrow \quad \dot{r} = \frac{\ell_3}{mr^2}r'(\theta) \ . \tag{1.37}$$

Of course, in the case $\ell_3 < 0$ we would arrive exactly at the same result, the sign of ℓ_3 determining only the direction of rotation (counter-clockwise if $\ell_3 > 0$). Now, instead of re-applying the chain rule and to express $\ddot{r}$ in terms of $r''(\theta)$, we follow another way, making use of the energy conservation law. The latter law, for equation (1.27), reads

$$\frac{m|\dot{x}|^2}{2} - \frac{k}{|x|} = E \ . \tag{1.38}$$

Substituting inside the latter $x = r\hat{e}_r$ and $\dot{x} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$, and $\dot{\theta}$ from (1.35) (the second equation), yields

$$\frac{m\dot{r}^2}{2} - \frac{k}{r} + \frac{\ell_3^2}{2mr^2} = E \ . \tag{1.39}$$

Observe that such an expression depends only on r and $\dot{r}$. Now, substituting $\dot{r}$ form (1.37), one gets

$$\frac{\ell_3^2}{2m}\left[\frac{(r')^2}{r^4} + \frac{1}{r^2}\right] - \frac{k}{r} = E \ . \tag{1.40}$$

The latter energy conservation law is extremely simplified if one rewrites it in terms of the variable

$$u(\theta) := \frac{1}{r(\theta)} \ , \tag{1.41}$$

which is due to Binet; one gets

$$\frac{\ell_3^2}{2m}\left[(u')^2 + u^2\right] - ku = E \ . \tag{1.42}$$

Taking the derivative of the latter equation with respect to θ, and taking into account that E does not depend on it, and that $u'(\theta)$ cannot be identically zero (i.e. zero on any interval), yields the *Binet equation*

$$u'' + u - \frac{km}{\ell_3^2} = 0 \ . \tag{1.43}$$

The general solution of the latter equation is easily checked to be given by

$$u(\theta) = \frac{km}{\ell_3^2} + A\cos(\theta - \theta_0) \ , \tag{1.44}$$

where A and θ_0 are arbitrary constants. We can choose, without any loss of generality, $A > 0$ and $\theta_0 = 0$ (convince yourself about it). Recalling that $u = 1/r$, we can rewrite (1.44) as

$$r(\theta) = \frac{\frac{\ell_3^2}{km}}{1 + \frac{\ell_3^2 A}{km}\cos\theta} \ , \tag{1.45}$$

i.e. a conic section in polar form, with parameter $p = \ell_3^2/(km)$ and eccentricity $\varepsilon = \ell_3^2 A/(km)$ (go back to Remark 1.6 and what said before it). In order to determine the value of the constant $A > 0$, and thus of the eccentricity in terms of E and ℓ_3 (and thus in terms of the initial conditions), we reinsert (1.44) into the energy conservation law (with $\theta_0 = 0$). Elementary calculations lead to determine

$$A^2 = \frac{k^2m^2}{\ell_3^4} + \frac{2mE}{\ell_3^2} \ ,$$

so that the eccentricity $\varepsilon = \ell_3^2 A/(km)$ of the orbit is given by

$$\varepsilon = \sqrt{1 + \frac{2\ell_3^2 E}{k^2 m}} \ . \tag{1.46}$$

Our first conclusion is that the orbits of the Kepler-Newton problem are conic sections of the form

$$r(\theta) = \frac{\frac{\ell_3^2}{km}}{1 + \sqrt{1 + \frac{2\ell_3^2 E}{k^2 m}}\ \cos\theta} \ . \tag{1.47}$$

For planetary motions ($k = GMm$) this implies the following.

- If $E = -km/(2\ell_3^2)$, $\varepsilon = 0$, $r(\theta) = \ell_3^2/(km)$ is constant and the orbit is circular; the angular velocity is also constant and given by $\dot\theta = \ell_3/(mr^2) = k^2 m/\ell_3^3$.

- If $-km/(2\ell_3^2) < E < 0$, $0 < \varepsilon < 1$ and the orbits are ellipses. This is the first Kepler law (recall that the in the polar form r refers to one of the foci).
- To $E = 0$, i.e. $\varepsilon = 1$, there corresponds a parabolic orbit.
- Finally, if $E > 0$, i.e. $\varepsilon > 1$, the orbits are hyperbolas.

The second Kepler law concerns the constancy of the areal velocity, which has been proven in general, for central forces. We here recall it, namely

$$\dot{A} = \frac{\ell_3}{2m} \quad \Leftrightarrow \quad A(t) = A(0) + \frac{\ell_3}{2m} t \; . \tag{1.48}$$

Consider now the case of elliptic motions. Imagine to set $t = 0$ when $\theta = 0$ (at the orbital perihelion), with $A(0) = 0$. After one complete revolution, denoting by T the revolution period, one has $A(T) = \ell_3 T/(2m)$, where $A(T)$ is the area of the ellipse. Thus

$$\begin{aligned} T &= \frac{2m}{\ell_3}\pi ab = \frac{2\pi m}{\ell_3} a\sqrt{a^2 - f^2} = \frac{2\pi m}{\ell_3} a^{3/2}\sqrt{\frac{a^2 - f^2}{a}} = \\ &= \frac{2\pi m}{\ell_3} a^{3/2}\sqrt{p} = 2\pi\sqrt{\frac{m}{k}}\; a^{3/2} \; . \end{aligned}$$

Recalling that $k = GMm$, we get

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \; , \tag{1.49}$$

i.e. the third Kepler law with an explicit constant which depends only on the mass of the star (the Sun, for example) and is independent of the mass of the planet.

Exercise 1.7. *The Binet equation (1.43) can be obtained by re-applying the chain rule (1.37) to first get $\ddot{r}$ as a function of r, r' and r'', and substitute it into the first of equations (1.35). Then, by setting $r = 1/u$ one obtains the result.*

1.3.3 The eccentricity vector

An alternative way to treat the Kepler-Newton problem rests on the introduction of another vector valued constant of motion, namely what is usually referred to as the Laplace-Runge-Lenz vector, whose discovery perhaps goes back to Laplace and Hamilton, who introduced it independently of each other; see the historical notes [19, 20]. The *eccentricity vector* e, as we will refer to in the sequel, is defined as follows:

$$\mathsf{e} := \frac{1}{k} v \times \ell - \hat{x} \; , \tag{1.50}$$

where reference is made to the Kepler-Newton equation (1.27): k is the Kepler constant, x, $v = \dot{x}$ and $\ell = x \times mv$ being the position, the velocity and the angular momentum of the particle. The following proposition holds.

Proposition 1.3. *The eccentricity vector* e *(1.50) is constant along the solutions of the Kepler Newton equation* $m\ddot{x} = -k\hat{x}/|x|^2$*;* $|\mathsf{e}| = \varepsilon$*, the latter being the orbital eccentricity; for elliptic orbits* e *lies on the focal axes, in the direction of the pericenter.*

Proof. We here recall that $\hat{x} = \hat{e}_r$, $m\dot{v} = -k\hat{e}_r/r^2$, $\ell = mr^2\dot{\theta}\hat{e}_3$, the latter being a constant vector, and $\hat{e}_r, \hat{e}_\theta, \hat{e}_3$ is a right-handed basis. We have

$$\frac{\mathrm{d}\mathsf{e}}{\mathrm{d}t} = \frac{1}{k}\left(-\frac{k}{mr^2}\hat{e}_r\right) \times (mr^2\dot{\theta}\hat{e}_3) - \dot{\theta}\hat{e}_\theta = (\dot{\theta} - \dot{\theta})\hat{e}_\theta \equiv 0 \ .$$

Now, taking into account that $v = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$, writing $\ell = \ell_3\hat{e}_3$, and eliminating $\dot{\theta}$ form $mr^2\dot{\theta} = \ell_3$, the vector e turns out to be given by

$$\mathsf{e} = \left(\frac{\ell_3 r\dot{\theta}}{k} - 1\right)\hat{e}_r - \left(\frac{\ell_3\dot{r}}{k}\right)\hat{e}_\theta = \left(\frac{\ell_3^2}{kmr} - 1\right)\hat{e}_r - \left(\frac{\ell_3\dot{r}}{k}\right)\hat{e}_\theta \ .$$

The square modulus of the latter vector is

$$\begin{aligned} |\mathsf{e}|^2 &= \left(\frac{\ell_3^2}{kmr} - 1\right)^2 + \left(\frac{\ell_3\dot{r}}{k}\right)^2 = \\ &= 1 + \frac{2\ell_3^2}{mk^2}\left(\frac{m\dot{r}^2}{2} - \frac{k}{r} + \frac{\ell_3^2}{2mr^2}\right) = 1 + \frac{2\ell_3^2 E}{mk^2} = \varepsilon^2 \ , \end{aligned}$$

where (1.39) and (1.46) have been used. This proves that $|\mathsf{e}| = \varepsilon$. Now, to end up the proof, we take the scalar product of the eccentricity vector (1.50) with the position vector $x = r\hat{e}_r$. On the left we get

$$\mathsf{e} \cdot x = |\mathsf{e}|\,|x|\cos\theta = \varepsilon\, r\cos\theta \ ,$$

where θ is the angle between e and x (we already give it the same name of the orbital angle because it turns out to be exactly the same). On the other hand,

$$\begin{aligned} \mathsf{e} \cdot x &= \frac{1}{k}(v \times \ell) \cdot x - x \cdot \hat{e}_r = \frac{1}{k}(x \times v) \cdot \ell - r = \\ &= \frac{1}{km}(x \times mv) \cdot \ell - r = \frac{\ell_3^2}{km} - r \ , \end{aligned}$$

where we used the known property of the mixed product $(a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b$ valid for any triple of vectors $a, b, c \in \mathbb{R}^3$. We thus get $\varepsilon\, r\cos\theta = \frac{\ell_3^2}{km} - r$, i.e.

$$r = \frac{\frac{\ell_3^2}{km}}{1 + \varepsilon\cos\theta} \ ,$$

identical to the orbital equation (1.47). This also proves that, having called θ the angle between e and the vector position $x(t)$, the constant eccentricity vector e lies on the focal axis of the ellipse, in the direction of the pericenter. □

1.3.4 Levi-Civita - Bohlin regularization

The 3D harmonic and hyperbolic oscillator, and the Kepler-Newton problem display orbits that are conic sections for all initial conditions. This is not by chance since, from a mathematical point of view, the two problems are one and the same. This was first obtained by Levi-Civita [31], and independently by Bohlin [8] (we here follow the subsequent, more synthetic treatment by Bohlin). Let us rewrite the Kepler-Newton equation in the form

$$\ddot{x} = -\frac{c}{|x|^3}x \ ; \quad c = k/m > 0 \ . \tag{1.51}$$

The energy conservation law reads

$$\frac{|\dot{x}|^2}{2} - \frac{c}{|x|} = \mathcal{E} \ ; \quad \mathcal{E} = E/m \ . \tag{1.52}$$

Let us introduce, in the former case (1.51), the complex variable $z = x_1 + \imath x_2$, where $\imath$ is the imaginary unit. One can then write (1.51) and (1.52) in the form

$$\ddot{z} = -\frac{c}{|z|^3}z \ ; \tag{1.53}$$

$$\frac{|\dot{z}|^2}{2} - \frac{c}{|z|} = \mathcal{E} \ , \tag{1.54}$$

with $z = x_1 + \imath x_2 \in \mathbb{C}$. Observe that $|z|^2 = x_1^2 + x_2^2 = |x|^2$. The Levi-Civita - Bohlin map $\mathbb{C} \to \mathbb{C} : (z, \bar{z}, t) \mapsto (w, \bar{w}, \tau)$ is defined by the relations

$$z = w^2 \quad ; \quad \mathrm{d}t = |w|^2 \mathrm{d}\tau \ . \tag{1.55}$$

By the chain rule, one gets

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{2ww'}{|w|^2} \ ,$$

where $w' = \mathrm{d}w/\mathrm{d}\tau$. Substituting the latter and $z = w^2$ into (1.54), yields

$$2|w'|^2 - \mathcal{E}|w|^2 - c = 0 \ . \tag{1.56}$$

Taking the derivative with respect to τ, and taking into account that both c and $\mathcal{E}$ do not depend on it, yields

$$(2w'' - \mathcal{E}w)\bar{w}' + (2\bar{w}'' - \mathcal{E}\bar{w})w' = 0 \ , \tag{1.57}$$

One easily proves that the above identity holds if and only if

$$w'' = \frac{\mathcal{E}}{2}w \ . \tag{1.58}$$

The latter complex equation, when we write $w = w_1 + \imath w_2$, is equivalent to the real system

$$\begin{cases} w_1'' = (\mathcal{E}/2)w_1 \\ w_2'' = (\mathcal{E}/2)w_2 \ . \end{cases} \tag{1.59}$$

The latter system represents a 2D harmonic oscillator if $\mathcal{E}/2 := -\omega^2 < 0$, and a 2D hyperbolic oscillator if $\mathcal{E}/2 := \omega^2 > 0$. Such a correspondence maps the Kepler-Newton elliptic/ hyperbolic motions into the harmonic/hyperbolic oscillator ones. See also [4] and [5].

Exercise 1.8. *Prove that identity (1.57) holds if and only if (1.58) holds. Hint: the if is trivial; for the only if start by examining the condition* $\mathrm{Re}(z_1\bar{z}_2) = 0$ *for two complex numbers* z_1, z_2.

The Kepler-Newton problem and the harmonic oscillator one are really special. Indeed, a theorem due to Bertrand [7] states that the sole central potentials $U(r)$ all of whose bounded motions consist of closed orbits are the harmonic potential $U(r) = kr^2/2$, and the Kepler one, $U(r) = -k/r$ ($k > 0$ in both cases); see [1] for a complete proof.

1.3.5 1D conservative Newtonian motions

One-dimensional (1D) conservative Newtonian motions are ruled by the Newton differential equation $m\ddot{x} = f(x) = -U'(x)$, where the potential energy $U(x) = -\int f(x)\mathrm{d}x$ always exists, since $x \in \mathbb{R}$. The Newton equation is equivalent to the system

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{1}{m}U'(x) \end{cases} . \tag{1.60}$$

The phase space of the system is the plane $\mathbb{R}^2 \ni (x, v)$.

The simplest (and relevant) solutions of (1.60) are the *equilibria*, or equilibrium points, namely the zeros of the vector field, i.e. those points of the phase space where the right hand side of (1.60) vanishes. Thus the equilibria of 1D conservative systems are of the form

$$(x, v) = (\bar{x}, 0) \ ; \quad f(\bar{x}) = -U'(\bar{x}) = 0 \ ,$$

i.e. they are the points of the x axis (zero velocity) whose abscissae are the critical points of the potential energy (zero force). To the single equilibrium point $(\bar{x}, 0)$ there corresponds the motion $t \mapsto (\bar{x}, 0)$ for all $t \in \mathbb{R}$, i.e. the unique solution of (1.60) corresponding to the initial condition $(\bar{x}, 0)$.

For system (1.60), the total energy $H(x, v)$ is preserved, namely

$$H(x, v) := \frac{mv^2}{2} + U(x) = E \ , \tag{1.61}$$

where $x(t)$ and $v(t)$ is understood in the latter expression. The constant value E of the function H along the *phase curve* $t \mapsto (x(t), v(t))$, i.e. the unique motion corresponding to a specific initial condition $(x(0), v(0))$, is determined by the the latter: $E = H(x(0), v(0))$. One thus concludes that the phase curves, i.e. the possible motions, belong to the E-level set of H, i.e.

$$H^{-1}(E) := \{(x, v) \in \mathbb{R}^2 : \ H(x, v) = E\} \ . \tag{1.62}$$

The latter set consists, when not empty, of one or more connected curves, some of them possibly degenerating into single points. The picture of the level sets (1.62), where one distinguishes the *traces* and the directions of the possible phase curves, or motions, corresponding to all possible values of E, is named *phase portrait, or diagram* of system (1.60). This is the most complete description of the dynamics of the 1D Newtonian system.

Example 1.1. *For the harmonic oscillator* $H = mv^2/2 + kx^2/2 = E \geq 0$. *To* $E = 0$ *there corresponds the origin* $(0,0)$ *of the phase plane, the only equilibrium point. The corresponding motion is the phase curve* $t \mapsto (0,0)$ *for all* $t \in \mathbb{R}$. *The level set for any* $E > 0$ *is an ellipse of canonical equation* $x^2/a^2 + v^2/b^2 = 1$, *where* $a := \sqrt{2E/k}$ *and* $b := \sqrt{2E/m}$ *are the length of the two semi-axes. This corresponds to the periodic motions of the harmonic oscillator, namely* $x(t) = x(0)\cos(\omega t) + [v(0)/\omega]\sin(\omega t)$, *where* $T = 2\pi/\omega$ *is the period of the motion and* $\omega := \sqrt{k/m}$ *the pulsation. The point mass oscillates on the real axis sweeping back and forth a segment of extremes* $x_\pm = \pm a = \pm\sqrt{2E/k}$, *the two* turning points, *where the particle stops and inverts the motion. The ellipse in the phase plane is swept clockwise (why?) once per period* T *of motion. Due to the structure of the phase portrait, the equilibrium point* $(0,0)$ *of the harmonic oscillator is referred to as an elliptic (equilibrium) point, or center.*

Example 1.2. *For the harmonic repulsor, or hyperbolic oscillator, defined by the Newton equation* $m\ddot{x} = +kx$ $(k > 0)$, *the energy function is* $H(x, v) = mv^2/2 - kx^2/2$. *The level set* $H^{-1}(E)$ *is non empty for any* $E \in \mathbb{R}$. *In particular,* $H^{-1}(0)$ *consists of the two lines* $v = \pm(k/m)x$. *Since such a set contain the equilibrium point* $(0,0)$, *by the uniqueness of the solutions of the Newton system,* $H^{-1}(0)$ *contains five distinct motions, or phase curves, namely the origin and four half-lines, two entering and two exiting the origin. For all values of* $E = -|E| < 0$, $H^{-1}(E)$ *consists of two distinct phase curves, namely the two branches (left and right) of hyperbola of equation* $x^2/a^2 - v^2/b^2 = 1$, *where* $a = \sqrt{2|E|/k}$, $b = \sqrt{2|E|/m}$. *On the other hand, for all values of* $E > 0$, $H^{-1}(E)$ *consists of the two branches (up and down) of hyperbola of equation* $x^2/a^2 - v^2/b^2 = -1$, *where* $a = \sqrt{2E/k}$, $b = \sqrt{2E/m}$. *Observe that* $H^{-1}(0)$ *consists, from a geometrical point of view, of the asymptotes of the hyperbolas. The equilibrium point* $(0,0)$ *in this case is referred to as a hyperbolic (equilibrium) point, or saddle point.*

Exercise 1.9. *Consider the* graph *of the energy function* $H(x, v)$, *i.e. the subset of* $\mathbb{R}^3$ *define by* $\{(x, v, z) : \ z = H(x, v)\}$. *The level sets of* H *are then the intersections of the graph of* H *with the horizontal planes* $z = E$. *Sketch the graphs of* H *for the harmonic and hyperbolic oscillator, realizing that one has a paraboloid with elliptic section in the former case, and a* mountain pass, *or* saddle-shaped *graph in the latter. The origin* $(0,0)$ *of the phase plane (the only equilibrium point in both cases) is called a* center *for the harmonic oscillator, and a* saddle *for the hyperbolic one. You can make use of a computer.*

The particular structure of the energy function (1.61) imposes some restrictions on the structure of the E-level sets of H.

1.3.6 Rules to draw a phase portrait

These are the general rules to draw a phase portrait.

►Suggestion 1.1. *Draw any graph of a function* $U(x)$ *displaying local minima and maxima, and understand each item below on the picture. Draw a corresponding sketch of the phase portrait on the* (x, v) *phase plane reported just below the plane with the graph of* $U(x)$, *with the two* x*-axes parallel.*

- Equilibria lie on the x-axis. The orientation of the phase curves is left-to-right in the upper half plane $\{(x,v): \ v>0\}$, and right to left in the lower half plane $\{(x,v): \ v<0\}$. This follows from $\dot{x}=v$, so that $x(t)$ is monotonically increasing in the upper half phase plane, and monotonically decreasing in the lower one. In particular, all closed curves are clockwise oriented.

- Since $mv^2/2 = E - U(x) \geq 0$, the possible values of x for motions of a given energy value E are those belonging to the E-sub-level set of U, i.e. the allowed values of x are such that $U(x) \leq E$.

- If $U(x)$ displays an absolute minimum value U_m, then E cannot assume values lower than it: for $E < U_m$ there are no possible motions. If U is unbounded from below, then E can take on any real value. Remark: U is defined up to an arbitrary constant, that must be chosen once and for all.

- H is symmetric in v, i.e. $H(x,-v) = H(x,v)$, which implies reflection symmetry of any level set with respect to the x-axis. More precisely, solving (1.61) for v, one gets

$$v_\pm(x) = \pm\sqrt{\frac{2}{m}\left[E-U(x)\right]}\ , \tag{1.63}$$

  which means that to any allowed x there correspond two values of v, $v_+ \geq 0$ and $v_- = -v_+ \leq 0$. The two branches $v_\pm$ may be either connected to each other or disconnected.

- The even symmetry of H in v corresponds to the so-called *time-reversal symmetry* of system (1.60): the equations are invariant by changing (x,v,t) into $(x,-v,-t)$

- The simple zeros of the equation $U(x) = E$ are called *turning points*. Indeed, if $U(\xi) = E$, and $U'(\xi) > 0$ (<0), at the turning point ξ, the particle coming from the left (right) of ξ stops there by the energy conservation law. Starting at the stop time with the initial condition $(\xi, 0)$, the force on the particle is $f(\xi) = -U'(\xi)/m < 0$ (>0), and the particle starts at rest and moves to the left (right).

- The previous point describes the *elastic smooth reflection* of a particle by a potential barrier: when the particle arrives at a turning point it rebounds inverting the direction of motion, assuming at the same positions equal and opposite velocities (this is implied by the $v \to -v$ symmetry of H).

- Taking the derivative of v_+, from (1.63), one gets

$$v'_+(x) = \frac{-U'(x)}{\sqrt{2m\left[E-U(x)\right]}}\ . \tag{1.64}$$

  From this formula one sees that $v_+ \to -\infty$ as $x \to \xi^-$, where ξ is a right turning point such that $U'(\xi) > 0$. Analogously, $v_+ \to +\infty$ as $x \to \xi^+$, where ξ is a left turning point such that $U'(\xi) < 0$. By the reflection symmetry $v \to -v$ it then follows that the tangent to a phase curve at the phase turning point $(\xi, 0)$ is vertical, and the curve is locally smooth.

- Another implication of (1.64) is that, for values of E larger than the local minimum (maximum) value of $U(x)$, the local maximum (minimum) points of $v_+(x)$ coincide with the local minimum (maximum) points of $U(x)$.

- Equilibrium points $(\bar{x}, 0)$ corresponding to nondegenerate critical points of $U(x)$ (i.e. those $\bar{x}$ such that $U'(\bar{x}) = 0$ and $U''(\bar{x}) \neq 0$) are locally elliptic if $U''(\bar{x}) > 0$, and locally hyperbolic if $U''(\bar{x}) < 0$. In other words, the phase portrait around $(\bar{x}, 0)$ resembles, in such two cases, either that of the harmonic or that of the hyperbolic oscillator. This easily follows by the energy conservation law (1.61), expanding $U(x)$ around $\bar{x}$ to second order in $x - \bar{x}$, and taking an energy value $E = U(\bar{x}) + \Delta E$. This yields

$$\frac{mv^2}{2} + \frac{U''(\bar{x})}{2}(x - \bar{x})^2 = \Delta E + O(|\Delta E|^{3/2}) \ .$$

  Neglecting the small remainder on the right hand side, when $|\Delta E|$ is very small, leads to the conclusion by one and the same analysis performed for the harmonic and the hyperbolic oscillator, with $U''(\bar{x})$ and ΔE in place of $\pm k$ and E, respectively.

- To an interval of allowed values of x with extrema two turning points $x_\pm(E)$, two consecutive simple zeros of $U(x) = E$, there corresponds a closed phase curve C_E, i.e. a periodic motion.

- To an interval of allowed values of x with extrema a turning point and a nondegenerate local maximum $\bar{x}$ of $U(x)$ there corresponds a closed curve connecting the equilibrium point $(\bar{x}, 0)$ to itself. Such a closed curve is called *homoclinic connection, or loop*. The time to reach the equilibrium along such a phase curve is infinite. The possible motions of the system in this case are two: the equilibrium one and the homoclinic motion.

- To an interval of allowed values of x with extrema two nondegenerate local maxima $\bar{x}_1$ and $\bar{x}_2$ of $U(x)$, there correspond two symmetric phase curves, one above and one below the x-axis, connecting the equilibria $(\bar{x}_1, 0)$ and $(\bar{x}_2, 0)$. Such curves are called *heteroclinic connections*. The possible motions of the system in this case are four: the two equilibria and the two heteroclinic motions.

1.3.7 Periods and flight times

Concerning the times the particle takes to move along a given phase curve, there are two interesting cases, namely the case of bounded closed curves (periodic motions), and that of unbounded curves.

Consider first a closed phase curve C_E, corresponding to a periodic motion. From the law of conservation of energy it follows that the period of such a motion $T(E)$ is given by

$$T(E) = \oint_{C_E} \frac{\mathrm{d}x}{v} = 2\int_{x_-(E)}^{x_+(E)} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}} \tag{1.65}$$

The area enclosed by the closed phase curve C_E in the (x, p) phase plane, where $p := mv$ is the momentum, is

$$A(E) = \oint_{C_E} p \, \mathrm{d}x = 2 \int_{x_-(E)}^{x_+(E)} \sqrt{2m\,[E - U(x)]} \, \mathrm{d}x \ . \tag{1.66}$$

The latter quantity is related to the period of motion (1.65) by the following fundamental relation:

$$T(E) = \frac{\mathrm{d}A}{\mathrm{d}E} \ , \tag{1.67}$$

which is easily proved by a direct calculation.

In the case of an unbounded phase curves (or motion), to be specific, suppose that $U(x) \to -\infty$ as $x \to +\infty$. Let ξ be such that $E - U(x) > 0$ for all $x \in [\xi, +\infty[$. The time the particle takes to reach $+\infty$, starting at ξ, is

$$\tau_\infty(E) = \int_\xi^{+\infty} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\,[E - U(x)]}} \ . \tag{1.68}$$

Such a time turns out to be finite (which means off to infinity in a finite time, i.e. a blow-up) if $|U| \to +\infty$ faster than x^2 as $x \to +\infty$.

1.3.8 The first quantum theory

In the first 20 years of the past century, in trying to establish a new theory in agreement with many experimental results that classical electrodynamics was unable to explain, Planck, Einstein, Bohr, Sommerfeld and Born [9] developed what is known today as the "first quantum theory". Quantum mechanics in its modern form was developed starting from 1925, by Heisenberg, Schrödinger, Pauli and Dirac, with the fundamental and definitive contribution of Von Neumann on the mathematical formulation of the theory.

The basis of the first quantum theory is the following. Consider a 1D conservative Newtonian system. For bounded periodic motions, consider the formulas (1.65), (1.66) and (1.67), expressing the period of the motion $T(E)$, the area $A(E)$ inside the closed phase curve C_E, and their relation, respectively. Then one observes that $A(E) = \oint_{C_E} p\mathrm{d}x$ has the dimension of an *action*, i.e. energy times time, or momentum times length. The principle of quantization set up by the founders of quantum physics requires that *for bounded periodic motions $A(E)$ must take on only integer multiple values of the Planck constant $h = 6.626\ 10^{-27} erg \cdot s$*, i.e.

$$A(E) = nh \ , \ n = 0, 1, 2 \ldots \tag{1.69}$$

An equivalent formulation of such a principle of quantization of the area is obtained by defining the *action variable*

$$I(E) := \frac{A(E)}{2\pi} = \frac{1}{2\pi} \oint_{C_E} p \, \mathrm{d}x \ . \tag{1.70}$$

According to (1.69), $I(E) = n\hbar$, where $\hbar := h/(2\pi) = 1.055\ 10^{-27} erg \cdot s$. Now, observing that $\mathrm{d}I/\mathrm{d}E = T/2\pi > 0$, one can invert (1.70) to get $E(I)$. One then finds the following consequence of the quantization principle: $I = n\hbar$ implies that the energy values of a bounded periodic motion are also quantized, the allowed values being $E_n := E(n\hbar)$. The sequence of such energy values is the so-called *energy spectrum* of the system.

Example 1.3. *For the harmonic oscillator the frequency ω does not depend on the energy. From $\mathrm{d}I/\mathrm{d}E = T/2\pi = 1/\omega$, one gets $E = \omega I$. Another way to get this result is to consider the law of conservation of energy in the (x, p) plane, namely*

$$\frac{p^2}{2m} + \frac{kx^2}{2} = E \ .$$

Then the phase curve corresponding to $E > 0$ is an ellipse of semi-axes $a = \sqrt{2E/k}$ and $b = \sqrt{2mE}$. The area of the ellipse is $A = \pi ab = 2\pi E/\omega$ (recall that $\omega = \sqrt{k/m}$), so that $I = E/\omega$. The energy spectrum of the quantum harmonic oscillator is then

$$E_n = \hbar\omega n \ , \ n = 0, 1, 2 \ldots . \tag{1.71}$$

Example 1.4. *Let us consider again the third Kepler law, namely*

$$T = 2\pi\sqrt{\frac{m}{k}}\, a^{3/2} \ . \tag{1.72}$$

Now, recalling that the parameter p of the ellipse is linked to a and f by the relation $p = \frac{a^2 - f^2}{a} = a(1 - \varepsilon^2)$, one gets

$$a = \frac{p}{1 - \varepsilon^2} = -\frac{k}{2E} \ .$$

This gives

$$\frac{T}{2\pi} = \frac{k\sqrt{m}}{2\sqrt{2}}(-E)^{-3/2} = \frac{\mathrm{d}I}{\mathrm{d}E} \ .$$

By a simple integration we get $I = k\sqrt{m}/\sqrt{-2E}$, and finally

$$E(I) = -\frac{mk^2}{2I^2} \ . \tag{1.73}$$

For the Hydrogen atom, $k = e^2$ and the principle of quantization yields the energy spectrum

$$E_n = -\frac{me^4}{2\hbar^2 n^2} \ , \ n = 1, 2, \ldots \tag{1.74}$$

first found by Bohr (1913).

Example 1.5. *Consider a particle in an infinitely high, rectangular box of basis 2ℓ. From the conservation of energy, at energy E the particle has velocity $v_\pm = \pm\sqrt{2E/m}$, and momentum $p_\pm = \pm\sqrt{2mE}$. Thus $A = 2p_+ 2\ell = 4\ell\sqrt{2mE}$, so that $E = A^2/(32m\ell^2) = \pi^2 I^2/(8m\ell^2)$. By quantizing the action I one gets the energy spectrum of the quantum particle in a box, namely*

$$E_n = \frac{\pi^2\hbar^2 n^2}{8m\ell^2} \ . \tag{1.75}$$

1.3.9 Two examples of constrained motions

In the sequel we treat two examples of constrained motions, namely the physical pendulum and the free motion of a particle on a sphere.

Physical pendulum

A "physical pendulum" of length ℓ is realized by letting a bead of mass m free to slide on a circular wire of radius ℓ. The Newton equation is

$$m\ddot{x} = -mg\hat{e}_2 + R \ , \tag{1.76}$$

where $x = x_1\hat{e}_1 + x_2\hat{e}_2 \in \mathbb{R}^2$, $(\hat{e}_1, \hat{e}_2)$ being the canonical basis, whereas R is the *unknown reaction force exerted by the wire* on the particle (bead). Observe that R is strictly necessary: if $R = 0$ the particle falls down. The problem is conveniently studied by introducing the polar coordinates

$$\begin{cases} x_1 = r\sin\theta \\ x_2 = -r\cos\theta \end{cases} , \tag{1.77}$$

θ being the angle between the vector x and the vertical axis, counter-clockwise oriented, and such that $x = -r\hat{e}_2$ when $\theta = 0$. The corresponding polar basis $(\hat{e}_r, \hat{e}_\theta)$ is defined by

$$\hat{e}_r(\theta) = \frac{\partial x}{\partial r} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \quad ; \quad \hat{e}_\theta(\theta) = \frac{1}{r}\frac{\partial x}{\partial \theta} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} .$$

The two polar vectors satisfy the relations $\mathrm{d}\hat{e}_r/\mathrm{d}\theta = \hat{e}_\theta$, $\mathrm{d}\hat{e}_\theta/\mathrm{d}\theta = -\hat{e}_r$. We can now rewrite the Newton equation (1.76) in polar coordinates, starting with $x(t) = r(t)\hat{e}_r(\theta(t))$ and setting $R = R_r\hat{e}_r + R_\theta\hat{e}_\theta$. The result is

$$\begin{cases} m(\ddot{r} - r\dot{\theta}^2) = -mg\sin\theta + R_r \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = mg\cos\theta + R_\theta \end{cases} .$$

Exercise 1.10. *Get the above system making all the computations.*

Now we insert the constraint: the bead has to move on the wire, so that $r(t) \equiv \ell$ (i.e. for all times), and, as a consequence, $\dot{r} \equiv 0$, $\ddot{r} \equiv 0$. We thus get

$$\begin{cases} m\ell\ddot{\theta} = -mg\sin\theta + R_\theta \\ -m\ell\dot{\theta}^2 = mg\cos\theta + R_r \end{cases} . \tag{1.78}$$

Here comes the fundamental point. In order to solve the above system, one has to make some hypothesis one the reaction force R. Due to the way the constraint is realized (a bead sliding on the wire), it is quite evident that the tangential reaction R_θ is due to to the friction exerted by the wire on the particle. One possibility, is to neglect such a friction force. Such a hypothesis of *ideal constraint*, i.e. $R_\theta = 0$, means that *the reaction force is, at each point, orthogonal to the* *constraint manifold* (the circle) on which the particle has to move, and it allows to completely solve the problem. Indeed, in this case system (1.78) becomes

$$\begin{cases} \ell\ddot{\theta} = -g\sin\theta \\ R_r = -m\ell\dot{\theta}^2 - mg\cos\theta \ . \end{cases} \quad (R_\theta = 0) \tag{1.79}$$

Here the first equation determines the motion $t \mapsto \theta(t)$ of the particle, whereas the second one yields the constraint reaction $R_r(t)$ at any time t along the given motion.

►Problem 1.6. *Solve the equation of the pendulum dynamics (the first of equations (1.79)) under the hypothesis of small oscillations:* $|\theta| \ll 1$, $|\dot{\theta}| \ll 1$ *(i.e.* θ *and* $\dot{\theta}$ *very small). Compute the corresponding reaction* R_r*. Hint:* $\sin\theta \simeq \theta$, $\cos\theta \simeq 1-\theta^2/2$ *to third and fourth order terms, respectively.*

►Problem 1.7. *Suppose* $|\dot{\theta}| \gg 1$ *(i.e.* $\dot{\theta}$ *very large), which is the case of fast rotations. Find the motion of the pendulum and the corresponding reaction. Hint: make use of the energy conservation law.*

"Free" particle on a sphere

As a second example, we consider the case of a particle of mass m constrained to move on the surface of a sphere of radius ρ, subject to no field of force. The Newton equation, together with the constraint one, read

$$\begin{cases} m\ddot{x} = R \\ |x| = \rho \end{cases} \ . \tag{1.80}$$

Here again, observe that the reaction R is necessary to keep the particle on the sphere, for otherwise the particle would move on uniform rectilinear motion. Now we apply the rule learnt in the previous example, and suppose that the constraint is ideal, i.e. that R is orthogonal to the surface of the sphere at each point. As a consequence, we also conclude that R must be directed towards the center of the sphere, for otherwise the particle would deviate outwards (think of it). Anyway, this means that the only force acting on the particle, namely the constraint reaction R, is a central force. The consequences are that the angular momentum $\ell = x \times mv$ is a constant vector, that the motion takes place on the plane orthogonal to ℓ of equation $\ell \cdot x = 0$, and that such a plane passes through the center of the sphere ($x = 0$ satisfies the equation $\ell \cdot x = 0$). Thus, the motion of the particle takes place on the intersection of the sphere with the plane of motion passing through its center, namely on $\{x : |x| = \rho\} \cap \{x : \ell \cdot x = 0\}$. The latter intersection consists of a circle of maximal radius ρ, i.e. an equator of the sphere, so that the particle moves on a circle of maximal radius ρ. Moreover, since $\ell = x \times mv$, and the instantaneous velocity $\dot{x} = v$ has to be tangent to the sphere (why?), then $|\ell| = m|x||v| = m\rho|v|$. Since ℓ is a constant vector, its modulus $|\ell|$ is also constant, so that $|v| = |\ell|/(m\rho)$ is constant too. Conclusion: the particle moves on an equator of the sphere performing a uniform circular motion. A particular case is $\ell = 0$, i.e. $v = 0$, in which case the particle stays at rest in the initial position.

The solution of the above problem, and that of the previous one, rest strongly on the hypothesis of ideal constraint, namely the hypothesis that the constraint reaction R is orthogonal to the constraint *manifold* (a surface here, a curve in the previous problem) at each point.

1.3.10 A guided list of problems

▶Problem 1.8. *Show that the energy conservation law for a particle moving in a central potentials $U(r)$, in the plane of motion, and eliminating the angular velocity from the conservation law of angular momentum, reads*

$$\frac{m\dot{r}^2}{2} + U_e(r) = E \; ; \quad U_e(r) := \frac{\ell_3^2}{2mr^2} + U(r) \; , \tag{1.81}$$

where $U_e(r)$ is called the "effective radial potential". Perform a detailed study of the phase portrait of the radial motion for the cases $U(r) = -k/r$, $k > 0$ and $k < 0$, and $U(r) = kr^2/2$, $k > 0$ (Kepler, attractive and repulsive, and the 2D harmonic oscillator).

▶Problem 1.9. *Study the phase portrait of the ideal pendulum of length ℓ, described by the equation*

$$\ell\ddot{\theta} = -g\sin\theta \; , \tag{1.82}$$

g being the gravitational acceleration. Call $\omega := \sqrt{g/l}$ and take $U(\theta) = \omega^2(1 - \cos\theta)$. Write the energy conservation law. Which is the frequency of the small oscillations around $\theta = 0$? Which is the energy value separating oscillations from rotations? Which is the frequency of fast rotations as $E \to +\infty$? (Answers: ω; $2\omega^2$; $|\dot{\theta}| \sim \sqrt{2E}$).

▶Problem 1.10. *A common potential mimicking certain chemical bonds in molecules (e.g. the hydrogen bonds linking complementary bases in DNA molecules) is the Morse one:*

$$U(x) = \frac{D}{2}(e^{-\alpha x} - 1)^2 \; . \tag{1.83}$$

Draw the phase portrait of $m\ddot{x} = -U'(x)$. Pay attention to the phase curve at $E = D/2$, separating bounded from unbounded motions. Which is the frequency of the small oscillations of the system as $E \to 0^+$?

▶Problem 1.11. *A* particle in a box *is described by the sequence of smooth potentials of the form*

$$U_n(x) = \epsilon\left(\frac{x}{\ell}\right)^{2n} \; , \; n = 1, 2, \ldots \tag{1.84}$$

1. *Compute the limit $U_\infty := \lim_{n\to\infty} U_n(x)$ for any $x \in \mathbb{R}$.*
2. *Draw the phase portrait of $m\ddot{x} = -U_n'(x)$ for increasing values of n. Which is the limit phase portrait as $n \to \infty$? Which are the corresponding physical motions of the particle?*
3. *Which is the period of the motion at a given energy E in the limit $n \to \infty$? Hint: use the previous results.*

▶Problem 1.12. *Study the phase portraits corresponding to the potentials $U(x) = kx^2/2 + \lambda x^4/4$ as k and λ vary in $\mathbb{R}$ (in an interval around zero).*

►**Problem 1.13.** *An important partial differential equation (PDE) in physics is the nonlinear Klein-Gordon equation*

$$u_{tt} = c^2 u_{xx} - \omega^2 u - \lambda u^3 \ , \tag{1.85}$$

where $u(x, t)$ is a certain real function of the two variables (x, t) (i.e. a scalar field defined on a 2-dimensional space-time). The equation is ruled by the three parameters c, ω and λ. Solving equation (1.85) in general is impossible, unless $\lambda = 0$ (i.e. the equation is linear). However, in many applications one can be interested in finding special solutions called solitary, or traveling waves, *namely special solutions of the form $u(x, t) = \varphi(x - vt)$, where $\varphi(\xi)$ is a function of one real variable and v is a parameter. Such solutions describe profiles that translate at the constant velocity v.*

1. *Determine the equation satisfied by $\varphi(\xi)$, where $\xi := x - vt$.*
2. *Distinguish the three cases $v^2 > c^2$, $v^2 = c^2$, and $v^2 < c^2$; call $m = v^2 - c^2$ in the first case, $m = 0$ in the second, and $m = c^2 - v^2$ in the third case. Interpret $\varphi(\xi)$ as the abscissa at "time" ξ of a particle of mass m moving on the real line and subject to a certain force.*
3. *Draw the phase portraits in the various cases (let λ vary around zero) and determine the possible forms of the traveling waves of (1.85).*

►**Problem 1.14.** *Study the phase portrait of the Newton system $m\ddot{x} = +kx^3$. Hint: start drawing the level set $H^{-1}(0)$. Compute the time of flight to infinity starting from $x(0) = \xi > 0$ with positive velocity and zero energy.*

Chapter 2

Lagrangian mechanics

We go on studying the motions of particle systems described by the Newton laws. As we shall show, such motions, under a certain restrictive hypothesis (the existence of a potential energy), are described by a set of equations invented by Lagrange and equivalent to the Newton ones, both in the case of ideal constrained systems and of unconstrained ones.

2.1 Potential Newtonian systems

Let us consider a system of n particles of masses $m_1, \ldots, m_n$ moving in the physical space $\mathbb{R}^d$ ($d = 1, 2, 3$). The Newton equations, describing (or defining) the dynamics of the system, read

$$m_i \ddot{x}^{(i)} = f^{(i)} \ , \ i = 1, \ldots, n \ , \tag{2.1}$$

where $x^{(i)}$ is the vector position of particle i and $f^{(i)}$ is the force acting on it, the latter being a specified function of the positions and velocities of all the particles of the system and of the time. This is the most general case. In the sequel, we make the following restriction.

Definition 2.1. *System (2.1) is said to be a potential (Newtonian) system if there exists a function*

$$U : \mathbb{R}^{nd} \times \mathbb{R} \to \mathbb{R} : (x^{(1)}, \ldots, x^{(n)}, t) \mapsto U(x^{(1)}, \ldots, x^{(n)}, t) \tag{2.2}$$

such that the force $f^{(i)}$ acting on the i-th particle is given by

$$f^{(i)} = -\frac{\partial U}{\partial x^{(i)}} := -\nabla_{x^{(i)}} U = \begin{pmatrix} \frac{\partial U}{\partial x_1^{(i)}} \\ \vdots \\ \frac{\partial U}{\partial x_d^{(i)}} \end{pmatrix} \ . \tag{2.3}$$

The function U is called the potential energy of the system.

We henceforth focus on potential systems, whose Newton equations of motion are

$$m_i \ddot{x}^{(i)} = -\nabla_{x^{(i)}} U \ , \ i = 1, \ldots, n \ . \tag{2.4}$$

If the potential energy U does not depend explicitly on time t, the system above is said to be conservative, for the following reason.

Proposition 2.1. *If $\partial U/\partial t = 0$, the the total energy function*

$$H := \sum_{i=1}^{n} \frac{m_i |\dot{x}^{(i)}|^2}{2} + U(x^{(1)}, \dots, x^{(n)}, t) \tag{2.5}$$

is a constant of motion, i.e. is preserved along the solutions of (2.4).

Proof. Taking the time derivative of H and making use of the chain rule we get

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_{i=1}^{n} \dot{x}^{(i)} \cdot \left(m_i \ddot{x}^{(i)} + \nabla_{x^{(i)}} U \right) + \frac{\partial U}{\partial t} \ .$$

By (2.4), each term of the sum vanishes, which implies $\dot{H} = \partial U/\partial t$, in general, and $\dot{H} = 0$ under the hypothesis made. □

To later purposes, and for mathematical convenience, we now introduce a more abstract notation. We start by defining the configuration, or position vector X of the system, defined by

$$X := \begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(n)} \end{pmatrix} \in \mathbb{R}^N \ , \ N := nd \ . \tag{2.6}$$

The space $\mathbb{R}^N$ is called the configuration space of the system: a point in such a space is uniquely determined by assigning the n vector positions $x^{(i)}$ of each particle of the system, and vice versa. In an analogous way, we define the velocity and acceleration vectors

$$\dot{X} := \begin{pmatrix} \dot{x}^{(1)} \\ \vdots \\ \dot{x}^{(n)} \end{pmatrix} \ ; \ \dot{X} := \begin{pmatrix} \ddot{x}^{(1)} \\ \vdots \\ \ddot{x}^{(n)} \end{pmatrix} \ , \tag{2.7}$$

and the gradient of the potential energy $U(X, t)$, namely

$$\frac{\partial U}{\partial X} := \nabla_X U = \begin{pmatrix} \nabla_{x^{(1)}} U \\ \vdots \\ \nabla_{x^{(n)}} U \end{pmatrix} \ , \tag{2.8}$$

Finally, we define the $N \times N$ *mass matrix*

$$\mathsf{M} := \mathrm{diag}(\underbrace{m_1, \dots, m_1}_{\text{d times}}, \dots, \underbrace{m_n, \dots, m_n}_{\text{d times}}) \ , \tag{2.9}$$

namely the $N \times N$ diagonal matrix with the first d entries equal to m_1, the second d entries equal to m_2, and so on so forth, up to the last d entries equal to m_n. It is now an easy exercise to check that the potential system

$$\begin{cases} m_1 \ddot{x}^{(1)} = -\nabla_{x^{(1)}} U \\ \vdots \\ m_n \ddot{x}^{(n)} = -\nabla_{x^{(n)}} U \end{cases}$$

can be rewritten in the compact form

$$\mathsf{M}\ddot{X} = -\nabla_X U \ . \tag{2.10}$$

The latter equation is the Newton equation of a potential system in the N-dimensional configuration space, which is also equivalent to the first order system

$$\begin{cases} \dot{X} = V \\ \dot{V} = -\mathsf{M}^{-1}\nabla_X U \end{cases} . \tag{2.11}$$

The space $\mathbb{R}^N \times \mathbb{R}^N \ni (X, V)$, where the initial conditions uniquely determine the motion, is called the *phase space* of the system.

►**Problem 2.1.** *Prove that the energy function (2.5) in the phase space notation reads*

$$H(X, V, t) = K + U = \frac{1}{2} V \cdot \mathsf{M} V + U(X, t) \ . \tag{2.12}$$

Making use of equations (2.11), prove the relation $\dot{H} = \partial U / \partial t$.

Our purpose, in the sequel, will be the study of the dynamics of system (2.10), or its equivalent (2.10), when we impose a restriction of geometrical character on the motions, namely that they are constrained on a certain differentiable sub-manifold embedded in the configuration space.

2.2 Differentiable manifolds

We here recall the notion of differentiable manifold, or, more precisely, of a differentiable manifold embedded in $\mathbb{R}^N$ [39].

Definition 2.2. *Given a function* $\Phi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^M : (X, t) \mapsto \Phi(X, t)$, *of class* C^k $(k \geq 1)$, *with* $M < N$, *its inverse image of* $\{0\} \subset \mathbb{R}^M$, *or zero level set*

$$\mathcal{M}_t := \Phi^{-1}(0) = \{X \in \mathbb{R}^N : \ \Phi(X, t) = 0\}$$

defines, for any fixed $t \in I \subseteq \mathbb{R}$, *a differentiable manifold embedded in* $\mathbb{R}^N$, *of class* C^k, *dimension* $L := N - M$, *and co-dimension* M, *if for any* $X \in \Phi^{-1}(0)$ *the Jacobian matrix*

$$\frac{\partial \Phi}{\partial X} = \frac{\partial(\Phi_1, \ldots, \Phi_M)}{\partial(X_1, \ldots, X_N)}$$

has maximal rank M.

In the above definition, the time t plays the role of a parameter: $t \mapsto \mathcal{M}_t$ describes the motion of the manifold. Recall that the rectangular $M \times N$ Jacobian $\partial\Phi/\partial X$ has maximal rank M if it contains an $M \times M$ nonsingular minor, which is also equivalent to the linear independence of the gradients $\nabla_X \Phi_i$, $i = 1, \dots, M$. In particular, none of such gradients can vanish (why?). Since $\nabla_X \Phi_i$ is orthogonal to $\{\Phi_i = 0\}$, $i = 1, \dots, M$, then at each $X \in \mathcal{M}_t$

$$N_X \mathcal{M}_t := \mathrm{span}\{\nabla_X \Phi_1, \dots, \nabla_X \Phi_M\} \tag{2.13}$$

is the normal space of $\mathcal{M}_t$ at X, a vector space of dimension M orthogonal to $\mathcal{M}_t$ at X (recall that $\mathrm{span}\{u, v, \dots\}$ is the vector sub-space generated by $u, v, \dots$, namely the set of vectors of the form $w = \alpha u + \beta v + \cdots$, for all real $\alpha, \beta, \dots$). This easily seen by considering a curve $s \mapsto X(s) \in \mathcal{M}_t$. Then $\Phi(X(s), t) \equiv 0$ (i.e. for each s) and, by taking the derivative with respect to s yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\Phi_i(X(s), t) = \sum_{k=1}^{N} \frac{\partial \Phi_i}{\partial X_k}\frac{\mathrm{d}X_k}{\mathrm{d}s} = \nabla_X \Phi_i \cdot \frac{\mathrm{d}X}{\mathrm{d}s}\ ,\ i = 1, \dots, M\ .$$

In other words, the tangent vector $\mathrm{d}X/\mathrm{d}s$ to an arbitrary curve on $\mathcal{M}_t$ is orthogonal to all the gradients $\nabla_X \Phi_i$.

Example 2.1. *The equation $\Phi_1(X, t) := X_1^2 + \cdots + X_N^2 - R^2(t) = 0$ defines a differentiable manifold of dimension $L = N - 1$, namely the surface of a sphere whose radius depends on time. The rank condition is expressed by $\nabla_X \Phi_1 = 2X \neq 0$ on $\Phi_1^{-1}(0)$, which is true for all t such that $R(t) > 0$. Notice that $\Phi_1 = 0$ can be rewritten as $|X| = R(t)$; when $R(t) = 0$ the sphere collapses to a point. Observe that $\nabla_X \Phi_1 = 2X$ is orthogonal to the sphere at each point. Here the normal space has dimension $M = 1$.*

Example 2.2. *In the case $N = 3$, $M = 2$, the set $\Phi^{-1}(0) = \{\Phi_1 = 0\} \cap \{\Phi_2 = 0\}$ defines a manifold of dimension $L = N - M = 1$, i.e. a curve, if the gradients $\nabla_X \Phi_1$ and $\nabla_X \Phi_2$ are not parallel to each other; in particular they must be different from zero (why?). Since $\nabla_X \Phi_i$ is orthogonal to $\{\Phi_i = 0\}$, $i = 1, 2$, it follows that* $\mathrm{span}\{\nabla_X \Phi_1, \nabla_X \Phi_2\}$ *is the plane orthogonal to the curve at each point.*

A direct consequence of the implicit function theorem by Dini [39] is that the differentiable manifold $\mathcal{M}_t := \Phi^{-1}(0)$ can always be locally represented as the Cartesian graph of a function. More precisely, there exists a function g such that, locally and up to a relabelling of the variables,

$$\begin{cases} X_1 = g_1(X_{M+1}, \dots, X_N, t) \\ \vdots \\ X_M = g_M(X_{M+1}, \dots, X_N, t) \end{cases} . \tag{2.14}$$

The above representation of the manifold implies that we can always choose a local parametric representation $h : \mathbb{R}^L \to \mathbb{R}^N : u \mapsto X = h(u, t)$. Indeed, by introducing any change of coordinates $(X_{M+1},\dots,X_N)\mapsto(u_1,\dots,u_L)$, whose inverse is defined by defined by $X_{M+i}=f_i(u_1,\dots,u_L)$, $i=1,\dots,L$, composing with (2.14) one gets

$$\begin{cases} X_1 = g_1(f_1(u),\dots,f_L(u),t) := h_1(u,t) \\ \vdots \\ X_M = g_M(f_1(u),\dots,f_L(u),t) := h_M(u,t) \\ X_{M+1} = f_1(u) := h_{M+1}(u,t) \\ \vdots \\ X_N = f_L(u) := h_N(u,t) \end{cases} .$$

The conclusion is that we can always think of our manifold $\mathcal{M}_t$ as locally defined in parametric form $\mathbb{R}^L \ni u \mapsto X(u,t)\in\mathbb{R}^N$ (with some abuse of notation we write $X(u,t)$ in place of $X=h(u,t)$). Notice that by construction, $L=N-M<N$. On the other hand, we will also consider below the case $L=N$. In such a case, the map $\mathbb{R}^N\ni u\mapsto X(u,t)\in\mathbb{R}^N$ defines a local change of coordinates, depending explicitly on time. In other words, we identify the manifold $\mathcal{M}_t$ with an open subset of $\mathbb{R}^N$.

Exercise 2.1. *Work out carefully the examples of the unit circle in $\mathbb{R}^2$ and of the (surface of the) unit sphere of in $\mathbb{R}^3$. Can you imagine a parametrization of the circle or of the sphere different from the polar one?*

The parametric representation of the manifold $\mathcal{M}_t$ provides a convenient way to build up the tangent vectors to the manifold at any given point. Indeed, let us consider $X(u,t)$, and let us fix a point $\bar{u}$, so that $X(\bar{u},t)$ is a fixed point on $\mathcal{M}_t$. Now, let us keep fixed all the u's but one, say u_1. Then, the map $u_1\mapsto X(u_1,\bar{u}_2,\dots,\bar{u}_L,t)$ defines a curve on $\mathcal{M}_t$ in the neighbourhood of $\bar{u}_1$ (in all these reasonings the time t is thought of as fixed). As a consequence, the vector

$$\frac{\partial}{\partial u_1}X(u_1,\bar{u}_2,\dots,\bar{u}_L,t)\Big|_{u_1=\bar{u}_1}$$

is tangent to $\mathcal{M}_t$ at $X(\bar{u},t)$. Repeating the same reasoning with $u_2,\dots,u_L$, one builds up exactly L vectors tangent to $\mathcal{M}_t$ at $X(\bar{u},t)$. It is again a consequence of the hypothesis of maximal rank of the Jacobian $\partial\Phi/\partial X$, and of the definition of the parametrization given above (as a change of the "free" coordinates), that the tangent vectors just built up are linearly independent. The conclusion is that

$$T_X\mathcal{M}_t := \mathrm{span}\left\{\frac{\partial X}{\partial u_1},\dots,\frac{\partial X}{\partial u_L}\right\} \tag{2.15}$$

is the tangent space of $\mathcal{M}_t$ at $X(u,t)$, a vector space of dimension L tangent to the manifold at the point X.

Example 2.3. *Consider a surface embedded in $\mathbb{R}^3$ expressed in parametric form. Draw a picture of what said above. Consider the unit sphere expressed in spherical polar coordinates. Build up the tangent plane of the sphere at any point and the normal space (line).*

2.3 Constrained potential systems

We are now ready to face the problem of constrained potential systems, which includes, as a limit case, that of expressing the equations of motion of the unconstrained system in arbitrary coordinates. The problem is defined as follows.

Definition 2.3. *A Newtonian potential system subject to holonomic, bilateral, ideal constraints is defined by the problem*

$$\begin{cases} \mathsf{M}\ddot{X} = -\nabla_X U(X,t) + R \\ X(t) \in \mathcal{M}_t \\ R \in N_X\mathcal{M}_t \ \forall X \in \mathcal{M}_t \end{cases} , \tag{2.16}$$

where $\mathcal{M}_t$ is a given, moving, constraint manifold of dimension L defined in the (local) parametric form $\mathbb{R}^L \ni q \mapsto X(q,t) \in \mathbb{R}^N$, with $L < N$. In the case $L = N$, $R \equiv 0$.

Let us explain the jargon. The constraint is called holonomic since the constraint manifold is embedded in the configuration space. Bilateral constraint refers to the fact that one requires $X(t) \in \mathcal{M}_t$: the point in configuration space cannot enter any of the local half-spaces on the two sides of the manifold. The ideality of the constraint is expressed by the requirement that the constraint reaction R is orthogonal to $\mathcal{M}_t$ at each point and each time t. We stress again the following fact: *when $L = N$ there is no constraint, the manifold is just an open subset of the configuration space $\mathbb{R}^N$ and we are just performing an arbitrary change of coordinates.* Observe that now, in order to attain to tradition, the parametric, or "free" coordinates are denoted by $q_1, \dots, q_L$. The following fundamental theorem, due to Lagrange, holds.

Theorem 2.1 (Lagrange)**.** *The dynamics of system (2.16) on $\mathcal{M}_t$ is described by the Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \frac{\partial \mathscr{L}}{\partial q_i} = 0 \ ; \ i = 1, \dots, L \ , \tag{2.17}$$

where $\mathscr{L} := (K-U)|_{\mathcal{M}_t}$ is the Lagrangian (or Lagrange function) of the system, $K := \dot{X}\cdot\mathsf{M}\dot{X}/2$ being the total kinetic energy. The constraint reaction R along the known motion $t \mapsto q(t)$ is determined a fortiori by $R(t) = (\mathsf{M}\ddot{X} + \nabla_X U)|_{\mathcal{M}_t}$. The Lagrange equations are invariant in form under any reparametrization of $\mathcal{M}_t$.

Proof. The condition $R \in N_X\mathcal{M}_t \ \forall X \in \mathcal{M}_t$ is equivalent to $R \perp T_x\mathcal{M}_t$ at each point $X \in \mathcal{M}_t$, and $T_X\mathcal{M}_t$ is generated by the L local tangent vectors $\partial X/\partial q_i$, $i = 1\dots, L$. Projecting the Newton equation $M\ddot{X} = -\nabla_X U + R$ onto $T_X\mathcal{M}_t$, i.e. scalarly multiplying it by $\partial X/\partial q_i$, and taking into account that $R \cdot \partial X/\partial q_i = 0$ for each $i = 1, \dots, L$, yields

$$M\ddot{X} \cdot \frac{\partial X}{\partial q_i} = -\nabla_X U \cdot \frac{\partial X}{\partial q_i} \ , \ \forall i = 1, \dots, L \ . \tag{2.18}$$

On the right hand side, by the chain rule, we obtain

$$-\nabla_X U \cdot \frac{\partial X}{\partial q_i} = -\frac{\partial}{\partial q_i} U(X(q,t),t) = -\frac{\partial}{\partial q_i} U|_{\mathcal{M}_t} \ , \tag{2.19}$$

where we have defined $U|_{\mathcal{M}_t} := U(X(q,t),t)$ is the restriction of the potential energy U on $\mathcal{M}_t$. Working out the right hand side of (2.18), we get

$$\mathsf{M}\ddot{X} \cdot \frac{\partial X}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathsf{M}\dot{X} \cdot \frac{\partial X}{\partial q_i}\right) - \mathsf{M}\dot{X} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X}{\partial q_i} \ . \tag{2.20}$$

Notice that here and henceforth, an over-dot is the same as a total derivative with respect to time t, and $\dot{X}$ means

$$\dot{X} = \frac{\mathrm{d}}{\mathrm{d}t} X(q(t),t) = \frac{\partial X}{\partial q_j}\dot{q}_j + \frac{\partial X}{\partial t} \ . \tag{2.21}$$

By means of the above relation, one easily gets

$$\frac{\partial X}{\partial q_i} = \frac{\partial \dot{X}}{\partial \dot{q}_i} \ ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X}{\partial q_i} = \frac{\partial \dot{X}}{\partial q_i} \ . \tag{2.22}$$

Using the latter identities, we rewrite (2.20) as follows

$$\mathsf{M}\ddot{X} \cdot \frac{\partial X}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathsf{M}\dot{X} \cdot \frac{\partial \dot{X}}{\partial \dot{q}_i}\right) - \mathsf{M}\dot{X} \cdot \frac{\partial \dot{X}}{\partial q_i} \tag{2.23}$$

Recalling that $K := (\dot{X} \cdot M\dot{X})/2$ is the kinetic energy of the system, we can write (2.23) in the simple form

$$\mathsf{M}\ddot{X} \cdot \frac{\partial X}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_i} K|_{\mathcal{M}_t}\right) - \frac{\partial}{\partial q_i} K|_{\mathcal{M}_t} \ , \tag{2.24}$$

where $K|_{\mathcal{M}_t} = \dot{X}(q,t) \cdot \mathsf{M}\dot{X}(q,t)/2$ is the restriction to $\mathcal{M}_t$ of the kinetic energy. By equating (2.24) with (2.19), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{q}_i} K|_{\mathcal{M}_t}\right) - \frac{\partial}{\partial q_i} K|_{\mathcal{M}_t} = -\frac{\partial}{\partial q_i} U|_{\mathcal{M}_t} \ . \tag{2.25}$$

Finally, by defining the Lagrange function, or Lagrangian

$$\mathscr{L}(q,\dot{q},t) := \frac{1}{2}\dot{X}(q,t) \cdot \mathsf{M}\dot{X}(q,t) - U(X(q,t),t) = (K-U)|_{\mathcal{M}_t} \ , \tag{2.26}$$

and observing the $\partial U/\partial \dot{q}_i = 0$, one easily checks that equations (2.25) can be rewritten in the final form of the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathscr{L}}{\partial q_i} = 0 \ ; \ i = 1, \ldots, L \ . \tag{2.27}$$

Once the motion on the constraint manifold $\mathcal{M}_t$ is obtained by solving the latter equations, the formula

$$R = (\mathsf{M}\ddot{X} + \nabla_X U)|_{\mathcal{M}_t} \tag{2.28}$$

follows by restricting to $\mathcal{M}_t$ the Newton equation. Observe that the right hand side must belong to the normal space $N_X\mathcal{M}_t$ by definition.

As a final remark, we observe that we never used a specific parametrization of the manifold $\mathcal{M}_t$ in order to deduce the Lagrange equations (2.27). It follows that, if imagine to (locally) re-parametrize $\mathcal{M}_t$ from the beginning, by means of new coordinates, say q', we must get the same result. More precisely, we will get the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}'}{\partial \dot{q}'_i}\right) - \frac{\partial \mathscr{L}'}{\partial q'_i} = 0 \ ; \ i = 1, \dots, L \ , \tag{2.29}$$

where the "new" Lagrangian $\mathscr{L}'$ is linked to the "old" one by

$$\mathscr{L}'(q', \dot{q}', t) = \mathscr{L}(g(q'), \dot{g}(q'), t) \ , \tag{2.30}$$

where g is the map defining the change of coordinates: $q' \mapsto q = g(q')$. $\square$

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