DYNAMICAL SYSTEMS

Lectures Notes 2021/2022

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The curve described by a simple molecule of air vapor is regulated in a manner just as certain as the planetary orbits; the only difference between them is that which comes from our ignorance. Probability is relative, in part to this ignorance, in part to our knowledge.

Pierre Simon de Laplace [10].

Predictability: Does the Flap of a Butterfly's Wings in Brazil Set off a Tornado in Texas?

Edward N. Lorenz [12][1].

[1]Title of a famous talk presented by E.N. Lorenz at the 139th meeting of the American Association for the Advancement of Science, session devoted to Global Atmospheric Research Program, Washington D.C., December 29, 1972

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Chapter 1

Introduction to Dynamical Systems

1.1 Abstract definition of dynamical system

All natural phenomena involve the change, or evolution in time of the state of a certain observed system. Think, for example, of the position and velocity of a planet in the solar system; the daily number of new individuals infected by a virus in a given region; the position and velocity of a car on a street; the number of particles of a given energy range arriving per unit time at the surface of a detector placed somewhere on the Earth (the so-called flow of cosmic rays); the position and form of a spot of dye in a given fluid; the velocity field of a fluid or gas in a given point; and so on so forth.

The mathematical theory of dynamical systems deals with the problem of *modeling* and *predicting* the behavior of a given class of phenomena, and of *evaluating the accuracy* of both models and their predictions in the light of the experiments. The cornerstone example of this field of research is represented by weather forecast and, as a matter of fact, its starting point as a discipline independent of celestial mechanics goes back to the pioneering work of the meteorologist Lorenz, in 1963 [18].

What one refers to as a dynamical system in mathematics is specified by the following quantities.

1. A $\mathscr{S}$ whose elements $x \in \mathscr{S}$ identify the possible states of the system. The set $\mathscr{S}$ is called the space of states, or **phase space**, of the system.

2. A **flow map**, or function, or simply a flow $\Phi^h : \mathscr{S} \to \mathscr{S} : x \to y = \Phi^h(x)$ determining the future state of the system y after a positive time h when the present state of the system x is known.

3. A **probability law** P on $\mathscr{S}$ specifying, for any $\mathscr{A} \subseteq \mathscr{S}$ in a certain class, the probability $0 \leq P(\mathscr{A}) \leq 1$ that, extracting a state x of the system, $x \in \mathscr{A}$.

Some comments are now in order.

– Concerning the phase space $\mathscr{S}$, this can be any set, containing finitely or infinitely many elements (possible states), and in the latter case such an infinity of states can be countable (discrete case) or uncountable (continuous case). In the continuous case, one usually needs to endow $\mathscr{S}$ with some structure, at least and typically that of vector space, which allows to measure distances (what is close to what), for example.

Example 1.1. *Neural networks, and in general all network systems characterized by N components, or elements, or nodes, whose state can be either* on *or* off*, have a phase space $\mathscr{S}$ which consists of 2^N points. The phase space of the single component can be represented by the set $\{0,1\}$, associating 1 to on and 0 to off. In this way $\mathscr{S} = \{0,1\} \times \cdots \times \{0,1\} := \{0,1\}^N$ is the set whose elements (states of the system) are all the 2^N possible binary strings $\sigma^{(1)} \cdots \sigma^{(N)}$, with $\sigma^{(j)} \in \{0,1\}$ for any $j = 1, \ldots, N$.*

Example 1.2. *For a point mass moving in space $\mathscr{S} = \mathbb{R}^3 \times \mathbb{R}^3$, whose generic element is an ordered pair of vectors $(\boldsymbol{x}, \boldsymbol{v})$ specifying the position and the velocity of the point, respectively.*

– The flow Φ^h depends on the time parameter h, the "natural time unit" of the system. Considering a fixed finite h or its limit $h \to 0$, is a matter of mathematical convenience (Nature is certainly discrete). In the case $\Phi^h(x)$ is an injective function, its inverse exists and one sets $x = (\Phi^h)^{-1}(y) := \Phi^{-h}(y)$, which determines the past state of the system x a time h before, when the present state y is known. The evolution law of a system characterized by a phase space $\mathscr{S}$ can be written in the form of the recurrence relation (or recurrent sequence)

$$x_{n+1} = \Phi^h(x_n) \quad ; \quad n \in \mathbb{N} \ , \tag{1.1}$$

where x_n is the state of the system at time nh. Initializing the sequence (1.1) at $n = 0$ one gets

$$x_1 = \Phi^h(x_0) \ ; \ x_2 = \Phi^h(x_1) = \Phi^h(\Phi^h(x_0)) \ ; \ x_3 = \Phi^h(x_2)) = \Phi^h(\Phi^h(\Phi^h(x_0))) \ \ldots \ ,$$

so that the state of the system at the arbitrary time Nh is given by

$$x_N = (\underbrace{\Phi^h \circ \cdots \circ \Phi^h}_{\text{N times}})(x_0) = \Phi^h(\cdots(\Phi^h(x_0))\cdots) := \Phi^{Nh}(x_0) \ , \tag{1.2}$$

where $\circ$ denotes the ordinary composition of functions. The solution (1.2) of the recurrence relation (1.1) holds for any initial condition $x_0 \in \mathscr{S}$, which allows to define the flow of the system at time Nh (in the future) as the iteration of the flow map N times, namely $\Phi^{Nh} := \Phi^h \circ \cdots \circ \Phi^h$. In the same way, if $(\Phi^h)^{-1} := \Phi^{-h}$ exists, one defines the flow at time Nh in the past as $\Phi^{-Nh} := \Phi^{-h} \circ \cdots \circ \Phi^{-h}$. Thus, the evolution of a discrete-time dynamical system is specified by an iterated flow map.

Example 1.3. *A possible recurrence relation specifying the flow map of a network with binary elements is of the form*

$$\sigma_{n+1}^{(j)} = \Theta\left(\sum_{k=1}^{N} J_{jk} \sigma_n^{(k)} - \tau_j\right) \ ,$$

where J_{jk} is the matrix of connections between the elements of the network, whereas τ_j is a certain threshold characterizing the activity of node j; finally $\Theta(x)$ is the step function which takes on value 1 if $x \geq 0$ and 0 if $x < 0$.

Example 1.4. *An evolution law for a point mass subject to a force $\boldsymbol{F}(\boldsymbol{x})$, in agreement with the Newton law $m\boldsymbol{a} = \boldsymbol{F}$, is given by the recurrence relation*

$$\begin{cases} \boldsymbol{x}_{n+1} = \boldsymbol{x}_n + h\boldsymbol{v}_n + \frac{h^2}{2}\boldsymbol{F}(\boldsymbol{x}_n) \\ \boldsymbol{v}_{n+1} = \boldsymbol{v}_n + \frac{h}{m}\boldsymbol{F}(\boldsymbol{x}_n) + \frac{h^2}{2}\boldsymbol{F}'(\boldsymbol{x}_n)\boldsymbol{v}_n \end{cases} .$$

This is a possible numerical scheme of integration for the Newton law.

The choice of the time unit h depends on the specific problem considered and on the space and time scales there involved.

Example 1.5. *A car on a street moves at $120Km/hour = 2Km/min \simeq 33m/sec$. Thus, the car displaces about 3.3 meters in $0.1sec$. Such a displacement is comparable with the size of the car, so that $h = 0.1sec$ is a good choice as a time unit.*

– The probability law, or measure, P on $\mathscr{S}$ is necessary for example when one thinks of choosing the initial state: in practical situations this is never known with infinite precision, and some states are more plausible than others. The probability law P is built up as follows. Consider the class of all subsets of $\mathscr{S}$ and denote it by $2^{\mathscr{S}}$. Let $\mathscr{A} \subseteq 2^{\mathscr{S}}$ be a subset of such class with the following properties [3] [13].

1. $\mathscr{S} \in \mathscr{A}$.

2. $A \in \mathscr{A}$ implies $A^c = \mathscr{S} \setminus A \in \mathscr{A}$;

3. For any (finite or infinite) sequence of sets $A_j \in \mathscr{A}$ also $\cup_j A_j \in \mathscr{A}$.

A subset $\mathscr{A}$ with such properties is called a sigma-algebra on $\mathscr{S}$. A probability law on $\mathscr{S}$ is then a function $P : \mathscr{A} \to [0,1] : A \mapsto P(A)$ such that $P(\mathscr{S}) = 1$ and, for any sequence of sets $A_j \in \mathscr{A}$, pairwise disjoint ($A_j \cap A_k = \emptyset$ for all $j \neq k$), $P(\cup_j A_j) = \sum_j P(A_j)$; the latter property is referred to as countable additivity of the probability law P. Often, but not always, P is chosen to be invariant with respect to the flow Φ^h, namely $P(\Phi^h(\mathscr{A})) = P(\mathscr{A})$, where $\Phi^h(\mathscr{A}) := \{y \in \mathscr{S} : \ y = \Phi^h(x), \ x \in \mathscr{A}\}$ is the evolution of $\mathscr{A}$ under the flow. In this way the probability to extract $x \in A$ is independent of time.

In summary, the abstract definition of dynamical system is a triple $(\mathscr{S}, \Phi^h, P)$ consisting of a set, the phase space, of a flow map, and of a probability law P on $\mathscr{S}$. In those applications where probability is not considered, the definition restricts to the pair $(\mathscr{S}, \Phi^h)$.

An important distinction must be made between deterministic dynamical systems and stochastic ones. If the flow map Φ^h is completely specified, the knowledge of the state of the system x with arbitrary precision allows, in principle, to determine the future state of the system $y = \Phi^h(x)$ with arbitrary precision. In this case the system is said to be deterministic. On the other hand, in some problems, the flow map Φ^h is not known exactly, which may be modeled by a dependence on some random parameters. In this case the future state of the system is a random variable, whose value cannot be determined exactly, and the system at hand is said to be stochastic.

Example 1.6. *1D Brownian motion (random walk). Consider a point moving on* $\mathscr{S} = \mathbb{R}$ *according to the evolution law* $x_{n+1} = x_n + a + \sigma_n$*, where* a *is a parameter while* σ_n *is a random variable which takes on the values* ± 1 *with equal probability* $1/2$ *(toss a fair coin at each step, and associate* $+1$ *to head and* -1 *to tail). First of all, it is convenient to eliminate the deterministic drift from the considered motion. This is done by changing variable, setting* $x_n = \xi_n + na$*, which yields the evolution law* $\xi_{n+1} = \xi_n + \sigma_n$*. Iterating the latter, with initial condition* ξ_0*, one gets* $\xi_N = \xi_0 + \sigma_0 + \sigma_1 + \cdots + \sigma_{N-1}$*. Without any loss of generality one can consider* $\xi_0 = 0$*. One thus finds*

$$x_N = \xi_N + Na = \sigma_0 + \cdots + \sigma_{N-1} + Na \ ,$$

i.e. at time N *the point is in a position corresponding to the uniform drift* Na *plus the sum of the* N *random displacement. Computing the average or expected value* $\langle x_N \rangle$ *and the mean square deviation, or variance,* $\langle (x_N - \langle x_N \rangle)^2 \rangle$*, one easily finds*

$$\langle x_N \rangle = Na \quad ; \quad \left\langle (x_N - \langle x_N \rangle)^2 \right\rangle = N \ .$$

1.2 From discrete to continuous time: ODEs

We now make two hypotheses. First, we suppose that the phase space $\mathscr{S}$ of the system has a structure of vector, or linear space of dimension n. This means that we can identify $\mathscr{S}$ with $\mathbb{R}^n$. Second, the flow map Φ^h is supposed to depend smoothly on $h \in I \subseteq \mathbb{R}$, where $I =]-a, a[$ is some open real interval. Of course, for consistency, one has $\Phi^0(x) = x$ for any $x \in \mathscr{S}$ (no displacement of the point in a null time interval). One can then Taylor expand $\Phi^h(x)$ with respect to h with center $h = 0$, to get

$$\Phi^h(x) = x + v(x)h + o(h) \ , \tag{1.3}$$

where we have defined the generating *vector field* $v(x)$ of the flow map Φ^h, namely

$$v(x) := \frac{\mathrm{d}}{\mathrm{d}h}\Phi^h(x)\Big|_{h=0} \ , \tag{1.4}$$

where $o(h)$ denotes a remainder such that $o(h)/h \to 0$ as $h \to 0$.

Remark 1.1. *The hypothesis on* $\mathscr{S}$ *is necessary to ensure that the linear combination on the right hand side of (1.3) be meaningful in* $\mathscr{S}$*.*

One has to think of $v : \mathscr{S} \to \mathscr{S} : x \mapsto v(x)$ as a field of vectors, associating to any point $x \in \mathscr{S}$ the vector $v(x)$, i.e. an oriented segment departing from the point, exactly the same way in a cornfield one has a plant at each specific point of the field. The picture to have in mind is the graph of v, namely the set $\{(x, y) \in \mathscr{S} \times \mathscr{S} : \ y = v(x)\}$. In dimension $n = 1$, this is the graph of the real valued (scalar) function of one real variable $v(x)$, the graph living in the cartesian plane $\mathbb{R}^2 = \mathscr{S} \times \mathscr{S}$. For $n = 2$ we cannot imagine a four dimensional space

$\mathscr{S} \times \mathscr{S}$. However, we can draw the vector $v(x)$ at any point x in the plane. The same holds for $n = 3$. That is exactly the way computer packages display the vector field $v(x)$.

Let $x(t)$ be the state of the system at time t. Then $\Phi^h(x(t)) = x(t+h)$ is the state of the system at time $t + h$. Substituting $x = x(t)$ into relation (1.3) one gets

$$x(t+h) = x(t) + v(x(t))h + o(h) \ . \tag{1.5}$$

Calling $\Delta x(t) := x(t+h) - x(t)$ and $\Delta t = h$, one can rewrite the latter relation as

$$\Delta x(t) = v((x(t))\Delta t + o(\Delta t) \ \Leftrightarrow \ \mathrm{d}x(t) = v(x(t)) \ \mathrm{d}t \ , \tag{1.6}$$

where on the right hand side of the equivalence above we symbolically wrote the same relation in terms of the infinitesimal quantities $\mathrm{d}x(t)$ and $\mathrm{d}t$ in place of the finite (small) ones $\Delta x(t)$ and Δt, without remainder. Dividing (1.5) by h and passing to the limit as $h \to 0$, one gets

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(x(t)) \ \Leftrightarrow \ \dot{x} = v(x) \ , \tag{1.7}$$

the right hand side of the equivalence being a shorthand notation for the left one, which will be often used in the sequel. One can interpret the latter equation, which is equivalent to (1.3), (1.5) and (1.6), in different ways. First of all, if Φ^h is known, then $x(t) = \Phi^t(x_0)$, is also known and defines the curve $I \to \mathscr{S} : t \mapsto x(t)$ passing through the point x_0 at $t = 0$ and such that its tangent vector at the point $x(t)$ is given by $v(x(t))$. In this case (1.7) appears as an interesting geometrical identity. On the other hand, suppose that $v(x)$ is known, and that the system at hand moves according to equation (1.7). The latter now appears as a *first order, autonomous, ordinary differential equation* (ODE, henceforth) in $\mathscr{S} = \mathbb{R}^n$, to be solved for the unknown $x(t)$. The geometrical interpretation of the equation of course is the same as before: $v(x(t_0))$ is equal to the tangent vector $\dot{x}(t_0)$ to the curve $t \mapsto x(t)$ at $t = t_0$.

Exercise 1.1. *Why do we say that* $\dot{x}(t_0) = \mathrm{d}x(t_0)/\mathrm{d}t$ *is the tangent vector to the curve* $t \mapsto x(t)$ *at* $t = t_0$*?*

The ODE (1.7) is said to be *of the first order* because it relates the first derivative of $x(t)$ to $x(t)$. A second order ODE in $\mathscr{S}$, relating the second derivative of $x(t)$ to the first one and to $x(t)$ itself, is of the form

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} = g\left(x(t), \frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) \ \Leftrightarrow \ \ddot{x} = g(x, \dot{x}) \ . \tag{1.8}$$

This can be written as an equivalent first order equation but in a space of double dimension, namely $\mathscr{S} \times \mathscr{S}$, as follows. Call $u(t) := \dot{x}$, and set $z := (x, u)^T \in \mathscr{S} \times \mathscr{S}$. Then the first order system

$$\begin{cases} \dot{x} = u \\ \dot{u} = g(x, u) \end{cases} \ \Leftrightarrow \ \dot{z} = G(z) \tag{1.9}$$

is clearly equivalent to equation (1.8).

Exercise 1.2. *Consider a third order ODE and write it as an equivalent system of the first order.*

The ODE (1.7) is said to be *autonomous* because the vector field $v(x)$ does not depend explicitly on time. Thus, a first order, non autonomous ODE is of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = v(x(t), t) \;\Leftrightarrow\; \dot{x} = v(x, t) \ . \tag{1.10}$$

To many purposes, but not all of them, one can use a trick to rewrite the latter equation as an autonomous one, but in a phase space of dimension $n+1$, as follows. Introduce the independent variable s such that $\mathrm{d}t(s)/\mathrm{d}s = 1$, plus the condition $t(0) = 0$ (so that $t = s$). Define also $y := (x, t)^T \in \mathscr{S} \times \mathbb{R}$. Then the first order system

$$\begin{cases} \mathrm{d}x(s)/\mathrm{d}s = v(x(s), t(s)) \\ \mathrm{d}t(s)/\mathrm{d}s = 1 \end{cases} \;\Leftrightarrow\; \mathrm{d}y(s)/\mathrm{d}s = w(y(s)) \tag{1.11}$$

is clearly equivalent to the equation (1.10). The extended phase space of the equivalent autonomous system is the space-time of the original one.

1.3 Classes of problems in the theory of ODEs

By solution of the ODE $\dot{x} = v(x)$ one means a function $t \mapsto x(t)$ (a curve in $\mathscr{S}$) such that $\dot{x}(t) - v(x(t)) \equiv 0$, i.e. identically zero for any t in the domain of $x(t)$. Notice that the latter domain is an unknown of the problem. However, in applications one needs not to determine all the solutions (if any) of the ODE, but just those curves passing through a specific point. Solving the system

$$\begin{cases} \dot{x} = v(x) \\ x(0) = x_0 \end{cases} , \tag{1.12}$$

consisting of an ODE and a specified initial datum (condition, or value) $x_0 \in D \subseteq \mathscr{S}$, is called the Cauchy, or initial value problem associated to the ODE defined by the vector field v. In general, i.e. but for a few specific cases, one is not able to explicitly provide a solution to a given Cauchy problem, which means giving a formula for $x(t)$. However, there is a class of solutions that can be easily found (and are also relevant to many applications). Such solutions correspond to the zeroes of the vector field v, i.e. those points $\xi \in \mathscr{S}$ such that $v(\xi) = 0$, and called equilibria, or stationary, or fixed points of the ODE $\dot{x} = v(x)$. The solution curve corresponding to a fixed point ξ is the constant curve $x(t) = \xi$ for all $t \in \mathbb{R}$. Indeed $\dot{x}(t) - v((x(t)) = \dot{\xi} - v(\xi) = 0 - 0 \equiv 0$. Such a function solves of course the Cauchy problem corresponding to the initial condition $x(0) = \xi$.

Remark 1.2. *For iterated maps equilibria are those points $\xi \in \mathscr{S}$ such that $\Phi^h(\xi) = \xi$, i.e. the fixed points of the map Φ^h.*

The theory of ODEs deals with four (overlapping) large classes or problems.

1. Well posedness, which means determining the conditions under which the solution to (1.12) exists, is unique, has a certain degree of regularity (e.g. is continuous, is continuously differentiable, and so on), how the solution depends on the initial datum and on the possible parameters entering the vector field v, and what is the maximal domain of existence of the solution itself.

2. Asymptotic behavior, which means understanding what happens to $x(t)$ at the limits of its interval of existence. For example the solution might blow-up (i.e. $x(t)$ or some of its higher order derivatives tend to infinity), or approach some subset of the phase space.

3. "Regularity" properties of the motion, which means determining whether the solution displays regular features (e.g. is constant, is simply periodic, is multi-periodic) or is characterized by some degree of "randomness". Notice that by regularity here one means something different from what is meant in well posedness.

4. Stability, concerning the general problem to determine the conditions under which, given the solution $x(t)$ of (1.12), and changing the initial condition by a small amount, say $x_0 \to \tilde{x}_0$, the resulting solution $\tilde{x}(t)$ stays close (in some sense) to the previous one, and how long. One could refer to such a point equally well as to: sensitivity to initial conditions.

Point 1 stays at the foundations of the whole theory, and one can reasonably state that well posedness for ODEs is almost completely understood. Points 2, 3 and 4 together constitute the backbone of the so-called "qualitative theory" of ODEs: understanding the relevant properties of the solutions without really knowing them. The role of the possible parameters entering the vector field v in the qualitative theory of ODEs is crucial: the sensitivity to small changes of the values of the parameters may change completely the behavior of the system.

Remark 1.3. *For iterated maps (discrete time), well posedness is somehow trivial. The other problems are at least as difficult as in the case of ODEs.*

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