Introduction to MATHEMATICAL PHYSICS

Lectures Notes 2025/2026

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"If a leaden ball, projected from the top of a mountain by the force of gunpowder with a given velocity, and in a direction parallel to the horizon, is carried in a curve line to the distance of two miles before it falls to the ground; the same, if the resistance of the air were taken away, with a double or decuple velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line, which it might describe, till at last it should fall at the distance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or lastly, so that it might never fall to the earth, but go forward into the celestial spaces, and proceed in its motion in infinitum. *And after the same manner that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the whole earth, the moon also, either by the force of gravity, if it is endued with gravity, or by any other force, that impels it towards the earth, may be perpetually drawn aside towards the earth, out of the rectilinear way, which by its innate force it would pursue; and would be made to revolve in the orbit which it now describes; nor could the moon without some such force, be retained in its orbit. If this force was too small, it would not sufficiently turn the moon out of a rectilinear course: if it was too great, it would turn it too much, and draw down the moon from its orbit towards the earth. It is necessary, that the force be of a just quantity, and it belongs to the mathematicians to find the force, that may serve exactly to retain a body in a given orbit, with a given velocity; and vice versa, to determine the curvilinear way, into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear way, by means of a given force."*

I. Newton, Book I of the *Philosophiae Naturalis Principia Mathematica*, 1687 [35].

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Preface

The present notes deal with classical mechanics and its mathematical methods, which means making use of advanced tools of analysis, geometry and algebra in order to solve classes of problems relevant to physics first, and, in so doing, to get a more abstract, higher level formulation of the physical laws of motion which is relevant far beyond mechanics (what is usually referred to as "formalisms").

The notes are meant to be an integration of, or complement to the class lectures, where most of the problems are worked out in a complete way. A complete preparation to the exam is based on the use of both of them. I intentionally avoid to write down the detailed solutions of exercises and problems in the notes, but for some examples, with the aim to stress the importance, for the students, to attend the class lectures and realize which kind of ideas and technical efforts are required, and how long it takes to solve specific problems up to the end, in a complete way. However, almost all exercises and problems are formulated in a step by step guided form. In the present notes I distinguish exercises, which are of a purely, maybe minor technical content, from problems, that have instead more physical relevance, require something new to be tackled, or constitute the starting point for the introduction of advanced tools. The topics in the notes are reported according to the sequence of those treated in the class lectures, although sometimes the correspondence can be a bit different, for the obvious reasons concerning the difference in between giving lectures at the blackboard and writing lecture notes.

The approach to the subject, in my course, is partially founded on *problem solving*. In tackling an interesting problem, one is required to understand completely what has to be done, to thus plan a strategy of solution, and to finally perform detailed computations; a complete check of the procedure and of the final formulas is required at the end. The method of solution of a specific problem is not unique, and the choice is a matter of convenience, time, and taste. In order to simply get the final answer one may certainly choose the shortest path. However, one can consider to follow also longer, alternative ways to the solution, where the use of, and the need to develop higher mathematics is required. This is the point where building up a general theory and its mathematical apparatus emerges, and this is exactly the way it works in research (with a nontrivial additional ingredient: also the problem is new there, and one does not even know whether it is solvable in advance). Although mathematics plays a dominant role in this course, I recommend the students to always make use of physical intuition during the preliminary analysis of the problem at hand, and in the final stage, when a result in closed mathematical form has been obtained. The dimensional correctness and the physical meaning of a formula, or a theorem, must always be considered.

Concerning the role and effectiveness of mathematics in physics (see e.g. [45]), and viceversa,

a remark is in order. The *truth value* in mathematics is given by the proof: a statement within a certain axiomatic framework is true if it is proven. On the other hand, the truth value in physics is given by the experimental evidence. Every experiment requires a reference, theoretical model defining the quantities to be measured and related to each other, and any theory is built up on, and must refer to experimental results, tests and facts. The relevance of any physical theory, whatever be the level of generality, rigour and difficulty involved, is based on its ability to explain experimental facts and to predict new ones. The history of theoretical and mathematical physics works this way. The mathematical apparatus of classical mechanics built up first by Newton, was founded on the experiments and data of Brahe, Kepler, Galilei, Boyle, Hooke and others, and was then extended by Euler, Laplace, Lagrange, Hamilton, Poincaré, Einstein (just to mention a few of them) to the impressive actual level leading to space explorations, to the prediction and measurement of gravitational waves, and to the evidence of dark matter. The overall process took, if we just start from Renaissance, five centuries, but almost four millennia if we start from the birth of astronomy within the Sumerian and Babylonian civilizations [29].

Padova, March 31, 2026

Antonio Ponno

Chapter 1

Problems of Newtonian dynamics

Part of the material contained in this chapter maight be known to the reader, at the level of a general physics course; a good reference is [26]. Newtonian mechanics is not only necessary as a basis for its subsequent developments, namely the Lagrangian and the Hamiltonian formulation, but also useful to solve fundamental problem of physics.

1.1 The Newton equation

The extreme synthesis of the monumental work of Newton [35] is encoded in the equation ruling the motion of a point mass, or particle, subject to a given force, i.e. $ma = f$, where m is the mass of the particle, a its acceleration, and f the force acting on it. The Newton equation, written in more explicit form, reads

$$\boxed{m\ddot{x}(t) = f(x(t), \dot{x}(t), t)} \ , \tag{1.1}$$

where over-dots denote total derivatives with respect to time t, one per dot. Here the curve $\mathbb{R} \ni t \mapsto x(t) \in \mathbb{R}^d$ is referred to as the trajectory, or motion of the particle, which gives the position of the latter in the physical space $\mathbb{R}^d$ ($d = 1, 2, 3$) at time t. The components of the vector position $x(t) = (x_1(t), \ldots, x_d(t))$, are determined with respect to a given reference frame (a basis in $\mathbb{R}^d$), and each of them is supposed to be smooth enough in t (class C^2, at least). We recall that the instantaneous velocity of the particle is the local tangent vector to the curve $t \mapsto x(t)$, namely

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \dot{x}(t) := \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \ ,$$

whereas the instantaneous acceleration is the local tangent vector to the velocity curve $t \mapsto v(t)$, namely

$$a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \dot{v}(t) = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \ddot{x}(t) \ .$$

The force acting on the particle, in the Newtonian scheme, is the *known*, vector valued function $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ appearing on the right hand side of (1.1). Once the form of the force

is specified, the Newton equation (1.1) becomes a vector valued, ordinary differential equation (ODE) of second order in normal form: the derivative of highest order - the second one - is given by a function of the derivatives of lower order and of time. Once specified the force, the unknown of the Newton equation is the motion $t \mapsto x(t)$, which is uniquely determined, locally in time, by assigning the initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$, i.e. the position and the velocity of the particle at some initial instant t_0, when f is smooth enough (e.g. C^1 in all arguments). Global existence - fundamental to physics - is instead ensured by confinement of motions in compact regions, which typically holds, as we shall see below, in presence of conservation laws.

►**Problem 1.1.** *Consider equation (1.1) in dimension $d = 1$, with initial conditions $x(t_0) := x_0$ and $\dot{x}(t_0) := v_0$. Show that $x(t)$ is uniquely determined in a suitable neighbourhood of t_0. Hint: assume $f(x, v, t)$ smooth enough and Taylor expand $x(t_0 + h)$ in h, to third order included (with a remainder of order h^4). Result:*

$$\begin{aligned} x(t_0+h) &= x_0 + v_0 h + \frac{1}{m} f(x_0, v_0, t_0)\frac{h^2}{2} + \frac{1}{m}\left[\frac{\partial f}{\partial x}(x_0, v_0, t_0)v_0 + \right. \\ &+ \left. \frac{\partial f}{\partial v}(x_0, v_0, t_0)\frac{1}{m} f(x_0, v_0, t_0) + \frac{\partial f}{\partial t}(x_0, v_0, t_0)\right]\frac{h^3}{6} + O(h^4) \ . \end{aligned} \tag{1.2}$$

Hint: start with the identity

$$x(t_0 + h) = x(t_0) + \int_{t_0}^{t_0+h} \dot{x}(s_1)\ \mathrm{d}s_1 \ ;$$

Go on by writing $\dot{x}(s_1) = \dot{x}(t_0) + \int_{t_0}^{s_1} \ddot{x}(s_2)\mathrm{d}s_2$. You need other two steps. Can you write down explicitly the remainder $O(h^4)$?

The Newton equation (1.1) can be rewritten in the equivalent first order form

$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m} f(x, v, t) \end{cases} . \tag{1.3}$$

The mathematical problem of solving (1.1), or (1.4), once the initial conditions $x(t_0)$ and $v(t_0) = \dot{x}(t_0)$ are specified, thus determining the motion of the particle, is named Cauchy problem, or initial value problem. The first order form (1.4) of the Newton equation clarifies that the state of a particle is determined by its position and its velocity at the same time: $(x(t), v(t))$. The space of states is thus $\mathbb{R}^d \times \mathbb{R}^d$, i.e. the space of ordered pairs position-velocity. The space of states is also called, in jargon, the *phase space*, and the curve $t \mapsto (x(t), v(t))$ is called *phase curve*, to be distinguished from the trajectory, or orbit, i.e. its projection $t \mapsto x(t)$ onto the physical space of positions. The right hand side of (1.4) is called the *vector field* of the system, which is the tangent vector to the phase curve passing through (x, v) at time t.

System (1.4), or the equivalent Newton equation (1.1), is said to be *autonomous* if f does not explicitly depend on time t: $\partial f/\partial t = 0$; it is said to be non-autonomous otherwise. In fact, the non-autonomous system (1.4) can be always brought into autonomous form by extending

the dimension of the phase space by one, introducing the auxiliary variable $s(t)$ such that $\dot{s} = 1$ and $s(0) = 0$, in terms of which (1.4) reads

$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{1}{m} f(x, v, s) \\ \dot{s} = 1 \end{cases} . \tag{1.4}$$

The $2d + 1$-dimensional space of points (x, v, s) is the extended phase space of the system, and the vector $(v, f/m, s)$ is tangent to the phase curve $t \mapsto (x(t), v(t), s(t))$ is the phase curve passing through (x, v, s). Observe the following: by the (local) existence and uniqueness theorem, there exist one and only one phase curve passing through a point of the phase space for autonomous systems, while the same conclusion holds for non-autonomous systems only in the extended phase space.

In dimension $d = 1$, the geometric interpretation of the vector field is useful to understand the behavior of the unknown solutions of the system. Indeed, in this case the phase space is the plane $\mathbb{R}^2$. One can then choose any portion of such a plane, select there a sample of points (x, v), and draw the vector $(v, f(x, v, t)/m)$ exiting any of them, at a fixed time t: this is relevant only in case $f(x, v, t)$ does explicitly depends o t. Since the local phase curve is tangent to $(v, f/m)$ at (x, v), and at time t, the denser is the sample in a given region, the better is the idea one gets on the behavior of local solutions there. Graphical applications usually sample the vector field by arbitrarily renormalizing the length of $(v, f/m)$ in order to avoid overlaps.

It may be useful to rewrite system (1.4) in the differential form

$$\begin{cases} \mathrm{d}x = v \ \mathrm{d}t \\ \mathrm{d}v = \frac{1}{m} f(x, v, t) \ \mathrm{d}t \end{cases} . \tag{1.5}$$

The meaning of (1.5) is the following: in a time $\mathrm{d}t$ the state (x, v) of the particle evolves to $(x', v') = (x + \mathrm{d}x, v + \mathrm{d}v) = (x + v\mathrm{d}t, v + (f/m)\mathrm{d}t)$.

Remark 1.1. *By $\mathrm{d}x(t) = v(t)\mathrm{d}t$ we mean here a shorthand notation for the standard Taylor expansion $\Delta x(t) := x(t+\Delta t) - x(t) = v(t)\Delta t + o(\Delta t)$ as $\Delta t \to 0$; the same for $\mathrm{d}v(t)$. Writing the Newton equation in the differential form (1.5) is not only convenient, but becomes mandatory when, for example, the force contains stochastic terms, as in the theory of Brownian motion, where derivatives become rigorously meaningless.*

Replacing the differential symbol d with the finite difference symbol Δ in (1.5), which is equivalent to neglect the $o(\Delta t)$ remainder in the first order Taylor expansion, one gets a (first, rough) numerical algorithm to be implemented on a computer in order to numerically solve the initial value problem. Consider, for $d = 1$, the finite difference approximation of system (1.5) by replacing $\mathrm{d}x$ and $\mathrm{d}t$ with Δx and $h := \Delta t$, respectively. You get

$$\begin{cases} x(t + h) = x(t) + v(t)h \\ v(t + h) = v(t) + \frac{1}{m} f(x(t), v(t), t)h \end{cases} ; \quad t = 0, h, \ldots, (n - 1)h . \tag{1.6}$$

This is the so called Euler algorithm, the simplest one to numerically integrate the Newton dynamics on a computer. The initial conditions are $x_0 := x(0)$ and $v_0 := v(0)$. Setting $t = 0$

in (1.6), you can determine $x_1 := x(h)$ and $v_1 := v(h)$. More in general, substituting into (1.6) $t = (k-1)h$, you can rewrite it as follows:

$$\begin{cases} x_k = x_{k-1} + v_{k-1}h \\ v_k = v_{k-1} + \frac{1}{m} f_{k-1} h \end{cases} \quad ; \quad k = 1, \ldots, n \ , \tag{1.7}$$

where $x_k := x(kh)$, $v_k := v(kh)$ and $f_{k-1} := f(x_{k-1}, v_{k-1}, (k-1)h)$. Now, system (1.7) is in the form of a recurrent sequence, or iterated map, namely $(x_k, v_k) = M(x_{k-1}, v_{k-1})$, which is what a computer is able to solve: compute the right hand side, get the new value, reinsert it on the right and side and so on for a finite number n of steps.

►**Problem 1.2.** *Fix the time-step h and the number of steps n such that $nh = T$, your "observation" time.*

1. *Solve analytically system (1.7) for the three cases: $f = 0$, $f = -mg$, $f = -kx$, i.e. get an explicit formula for x_k and v_k.*

2. *You could make use of a computer to solve (1.7). Play the game choosing h small enough and n large enough. Plot the sequence $\{x_k, v_k\}_{k=1,\ldots,n}$ in the phase plane (x, v), for the three cases. What about the third one?*

3. *Solve the "actual" Newton equation (1.1), or the equivalent system (1.4), in the three cases, getting explicit formulas for $x_a(t)$ and $v_a(t)$ in terms of the initial conditions x_0 and v_0 (the subscript "a" means "actual").*

4. *Compute the exact errors $E_x = |x_a(T) - x_n|$ and $E_v = |v_a - v_n|$ for the three cases. What general conclusion can be drawn? What about the limit of the errors as $n \to \infty$, or $h \to 0$ at fixed $T = nh$?*

Hint: in the third case, $f = -kx$, you might consider to write the recurrent map for the complex variable $z_k := \omega x_k + \imath v_k$, where $\omega := \sqrt{k/m}$ and $\imath$ is the imaginary unit.

A better algorithm to numerically solve the Newton equation (1.1) is the leap-frog one, due to Verlet [44], which looks like a mild correction to (1.6):

$$\begin{cases} x(t+h) = x(t) + v(t+h)h \\ v(t+h) = v(t) + \frac{1}{m} f(x(t), v(t), t)h \end{cases} \quad ; \ t = 0, \ldots, (n-1)h \ . \tag{1.8}$$

►**Problem 1.3.** *Re-analyze the problem of the harmonic oscillator, $f = -kx$ using the Verlet algorithm (1.8).*

1. *Show that (1.8) is equivalent to the discretized Newton equation*

$$\frac{x(t+h) + x(t-h) - 2x(t)}{h^2} = \frac{1}{m} f(x(t)) = -\frac{k}{m} x(t) \ .$$

2. *Look for solutions of the above equation in the form* $A\cos(\Omega t)$, *or* $B\sin(\Omega t)$, *or* $Ce^{\pm\imath\Omega t}$. *Answer: such solutions exist iff* Ω *satisfies the equation*
$$\frac{4\sin^2(\Omega h/2)}{h^2} = \frac{k}{m} \ .$$
3. *Show that* $\Omega = \omega + O(h^2)$, *with* $\omega = \sqrt{k/m}$, *explicitly determining the* $O(h^2)$ *correction.*

Excellent references on ODEs and much of the mathematics around them are [2] and [24]. Perhaps the best reference on numerical algorithms for ODEs is [23].

Remark 1.2. *The fact that the force* f *depends on the position and the velocity of the particle is an assumption, which allows to solve the problem of predicting the motion and turns out to be in agreement with a wide experimental evidence. However, there are situations where such an assumption does not work. An example is found in classical electrodynamics of point charges: the force exerted on a charged particle by the electromagnetic field generated by the particle itself is proportional to the derivative of the acceleration. Starting with a finite size particle, a non trivial limit procedure, which also requires a mass renormalization, leads to the Abraham-Lorentz radiation reaction force [16, 25, 37]*

$$f_{rad} = \frac{2q^2}{3c^3}\,\dot{a} = \frac{2q^2}{3c^3}\,\dddot{x} \ , \tag{1.9}$$

where q *is the charge of the particle and* c *is the speed of light. Which is the general solution of the non Newtonian equation* $ma = f_{rad}$ *for the free charge? This is known as the runaway problem. Hint: look for solutions of the form* $x(t) = e^{\lambda t}\xi$, *where* ξ *is a constant vector. If interested, take a look at chapter 11 of [16], and problems/references therein.*

1.2 Force models and their mathematical properties

Solving "explicitly" the Newton equation (1.1) is certainly not possible in the general case. However, in certain special cases where the force displays a particular dependence on its arguments. As an example, if the force f does not depend explicitly on time, i.e. $\partial f/\partial t = 0$, the Newton equation (1.1), or (1.4), is said to be *autonomous*, as opposed to the general, non-autonomous case. Such a uniformity of the force with respect to time may help solving some problems in some particular cases, when the force is also independent of the position x, or when the force is independent of the velocity, or when the dimension of the problem is $d = 1$. In such cases, also depending on the space dimension d of the problem, one can determine, completely or in part, the main features of the motions. Some relevant cases are analyzed below.

1.2.1 Spatially uniform forces

This is the case of forces dependent only on the velocity (and time). In such a case, the second of Newton equations becomes a vector valued, first order ODE for the velocity: $m\dot{v} = f(v, t)$.

If the latter equation can be solved for $v(t)$, the motion is then determined by an integration with respect to time: $x(t) = x(t_0) + \int_{t_0}^{t} v(s)\mathrm{d}s$. Such a procedure works, for example, when $f(v,t)$ is a linear (non homogeneous, in general) function of the velocity. An interesting solvable sub-case is the autonomous one-dimensional case: $f(v)$, $d = 1$. In the latter case the velocity equation can be solved by separation of variables:

$$\dot{v} = \frac{1}{m} f(v) := g(v) \ ; \quad \Leftrightarrow \quad \frac{\mathrm{d}v}{g(v)} = \mathrm{d}t \ ; \quad \Leftrightarrow \quad F(v) = t + c \ ,$$

the last step being obtained by indefinite integration of both sides, with F a primitive of $1/g$. In those intervals where F is monotonic, one finally gets $v(t) = F^{-1}(t+c)$. However, in general one will not be able to find the function F or, even if able, will not be able to invert it. Moreover, also supposing that a closed formula in terms of elementary functions has been obtained, this will be unreadable but for a few trivial cases, and extracting simple informations concerning the behaviour of the solution may turn out to be very difficult.

Remark 1.3. *In fact, a much more effective technique works here. Indeed, it is sufficient to draw the graph of $g(v) = f(v)/m$. Then, the equation $\dot{v} = f(v)/m$ tells us that $v(t)$ is a monotonically increasing (decreasing) function of time where $f(v) > 0$ ($f(v) < 0$, respectively), and that the zeroes of f are constant velocity motions. In particular, if v_* is a zero of f such that $f(v) > 0$ for $v < v_*$, and $f(v) < 0$ for $v > v_*$, then v_* is locally attracting, i.e. $\lim_{t\to+\infty} v(t) = v_*$ for all initial conditions $v(0)$ close enough to v_*.*

►**Problem 1.4.** *Study in detail the case of falling bodies in presence of resistance of the air, modelled by the force*

$$f(v) = -\gamma_0 v - \gamma_1 |v| v + mg = \begin{cases} -\gamma_0 v - \gamma_1 v^2 + mg, & v \geq 0 \\ -\gamma_0 v + \gamma_1 v^2 + mg, & v \leq 0 \end{cases} \ ,$$

the vertical axis being oriented downward.

1.2.2 Positional, potential and conservative forces

Positional forces are independent of the velocity. In such a case, the Newton equation reads $m\ddot{x} = f(x,t)$. Here too, if $f(x,t)$ is a linear (non homogenous) function of x, for example, the problem can be solved. Relevant sub-cases are the following.

- *Potential forces* are those positional forces that are minus the gradient of a given scalar function $U(x,t)$:

$$f(x,t) = -\nabla_x U(x,t) = -\frac{\partial U}{\partial x}(x,t) \ . \tag{1.10}$$

  The scalar function U is named *potential energy*. Such a condition is not sufficient, in general, to solve the Newton equation. However, large part of physics rests on such an assumption, which is the standard one in both classical and quantum mechanics.

- *Conservative forces* are potential forces independent of time:

$$f(x) = -\nabla_x U(x) \ . \tag{1.11}$$

The jargon is due to the fact that in such a case there exists a conservation law, namely the *energy conservation law*. Indeed, the following important theorem holds.

Proposition 1.1. *The* total energy function

$$H(x, \dot{x}) := K + U := \frac{m|\dot{x}|^2}{2} + U(x) \ , \tag{1.12}$$

sum of the kinetic energy $K := m|v|^2/2$ and of the potential energy $U(x)$, is a constant of motion, i.e. is constant along any solution of the Newton equation $m\ddot{x} = -\nabla_x U$.

Proof. The time derivative of $H(x(t), \dot{x}(t))$, by the chain rule and making use of the Newton equation $m\ddot{x} = f$, reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(x(t), \dot{x}(t)) &= \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial \dot{x}} \cdot \ddot{x} = \frac{\partial U}{\partial x} \cdot \dot{x} + m\dot{x} \cdot \frac{1}{m} f = \\ &= \dot{x} \cdot \left(f + \frac{\partial U}{\partial x} \right) \equiv 0 \end{aligned} \tag{1.13}$$

under the hypotheses made. Thus $H(x(t), \dot{x}(t)) = H(x(0), \dot{x}(0)) := E$. □

The geometric consequence of this theorem is that the phase curve of a particle subject to a conservative force lies on the sub-manifold, or hyper-surface, or surface of the phase space determined by the equation $H(x, v) = E$, i.e. the set $H^{-1}(E) = \{(x, v) : H(x, v) = E\}$, the value E of the energy being determined by the initial condition: $E = H(x(0), \dot{x}(0))$. Such a conclusion can be also understood by checking that the gradient of H is orthogonal to the vector field of the Newton system, namely

$$\begin{pmatrix} \nabla_x U \\ mv \end{pmatrix} \cdot \begin{pmatrix} v \\ -\frac{1}{m} \nabla_x U \end{pmatrix} = 0 \ .$$

so that the Newtonian vector field is point-wise tangent to $H^{-1}(E)$. The constant energy manifold, embedded in the $2d$-dimensional phase space, has dimension $2d - 1$, or, which is the same, co-dimension 1.

Remark 1.4. *In particular, if $d = 1$, the constant energy manifold $H^{-1}(E)$ consists of (the union of) curves, which implies that motions can be determined geometrically. In such a case, $H^{-1}(E)$ consists of the union of the curves*

$$v_{\pm}(x) = \pm \sqrt{\frac{2}{m}(E - U(x))} \ , \tag{1.14}$$

defined for all x such that $U(x) \leq E$.

1.2.3 Gyroscopic forces ($d = 3$)

From the proof of Proposition 1.1, last line of (1.13), it follows that

Proposition 1.2. *The energy (1.12) is a constant of motion of the Newton equation $m\ddot{x} = f$ for any force of the form $f = -\nabla_x U(x) + b \times \dot{x}$, where $b(x, \dot{x}, t)$ is any vector valued function of its arguments.*

Force terms orthogonal to the velocity of the particle are called *gyroscopic*: their instantaneous effect, taken alone, is to make the particle turn around the vector b. One example is the magnetic part of the Lorentz force, i.e. $q(v \times B)/c$, acting on a charged particle (charge q) moving in a given magnetic field B. Another example is the Coriolis "apparent" force, i.e. $2m(v \times \omega)$, acting on a particle moving in a non inertial frame rotating with instantaneous angular velocity ω. In both cases B and ω may depend on both x and t.

1.2.4 Central forces: angular momentum; areal velocity ($d = 3$)

Central forces are those forces parallel to the vector position x when this is referred to a particular origin O, called the center of the force, namely:

$$f(x, v, t) = \varphi(x, v, t)\hat{x} \ , \tag{1.15}$$

where $\varphi : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given scalar function, and $\hat{x} = x/|x|$ denotes the unit vector in the direction of x. In such a case there exists another conservation law.

Proposition 1.3. *The vector valued function* angular momentum

$$\ell(x, \dot{x}) := x \times m\dot{x} \tag{1.16}$$

is constant along the solutions of the Newton equation $m\ddot{x} = \varphi\hat{x}$. In such a case the trajectory of the particle in the physical space $\mathbb{R}^d$ lies on the plane orthogonal to $\ell = x(0) \times m\dot{x}(0)$ if the latter is different from zero. The invariant plane of motion has equation $\ell \cdot x = 0$. If $x(0)$ is parallel to $\dot{x}(0)$, then $\ell = 0$ for all times, and $x(t)$ is parallel to $\dot{x}(t)$ for any t, the motion taking place on a line.

Proof. Taking the time derivative of the angular momentum (1.16) along the solutions of the Newton equation $m\ddot{x} = f$ (any f for the moment), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell(x(t), \dot{x}(t)) = \dot{x} \times m\dot{x} + x \times m\ddot{x} = x \times f \ .$$

The latter quantity, called the moment of the force, is zero if (and actually only if) f is a central force, i.e. $f = \varphi\hat{x}$. Since $x(t) \cdot \ell(0) = x(t) \cdot \ell(t) \equiv 0$, the orbit lies on the plane of equation $\ell \cdot x = 0$, orthogonal to ℓ and passing through the center of force, if $\ell \neq 0$. If $\ell = 0$, then $x(t)$ is parallel to $\dot{x}(t)$ for all times, which means that

the motion takes plane on the line connecting the particle with the center of force at the initial time. □

A significant property of central forces is that the vector position $x(t)$ sweeps equal areas in equal times, or, which is the same, that the area swept by $x(t)$ increases linearly with time. When referred to planetary motions, this is called the second Kepler law, but is actually a property of all central motions, i.e. motions of particles subject to a central force, and is a consequence of the law of conservation of angular momentum.

►**Problem 1.5.** *Show that the area $A(t)$ swept by the vector position satisfies*

$$\dot{A}(t) = \frac{|\ell(t)|}{2m} \ , \tag{1.17}$$

whatever the motion of the particle, i.e. whatever be the force f. Then, in particular, for central forces $\ell(t) = \ell(0)$ and the areal velocity $\dot{A}$ is constant, i.e. $A(t) = A(0) + \frac{|\ell|}{2m}t$. Hint: start by observing that, by the definition of cross product, the area swept by the vector position in a time $\mathrm{d}t$ *is* $\mathrm{d}A = |x \times \mathrm{d}x|/2$.

1.2.5 Central conservative forces ($d = 3$)

Among the central forces, a special role is played by the conservative ones, i.e. those central forces such that

$$f(x) = \varphi(|x|)\hat{x} = -\nabla U(|x|) \ ; \quad U(s) = -\int \varphi(s)\ \mathrm{d}s \ . \tag{1.18}$$

The most important forces of this kind are the following:

- the Newton gravitational force $\varphi(r) = -GMm/r^2$, where M is the mass of the star placed at the center of force, m the mass of the planet, and G the Newton gravitational constant;
- the Coulomb electrostatic force $\varphi(r) = qQ/r^2$, where q is the charge of the particle and Q is the other charge placed at the center of force;
- the Hooke elastic force $\varphi(r) = -kr$, k being the elastic constant.

The spherical symmetry of the force (1.18) implies that the Newton equation $m\ddot{x} = f(x)$ is invariant under any rotation R: if $x = \mathsf{R}\xi$, then $f(\mathsf{R}\xi) = \mathsf{R}f(\xi)$, so that $\ddot{\xi} = f(\xi)$. Recall that $\mathsf{R}^T\mathsf{R} = \mathbf{1}_3$, where the superscript T denotes transposition, whereas $\mathbf{1}_d = \mathrm{diag}(1,1,1)$ is the 3×3 identity.

Exercise 1.1. *Prove the right hand side of (1.18), i.e. that if $f(x) = \varphi(|x|)\hat{x}$ then there exists $U(|x|)$ such that $f = -\nabla U$. Hint: show first that $\nabla U(|x|) = U'(|x|)\hat{x}$; recall that $|x|^2 = x_1^2 + \cdots + x_d^2$.*

Exercise 1.2. *Show that if a central, positional, autonomous force is conservative, then it is necessarily spherically symmetric, i.e. that $\varphi(x)\hat{x} = -\nabla U$ implies that φ, and thus U, are functions of $|x|$ only. Hint: pass to spherical polar coordinates (r, θ, ϕ) and show that $U = U(r)$, where $r = |x|$.*

Central conservative forces are extremely interesting because in this case both energy and angular momentum are constants of motion, and this in turn allows to reduce the study of the motion to a one-dimensional Newtonian problem (that of the radial motion). Thus, such motions are "integrable", i.e. solvable (in a geometric sense, in general, i.e. without explicitly solving the Newton equation).

1.3 Problems of single particle motion

In this Section some physically relevant Newtonian problems are presented, grouped together according to the classification made above.

1.3.1 The 3D harmonic oscillator

The first important example of central conservative motion is that of a particle of mass m attracted by the center of force by the Hooke elastic force, i.e. connected to the origin by an ideal spring of zero rest length. The Newton equation is

$$m\ddot{x} = -kx \ , \quad x \in \mathbb{R}^3 \ , \tag{1.19}$$

where $k > 0$ is the elastic constant (we will also comment about the case $k < 0$; see below). In the notations introduced above, the force is of the form $f = \varphi(|x|)\hat{x}$, where $\varphi(|x|) = -k|x|$ (recall that $x = |x|\hat{x}$). By defining $\omega = \sqrt{k/m}$ one can rewrite (1.19) in the form $\ddot{x} = -\omega^2 x$, i.e. the 3D version of the harmonic oscillator equation. The problem we want to solve here, is to determine the form of the orbit, i.e. the form of the curve $t \mapsto x(t)$.

Due to the conservation of the angular momentum $\ell = x \times m\dot{x}$, the motion takes place on the plane of equation $\ell \cdot x = 0$. We can then rotate the axes of the reference frame in such a way to bring the unit vector $\hat{e}_3$ to coincide with $\hat{\ell}$. Thus, x and $\dot{x}$ have two components only (with respect to an arbitrary basis $\hat{e}_1, \hat{e}_2$ on the plane of motion), and (1.19), by components, reads

$$\begin{cases} \ddot{x}_1 = -\omega^2 x_1 \\ \ddot{x}_2 = -\omega^2 x_2 \end{cases} \ ; \quad \omega = \sqrt{k/m} \ . \tag{1.20}$$

These are two equations of 1D harmonic oscillator, and their general solutions read

$$\begin{cases} x_1(t) = a\cos(\omega t + \varphi) \\ x_2(t) = b\cos(\omega t + \psi) \end{cases} \ , \tag{1.21}$$

where a, b, φ, ψ are arbitrary constants. With no loss of generality we can choose the amplitudes $a, b > 0$. Now, defining $\delta := \psi - \varphi$, and $\theta := \omega t + \varphi$, we rewrite (1.21) as

$$\begin{cases} x_1(t) = a\cos(\theta) \\ x_2(t) = b\cos(\theta + \delta) \end{cases} \ . \tag{1.22}$$

This is clearly the equation of a 2π-periodic plane curve in parametric form (the parameter is the angle θ). We are going to show that such a curve is precisely an ellipse whose axes are rotated, in general, with respect to the canonical ones.

As a preliminary remark, we observe that when $\delta = 0$, (1.21) describes a segment on the line of Cartesian equation $x_2 = (b/a)x_1$, the segment being determined by the condition $|x_1| \leq a$, which implies $|x_2| \leq b$; both of them being due to $|\cos\theta| \leq 1$. Thus the motion corresponding to $\delta = 0$ is linear. The same holds for $\delta = \pi$, where the line equation becomes $x_2 = -(b/a)x_1$. The cases of linear motion $\delta = 0, \pi$ (and thus $\delta = n\pi$, for any integer n) corresponds to $\ell = 0$, when the 3D equation (1.19) degenerates into a 1D equation.

In the sequel we suppose $\ell \neq 0$. By eliminating $\cos\theta$ and $\sin\theta$ in (1.22) we get

$$\frac{x_1^2}{a^2\sin^2\delta} + \frac{x_2^2}{b^2\sin^2\delta} - \frac{2x_1x_2\cos\delta}{ab\sin^2\delta} = 1 \ , \tag{1.23}$$

that can be rewritten in the form $x \cdot \mathsf{M}x = x^T\mathsf{M}x = 1$; explicitly

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} \frac{1}{a^2\sin^2\delta} & \frac{-\cos\delta}{ab\sin^2\delta} \\ \frac{-\cos\delta}{ab\sin^2\delta} & \frac{1}{b^2\sin^2\delta} \end{pmatrix}}_{\mathsf{M}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \ . \tag{1.24}$$

The matrix M defining the homogeneous quadratic form on the right hand side is real symmetric, and by the spectral theorem [15] is orthogonally similar to a real diagonal one: there exists an orthogonal 2×2 matrix R, satisfying $\mathsf{R}^T\mathsf{R} = \mathsf{1}_2$, such that $\mathsf{R}^T\mathsf{M}\mathsf{R} = \mathrm{diag}(\lambda_-, \lambda_+)$, where λ_- and λ_+ are the two real eigenvalues of M, whereas the two columns of R are the corresponding, mutually orthogonal, unit eigenvectors of u^- and u^+ (in this order). As a consequence, the change of coordinates $y = \mathsf{R}^T x$ diagonalizes the quadratic form $x \cdot \mathsf{M}x$. Indeed, upon substituting $x = \mathsf{R}y$ in the latter expression, we get

$$(\mathsf{R}y) \cdot \mathsf{M}\mathsf{R}y = y \cdot (\mathsf{R}^T\mathsf{M}\mathsf{R})y = y \cdot \mathrm{diag}(\lambda_-, \lambda_+)y \ ,$$

i.e. we map equation (1.24) into

$$\lambda_- y_1^2 + \lambda_+ y_2^2 = 1 \ . \tag{1.25}$$

This is the Cartesian canonical equation of a conic section. The possible cases are: an ellipse, if λ_- and λ_+ are both positive, or a hyperbola, if one is positive and the other one is negative (both negative is impossible). The latter case is not expected on physical grounds: the force is attractive and grows with the distance from the center, which excludes unbounded motions. In fact, by a quick inspection of the matrix M, one gets $\mathrm{tr}(\mathsf{M}) = \lambda_- + \lambda_+ > 0$, $\det(\mathsf{M}) = \lambda_-\lambda_+ > 0$, which imply $0 < \lambda_- < \lambda_+$, so that (1.25) is the equation of an ellipse whose lengths of the major and minor semi-axes are $1/\sqrt{\lambda_-}$ and $1/\sqrt{\lambda_+}$, respectively. The explicit computation of the eigenvalues yields

$$\lambda_\pm = \frac{a^2 + b^2 \pm \sqrt{(a^2+b^2)^2 - 4a^2b^2\sin^2\delta}}{2a^2b^2\sin^2\delta} \ . \tag{1.26}$$

Exercise 1.3. *Check that $\lambda_+ > \lambda_- > 0$.*

Remark 1.5. *Concerning the form of the matrix* R *whose (ordered) columns are the eigenvectors* u^- *and* u^+*, one can obviously compute it. On the other hand, the unit vector* $u^- \in \mathbb{R}^2$ *can always be written in the form* $u^- = (\cos\phi, \sin\phi)^T$*. The only choices for the unit vector* u^+*, orthogonal to* u^-*, are* $u^+ = \pm(-\sin\phi, \cos\phi)^T$ *(check it). The two choices of sign yield*

$$\mathsf{R}_+(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \quad , \quad \mathsf{R}_-(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} .$$

One has $\det(\mathsf{R}_+) = +1$ *and, in particular,* $\mathsf{R}_+(0) = \mathsf{1}_2 = \mathrm{diag}(1,1)$*. On the other hand,* $\det(\mathsf{R}_-) = -1$ *and* $\mathsf{R}_-(\phi)$ *does not coincide with the identity for any value of* ϕ*; in particular,* $\mathsf{R}_-(0) = \mathrm{diag}(1,-1)$*. Recall that the relation* $\mathsf{R}^T\mathsf{R} = \mathsf{1}_2$ *implies* $\det(\mathsf{R}) = \pm 1$*. Thus, proper rotations are orthogonal matrices with determinant* $+1$*: they rotate the Cartesian axes preserving their mutual initial orientation. Orthogonal matrices with determinant* -1 *correspond to the initial inversion of an axis (the second one), followed by a proper rotation:* $\mathsf{R}_-(\phi) = \mathsf{R}_+(\phi)\mathrm{diag}(1,-1)$*. The set of all orthogonal matrices* 2×2 *is denoted by* $O(2)$*. The subset of orthogonal matrices of unit determinant, i.e. the component of* $O(2)$ *connected to (containing) the identity, is denoted by* $SO(2)$*.* $O(2)$ *is a group with respect to matrix multiplication (check it), and is called the* 2×2 *"orthogonal group".* $SO(2)$ *is a sub-group of* $O(2)$ *and is called the* 2×2 *"special orthogonal group". You can prove that the set of matrices* R_- *and* R_+ *introduced above, and denote respectively by* $O^-(2)$ *and* $O^+(2) = SO(2)$*, are such that* $O^-(2) \cup O^+(2) = O(2)$*,* $O^-(2) \cap O^+(2) = \emptyset$*. Such a decomposition holds replacing* 2 *by any dimension* n*. See [6].*

►**Problem 1.6.** *Prove that unbounded solutions of (1.19) are forbidden, a priori, by the energy conservation law (reduction to plane motion can be used, but is not necessary).*

►**Problem 1.7.** *Suppose to reverse the sign of the elastic constant* k *in (1.19). This is the equation of the 3D hyperbolic oscillator (but nothing oscillates), or harmonic repulsor (but there is nothing harmonic here). Set* $k = -|k|$ *and call* $\omega = \sqrt{|k|/m}$*. You arrive at (1.20) with a plus sign on the right hand side. The solution of the two equations can be written in the form* $x_i(t) = a_i e^{\omega t} + b_i e^{-\omega t}$*,* $i = 1,2$ *(check it). Eliminate the exponentials and get an equation of the form* $x \cdot \mathsf{M}x = 1$*. Prove that this can only be diagonalized to the canonical equation of a hyperbola. Hint:* $e^{\omega t}e^{-\omega t} = 1$*. What about the energy conservation law in this case?*

The conclusion of this section is the following: *the orbits of problem (1.19), with* $\ell \neq 0$*, are ellipses if* $k > 0$ *and hyperbolas if* $k < 0$*.*

1.3.2 An excursus on conic sections

Conic sections are so called because are the curves obtained by intersecting a cone with a plane transversal to its axis. Depending on the inclination of the plane, the intersection curve is either close, in which case it is an ellipse, or open, in which case it is a hyperbola, the critical open curve separating the two cases being a parabola.

The orbits of the 3D harmonic oscillator are conic sections, and the same holds for Keplerian motions, such a coincidence being not by chance, as we shall see below. We recall in the sequel the basic mathematics of conic sections, with a particular focus on the ellipse.

Exercise 1.4. *Start from the geometric definition of ellipse as the locus of points of the plane such that the sum of their distances d_1 and d_2 from two fixed points F_1 and F_2, called foci, is constant: $d_1 + d_2 = 2a$, with the focal distance $\overline{F_1F_2} = 2f$. Set the foci F_2 and F_1 on the x axis of the Cartesian plane at $(f, 0)$ and $(-f, 0)$, respectively. First, check that when the point is on the y axis its distance b from the origin is given by $b^2 = a^2 - f^2$. Now, check that the distances of a point $P = (x, y)$ on the locus are given by $d_1^2 = (x - f)^2 + y^2$, $d_2^2 = (x + f)^2 + y^2$, respectively. Now obtain the Cartesian canonical equation of the ellipse, namely*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ , \tag{1.27}$$

where a and b are the lengths of the major and minor semi-axes, respectively. Hint: square $d_1 + d_2 = 2a$, isolate $2d_1d_2$ and square again.

Exercise 1.5. *Set $x = f + r\cos\theta$ and $y = r\sin\theta$ into (1.27), so that r is the distance from the focus F_1 and θ the inclination angle of F_1P with respect to the x axis (counterclockwise). Now obtain the polar equation of the ellipse, namely*

$$r = \frac{p}{1 + \varepsilon\cos\theta} \ , \tag{1.28}$$

where the quantities

$$\varepsilon := \frac{f}{a} \quad ; \quad p := \frac{a^2 - f^2}{a} = \frac{b^2}{a} \tag{1.29}$$

are called eccentricity (ε) and parameter (p) of the ellipse, respectively. Hint: start from (1.27) written in the form $b^2x^2 + a^2y^2 = a^2b^2$ and substitute there $x = f + r\cos\theta$, $y = r\sin\theta$; solve for r, recalling that $r > 0$.

Remark 1.6. *Equation (1.28) defines all the conic sections. When $\varepsilon = 0$, (1.28) is the equation of a circle: $r(\theta) = p$. For proper ellipses, $0 < \varepsilon = f/a < 1$. When $\varepsilon = 1$, (1.28) is the polar equation of a parabola. Finally, if $\varepsilon > 1$ (1.28) defines hyperbolas. Observe that when $\varepsilon \geq 1$ the denominator on the right hand side of (1.28) vanishes at at two finite angles: $\pm\pi$ at $\varepsilon = 1$ and $\pm\hat{\theta}(\varepsilon)$ at $\varepsilon > 1$. In the latter case the angle $\pm\hat{\theta}$ are the inclination angles of the asymptotes of the hyperbola (one branch). See [14] for further details.*

For a complete introduction to the geometry and physical relevance of the conic sections (and much more) see [4].

1.3.3 The Kepler-Newton problem

The so called Kepler-Newton problem consists in the study of the motion of a particle of mass m subject to a conservative central force inversely proportional to the square of the distance from the center, namely

$$m\ddot{x} = \varphi(|x|)\hat{x} \ ; \quad \varphi(|x|) := -\frac{k}{|x|^2} \ , \quad x \in \mathbb{R}^3 \ , \tag{1.30}$$

where $k \neq 0$ is a constant. The physically relevant cases of attractive force, with $k > 0$, are the gravitational one, where $k = GMm$, G being the gravitational Newton constant, m the mass of a planet and M the mass of a star (at rest in the center), or the Coulomb one, where, for example, $k = e^2$, e being the common charge of the electron and the proton, which is the case of the Hydrogen atom (with the proton at rest in the center). A repulsive force, with $k < 0$, rules the dynamics of a particle of given charge q repelled by a center of equal charge q; here $k = -q^2$.

The (invention and) study of problem (1.30) was motivated by the three empirical laws induced by Kepler on the basis of the analysis of the data collected by Brahe. The three Kepler laws are the following.

1. Planets move on elliptical orbits, with the Sun in one of the foci.
2. The vector Sun-planet sweeps equal areas in equal times (constant areal velocity).
3. If, for any planet, T and a denote the revolution period and the length of the orbital major semi-axis, then the ratio T^2/a^3 is a constant, independent of the planet.

Some data relative to the planetary orbits in the solar system are reported in Table **??** below. In what follows, we study equation (1.30) with $k > 0$ under the hypothesis $\ell \neq 0$, with the aim of deducing (and extend) the three Kepler laws. As in the case of the 3D harmonic oscillator, we rotate the reference system in such a way that $\hat{e}_3 = \hat{\ell}$. On the plane of motion, orthogonal to ℓ, we could rewrite equation (1.30) in Cartesian (two) components. However, to our purposes here, this is not the convenient choice. Instead, we pass to plane polar coordinates. We attach to each point in the plane a local, orthonormal polar basis, defined as follows. Consider the transformation to plane polar coordinates defined by

$$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases} . \tag{1.31}$$

Then the unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ tangent to the coordinate lines $\theta = constant$ and $r = constant$, respectively, are given by

$$\hat{e}_r := \frac{\partial x}{\partial r} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \ ; \ \hat{e}_\theta = \frac{1}{r}\frac{\partial x}{\partial \theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} . \tag{1.32}$$

Observe that the vectors $\hat{e}_r$ and $\hat{e}_\theta$ depend only on the angle θ; moreover, the the polar basis $\hat{e}_r, \hat{e}_\theta$ preserves the orientation of the plane: $\hat{e}_r(0) = \hat{e}_1 = (1,0)^T$, $\hat{e}_\theta(0) = \hat{e}_2 = (0,1)^T$. One

easily checks the fundamental relations:

$$\frac{\mathrm{d}\hat{e}_r}{\mathrm{d}\theta} = \hat{e}_\theta \quad ; \quad \frac{\mathrm{d}\hat{e}_\theta}{\mathrm{d}\theta} = -\hat{e}_r \ . \tag{1.33}$$

We now start from the obvious relation $x = r\hat{e}_r(\theta)$ (we are just rewriting (1.31) making use of the expression (1.32) of $\hat{e}_r$). Along the motions, i.e. the solutions of (1.30), r and θ depend on time. Thus, starting from $x(t) = r(t)\hat{e}_r(\theta(t))$, one easily finds

$$\dot{x} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \ ; \tag{1.34}$$

$$\ddot{x} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta \ . \tag{1.35}$$

Such a computation makes use of the chain rule together with repeated use of (1.33). Upon substitution of (1.35) into the left hand side of (1.30), now thought of in the plane ($x \in \mathbb{R}^2$), one gets

$$m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta = \varphi(r)\ \hat{e}_r \ ,$$

i.e., since $\hat{e}_r$ and $\hat{e}_\theta$ are linearly independent,

$$\begin{cases} m(\ddot{r} - r\dot{\theta}^2) = \varphi(r) \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \end{cases} \ . \tag{1.36}$$

This is equation (1.30) written on the plane of motion in plane polar coordinates. The second equation, i.e. the θ component, expresses nothing but the conservation of the angular momentum. Indeed, multiplying it by r, one gets

$$0 = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) \ .$$

Planet	a [A.U.]	T (years)	ε	α	m_P/m_S	T^2/a^3
Mercury	0.39	0.24	0.205	7^o00	$1.6\ 10^{-7}$	0.97
Venus	0.72	0.61	0.006	$3^o23'$	$2.4\ 10^{-6}$	0.99
Earth	1	1	0.016	$0^o00'$	$3.0\ 10^{-6}$	1
Mars	1.52	1.88	0.093	$1^o51'$	$3.2\ 10^{-7}$	1.00
Jupiter	5.20	11.83	0.048	$1^o18'$	$0.9\ 10^{-3}$	0.99
Saturn	9.55	29.43	0.055	$2^o29'$	$2.8\ 10^{-4}$	0.99
Uranus	19.22	84.18	0.046	$0^o46'$	$4.4\ 10^{-5}$	0.99
Neptune	30.11	164.56	0.008	$1^o46'$	$0.5\ 10^{-4}$	0.99
Pluto	39.60	247.47	0.246	$17^o07'$	$1.7\ 10^{-6}$	0.98

Table 1.1: In the present table, for each planet, a is the major semi-axis length, in astronomic units (A.U.), i.e. referred to the value of a for the Earth; T is the revolution period, in years (again referred to the value of T for the Earth); ε is the eccentricity, α the inclination angle of the orbital plane with respect to that of the Earth; m_P/m_S is the ratio mass of the planet to mass of the Sun. In the last column, the 3rd Kepler law is checked. Source of data: [26].

On the other hand, the angular momentum in polar coordinates reads

$$\ell = x \times m\dot{x} = r\hat{e}_r \times m(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = mr^2\dot{\theta}\hat{e}^3 \ , \tag{1.37}$$

where $\hat{e}_3 = \hat{\ell} = \hat{e}_r \times \hat{e}_\theta$. Thus, $mr^2\dot{\theta} = \ell_3$, and system (1.36) can be rewritten in the form

$$\begin{cases} m\ddot{r} = \frac{\ell_3^2}{mr^3} + \varphi(r) \\ \dot{\theta} = \frac{\ell_3}{mr^2} \end{cases} \ . \tag{1.38}$$

Remark 1.7. *In what did up to now we have never made use of the specific form of the force $\varphi(r)$, so that (1.38) actually holds for any particle motion in a central, conservative force field.*

Now the radial equation (the first one) in (1.38) involves only the variable $r(t)$, so that the problem has been reduced to study the one-dimensional radial motion (we will com back below to its qualitative analysis). In principle, solving for $r(t)$, one then determines $\theta(t)$ from the second of equations (1.38). Such a procedure is possible but does not lead to a solution expressed in simple closed form. However, the form of the orbit can be explicitly determined in the polar form $\theta \mapsto r(\theta)$ since, if $\ell_3 > 0$, for example, then $\dot{\theta} > 0$ and the function $t \mapsto \theta(t)$ is invertible, so that $t(\theta)$ exists for all $\theta \in \mathbb{R}$. By the chain rule, and identifying $r(\theta) = r(t(\theta))$, we get

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{mr^2}{\ell_3}\dot{r} \quad \Leftrightarrow \quad \dot{r} = \frac{\ell_3}{mr^2}r'(\theta) \ . \tag{1.39}$$

Of course, in the case $\ell_3 < 0$ we would arrive exactly at the same result, the sign of ℓ_3 determining only the direction of rotation (counter-clockwise if $\ell_3 > 0$). Now, instead of re-applying the chain rule and to express $\ddot{r}$ in terms of $r''(\theta)$, we follow another way, making use of the energy conservation law. The latter law, for equation (1.30), reads

$$\frac{m|\dot{x}|^2}{2} - \frac{k}{|x|} = E \ . \tag{1.40}$$

Substituting inside the latter $x = r\hat{e}_r$ and $\dot{x} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$, and $\dot{\theta}$ from (1.38) (the second equation), yields

$$\frac{m\dot{r}^2}{2} - \frac{k}{r} + \frac{\ell_3^2}{2mr^2} = E \ . \tag{1.41}$$

Remark 1.8. *The energy conservation law for the radial motion, governed by the first of equations (1.38), has the general expression*

$$\frac{m\dot{r}^2}{2} + U_e(r) = E \ ; \quad U_e(r) := U(r) + \frac{\ell_3^2}{2mr^2} \ , \tag{1.42}$$

where $U(r) = -\int \varphi(r)\mathrm{d}r$ is the potential energy of the force. The quantity $U_e(r)$ is called the "effective potential energy" of the central motion. In the Kepler problem $U(r) = -k/r$, and (1.42) coincides with (1.41). In general, for any central motion, the energy conservation law of the radial motion coincides with the total energy conservation law.

Now, substituting $\dot{r}$ from (1.39) into (1.41), one gets

$$\frac{\ell_3^2}{2m}\left[\frac{(r')^2}{r^4}+\frac{1}{r^2}\right]-\frac{k}{r}=E\ . \tag{1.43}$$

The latter energy conservation law is extremely simplified if one rewrites it in terms of the variable

$$u(\theta):=\frac{1}{r(\theta)}\ , \tag{1.44}$$

which is due to Binet; one gets

$$\frac{\ell_3^2}{2m}\left[(u')^2+u^2\right]-ku=E\ . \tag{1.45}$$

Taking the derivative of the latter equation with respect to θ, and taking into account that E does not depend on it, and excluding the case of $u'(\theta)$ identically vanishing (corresponding to circular orbits), yields the *Binet equation*

$$u''+u-\frac{km}{\ell_3^2}=0\ . \tag{1.46}$$

The general solution of the latter equation is easily checked to be given by

$$u(\theta)=\frac{km}{\ell_3^2}+A\cos(\theta-\theta_0)\ , \tag{1.47}$$

where A and θ_0 are arbitrary constants. We can choose, without any loss of generality, $A>0$ and $\theta_0=0$ (convince yourself about it). Recalling that $u=1/r$, we can rewrite (1.47) as

$$r(\theta)=\frac{\frac{\ell_3^2}{km}}{1+\frac{\ell_3^2 A}{km}\cos\theta}\ , \tag{1.48}$$

i.e. a conic section in polar form, with parameter $p=\ell_3^2/(km)$ and eccentricity $\varepsilon=\ell_3^2 A/(km)$ (go back to Remark 1.6 and what said before it). In order to determine the value of the constant $A>0$, and thus of the eccentricity in terms of E and ℓ_3 (and thus in terms of the initial conditions), we reinsert (1.47) into the energy conservation law (with $\theta_0=0$). Elementary calculations lead to determine

$$A^2=\frac{k^2m^2}{\ell_3^4}+\frac{2mE}{\ell_3^2}\ ,$$

so that the eccentricity $\varepsilon=\ell_3^2 A/(km)$ of the orbit is given by

$$\varepsilon=\sqrt{1+\frac{2\ell_3^2 E}{k^2m}}\ . \tag{1.49}$$

Our first conclusion is that the orbits of the Kepler-Newton problem are conic sections of the form

$$r(\theta)=\frac{\frac{\ell_3^2}{km}}{1+\sqrt{1+\frac{2\ell_3^2 E}{k^2m}}\ \cos\theta}\ . \tag{1.50}$$

For planetary motions ($k=GMm$) this implies the following.

- If $E = -km/(2\ell_3^2)$, $\varepsilon = 0$, $r(\theta) = \ell_3^2/(km)$ is constant and the orbit is circular; the angular velocity is also constant and given by $\dot{\theta} = \ell_3/(mr^2) = k^2 m/\ell_3^3$.
- If $-km/(2\ell_3^2) < E < 0$, $0 < \varepsilon < 1$ and the orbits are ellipses. This is the first Kepler law (recall that the in the polar form r refers to one of the foci).
- To $E = 0$, i.e. $\varepsilon = 1$, there corresponds a parabolic orbit.
- Finally, if $E > 0$, i.e. $\varepsilon > 1$, the orbits are hyperbolas.

The second Kepler law concerns the constancy of the areal velocity, which has been proven in general, for central forces. We here recall it, namely

$$\dot{A} = \frac{\ell_3}{2m} \quad \Leftrightarrow \quad A(t) = A(0) + \frac{\ell_3}{2m}\, t \ . \tag{1.51}$$

Consider now the case of elliptic motions. Imagine to set $t = 0$ when $\theta = 0$ (at the orbital perihelion), with $A(0) = 0$. After one complete revolution, denoting by T the revolution period, one has $A(T) = \ell_3 T/(2m)$, where $A(T)$ is the area of the ellipse. Thus

$$\begin{aligned} T &= \frac{2m}{\ell_3}\pi ab = \frac{2\pi m}{\ell_3} a\sqrt{a^2 - f^2} = \frac{2\pi m}{\ell_3} a^{3/2}\sqrt{\frac{a^2 - f^2}{a}} = \\ &= \frac{2\pi m}{\ell_3} a^{3/2}\sqrt{p} = 2\pi\sqrt{\frac{m}{k}}\ a^{3/2} \ . \end{aligned}$$

Recalling that $k = GMm$, we get

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \ , \tag{1.52}$$

i.e. the third Kepler law with an explicit constant which depends only on the mass of the star (the Sun, for example) and is independent of the mass of the planet.

Exercise 1.6. *The Binet equation (1.46) can be obtained by re-applying the chain rule (1.39) to first get $\ddot{r}$ as a function of r, r' and r'', and substitute it into the first of equations (1.38). Then, by setting $r = 1/u$ one obtains the result.*

1.3.4 The eccentricity vector

An alternative way to treat the Kepler-Newton problem rests on the introduction of another vector valued constant of motion, namely what is usually referred to as the Laplace-Runge-Lenz vector, whose discovery perhaps goes back to Laplace and Hamilton, who introduced it independently of each other; see the historical notes [20, 21]. The *eccentricity vector* e, as we will refer to in the sequel, is defined as follows:

$$\mathsf{e} := \frac{1}{k} v \times \ell - \hat{x} \ , \tag{1.53}$$

where reference is made to the Kepler-Newton equation (1.30): k is the Kepler constant, x, $v = \dot{x}$ and $\ell = x \times mv$ being the position, the velocity and the angular momentum of the particle. The following proposition holds.

Proposition 1.4. *The eccentricity vector* e *(1.53) is constant along the solutions of the Kepler Newton equation* $m\ddot{x} = -k\hat{x}/|x|^2$*;* $|\mathsf{e}| = \varepsilon$*, the latter being the orbital eccentricity; for elliptic orbits* e *lies on the focal axes, in the direction of the pericenter.*

Proof. We here recall that $\hat{x} = \hat{e}_r$, $m\dot{v} = -k\hat{e}_r/r^2$, $\ell = mr^2\dot{\theta}\hat{e}_3$, the latter being a constant vector, and $\hat{e}_r, \hat{e}_\theta, \hat{e}_3$ is a right-handed basis. We have

$$\frac{\mathrm{d}\mathsf{e}}{\mathrm{d}t} = \frac{1}{k}\left(-\frac{k}{mr^2}\hat{e}_r\right) \times (mr^2\dot{\theta}\hat{e}_3) - \dot{\theta}\hat{e}_\theta = (\dot{\theta} - \dot{\theta})\hat{e}_\theta \equiv 0 \ .$$

Now, taking into account that $v = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$, writing $\ell = \ell_3\hat{e}_3$, and eliminating $\dot{\theta}$ from $mr^2\dot{\theta} = \ell_3$, the vector e turns out to be given by

$$\mathsf{e} = \left(\frac{\ell_3 r\dot{\theta}}{k} - 1\right)\hat{e}_r - \left(\frac{\ell_3\dot{r}}{k}\right)\hat{e}_\theta = \left(\frac{\ell_3^2}{kmr} - 1\right)\hat{e}_r - \left(\frac{\ell_3\dot{r}}{k}\right)\hat{e}_\theta \ .$$

The square modulus of the latter vector is

$$\begin{aligned} |\mathsf{e}|^2 &= \left(\frac{\ell_3^2}{kmr} - 1\right)^2 + \left(\frac{\ell_3\dot{r}}{k}\right)^2 = \\ &= 1 + \frac{2\ell_3^2}{mk^2}\left(\frac{m\dot{r}^2}{2} - \frac{k}{r} + \frac{\ell_3^2}{2mr^2}\right) = 1 + \frac{2\ell_3^2 E}{mk^2} = \varepsilon^2 \ , \end{aligned}$$

where (1.41) and (1.49) have been used. This proves that $|\mathsf{e}| = \varepsilon$. Now, to end up the proof, we take the scalar product of the eccentricity vector (1.53) with the position vector $x = r\hat{e}_r$. On the left we get

$$\mathsf{e} \cdot x = |\mathsf{e}|\,|x| \cos\theta = \varepsilon\, r \cos\theta \ ,$$

where θ is the angle between e and x (we already give it the same name of the orbital angle because it turns out to be exactly the same). On the other hand,

$$\begin{aligned} \mathsf{e} \cdot x &= \frac{1}{k}(v \times \ell) \cdot x - x \cdot \hat{e}_r = \frac{1}{k}(x \times v) \cdot \ell - r = \\ &= \frac{1}{km}(x \times mv) \cdot \ell - r = \frac{\ell_3^2}{km} - r \ , \end{aligned}$$

where we used the known property of the mixed product $(a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b$ valid for any triple of vectors $a, b, c \in \mathbb{R}^3$. We thus get $\varepsilon\, r \cos\theta = \frac{\ell_3^2}{km} - r$, i.e.

$$r = \frac{\frac{\ell_3^2}{km}}{1 + \varepsilon\cos\theta} \ ,$$

identical to the orbital equation (1.50). This also proves that, having called θ the angle between e and the vector position $x(t)$, the constant eccentricity vector e lies on the focal axis of the ellipse, in the direction of the pericenter. □

1.3.5 Levi-Civita - Bohlin regularization

The 3D harmonic and hyperbolic oscillator, and the Kepler-Newton problem display orbits that are conic sections for all initial conditions. This is not by chance since, from a mathematical point of view, the two problems are one and the same. This was first obtained by Levi-Civita [32], and independently by Bohlin [8] (we here follow the subsequent, more synthetic treatment by Bohlin). Let us rewrite the Kepler-Newton equation in the form

$$\ddot{x} = -\frac{c}{|x|^3}x \ ; \quad c = k/m > 0 \ . \tag{1.54}$$

The energy conservation law reads

$$\frac{|\dot{x}|^2}{2} - \frac{c}{|x|} = \mathcal{E} \ ; \quad \mathcal{E} = E/m \ . \tag{1.55}$$

Let us introduce the complex variable $z = x_1 + \imath x_2 \in \mathbb{C}$, where $\imath$ is the imaginary unit. One can then write (1.54) and (1.55) in the form

$$\ddot{z} = -\frac{c}{|z|^3}z \ ; \tag{1.56}$$

$$\frac{|\dot{z}|^2}{2} - \frac{c}{|z|} = \mathcal{E} \ , \tag{1.57}$$

with $z = x_1 + \imath x_2 \in \mathbb{C}$. Observe that $|z|^2 = \bar{z}z = x_1^2 + x_2^2 = |x|^2$ (where $\bar{z} = x_1 - \imath x_2$ is the complex conjugate of z). The Levi-Civita - Bohlin map $\mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R} : (z, t) \mapsto (w, \tau)$ is defined by the relations

$$z = w^2 \ \ ; \ \ \mathrm{d}t = |w|^2 \mathrm{d}\tau \ . \tag{1.58}$$

By the chain rule, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{1}{|w|^2}\frac{\partial}{\partial\tau} \ ,$$

so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{2ww'}{|w|^2} = \frac{2w'}{\bar{w}} \ ; \tag{1.59}$$

Substituting the latter and $z = w^2$ into (1.57), yields

$$2|w'|^2 = c + \mathcal{E}|w|^2 \ . \tag{1.60}$$

Concerning the acceleration, taking also into account the latter relation, we find

$$\begin{aligned} \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} &= \frac{1}{|w|^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{2w'}{\bar{w}} = \frac{2w''}{|w|^2\bar{w}} - \frac{2|w'|^2}{|w|^2\bar{w}^2} = \\ &= \frac{2w''}{|w|^2\bar{w}} - \frac{c}{|w|^2\bar{w}^2} - \frac{\mathcal{E}}{\bar{w}^2} \ . \end{aligned} \tag{1.61}$$

The Newton equation (1.56) now reads

$$\begin{aligned} 0 &= \ddot{z} + \frac{c}{|z|^3} z = \frac{2w''}{|w|^2 \bar{w}} - \frac{c}{|w|^2 \bar{w}^2} - \frac{\mathcal{E}}{\bar{w}^2} + \frac{c}{|w|^6} w^2 = \\ &= \frac{2w''}{w\bar{w}^2} - \frac{\mathcal{E}}{\bar{w}^2} \ , \end{aligned} \tag{1.62}$$

just recalling the $|w|^2 = \bar{w}w$. The above identity is equivalent to

$$w'' = \frac{\mathcal{E}}{2} w \ , \tag{1.63}$$

and the latter complex equation, when we write $w = w_1 + \imath w_2$, is in turn equivalent to the real system

$$\begin{cases} w_1'' = (\mathcal{E}/2) w_1 \\ w_2'' = (\mathcal{E}/2) w_2 \ . \end{cases} \tag{1.64}$$

Such a system represents a 2D harmonic oscillator if $\mathcal{E}/2 := -\omega^2 < 0$, and a 2D hyperbolic oscillator if $\mathcal{E}/2 := \omega^2 > 0$. Such a correspondence maps the Kepler-Newton elliptic/ hyperbolic motions into the harmonic/hyperbolic oscillator ones. See also [4] and [5].

The Kepler-Newton problem and the harmonic oscillator one are really special. Indeed, a theorem due to Bertrand [7] states that the sole central potentials $U(r)$ displaying closed orbits for initial conditions on open sets are the harmonic potential $U(r) = kr^2/2$, and the Kepler one, $U(r) = -k/r$ ($k > 0$ in both cases); see [1] for a precise statement and a complete proof.

1.3.6 1D conservative Newtonian motions

One-dimensional (1D) conservative Newtonian motions are ruled by the Newton differential equation $m\ddot{x} = f(x) = -U'(x)$, where the potential energy $U(x) = -\int f(x)\mathrm{d}x$ always exists, since $x \in \mathbb{R}$. The Newton equation is equivalent to the system

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{1}{m} U'(x) \end{cases} \ . \tag{1.65}$$

The phase space of the system is the plane $\mathbb{R}^2 \ni (x, v)$.

The simplest (and relevant) solutions of (1.65) are the *equilibria*, or equilibrium points, namely the zeros of the vector field, i.e. those points of the phase space where the right hand side of (1.65) vanishes. Thus the equilibria of 1D conservative systems are of the form

$$(x, v) = (\bar{x}, 0) \ ; \quad f(\bar{x}) = -U'(\bar{x}) = 0 \ ,$$

i.e. they are the points of the x axis (zero velocity) whose abscissae are the critical points of the potential energy (zero force). To the single equilibrium point $(\bar{x}, 0)$ there corresponds the motion $t \mapsto (\bar{x}, 0)$ for all $t \in \mathbb{R}$, i.e. the unique solution of (1.65) corresponding to the initial condition $(\bar{x}, 0)$.

For system (1.65), the total energy $H(x, v)$ is preserved, namely

$$H(x, v) := \frac{mv^2}{2} + U(x) = E \ , \tag{1.66}$$

where $x(t)$ and $v(t)$ is understood in the latter expression. The constant value E of the function H along the *phase curve* $t \mapsto (x(t), v(t))$, i.e. the unique motion corresponding to a specific initial condition $(x(0), v(0))$, is determined by the the latter: $E = H(x(0), v(0))$. One thus concludes that the phase curves, i.e. the possible motions, belong to the E-level set of H, i.e.

$$H^{-1}(E) := \{(x, v) \in \mathbb{R}^2 : \ H(x, v) = E\} \ . \tag{1.67}$$

The latter set, by the energy conservation law (1.66), solved for v (two branches) can be represented as the union of two graphs, namely

$$H^{-1}(E) := \left\{(x, v) : \ v = +\sqrt{\frac{2}{m}(E - U(x))}\right\} \cup \left\{(x, v) : \ v = -\sqrt{\frac{2}{m}(E - U(x))}\right\} \ , \tag{1.68}$$

the domain of the functions $v_\pm(x) := \pm\sqrt{\frac{2}{m}(E - U(x))}$ being given by the condition $U(x) \leq E$.

The set (1.67), or (1.68) consists, when not empty, of one or more connected curves, some of them possibly degenerating into single points. The picture of the level sets (1.67), where one distinguishes the *traces* and the directions of the possible phase curves, or motions, corresponding to all possible values of E, is named *phase portrait, or diagram* of system (1.65). This is the most complete description of the dynamics of the 1D Newtonian system.

Example 1.1. *For the harmonic oscillator $H = mv^2/2 + kx^2/2 = E \geq 0$. To $E = 0$ there corresponds the origin $(0, 0)$ of the phase plane, the only equilibrium point. The corresponding motion is the phase curve $t \mapsto (0, 0)$ for all $t \in \mathbb{R}$. The level set for any $E > 0$ is an ellipse of canonical equation $x^2/a^2 + v^2/b^2 = 1$, where $a := \sqrt{2E/k}$ and $b := \sqrt{2E/m}$ are the length of the two semi-axes. This corresponds to the periodic motions of the harmonic oscillator, namely $x(t) = x(0)\cos(\omega t) + [v(0)/\omega]\sin(\omega t)$, where $T = 2\pi/\omega$ is the period of the motion and $\omega := \sqrt{k/m}$ the pulsation. The point mass oscillates on the real axis sweeping back and forth a segment of extremes $x_\pm = \pm a = \pm\sqrt{2E/k}$, the two* turning points*, where the particle stops and inverts the motion. The ellipse in the phase plane is swept clockwise (why?) once per period T of motion. Due to the structure of the phase portrait, the equilibrium point $(0, 0)$ of the harmonic oscillator is referred to as an elliptic (equilibrium) point, or center.*

Example 1.2. *For the harmonic repulsor, or hyperbolic oscillator, defined by the Newton equation $m\ddot{x} = +kx$ $(k > 0)$, the energy function is $H(x, v) = mv^2/2 - kx^2/2$. The level set $H^{-1}(E)$ is non empty for any $E \in \mathbb{R}$. In particular, $H^{-1}(0)$ consists of the two lines $v = \pm(k/m)x$. Since such a set contain the equilibrium point $(0, 0)$, by the uniqueness of the solutions of the Newton system, $H^{-1}(0)$ contains five distinct motions, or phase curves, namely the origin and four half-lines, two entering and two exiting the origin. For all values of $E = -|E| < 0$, $H^{-1}(E)$ consists of two distinct phase curves, namely the two branches (left*

and right) of hyperbola of equation $x^2/a^2 - v^2/b^2 = 1$*, where* $a = \sqrt{2|E|/k}$*,* $b = \sqrt{2|E|/m}$*. On the other hand, for all values of* $E > 0$*,* $H^{-1}(E)$ *consists of the two branches (up and down) of hyperbola of equation* $x^2/a^2 - v^2/b^2 = -1$*, where* $a = \sqrt{2E/k}$*,* $b = \sqrt{2E/m}$*. Observe that* $H^{-1}(0)$ *consists, from a geometrical point of view, of the asymptotes of the hyperbolas. The equilibrium point* $(0,0)$ *in this case is referred to as a hyperbolic (equilibrium) point, or saddle point.*

Exercise 1.7. *Consider the* graph *of the energy function* $H(x,v)$*, i.e. the subset of* $\mathbb{R}^3$ *define by* $\{(x,v,z) : \ z = H(x,v)\}$*. The level sets of* H *are then the intersections of the graph of* H *with the horizontal planes* $z = E$*. Sketch the graphs of* H *for the harmonic and hyperbolic oscillator, realizing that one has a paraboloid with elliptic section in the former case, and a* mountain pass*, or* saddle-shaped *graph in the latter. The origin* $(0,0)$ *of the phase plane (the only equilibrium point in both cases) is called a* center *for the harmonic oscillator, and a* saddle *for the hyperbolic one. You can make use of a computer.*

The particular structure of the energy function (1.66) imposes some restrictions on the structure of the E-level sets of H.

1.3.7 Recipe for drawing a phase portrait

These are the general rules to draw a phase portrait.

►**Suggestion 1.1.** *Draw any graph of a function* $U(x)$ *displaying local minima and maxima, and understand each item below on the picture. Draw a corresponding sketch of the phase portrait on the* (x,v) *phase plane reported just below the plane with the graph of* $U(x)$*, with the two* x*-axes parallel.*

- Equilibria lie on the x-axis. The orientation of the phase curves is left-to-right in the upper half plane $\{(x,v) : \ v > 0\}$, and right to left in the lower half plane $\{(x,v) : \ v < 0\}$. This follows from $\dot{x} = v$, so that $x(t)$ is monotonically increasing in the upper half phase plane, and monotonically decreasing in the lower one. In particular, all closed curves are clockwise oriented.

- Since $mv^2/2 = E - U(x) \geq 0$, the possible values of x for motions of a given energy value E are those belonging to the E-sub-level set of U, i.e. the allowed values of x are such that $U(x) \leq E$.

- If $U(x)$ displays an absolute minimum value U_m, then E cannot assume values lower than it: for $E < U_m$ there are no possible motions. If U is unbounded from below, then E can take on any real value. Remark: U is defined up to an arbitrary constant, that must be chosen once and for all.

- H is symmetric in v, i.e. $H(x,-v) = H(x,v)$, which implies reflection symmetry of any level set with respect to the x-axis. More precisely, solving (1.66) for v, one gets

$$v_{\pm}(x) = \pm\sqrt{\frac{2}{m}\left[E - U(x)\right]}\ , \tag{1.69}$$

which means that to any allowed value of x, i.e. such that $U(x) \leq E$, there correspond two values of the velocity v, namely $v_+(x) \geq 0$ and $v_-(x) = -v_+(x) \leq 0$. The two branches $v_\pm$ may be either connected to each other or disconnected.

- The even symmetry of H in v corresponds to the so-called *time-reversal symmetry* of system (1.65): the equations are invariant by changing (x, v, t) into $(x, -v, -t)$

- The simple zeros of the equation $U(x) = E$ are called *turning points*. Indeed, if $U(x_+) = E$, and $U'(x_+) > 0$ (< 0), at the turning point $x_+(E)$, the particle coming from the left of x_+ stops there by the energy conservation law. Starting at the stop time with the initial condition $(x_+, 0)$, the force on the particle is $f(x_+) = -U'(x_+)/m < 0$, and the particle re-starts moveing to the left. Likewise, a particle approaching from the right the turning point $x_-(E)$, such that $U(x_-) = E$ and $U'(x_-) < 0$, stops there and restarts moving to the right.

- The previous point describes the *elastic smooth reflection* of a particle by a potential barrier: when the particle arrives at a turning point it rebounds inverting the direction of motion, assuming at the same positions equal and opposite velocities (this is implied by the $v \to -v$ symmetry of H).

- Taking the derivative of v_+, from (1.69), one gets

$$v'_+(x) = \frac{-U'(x)}{\sqrt{2m\,[E - U(x)]}} \ . \tag{1.70}$$

  From this formula one sees that $v_+ \to -\infty$ as $x \to x_+^-$, where x_+ is a right turning point such that $U'(x_+) > 0$. Analogously, $v_+ \to +\infty$ as $x \to x_-^+$, where x_- is a left turning point such that $U'(x_-) < 0$. By the reflection symmetry $v \to -v$ it then follows that, at the the turning points $(x_\pm, 0)$, the tangent to a phase curve is vertical, the two branches v_+ and v_- connect to each other and their union describes one and the same phase curve, or motion.

- A consequence of the previous analysis is that to any allowed interval of x bounded by two turning points, i.e. $[x_-(E), x_+(E)]$, there corresponds a single closed phase curve C_E, i.e. a periodic motion.

- Another implication of (1.70) is that, for values of E larger than the local minimum (maximum) value of $U(x)$, the local maximum (minimum) points of $v_+(x)$ coincide with the local minimum (maximum) points of $U(x)$.

- Equilibrium points $(\bar{x}, 0)$ corresponding to non-degenerate critical points of $U(x)$ (i.e. those $\bar{x}$ such that $U'(\bar{x}) = 0$ and $U''(\bar{x}) \neq 0$) are locally elliptic if $U''(\bar{x}) > 0$, and locally hyperbolic if $U''(\bar{x}) < 0$. In other words, the phase portrait around $(\bar{x}, 0)$ resembles, in such two cases, either that of the harmonic or that of the hyperbolic oscillator. This

easily follows by the energy conservation law (1.66), expanding $U(x)$ around $\bar{x}$ to second order in $x - \bar{x}$, and taking an energy value $E = U(\bar{x}) + \Delta E$. This yields

$$\frac{mv^2}{2} + \frac{U''(\bar{x})}{2}(x - \bar{x})^2 = \Delta E + O(|\Delta E|^{3/2}) \ .$$

Neglecting the small remainder on the right hand side, when $|\Delta E|$ is very small, leads to the conclusion by one and the same analysis performed for the harmonic and the hyperbolic oscillator, with $U''(\bar{x})$ and ΔE in place of $\pm k$ and E, respectively.

- To an interval of allowed values of x with extrema a turning point and a nondegenerate local maximum $\bar{x}$ of $U(x)$ there corresponds a closed curve connecting the equilibrium point $(\bar{x}, 0)$ to itself. Such a closed curve is called *homoclinic connection, or loop*. The time to reach the equilibrium along such a phase curve is infinite. The possible motions of the system in this case are two: the equilibrium one and the homoclinic motion.

- To an interval of allowed values of x with extrema two nondegenerate local maxima $\bar{x}_1$ and $\bar{x}_2$ of $U(x)$, there correspond two symmetric phase curves, one above and one below the x-axis, connecting the equilibria $(\bar{x}_1, 0)$ and $(\bar{x}_2, 0)$. Such curves are called *heteroclinic connections*. The possible motions of the system in this case are four: the two equilibria and the two heteroclinic motions.

1.3.8 Linearization around non-degenerate equilibria

Let us consider the Newton system (1.65), or its equivalent Newton equation $m\ddot{x} = -U'(x)$. Let $\bar{x}$ be a non-degenerate equilibrium point, i.e. a non-degenerate critical point of U: $U'(\bar{x}) = 0$; $U''(\bar{x}) \neq 0$. Let us now define the new position variable $\xi(t) := x(t) - \bar{x}$, describing the displacement (with sign) of the particle with respect to its equilibrium value. By substituting $x(t) = \bar{x} + \xi(t)$ into the Newton equation, yields

$$m\ddot{\xi}(t) = -U'(\bar{x} + \xi(t)) = -U''(\bar{x})\xi(t) + O(\xi^2(t)) \ , \tag{1.71}$$

where we have expanded U' to first order at $\bar{x}$ with increment $\xi(t)$. The linearization procedure consists now in neglecting the remainder $O(\xi^2)$, under the hypothesis that $|\xi(t)|$ is small enough, which has to be checked a fortiori. This yields the *linearized Newton equation*

$$m\ddot{\xi} = -U''(\bar{x})\xi \ . \tag{1.72}$$

Due to the non degeneracy hypothesis on $\bar{x}$, one of the following two hypotheses is possible.

- $U''(\bar{x}) = k > 0$, equation (1.72) reads $m\ddot{\xi} = -k\xi$, i.e. the equation of a harmonic oscillator; the deviation $\xi(t) = a\cos(\omega t) + b\sin(\omega t)$ oscillates at the frequency

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(\bar{x})}{m}} \tag{1.73}$$

and stays small if initially small.

- $U''(\bar{x}) = -k < 0$, equation (1.72) reads $m\ddot{\xi} = +k\xi$, i.e. the equation of a repulsor; the deviation $\xi(t) = ae^{\omega t} + be^{-\omega t}$ grows exponentially fast on a time scale

$$\frac{1}{\omega} = \sqrt{\frac{m}{k}} = \sqrt{\frac{m}{-U''(\bar{x})}} \ . \tag{1.74}$$

Conclusion: the linearized Newton equation is compatible with the local analysis of the phase portrait around non-degenerate equilibria. The linear approximation is good for all times around local minima, while it becomes bad around local maxima, the faster the larger is ω.

1.3.9 Periods of oscillation and flight times

Concerning the times the particle takes to move along a given phase curve, there are two interesting cases, namely the case of bounded closed curves (periodic motions), and that of unbounded curves.

Consider first a closed phase curve C_E, corresponding to a periodic motion. From the law of conservation of energy, with $v = \mathrm{d}x/\mathrm{d}t$, it follows that the period of such a motion $T(E)$ is given by

$$T(E) = \oint_{C_E} \frac{\mathrm{d}x}{v} = 2\int_{x_-(E)}^{x_+(E)} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}} \ , \tag{1.75}$$

where $x_\pm(E)$ are the two turning points, i.e. the two solutions of $U(x) = E$. The area enclosed by the closed phase curve C_E in the (x, p) phase plane, where $p := mv$ is the momentum, is

$$A(E) = \oint_{C_E} p \ \mathrm{d}x = 2\int_{x_-(E)}^{x_+(E)} \sqrt{2m\left[E - U(x)\right]} \ \mathrm{d}x \ . \tag{1.76}$$

The latter quantity is related to the period of motion (1.75) by the following fundamental relation:

$$T(E) = \frac{\mathrm{d}A}{\mathrm{d}E} \ , \tag{1.77}$$

which is easily proved by a direct calculation. The inverse of the frequency of motion is obtained by dividing (1.77) by 2π, i.e. $1/\omega(E) = \mathrm{d}(A/2\pi)/\mathrm{d}E$. The latter formula motivates the introduction of the so-called *action variable*

$$I(E) := \frac{A(E)}{2\pi} = \frac{1}{2\pi}\oint_{C_E} p \ \mathrm{d}x \ , \tag{1.78}$$

so that $\omega^{-1} = \mathrm{d}I/\mathrm{d}E$. Now, observing that I is a monotonically increasing function of E, so that the inverse function $E(I)$ exists, we can write the important formula

$$\omega(I) = \frac{\mathrm{d}E}{\mathrm{d}I} \ . \tag{1.79}$$

Observe that since E is a constant of motion, $I(E)$ is also a constant of motion, and the same holds for the frequency (1.79) of the periodic motion (as is quite natural, from a physical point

of view). One can then associate to the frequency $\omega(I)$ and angle $\varphi(t)$ such that $\dot{\varphi} = \omega(I)$. In such a way, the continuous family of closed phase-curve C_E in the phase plane $\mathbb{R}^2 \ni (x, v)$, labelled by the energy E, is mapped into the circular motions on $S^1 \times \mathbb{R}^+ \ni (\varphi, I)$, labelled by the action variable I. Here $S^1 := \mathbb{R}/2\pi\mathbb{Z}$ denotes the real numbers modulo 2π, i.e. the circle.

In the case of an unbounded phase curves (or motion), to be specific, suppose that $U(x) \to -\infty$ as $x \to +\infty$. Let ξ be such that $E - U(x) > 0$ for all $x \in [\xi, +\infty[$. The time the particle takes to reach $+\infty$, starting at ξ, is

$$\tau_\infty(E) = \int_\xi^{+\infty} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}} \ . \tag{1.80}$$

Such a time turns out to be finite (which means off to infinity in a finite time, i.e. a blow-up) if $|U| \to +\infty$ faster than x^2 as $x \to +\infty$. Examples are $U \sim -cx^3$ and $U \sim -cx^4$ as $x \to +\infty$, where $c > 0$.

1.3.10 A guided list of problems

►**Problem 1.8.** *Study the phase portraits corresponding to the potentials $U(x) = x^2/2 + \alpha x^3/3$ as α varies in $\mathbb{R}$ (in an interval around zero). There are two cases.*

►**Problem 1.9.** *Study the phase portraits corresponding to the potentials $U(x) = kx^2/2 + \lambda x^4/4$ as $k \neq 0$ and $\lambda \neq 0$ vary in $\mathbb{R}$ (in an interval around zero). There are four cases.*

►**Problem 1.10.** *Study the phase portrait of the Newton equations $m\ddot{x} = \pm kx^3$, where $k > 0$ (hint: in the case $+kx^3$, start drawing the level set $H^{-1}(0)$). In the case $+kx^3$, compute the time of flight to infinity starting from $x(0) = \xi > 0$ with positive velocity and zero energy. In the case $-kx^3$, determine the dependence of the oscillation period $T(E)$ on the energy E.*

►**Problem 1.11.** *Exploiting the energy conservation law (1.42), perform a detailed study of the phase portrait of the radial motion for the cases $U(r) = -k/r$, and $U(r) = kr^2/2$, $k > 0$ (Kepler problem and the 2D harmonic oscillator). Which kind of orbits do you expect in the general case, where they are not close? Hint: make use of the evolution equation for the angle θ.*

►**Problem 1.12.** *Study the phase portrait of the ideal pendulum of length ℓ, described by the equation*

$$\ell\ddot{\theta} = -g \sin\theta \ , \tag{1.81}$$

g being the gravitational acceleration. Call $\omega := \sqrt{g/l}$ and take $U(\theta) = \omega^2(1 - \cos\theta)$; the energy conservation law reads $\dot{\theta}^2/2 + U(\theta) = E$. Which is the frequency of the small oscillations around $\theta = 0$? Which is the energy value separating oscillations from rotations? Which is the frequency of fast rotations as $E \to +\infty$? (Answers: ω; $2\omega^2$; $|\dot{\theta}| \sim \sqrt{2E}$).

Since the position variable is the angle θ, show that the natural phase space of the system is $S^1 \times \mathbb{R} \ni (\theta, \dot{\theta})$. Try to draw a picture, or bend your phase portrait bringing $-\pi$ and π, on the θ-axis, to coincide.

►**Problem 1.13.** *The period of the radial oscillation of central motions is given by*

$$T(E) = 2\int_{r_-(E)}^{r_+(E)} \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}[E - U_e(r)]}} \ . \tag{1.82}$$

Prove that, in one period of radial oscillation, the angle θ undergoes a variation given by

$$\Delta\theta(T) = 2\int_{r_-(E)}^{r_+(E)} \frac{\ell_3}{r^2}\frac{\mathrm{d}r}{\sqrt{2m[E - U_e(r)]}} \ . \tag{1.83}$$

Show that the condition of closure of the orbits is $\Delta\theta = 2\pi q$, where q is a rational number. What is the value of q for the Kepler problem and for the 2D harmonic oscillator? (Answer: $q = 1$ and $q = 1/2$, respectively; explain why).

►**Problem 1.14.** *A common potential mimicking certain chemical bonds in molecules (e.g. the hydrogen bonds linking complementary bases in DNA molecules) is the Morse one:*

$$U(x) = \frac{D}{2}(e^{-\alpha x} - 1)^2 \ . \tag{1.84}$$

Draw the phase portrait of $m\ddot{x} = -U'(x)$. Pay attention to the phase curve at $E = D/2$, separating bounded from unbounded motions. Which is the frequency of the small oscillations of the system as $E \to 0^+$?

►**Problem 1.15.** *A* particle in a box *is described by the sequence of smooth potentials of the form*

$$U_n(x) = \epsilon\left(\frac{x}{\ell}\right)^{2n} \ , \ n = 1, 2, \ldots \tag{1.85}$$

1. *Compute the limit $U_\infty := \lim_{n\to\infty} U_n(x)$ for any $x \in \mathbb{R}$.*
2. *Draw the phase portrait of $m\ddot{x} = -U_n'(x)$ for increasing values of n. Which is the limit phase portrait as $n \to \infty$? Which are the corresponding physical motions of the particle?*
3. *Which is the period of the motion at a given energy E in the limit $n \to \infty$? Hint: use the previous results.*
4. *Write down in detail the formula (1.75) for the period of the motions $T_n(E)$ for any finite n. Determine the exact dependence of $T_n(E)$ on E. Finally, compute the limit $T_\infty(E) = \lim_{n\to\infty} T_n(E)$.*

1.4 A historical note: the first quantum theory

In the first 20 years of the past century, in trying to establish a new theory in agreement with many experimental results that classical electrodynamics was unable to explain, Planck, Einstein, Bohr, Sommerfeld and Born [9] developed what is known today as the "first quantum theory". Quantum mechanics in its modern form was developed starting from 1925, by Heisenberg, Schrödinger, Pauli and Dirac, with the fundamental and definitive contribution of Von Neumann on the mathematical formulation of the theory.

The basis of the first quantum theory is the following. Consider a 1D conservative Newtonian system. For bounded periodic motions, consider the formulas (1.75), (1.76) and (1.77), expressing the period of the motion $T(E)$, the area $A(E)$ inside the closed phase curve C_E, and their relation, respectively. Then one observes that $A(E) = \oint_{C_E} p\mathrm{d}x$ has the dimension of an *action*, i.e. energy times time, or momentum times length. The principle of quantization set up by the founders of quantum physics requires that *for bounded periodic motions $A(E)$ must take on only integer multiple values of the Planck constant $h = 6.626\ 10^{-27} erg \cdot s$*, i.e.

$$A(E) = nh \ , \ n \in \mathbb{N} \ , \tag{1.86}$$

with the possible exclusion of $n = 0$, depending on the problem. Such a principle of quantization of the area implies an equivalent principle of quantization of the action variable (1.78), namely

$$I(E) := \frac{A(E)}{2\pi} = n\hbar \ , \tag{1.87}$$

where $\hbar := h/(2\pi) = 1.055\ 10^{-27} erg \cdot s$. Now, since, as already observed above, $\mathrm{d}I/\mathrm{d}E > 0$, one can invert (1.87) to get $E(I)$. One then finds the following consequence of the quantization principle: $I = n\hbar$ implies that the energy values of a bounded periodic motion are also quantized, the allowed values being $E_n := I^{-1}(n\hbar)$. The sequence of such energy values is the so-called *energy spectrum* of the system. Observe that energy quantization, i.e. the presence of a discrete set of allowed values of the energy, is a consequence of the quantization of the action, the latter being well defined because of the existence of a quantum of action, i.e. $\hbar$, a fundamental constant of nature.

Example 1.3. *For the harmonic oscillator the frequency ω does not depend on the energy. From $\mathrm{d}I/\mathrm{d}E = T/2\pi = 1/\omega$, one gets $E = \omega I$. Another way to get this result is to consider the law of conservation of energy in the (x, p) plane, namely*

$$\frac{p^2}{2m} + \frac{kx^2}{2} = E \ .$$

Then the phase curve corresponding to $E > 0$ is an ellipse of semi-axes $a = \sqrt{2E/k}$ and $b = \sqrt{2mE}$. The area of the ellipse is $A = \pi ab = 2\pi E/\omega$ (recall that $\omega = \sqrt{k/m}$), so that $I = E/\omega$. The energy spectrum of the quantum harmonic oscillator is then

$$E_n = \hbar\omega n \ , \ n = 0, 1, 2 \ldots . \tag{1.88}$$

Example 1.4. *Let us consider again the third Kepler law, namely*

$$T = 2\pi\sqrt{\frac{m}{k}}\, a^{3/2} \ . \tag{1.89}$$

Now, recalling that the parameter p of the ellipse is linked to a and f by the relation $p = \frac{a^2-f^2}{a} = a(1-\varepsilon^2)$, one gets

$$a = \frac{p}{1-\varepsilon^2} = -\frac{k}{2E} \ .$$

This gives

$$\frac{T}{2\pi} = \frac{k\sqrt{m}}{2\sqrt{2}}(-E)^{-3/2} = \frac{\mathrm{d}I}{\mathrm{d}E} \ .$$

By a simple integration we get $I = k\sqrt{m}/\sqrt{-2E}$, and finally

$$E(I) = -\frac{mk^2}{2I^2} \ . \tag{1.90}$$

For the Hydrogen atom, $k = e^2$ and the principle of quantization yields the energy spectrum

$$E_n = -\frac{me^4}{2\hbar^2 n^2} \ , \ n = 1, 2, \dots \ , \tag{1.91}$$

first found by Bohr (1913).

Example 1.5. *Consider a particle in an infinitely high, rectangular box of basis 2ℓ. From the conservation of energy, at energy E the particle has velocity $v_\pm = \pm\sqrt{2E/m}$, and momentum $p_\pm = \pm\sqrt{2mE}$. Thus $A = 2p_+ 2\ell = 4\ell\sqrt{2mE}$, so that $E = A^2/(32m\ell^2) = \pi^2 I^2/(8m\ell^2)$. By quantizing the action I one gets the energy spectrum of the quantum particle in a box, namely*

$$E_n = \frac{\pi^2\hbar^2 n^2}{8m\ell^2} \ , \ n = 1, 2, \dots \ . \tag{1.92}$$

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Nota: il saggio si compone di due parti. La prima riporta in versione integrale la Storia della Astronomia che Giacomo Leopardi scrisse nel 1813 all'età di 15 anni, ancora oggi una delle fonti bibliografiche più importanti sul tema. Margherita Hack completa poi la Storia dal 1813 al 2002.

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