On the stability of spatially uniform Langmuir oscillations of
electronic plasmas

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Abstract

We investigate the stability of spatially uniform, time-periodic solutions of the one-dimensional Vlasov–Maxwell system describing the longitudinal oscillations of an electronic plasma in an uniform neutralizing ion background. We show that such a stability problem can be trivially solved since the zero wave number mode of the electric field, i.e. its space average, performs pure Langmuir oscillations independently of the other modes. We however point out that such oscillations do affect on time average the evolution of the velocity distribution function in the frame at rest.

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In the present contribution we consider the longitudinal oscillations of an electronic plasma in an uniform neutralizing ion background, described by the one-dimensional Vlasov–Maxwell model (in absence of magnetic field). In particular, we study the stability of a particular family of spatially uniform, time-dependent exact solutions of the latter model, namely those corresponding to pure Langmuir oscillations of the electric field at the electron plasma frequency, followed by the rigid oscillations of the initial velocity distribution function (VDF) profile at the same frequency. Such solutions are physically plausible since almost homogeneous plasmas may display, in general, both (displacement) current carrying VDFs and a finite local electric field. Though the stability analysis of periodic solutions is expected to be, in general, a very difficult task (due for example to the possible appearance of parametric resonances), we show that by means of a suitable time-dependent change of coordinates in the single-particle phase space, the full nonlinear problem is mapped into an easier one, namely that concerning the stability of an effective stationary solution, just of the kind investigated by Landau and others [1–3]. Thus, conclusions can be drawn on both linear stability and

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quasi-linear saturation. In particular, the linear stability of the periodic solutions is decided only by the Landau dispersion function computed on the initial VDF (including the initial space average of the perturbation).

The reason for such an unexpected easy solution to the problem is shown to be due to the fact that the vibrations of the zero wave number mode, or space averaged component, of the electric field turn out to be completely independent of the dynamics of all the other modes. Such nondispersive oscillations at the electron vibrations of the zero wave number mode, or space averaged component, of the electric field turn out to be a trivial decoupling of Langmuir oscillations is no longer true [5].

In the case of ions distributed with a space-dependent profile, or in the more realistic case of moving ions, such out that such a “miracle” takes place only due to the fact that ions are supposed to be uniformly distributed. Thus the dynamics of \( E \) given by

\[
\begin{align*}
\frac{dt}{f_t + v f_x - E f_t = 0,} \\
E_t = \int f v \, dv, \\
E_x(x, 0) + \int f(x, v, 0) dv = 1,
\end{align*}
\]

where \( f(x, v, t) \) and \( E(x, t) \) denote the single-particle distribution function and the electric field component along the \( x \)-direction, respectively. All the equations above are written in suitable dimensionless units. We consider the case of space periodic boundary conditions with period \( L \), whereas \( f \) is assumed to depend on the velocity \( v \) in such a way that all the needed integrals exist. The evolution equations (1) and (2) are complemented by the initial constraint (3), which then holds for all the times.

System (1)–(3) admits infinitely many stationary solutions of the form \( (f, E) = (f_0(v), 0) \), where \( f_0 \) is any positive, normalized VDF with vanishing first moment. The problem of the linear stability of such equilibria was first set up and solved by Landau [1], who in particular proved the linear stability of MB distributions due to the damping of the electric field fluctuations. It is known [5] that spatially uniform equilibria are part of a family of spatially uniform periodic solution of system (1)–(3). Indeed, by setting \( f_x = 0 \) and \( E_x = 0 \) in the latter equations one gets the system

\[
\begin{align*}
f_t(v, t) - E(t)f_x(v, t) = 0, \\
\dot{E}(t) = \int f(v, t) v \, dv, \\
\int f(v, 0) \, dv = 1.
\end{align*}
\]

The latter system can be easily solved by differentiating Eq. (5) with respect to time, and making use first of Eq. (4) and then of Eq. (6) (that holds at any time), thus getting

\[
\dot{E} = \int f v \, dv = E \int f_t v \, dv = -E \int f \, dv = -E.
\]

Thus the dynamics of \( E(t) \) is that of a simple harmonic oscillator. The initial data for such an equation are given by \( E_0 = E(0) \) and \( \dot{E}(0) \), the latter being determined once the initial distribution function \( f_0(v) = f(v, 0) \) to be inserted into the right hand side of Eq. (5) is known. Given \( E(t) \), the linear partial differential equation (4) is easily integrated along its characteristics. The solution of system (4)–(6), corresponding to the initial datum \( (f_0(v), E_0) \), is thus given by
\[ f(v, t) = f_0(v + \int_0^t E(s) \, ds), \quad E(t) = E_0 \cos t + \left[ \int f_0(v) \, dv \right] \sin t. \] (7)

Notice that here \( f_0 \) can be any normalized VDF. When \( E_0 = 0 \) and the first moment of \( f_0 \) vanishes, the above periodic solution degenerates into a stationary one. In order to study the stability of the periodic solutions (7) we preliminarily show how the Vlasov–Maxwell equations split up, in general, into a form which turns out to be useful to our analysis.

The two unknowns \( f \) and \( E \) admit a natural decomposition into a spatially homogeneous (independent of \( x \)) and a fluctuating (space-dependent) part, namely

\[ f(x, v, t) = \mathcal{F}(v, t) + \delta f(x, v, t), \quad E(x, t) = \mathcal{E}(t) + \delta E(x, t), \] (8)

\[ \mathcal{F}(v, t) = \frac{1}{L} \int f(x, v, t) \, dx \equiv \langle f \rangle(v, t), \] (9)

\[ \mathcal{E}(t) = \frac{1}{L} \int E(x, t) \, dx \equiv \langle E \rangle(t), \] (10)

\[ \langle \cdot \rangle = \int_0^L (\cdot) \, dx / L \] henceforth denoting space averaging. By the definitions (8)–(10) it follows that \( \langle \delta f \rangle = \langle \delta E \rangle = 0 \). Notice also that both the derivative with respect to the time \( t \) and the derivative with respect to the velocity \( v \) “commute” with the space average operation \( \langle \cdot \rangle \), so that, for example, \( \langle \delta f_t \rangle = \langle \delta f \rangle_t = 0 \) holds.

By inserting the decomposition (8) into Eqs. (1)–(3), taking the space average and keeping in mind the above remarks, one gets the system

\[ \begin{align*}
\mathcal{F}_t - \delta \mathcal{F}_v &= \langle \delta E \delta f_v \rangle, \\
\mathcal{E}_t &= \int \mathcal{F}_v \, dv, \\
\int \mathcal{F}(v, 0) \, dv &= 1.
\end{align*} \] (11)

The last condition in the above system, if true at time \( t = 0 \), actually holds at any time \( t \), which follows from the first two equations. By subtracting the three equations of system (11) from Eqs. (1)–(3) one gets the equations for the fluctuations \( \delta f \) and \( \delta E \), namely

\[ \begin{align*}
\delta f_t + v \delta f_v - \delta E \mathcal{F}_v - \mathcal{E} \delta f_v &= \delta E \delta f_v = \langle \delta E \delta f_v \rangle, \\
\delta E_t &= \int \delta \mathcal{F}_v \, dv, \\
\delta E_v(x, 0) + \int \delta f(x, v, 0) \, dv &= 0.
\end{align*} \] (12)

Again, as a consequence of the first two equations, the last condition in the above system holds at any time \( t \). The above two systems (11) and (12) are exact and equivalent to system (1)–(3). It follows from system (11) and from the last of the three equations of system (12) that

\[ \mathcal{E} + \mathcal{E} = 0, \] (13)

which is what stated in the introduction above: the spatial average of the electric field performs pure harmonic oscillations at the electron plasma frequency (namely one, in present dimensionless units). The validity of Eq. (13) can be proven as follows. By differentiating with respect to time the second equation of system (11) and taking into account the other two equations one gets

\[ \begin{align*}
\mathcal{E} &= \int \mathcal{F}_v \, dv = \mathcal{E} \int \mathcal{F}_v \, dv + \int \langle \delta E \delta f_v \rangle \, dv = -\mathcal{E} \int \mathcal{F}_v \, dv + \left[ \langle \delta E \int \delta f_v \, dv \rangle \right] = -\mathcal{E} - \left[ \langle \delta E \int \delta f \, dv \rangle \right].
\end{align*} \] (14)
Now, due to the last of the three equations (12) and to the periodic boundary conditions chosen, the last term of the second line of (14) exactly vanishes, so that Eq. (13) holds. Indeed, recalling that $\delta E_e + \int \delta f \, dv = 0$, if valid at $t = 0$, holds at any time, one has

$$\left< \delta E \int \delta f \, dv \right> = -\left< \delta E \delta E_e \right> = -\frac{1}{2L} \int_0^L \frac{d(\delta E)^2}{dx} \, dx = 0.$$  

Eq. (13) suggests us to introduce the following change of variables:

$$X = x + \int_0^t \left( \int_0^s \delta(\tau) \, d\tau \right) \, dx, \quad V = v + \int_0^t \delta(s) \, ds, \quad (15)$$

$$\mathcal{F}(v,t) \equiv \Phi(V,t), \quad \delta f(x,v,t) \equiv \varphi(X,V,t), \quad \delta E(x,t) \equiv \varepsilon(X,t). \quad (16)$$

In terms of the new variables, systems (11) and (12) transform together into the system

$$\Phi_t = \langle v \varphi \rangle, \quad (17)$$

$$\varphi_t + V \varphi_x - \varepsilon \Phi_V = v \varphi_x - \langle v \varphi \rangle, \quad (18)$$

$$\varepsilon_t = \int \varphi V \, dV, \quad (19)$$

$$\varepsilon_x(X,0) + \int \varphi(X,V,0) \, dV = 0, \quad \int \Phi(V,0) \, dV = 1. \quad (20)$$

By the definitions (15) and (16), it follows that the initial data for the latter system are given by

$$\Phi(V,0) = \mathcal{F}(V,0), \quad \varphi(X,V,0) = \delta f(X,V,0), \quad \varepsilon(X,0) = \delta E(X,0). \quad (21)$$

Notice that a transformation of the Vlasov–Maxwell system similar to (15) and (16) was first introduced by Klimas and Cooper [6]. At this stage we stress that when Eqs. (17) and (18) are linearized, with the hypothesis that the fluctuations $\varphi$ and $\varepsilon$ are small, taking into account that the first of Eq. (20) holds for all the times and making use of the first of (21), one is left with the system

$$\varphi_t + V \varphi_x - \varepsilon \mathcal{F}_V(V,0) = 0, \quad (22)$$

$$\varepsilon_x + \int \varphi \, dV = 0, \quad (23)$$

i.e. the linearized Vlasov–Poisson system usually considered in the literature to investigate the stability of the stationary states $\mathcal{F}(V) \equiv \mathcal{F}(V,0), \varepsilon = 0$.

Let us now come back to the problem concerning the stability of the periodic solution (7) of the Vlasov–Maxwell system (1)–(3). To this end we consider the displaced initial datum

$$f_0(v) + \delta f_0(v) + \delta f(x,v,0), \quad E_0 + \delta E_0 + \delta E(x,0), \quad (24)$$

close to the initial datum $(f_0(v),E_0)$ giving rise to (7). Notice that perturbations of the reference initial datum consisting both of a mean and of a fluctuating component are introduced. Of course, since $\langle \delta f \rangle = 0$ and $\langle \delta E \rangle = 0$, $f_0 + \delta f_0$ has to be nonnegative and normalized to one. The initial VDF and average electric field are given by $\mathcal{F}(v,0) = f_0(v) + \delta f_0(v)$ and $\varepsilon(0) = E_0 + \delta E_0$, respectively, and they in turn give rise to the exact Langmuir oscillation

$$\delta(t) = (E_0 + \delta E_0) \cos t + \int [f_0(v) + \delta f_0(v)] \, dv \, \sin t. \quad (25)$$

By making use of such an expression to define the change of variables (15), (16) and proceeding as above, after linearization with respect to $\varphi$ and $\varepsilon$, one is left with system (22) and (23), with $\mathcal{F}_V(V,0) = d [f_0(V) + \delta f_0(V)] / dV$. As is well known, upon inserting the latter function into the Landau dispersion function [1], one determines the asymptotic behaviour of the Fourier–Laplace modes of the solution of system (22) and (23). Thus, the linear stability analysis of the periodic solution (7) is reduced to the standard Landau analysis of the effective stationary VDF $f_0 + \delta f_0$, as stated above. It might be shown that, in absence of instabilities, and at a linear level, the difference between the perturbed solution corresponding to the slightly displaced initial
datum (24), and solution (7) is small, uniformly in time; in other words the reference periodic solution (7) is Lyapunov stable.

In order to understand which is the effect of the Langmuir oscillations of the averaged electric field on the VDF in the reference frame at rest, one can imagine to initialize the dynamics of the Vlasov–Maxwell system with an initial datum of the form $f(x,v,0) = f_0(v) + \delta f(x,v,0), E(x,0) = E_0 + \delta E(x,0)$, with $f_0(v) = e^{-v^2/2}/\sqrt{2\pi}$, i.e. a MB distribution. In the linear approximation, the time average of the VDF (in the frame at rest) is given by

$$f_0(v; E_0) = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} e^{-\frac{1}{2}(v+E_0 \sin \theta)^2} d\theta,$$

which is nothing but the time average, over one electron plasma period, of the rigidly oscillating initial Gaussian VDF. One easily realizes (analytically or by means of a simple numerical integration) that if $E_0$ is small, the profile of $f_0$ resembles that of the initial Gaussian VDF, i.e. is single humped, with a maximum at $v = 0$. However, when $E_0$ exceeds the critical value of about $E_{0c} \approx 1.25$, $f_0(v; E_0)$ becomes double humped, displaying a minimum at $v = 0$ and two symmetric maxima placed at $v \approx \pm E_0$. Such a double humped profile was already found in a different context by Krapchev [7], who observed that, though double humped, the supercritical $f_0$, when used as an equilibrium function, does not give rise to any instability. Such a nontrivial result can be checked by means of the Penrose criterion [2]. Of course, the real time averaged VDF profile will be modified by nonlinear resonant flattening effects. However, as appears from some numerical simulations [8,9], the latter effects, in the supercritical case, might be less important than the topological deformation induced by the Langmuir oscillations.

To end with, we stress that in the full three-dimensional Vlasov–Maxwell problem (including the magnetic field), exact spatially uniform (quasi-periodic in time) solutions exist. An analysis similar to that presented here should be possible, and deserves to be carried out.

References


