Role of the initial conditions in the FPU problem

MPIPKS, Dresden - December 3, 2008

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most of results from a preprint - under review, JSP - in collaboration with G. Benettin (Padova) and R. Livi (Firenze)

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THANKS!

Outline part I

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- The motivation of the study goes back to previous works of Poincaré and Fermi himself on the subject.

(2) Model, relevant quantities, numerics

Study the *prototype* Hamiltonian problem

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- the system eventually relaxes to equilibrium, characterized by modal energy equipartition: $E_1 = \cdots = E_{N-1} \simeq \varepsilon$;
- a *metastable state*, or "quasi-state" characterized by an exponentially decaying energy spectrum - $E_k \sim C \left(\frac{k}{N}\right)^{\rho} e^{-\gamma(\varepsilon)k/N}$ - persists for very long times

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One defines the (normalized, time-averged) modal energy spectrum:

$$\boxed{\frac{\overline{E}_k(t)}{\varepsilon} = \frac{1}{\varepsilon t}\int_0^t E_k(s)ds \quad \text{vs.} \quad k/N}$$

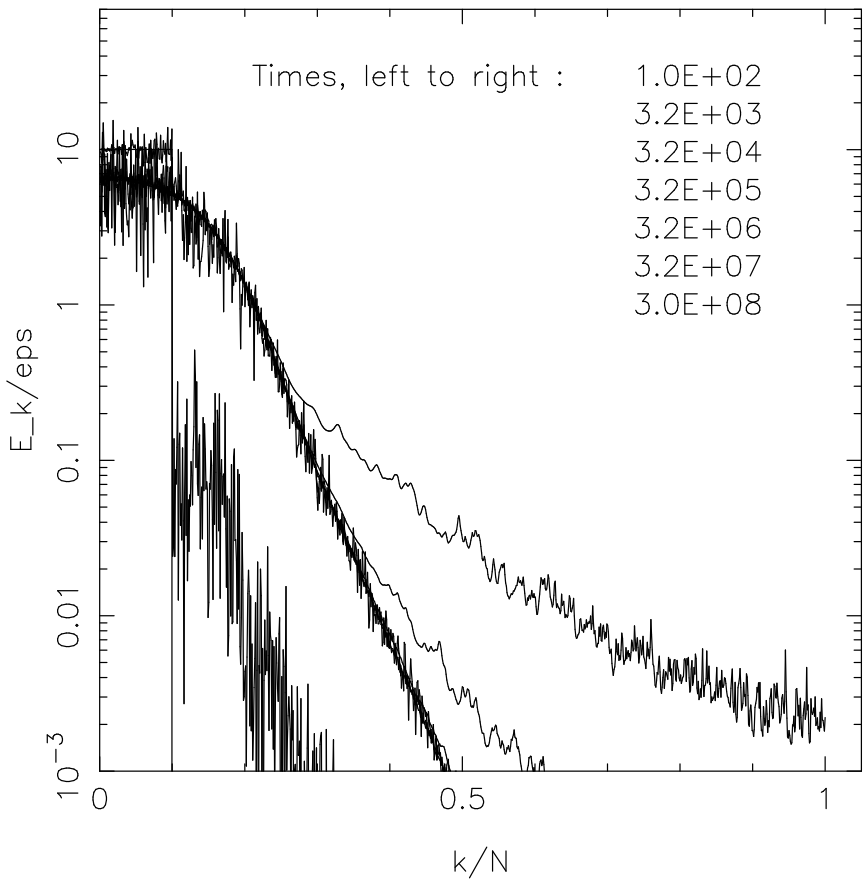

Figure: Snapshots of a modal energy spectrum: $N = 1024$, $\varepsilon = 2.5 \cdot 10^{-4}$, 10% of modes initially excited, random phases; notice the persistence of the exponential tail

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Effective density $f(t)$ of active modes:

$$\boxed{f(t) = \frac{M(t)}{N} = \frac{e^{S(t)}}{N}}$$

$\mathcal{P}_k = 1/N$ at equilibrium, and $f_{eq} = 1$.

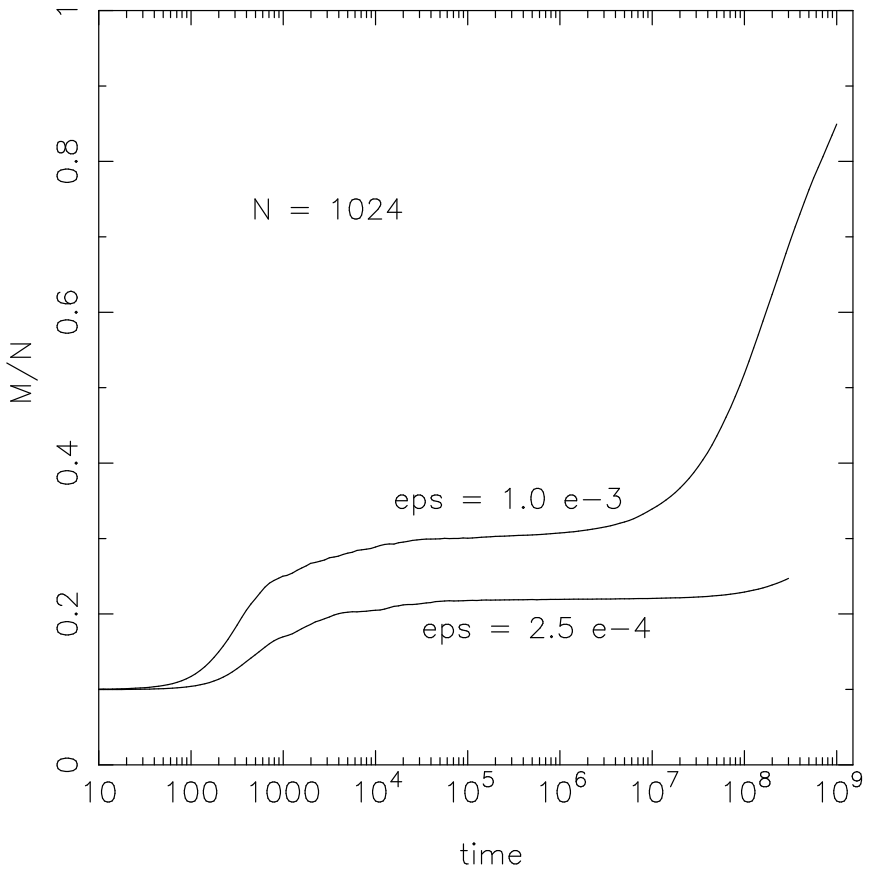

Figure: Effective density $f = M/N$ of active modes vs. time; $N = 1024$, $\varepsilon = 2.5 \cdot 10^{-4}$ and $\varepsilon = 10^{-3}$, $f(0) = 0.1$

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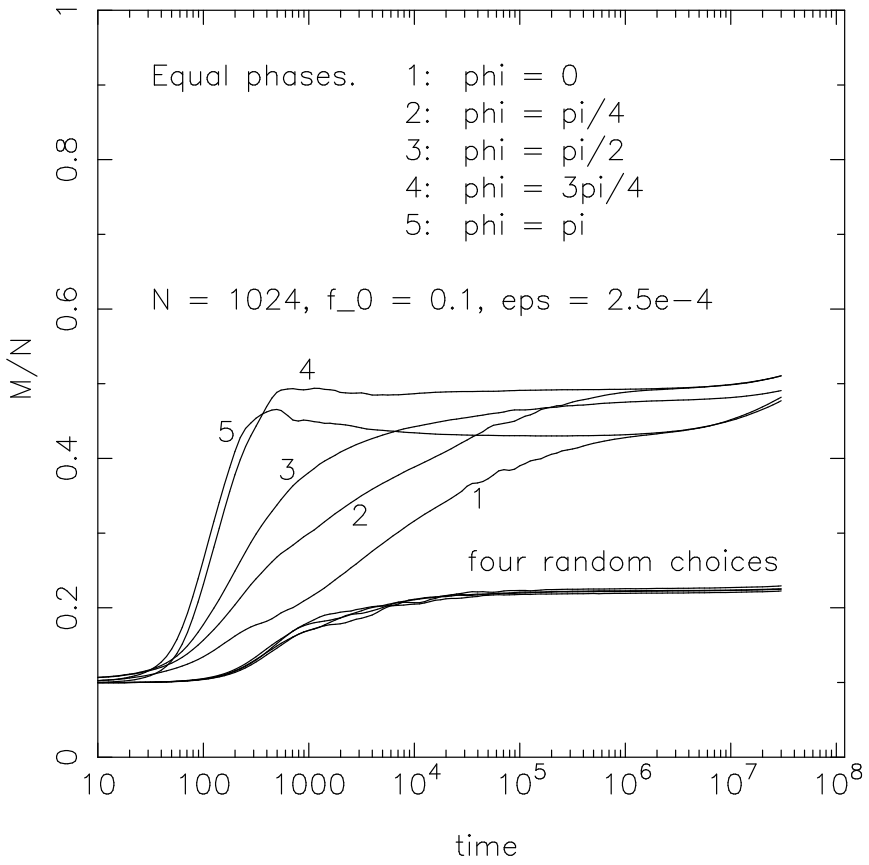

Figure: Effective density $f = M/N$ of active modes vs. time; $N = 1024$, $\varepsilon = 2.5 \cdot 10^{-4}$, $f(0) = 0.1$ ($M_0 = 102$); six different choices of phases

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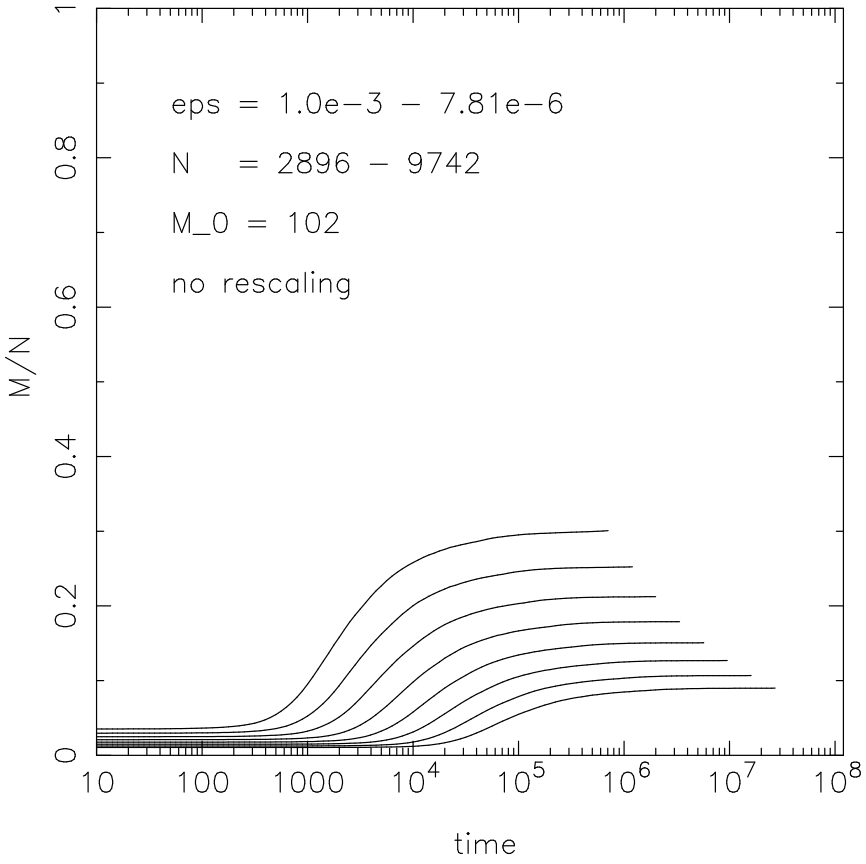
(S1) can be predicted exactly from KdV scaling; (S2) has no clear explanation (but for some *a fortiori* heuristic estimate).

(3) Numerical results



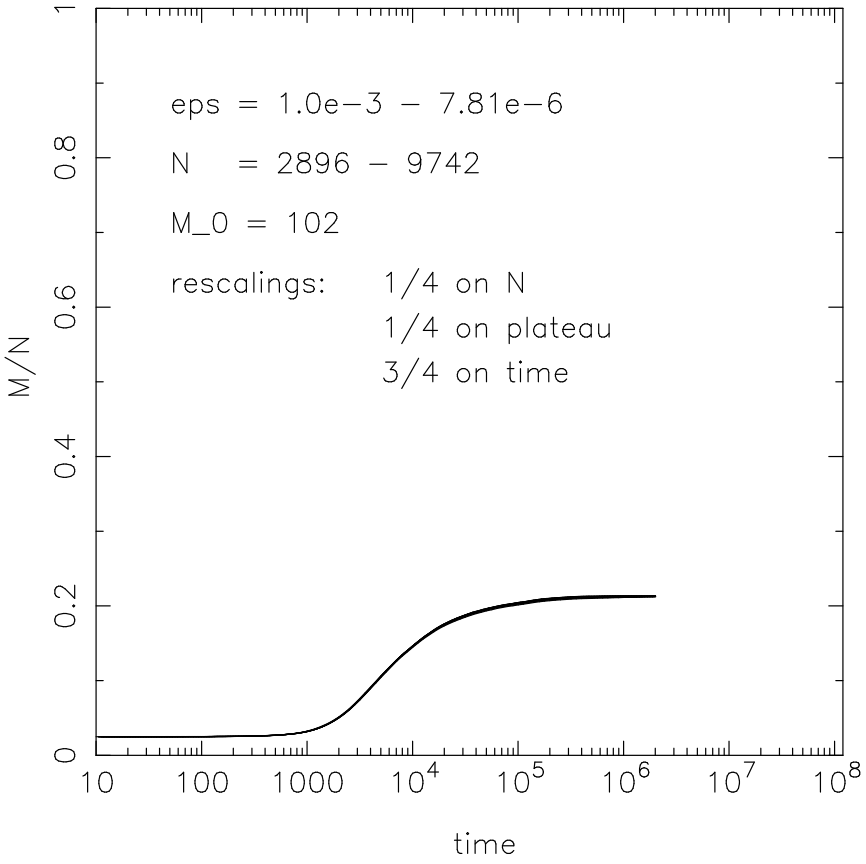
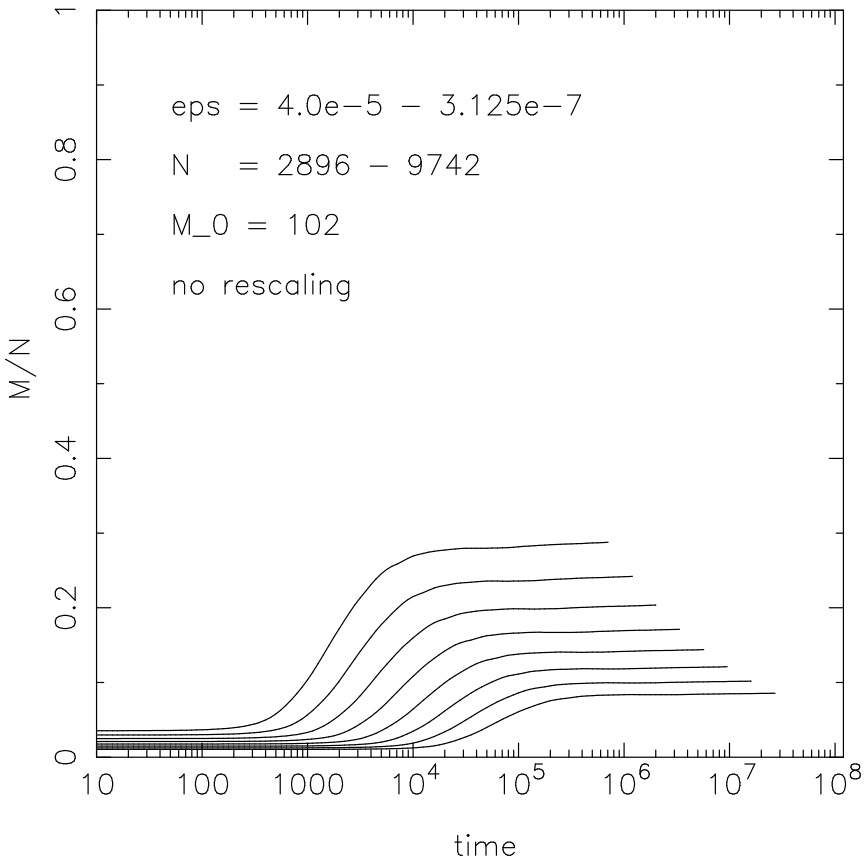


Figure: Check of scaling (S1); $\lambda_j = 2^{-j+2}$, $j = 0, \ldots 7$, $\varepsilon_{ref} = 2.5 \cdot 10^{-4}$, $N_{ref} = 4096$; random phases.



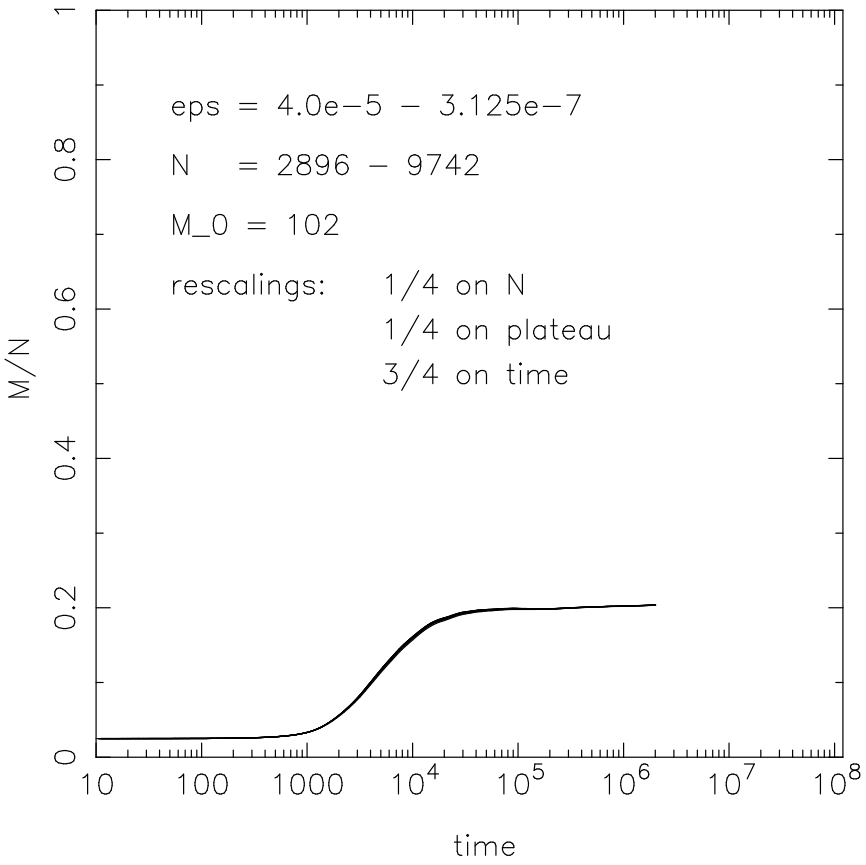


Figure: Check of scaling (S1); $\lambda_j = 2^{-j+2}$, $j = 0, \ldots 7$, $\varepsilon_{ref} = 2.5 \cdot 10^{-4}$, $N_{ref} = 4096$; equal phases: kick-like initial conditon.

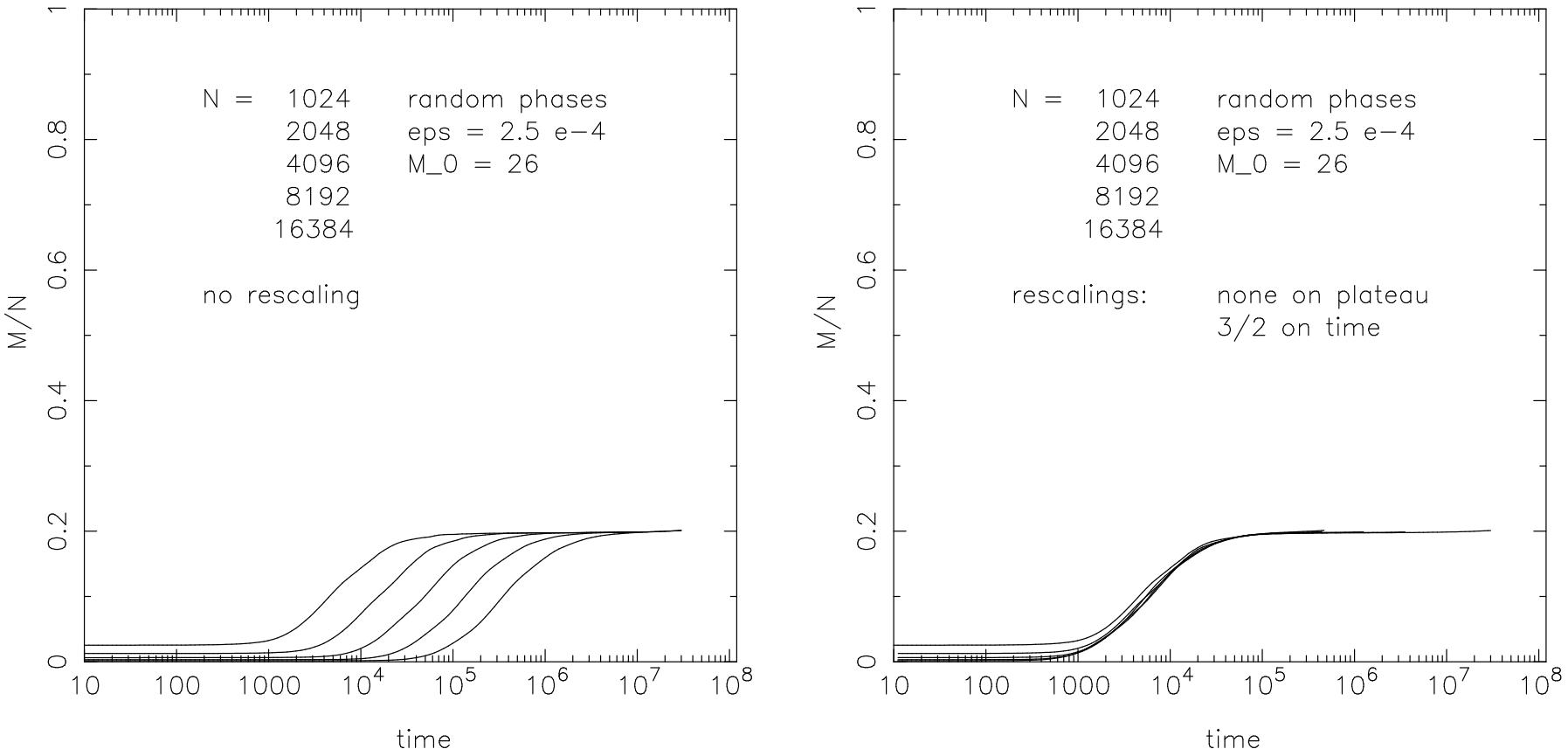


Figure: Check of scaling (S2); $\mu_j = 2^j$, $j = 0, \ldots 4$, $\varepsilon = 2.5 \cdot 10^{-4}$, $N_{ref} = 1024$, $M_0 = 26$ random phases.

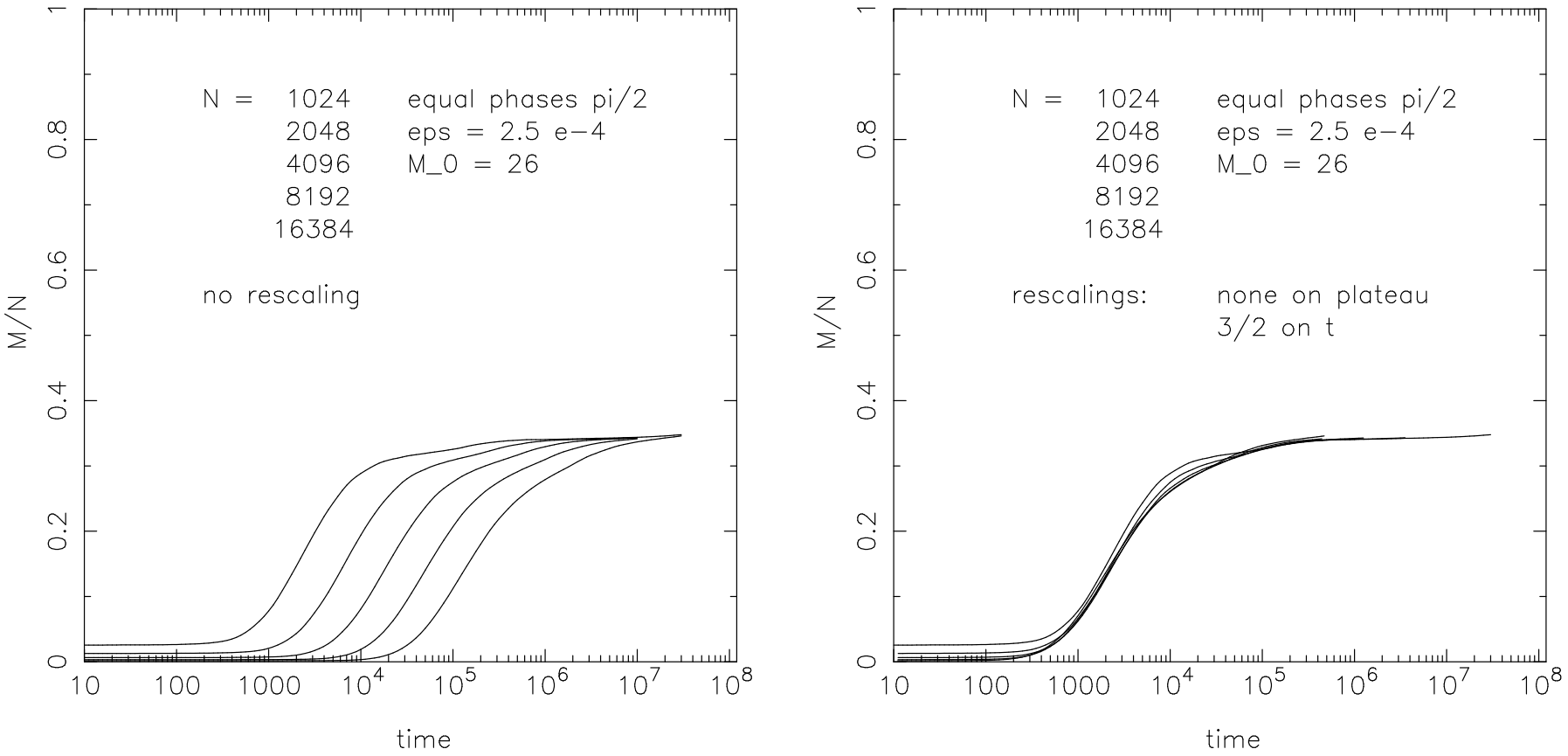


Figure: Check of scaling (S2); $\mu_j = 2^j$, $j = 0, \ldots 4$, $\varepsilon = 2.5 \cdot 10^{-4}$, $N_{ref} = 1024$, $M_0 = 26$ equal phases: zero initial velocities.

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In other words: *if M_0 is kept fixed - independent of N - then $f \sim \varepsilon^{1/4}$ on a time-scale $\tau_1 \sim N^{3/2} \varepsilon^{-3/8}$.*

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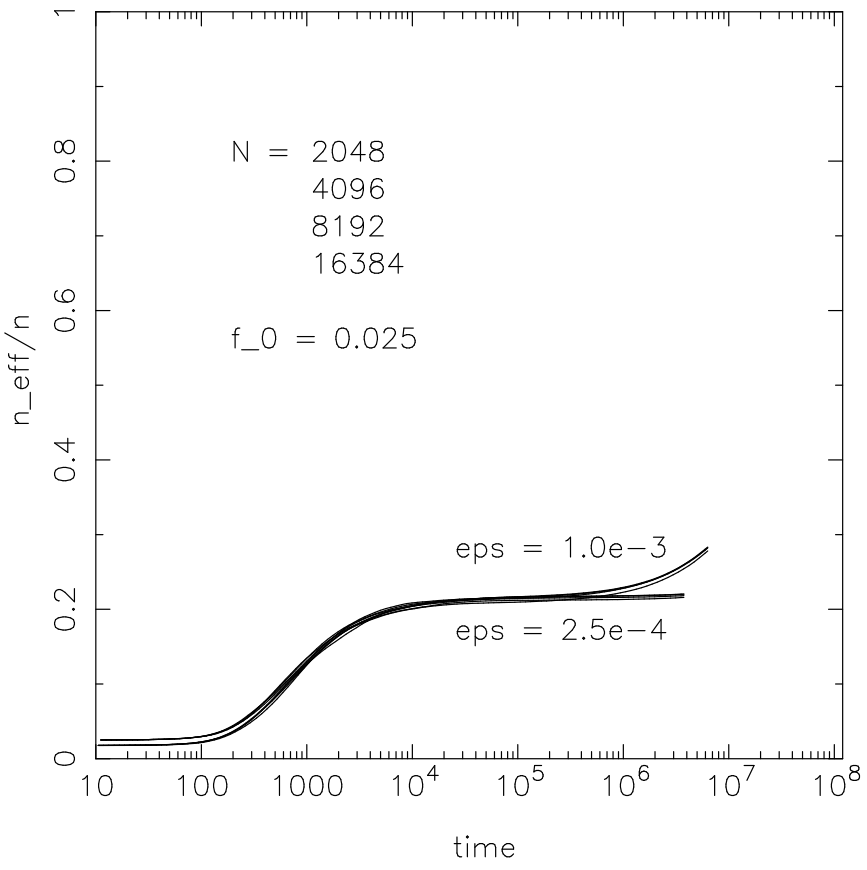

Figure: Check of scaling (RP), two values of ε, four values of N.

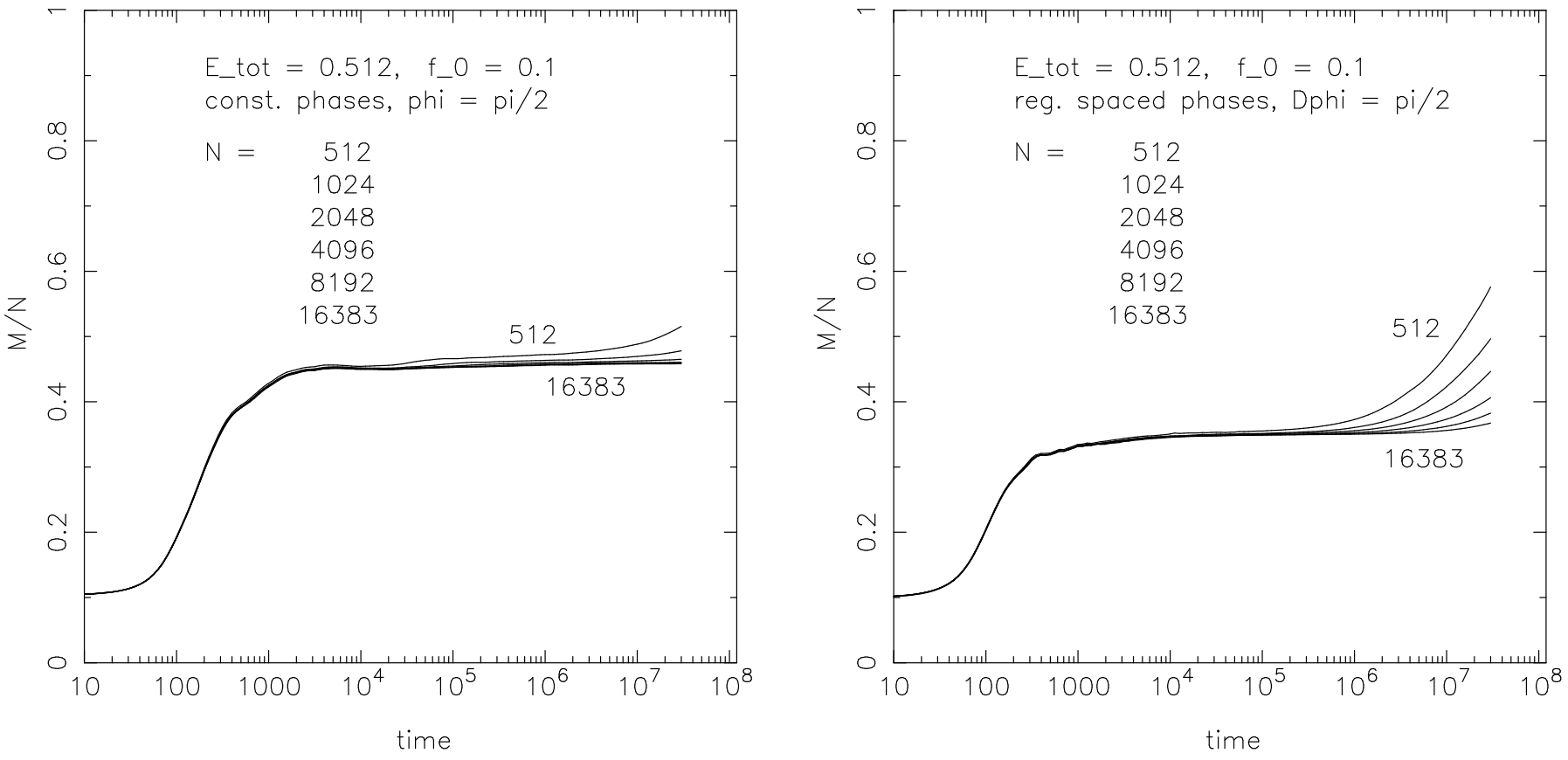

Figure: Check of scaling (CP); left: $\phi_k = \pi/2$; right: $\phi_k = k\pi/2$

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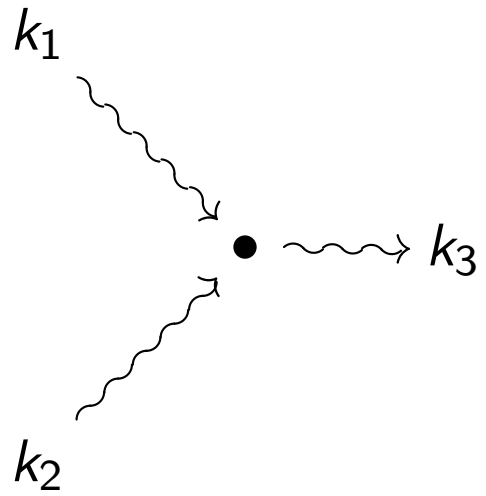
- No t.d. limit persistence is expected to be possible with extended excitations and coherent phases

THEORY I

Mode-coupling is ruled by the selector

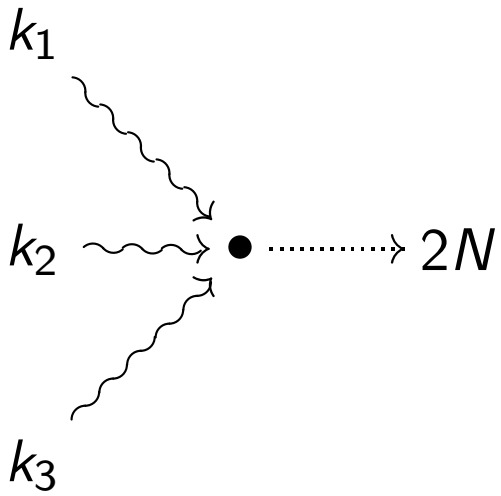
$$\Delta_{k_1,k_2,k_3} = \delta_{k_1+k_2,k_3} + \delta_{k_2+k_3,k_1} + \delta_{k_3+k_1,k_2} - \delta_{k_1+k_2+k_3,2N}$$

with corresponding processes



small denominators:

$\omega_{k_1} + \omega_{k_2} \simeq \omega_{k_3=k_1+k_2}$



forbidden to low modes

STRATEGY

Resonant normal form construction

STRATEGY

$$
\begin{tikzcd}
(q,p)_{FPU} \arrow[r, "\text{Perturbation theory}"] & (\xi,\xi^*)_{FPU} \arrow[d, "\text{Normal form}"] \\
U_{KdV} \arrow[u, "\text{analyticity}"] & (u,u^*)_{FG-KdV} \arrow[l, "\text{reconstruction}"]
\end{tikzcd}
$$

Perform the canonical rescaling $(z, t, H_N) \mapsto (\zeta, \tau, K_N)$ defined by

$$\zeta_k = \frac{z_k}{N\sqrt{2\epsilon}} \quad , \quad \tau = \frac{t}{2N} \quad , \quad K_N = \frac{H_N}{N\epsilon}$$

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where $0 < \epsilon = \varepsilon + O(\varepsilon^{3/2})$. Define $\boxed{\mu = \alpha\sqrt{\epsilon}}$, $\boxed{L = 2N\sqrt{\mu}}$ ($\alpha > 0$ w.l.g.) and expand ω_k.

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where

$$\boxed{W = -\sum_{k=1}^{N-1} \frac{(2\pi k)^3}{24L^2}|\zeta_k|^2 + \frac{2}{3} \sum_{k_1,k_2,k_3=1}^{N-1} \Delta_{k_1,k_2,k_3} \prod_{s=1}^{3} \sqrt{2\pi k_s}\ \mathrm{Re}(\zeta_{k_s})}$$

Unperturbed ($\mu = 0$) motion: $\zeta = e^{-i2\pi\mathcal{J}\tau}\zeta(0)$, 1-periodic in time, $\mathcal{J} = \mathrm{diag}(1, \ldots, N-1)$ (i.e. $\zeta_k = e^{-i2\pi k\tau}\zeta_k(0)$).

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Proposition (Averaging)

There exists a canonical transformation $(\zeta, \zeta^*) \mapsto (\xi, \xi^*)$, μ-close to the identity, such that

$$K_N(\xi, \xi^*) = \sum_{k=1}^{N-1} (2\pi k)|\xi_k|^2 + \mu\overline{W}(\xi, \xi^*) + O(\mu^2)$$

where

$$\overline{W}(\xi, \xi^*) = \int_0^1 W(e^{-i2\pi\mathcal{J}\tau}\xi, e^{i2\pi\mathcal{J}\tau}\xi^*)\, d\tau$$

Introducing the time-dependent transformation to noncanonical co-rotating coordinates

$$(\xi, \tau, K_N) \mapsto (u, T, \overline{W} + O(\mu))$$

$$u = \sqrt{2\pi\mathcal{J}}\ e^{i2\pi\mathcal{J}\tau}\xi \quad , \quad T = \mu\tau$$

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up to a remainder $O(\mu)$. The equations of motion are

$$\boxed{\frac{du_k}{dT} = -i(2\pi k)\frac{\partial \overline{W}}{\partial u^*_k}}$$

Proposition

The Hamilton equations of $\overline{W}(u, u^*)$ *coincide with the Fourier-Galerkin truncation to the first* $N-1$ *modes of the KdV equation*

$$U_T = \frac{1}{24L^2} U_{XXX} + \frac{1}{2} U U_X \quad , \quad X \in \mathbb{T}(= \mathbb{R}/\mathbb{Z})$$

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Thus the KdV equation in the small dispersion regime ($L \propto N\epsilon^{1/4}$ very large) "formally" describes the dynamics of the FPU problem on the time-scale $t \sim N/\sqrt{\epsilon}$ (recall that $T = t\sqrt{\epsilon}/N$).

The Fourier-Galerkin projection operator $\mathbf{P}^N$ is defined by

$$(\mathbf{P}^N U)(X,T) = \sum_{k=-N+1}^{N-1} \hat{U}_k(T) e^{-i2\pi kX} \ .$$

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Scaling properties

Recall that $U_T = \frac{1}{24L^2} U_{XXX} + \frac{1}{2} U U_X$ and $L \propto N\epsilon^{1/4}$, $T \propto \sqrt{\epsilon} t/N$.

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Regularity of solutions

Theorem (Kappeler-Pöschel 07)

Consider the KdV equation $U_T = \delta U_{XXX} + \frac{1}{2} U U_X$.
If the initial datum $U(X,0)$, $X \in \mathbb{T}$, *is analytic in the (maximal) complex strip* $\{|Im(z)| \leq a\}$, *then there exists* $0 < \rho(\delta) \leq 1$ *s.t. the corresponding solution* $U(X,T)$ *is analytic in the complex strip* $\{|Im(z)| \leq a\rho(\delta)\}$ *for any* $T \in \mathbb{R}$.

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No estimate of $\rho(\delta)$ available:
we are interested in the limit $\delta \to 0$!

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Notice that as $L \to \infty$ the KdV eqn approaches the Burgers-Hopf eqn $U_T = \frac{1}{2} U U_X$, whose solutions display vertical slopes in a finite critical time $T_c = O(1)$.

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HEURISTIC ARGUMENTS:

- for any fixed $T > T_c$, $\sigma(T;L) \to 0$ as $L \to \infty$;
- requiring, in the limit, constant ratio of the above Cauchy upper bounds yields

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Theorem (Kalisch 05)

Suppose $U(X,0) \in \mathcal{H}^{a_0,s}$, where $a_0 > 0$ and $s > 0$; then, for any fixed $T > 0$ there exist two positive constants $\lambda(T)$ and $a(T)$ s.t.

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That is to say: u^N and U can be made arbitrarily close on any <u>fixed time-interval</u> if N is large enough.

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The r.h.s. is evaluated on the initial condition

$$U(X,0) = \frac{2}{\sqrt{M_0}} \sum_{k=1}^{M_0} cos(2\pi kX + \phi_k)$$

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One gets

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This is a rough, though eventually correct, explanation of the observed dependence of the dynamics on the phases.

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both being lower bounds to τ_1!

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- Second stage of the story: breakdown of KdV approximation and estimate of the time-scale to equipartition - difficult