On the role of KdV equation in the FPU problem

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November 29, 2013 - SISSA, Trieste

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* Joint work with G. Benettin (Padova), H. Christodoulidi (Patras) and B. Rink (Amsterdam)

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Statistical mechanical purpose:*"The ergodic behavior of such systems was studied with the primary aim of establishing, experimentally, the rate of approach to the equipartition of energy among the various degrees of freedom of the system."*

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"Instead of a gradual increase of all the higher modes, the energy is exchanged, essentially, among only a certain few. [...] Certainly, there seems to be very little, if any, tendency towards equipartition of energy among all degrees of freedom at a given time. [...] What is suggested by these special results is that in certain problems which are approximately linear, the existence of quasi states may be conjectured."

Historical motivation

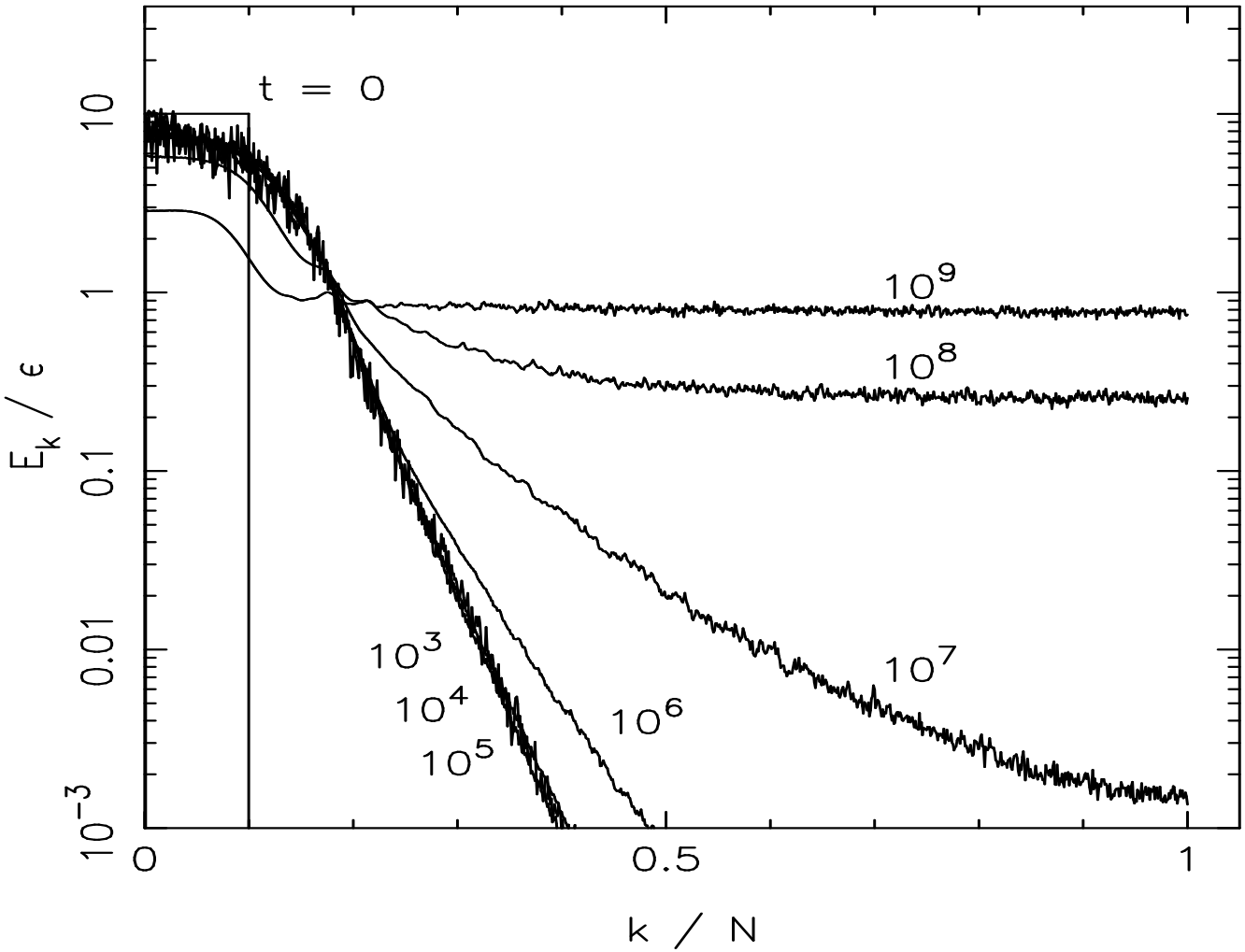

Figure: Recent figure: N=1024, 10% of modes initially excited

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Result: the first numerical observation of "solitons", and a phenomenological description of the recurrence observed by FPU in terms of the latters.

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- Conclusions

Hamiltonian formulation

- Hamiltonian:

$$\mathscr{H}(q,p) = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \left[\frac{p_j^2}{2} + \phi(q_{j+1} - q_j) \right]$$

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- Trivial conservation laws:

$$\sum_j q_j = 0 = \sum_j p_j \qquad (w.l.o.g.)$$

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Thermodynamic limit: large size limit with ε small kept fixed.

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Proposition

The FPU equations $\dot{q}_j = \partial \mathscr{H} / \partial p_j$, $\dot{p}_j = -\partial \mathscr{H} / \partial q_j$ *are the Hamilton equations* $Q_\tau = \delta H / \delta P$, $P_\tau = -\delta H / \delta Q$, *associated to*

$$H[Q,P] = \int_{\mathbb{T}} \left[\frac{P^2}{2} + \frac{\phi(\sqrt{\varepsilon}\hat{D}_h Q)}{\varepsilon} \right] \mathrm{d}x$$

and restricted to $\tau = ht$ *and* $x = hj$, $j \in \mathbb{Z}/N\mathbb{Z}$.

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$$\begin{aligned} H &= \int_{\mathbb{T}} \left[\frac{P^2 + (Q_x)^2}{2} \right] \mathrm{d}x + \int_{\mathbb{T}} \left[-\frac{h^2}{24} (Q_{xx})^2 + \frac{\alpha\sqrt{\varepsilon}}{3} (Q_x)^3 \right] \mathrm{d}x \\ &+ \quad \dots \end{aligned}$$

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$$Q_{\tau\tau} = Q_{xx} + \frac{h^2}{12} Q_{xxxx} + 2\alpha\sqrt{\varepsilon} Q_x Q_{xx}$$

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$$\underbrace{Q_{\tau\tau} = Q_{xx} + \frac{h^2}{12} Q_{xxxx} + 2\alpha\sqrt{\varepsilon} Q_x Q_{xx}}_{\text{Integrable Boussinesq equation}} + (h.o.t.)$$

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Noncanonical change of variables $(Q, P) \to (u, v)$:

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The Hamilton equations take on a Gardner-like structure

$$\begin{pmatrix} u \\ v \end{pmatrix}_\tau = \begin{pmatrix} \partial/\partial x & 0 \\ 0 & -\partial/\partial x \end{pmatrix} \begin{pmatrix} \delta H/\delta u \\ \delta H/\delta v \end{pmatrix}$$

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$$m_{\sigma_0} = \max_{S_{\sigma_0}} \{|u_0|, |v_0|\}$$

Proposition (existence of ordering)

Suppose that $h^2/\sigma_0^2 = O(\sqrt{\varepsilon} m_{\sigma_0})$; *then*

$$H[u_0, v_0] = \sum_{j \geq 0} H_j[u_0, v_0]$$

with $|H_j[u_0, v_0]| = O(\varepsilon^{j/2} m_{\sigma_0}^{j+2})$

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$$\Phi^{\tau}_{H_0}(u_0, v_0)(x) = (u_0(x+\tau), v_0(x-\tau))$$

Hamiltonian formulation

- Resonant averaging

Proposition

There exists a canonical transformation (flow) $e^{-\{\ ,G\}} : (u, v) \mapsto (\tilde{u}, \tilde{v})$ *s.t.* $H \circ e^{\{\ ,G\}} = H_0 + \overline{H}_1 + R'$, *where*

$$\overline{H}_1 = \int_0^1 H_1 \circ \Phi^s_{H_0} \, \mathrm{d}s$$

$$G = \int_0^1 s(H_1 - \overline{H}_1) \circ \Phi^s_{H_0} \mathrm{d}s$$

Hamiltonian formulation

Neglecting the reminder R', the resonant normal form Hamiltonian reads

$$H_0[\tilde{u}, \tilde{v}] + \overline{H}_1[\tilde{u}, \tilde{v}] = K[\tilde{u}] + K[\tilde{v}]$$

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The dynamics, to leading order, is described by two noninteracting KdV equations:
$\tilde{u}_\tau = (\delta K/\delta\tilde{u})_x$, $\tilde{v}_\tau = -(\delta K/\delta\tilde{v})_x$

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As $\delta \to 0$, $\sigma \to 0$ and m_σ becomes asymptotically independent of σ. Thus $\boxed{\sigma \sim \delta}$, independent of time

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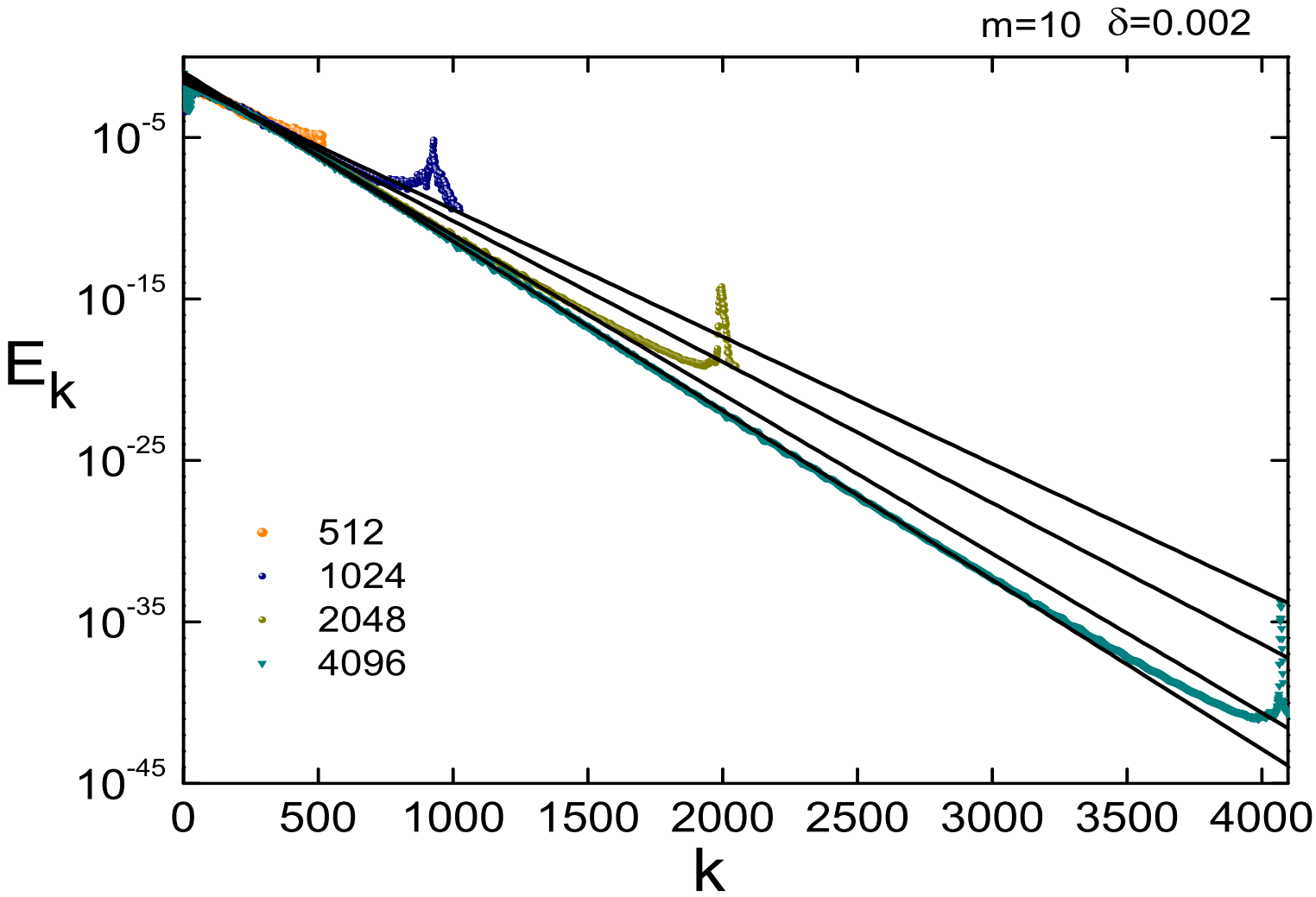

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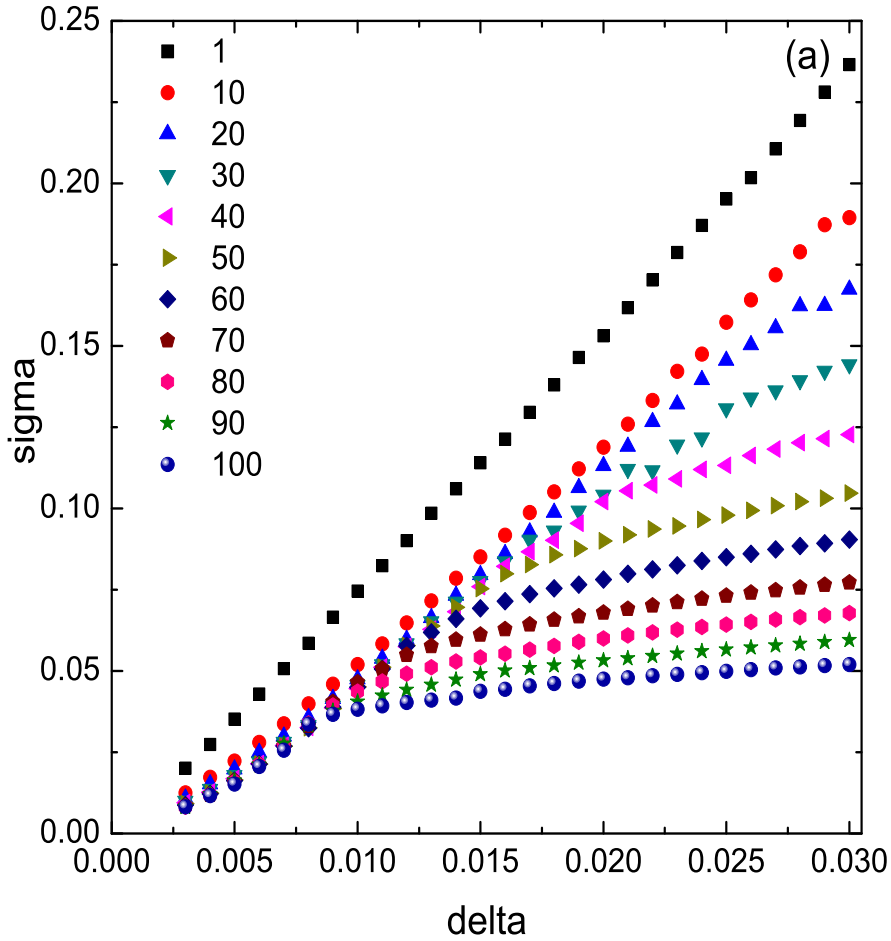

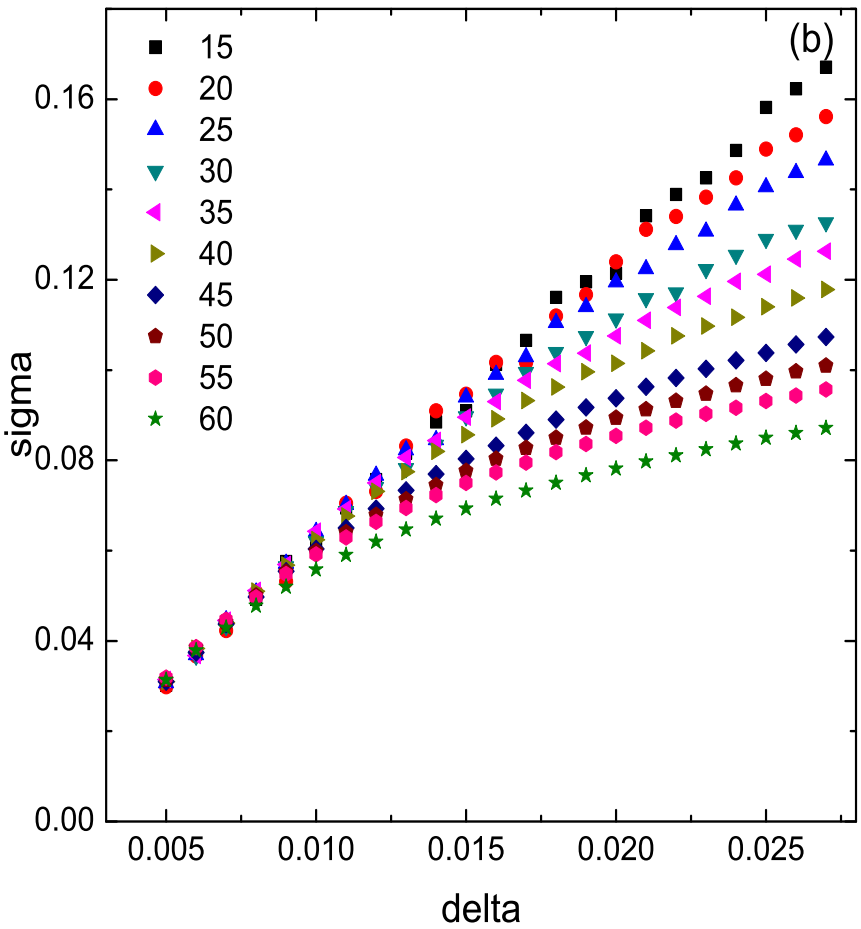

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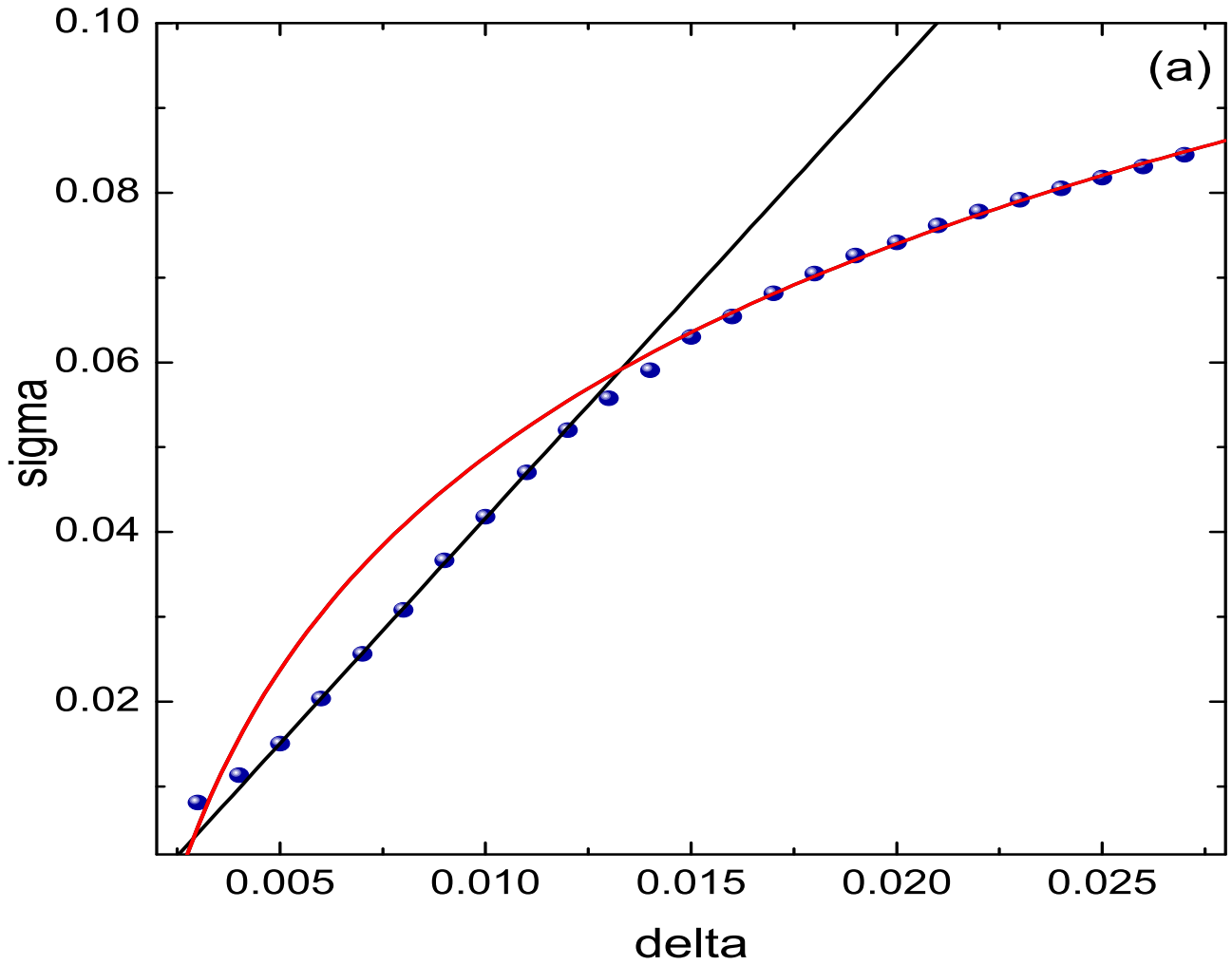

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The KdV equations

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are invariant under the re-scaling

$$\varepsilon \to \lambda\varepsilon \; ; \; N \to \lambda^{-1/4} N \; ; \; t \to \lambda^{-3/4} t$$

Properties of the KdV

By construction:

$$E_k^{FPU} \propto |U_k|^2 \sim e^{-\sigma k} = e^{-c\delta k} = \exp\left(-c' \frac{k/N}{\sqrt{\alpha}\varepsilon^{1/4}}\right)$$

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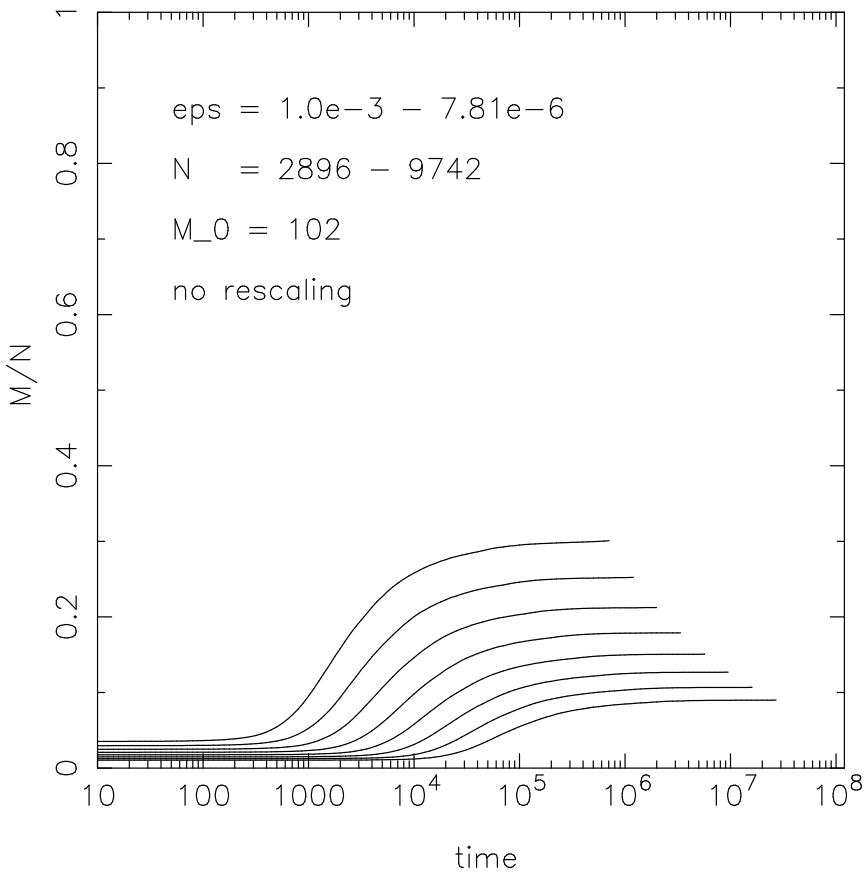
Thus, for the fraction $f = M/N$ of active modes one expects $f \to \lambda^{1/4} f$ under the λ-rescaling.

Properties of the KdV



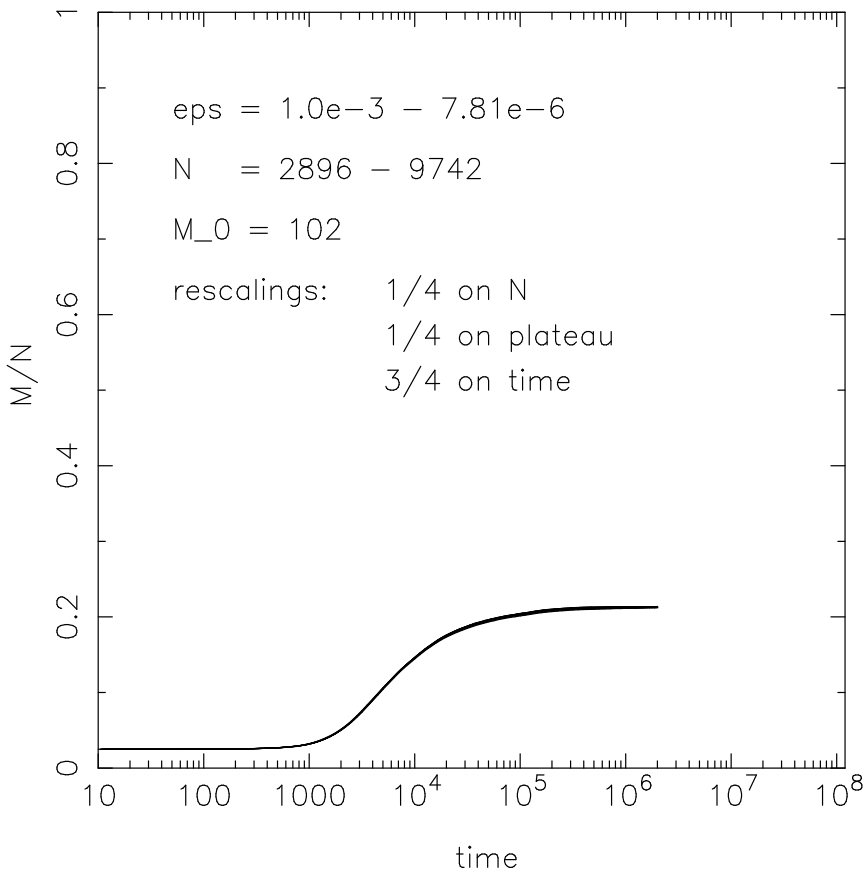


Figure: Check of the KdV scaling-law
$f(\lambda^{-3/4}t, \lambda^{-1/4}N, \lambda\varepsilon, M_0) = \lambda^{1/4} f(t, N, \varepsilon, M_0)$ in FPU

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  - $\lim_{N\to\infty} 1/(N^3 E) = 0$: KdV in the zero dispersion limit. This regime includes, as a particular case, the thermodynamic limit ($E/N\varepsilon$ fixed).

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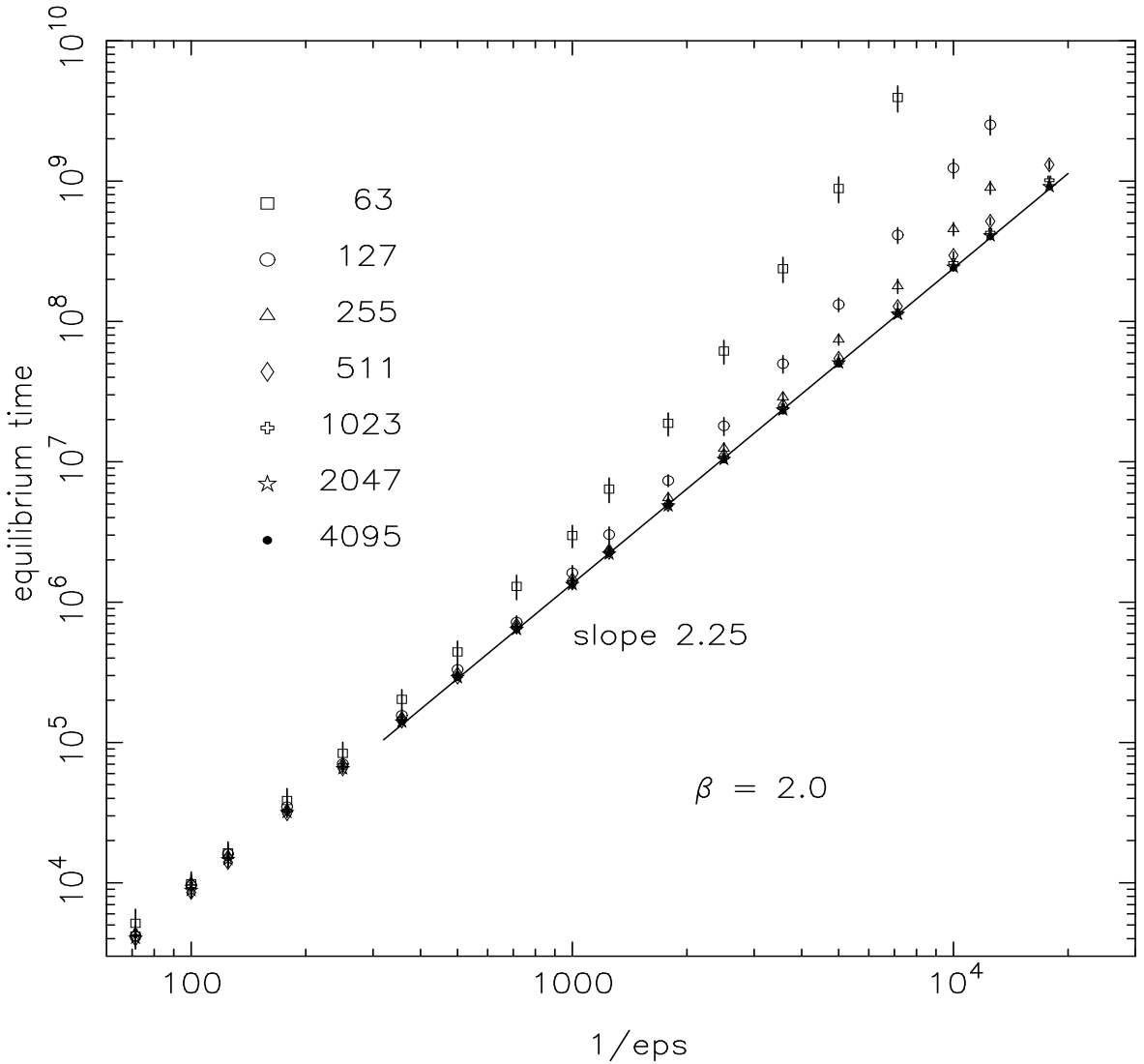

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- Besides KdV: it is *conjectured* that

$$H_{FPU} = KdV_3 + KdV_5 + R$$

THANKS