Construction of discrete time models admitting a finite dimensional filter : an approach based on the inverse Laplace transform

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Abstract

We study the problem of existence of finite-dimensional filters in discrete time. Discrete time filtering problems can be seen as dynamic generalizations of Bayesian statistics, when the unknown parameter becomes an unobserved (state/signal) process. The problem of existence of finite-dimensional filters corresponds, in Bayesian statistics, to the existence to a finitely parametrized family of conjugate distributions and this makes the exponential class of distributions relevant also for filtering. Given an observation model with distribution belonging to the exponential class and a conjugate family of filter distributions together with its dynamics, we give conditions under which there exists an unobserved state/signal process leading to the given filter distribution with its dynamics. We also present an approach to actually determine the state process dynamics. The methodology is based on the inverse Laplace transform.

Key Words: Dynamic Bayes formula, exponential families, finite dimensional filters, infinitely divisible distributions, inverse Laplace transform.

Running title : Discrete time finite dimensional filters.

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1 Introduction

The discrete time filtering problem is a natural dynamic generalization of the classical Bayesian inference problem. In its simplest form, the Bayesian inference problem concerns in fact an unknown parameter, let us call it here X, and a sequence of observations $y_1, y_2, \cdots$ that are realizations of corresponding random variables $Y_1, Y_2, \cdots$, whose joint distribution depends on the value x of X; in most cases one makes the conditional independence assumption : conditional on x the random variables Y_n are independent (the Y_n's are exchangeable). Given a prior distribution on X, the problem is to update recursively this distribution on the basis of the current (most recent) observation.

In the discrete-time filtering problem, instead of a fixed unknown parameter, one considers an unobserved process $\{X_n\}$, for which an apriori probabilistic description is given. More precisely, one considers a partially observable process (X_n, Y_n), where $\{Y_n\}$ is the observable and $\{X_n\}$ the unobservable component; a realization of the latter determines the stochastic evolution of $\{Y_n\}$ according to a relation satisfying the following conditional independence property

$$p(y_n \,|\, x_0, \cdots, x_n; y_0, \cdots, y_{n-1}) = p(y_n \,|\, x_n) \tag{1}$$

with p denoting generically the "distribution". Filtering problems have a wide range of applications, namely every time one has a state or signal process $\{X_n\}$, that can be observed only indirectly through an observation process $\{Y_n\}$ and one wants to make an inference on the current value x_n of the state/signal on the basis of the past and current observations $y^n := \{y_0, \cdots, y_n\}$.

Assuming that the process $\{X_n\}$ is Markov and that, for simplicity, all distributions we shall be dealing with are absolutely continuous with respect to a same dominating measure (the Lebesgue measure), the elements characterizing a filtering problem for the partially observable process $\{X_n, Y_n\}$ with $X_n \in \mathcal{X}$, $Y_n \in \mathcal{Y}$ are, as far as the filtering model is concerned :

$$\begin{array}{lcl} \pi_0(x) & : & \text{density of } x_0 \text{ (initial density)} \\ p(x_n | \, x_{n-1}) & : & \text{transition kernel} \\ f(y| \, x) & : & \text{observation density} \end{array} \tag{2}$$

and, concerning the filter solution :

$$\begin{array}{lcl} \pi_n(x_n | \, y^n) & : & \text{filtering density} \\ p_n(x_n | \, y^{n-1}) & : & \text{prediction density} \end{array} \tag{3}$$

For simplicity of exposition, in this paper we shall assume $\mathcal{X}$ and $\mathcal{Y}$ to be scalar sets ($\mathcal{X}, \mathcal{Y} \subset \mathbb{R}$).

Analogously to what is the main tool in Bayesian statistical inference, namely Bayes' formula, for a discrete time filtering problem the main tool is the recursive/dynamic Bayes formula, that (see [8]) corresponds to the Zakai equation in continuous time filtering problems (for the latter see e.g. [7]). More precisely, given the filtering model $(\mathcal{X}, \mathcal{Y}, \pi_0(x), p(x_n | x_{n-1}), f(y_n | x_n))$, by this formula the filtering and prediction densities $\pi_n(x_n | y^n)$, $p_n(x_n | y^{n-1})$ satisfy the recursive relations

$$\pi_n(x_n | y^n) \propto f(y_n | x_n) p_n(x_n | y^{n-1}) \tag{4}$$

$$p_n(x_n | y^{n-1}) = \int_{\mathcal{X}} p(x_n | x_{n-1}) \pi_{n-1}(x_{n-1} | y^{n-1}) dx_{n-1} \tag{5}$$

where (4) can be characterized as *observation updating step* and (5) as *prediction step.*

Recall now a basic concept from Bayesian statistics

Definition 1.1 *$P := \{p_\theta(x)\}_{\theta \in \Theta}$ is a family of densities conjugate to $\{f(y | x)\}$, if there exists $\theta' : \Theta \times \mathcal{Y} \to \Theta$ such that for all $\theta \in \Theta$, $y \in \mathcal{Y}$*

$$p_\theta(x) f(y \,|\, x) \propto p_{\theta'(\theta, y)}(x) \tag{6}$$

Analogously we have (see [14])

Definition 1.2 *Let the pair $\{f(y | x), p(x | x')\}$ of densities be given. $P := \{p_\theta(x)\}_{\theta \in \Theta}$ is a family of densities* **filter-conjugate** *(f-conjugate) to $\{f(y | x), p(x | x')\}$ if there exists $\theta' : \Theta \times \mathcal{Y} \to \Theta$ such that, $\forall \theta \in \Theta$, $y \in \mathcal{Y}$*

$$f(y \,|\, x) \int_{\mathcal{X}} p_\theta(x') p(x \,|\, x') dx' \propto p_{\theta'(\theta, y)}(x) \tag{7}$$

Remark 1.3 If a filter-conjugate family exists for a pair $\{f(y | x), p(x | x')\}$, then, if the initial distribution $\pi_0(x)$ belongs to this f-conjugate family, at every step the filter distribution $\pi_n(x_n | y^n)$ also belongs to this family. Consequently, the updating of the filter distribution from one step to the next can be accomplished by a simple transformation of the indexing parameter θ, i.e. by the mapping $\theta'(\theta, y) : \Theta \times \mathcal{Y} \to \Theta$. If Θ is finite-dimensional, in accordance with a more engineering-type terminology we shall then say that $\{f(y | x), p(x | x')\}$ admits a finite-dimensional filter. Finite-dimensional filters are obviously of great practical importance, since they are the only ones that can actually be computed.

Remark 1.4 The definition of f-conjugate family can be extended, in a natural way, to the case when the transition kernels depend on the step n and, possibly, on the observation history : for such a family of transition kernels $\{p_n(x_n \,|\, x_{n-1}; y^{n-1})\}$ we say that $\{p_\theta(x)\}_{\theta \in \Theta}$ is f-conjugate to

$\{f(y_n \mid x_n), \{p_n(x_n \mid x_{n-1}; y^{n-1})\}\}$ if there exists $\theta' : \Theta \times \mathcal{Y}^\infty \times \mathbb{N} \to \Theta$ such that $\forall \theta \in \Theta$, $y_n \in \mathcal{Y}$, $n \in \mathbb{N}$, $y^{n-1} \in \mathcal{Y}^{n-1}$ we have

$$f(y_n \mid x_n) \int_{\mathcal{X}} p_\theta(x_{n-1}) p_n(x_n \mid x_{n-1}, y^{n-1}) dx_{n-1} \propto p_{\theta'(\theta, y_n, n, y^{n-1})}(x_n) \tag{8}$$

For the time being, we shall for simplicity keep the observation distribution independent of n; some form of dependence on n can however be admitted without any substantial change in our method (see remark 2.4)

For sake of completeness we recall also the following notion in the form that we shall use.

Definition 1.5 $P := \{p_\theta(\cdot)\}_{\theta \in \Theta}$ *is an exponential family/class if, for a pair* $R(\cdot), Q(\cdot)$

$$p_\theta(\cdot) = R(\cdot) B(\theta) \exp\{\langle \theta, Q(\cdot) \rangle\} \tag{9}$$

where $B(\theta)$, being a normalization factor, depends on the choice of $R(\cdot)$ and $Q(\cdot)$. The order of P is $dim\, \theta = dim\, Q$.

In Bayesian inference it is well-known (see e.g. [1], [2]) that, under regularity assumptions, $\{f(y \mid x)\}$ admits a conjugate family of finite order if and only if it is an exponential family and in this case also the conjugate family is exponential. Correspondingly, for discrete-time filtering problems we have (see [11], [15]) that, under regularity assumptions, $\{f(y_n \mid x_n), \{p_n(x_n \mid x_{n-1}; y^{n-1})\}\}$ admits a filter conjugate family of finite order only if $\{f(y_n \mid x_n)\}$ is an exponential family. If this holds true for all $n = 0, 1, \cdots$, then $\pi_n(x_n \mid y^n)$ and $p_n(x_n \mid y^{n-1})$ are exponential families as well.

The purpose of the paper is to study the following problem : Given $f(y \mid x)$ belonging to an exponential class, determine transition kernels $p_n(x_n \mid x_{n-1}; y^{n-1})$ such that, for each $n = 0, 1, \cdots$, $\{f(y_n \mid x_n), \{p_n(x_n \mid x_{n-1}; y^{n-1})\}\}$ admits a finite-dimensional filter (a filter-conjugate family of finite order).

The problem of determining state/signal transition kernels $p_n(x_n \mid x_{n-1}; y^{n-1})$ that, combined with a given observation model $f(y \mid x)$, lead to filtering models admitting a finite-dimensional filter is evidently of great practical interest. To the best of our knowledge, only necessary conditions on the transition kernels have so far been obtained (see [12]). In this paper we present a method that, in addition to guaranteeing existence, allows also to determine explicitly the kernels (with obvious limitations due to the possibility of actually carrying out the calculations required for the inverse Laplace transform). Our problem is thus in a sense the inverse of the usual filtering problem where, given an evolution model for (X_n, Y_n), one determines the filter distribution and its dynamics; here, given a model for Y_n and having chosen a family of filter distributions and its dynamics, we determine a model for X_n so that the filtering problem for the pair (X_n, Y_n) admits as solution the given filter distribution with its dynamics. Under such form our problem can also be seen as filter analogon of the Bayesian statistics problem studied in [4].

More precisely, our method proceeds as follows : We fix an observation model with distribution $f(y \,|\, x)$ of the exponential class, namely

$$f(y \,|\, x) = a(x)b(y)\exp\{\langle d(y), q(x)\rangle\} \tag{10}$$

and, correspondingly, consider a conjugate family $\mathcal{P}_{A,Z}(a,q) \equiv \{\pi_{\alpha,z}(\cdot)\}_{x\in A, z\in Z}$ where

$$\pi_{\alpha,z}(x) \propto a^{\alpha}(x)\exp\{\langle z, q(x)\rangle\}, \qquad \alpha \in A,\ z \in Z \tag{11}$$

with A and Z given finite-dimensional sets, chosen so that integrability in x holds in (11); the integrability requirement also determines the support set $\mathcal{X}$ for x. (We thus have $\pi_\theta(x)$ with $\theta = (\alpha, z) \in \Theta = A \times Z$). Since the pair (a, q) is fixed, the family of densities $\mathcal{P}_{A,Z}(a,q)$ will simply be denoted by $\mathcal{P}_{A,Z}$.

We aim at finding transition kernels (possibly depending on the observation history through the parameters α, z) such that $\mathcal{P}_{A,Z}$ turns out to be f-conjugate, in the sense of definition 1.2 and remark 1.4. More precisely, we want a family of kernels $\{p_{\alpha,z}(x \,|\, x')\}_{\alpha\in A, z\in Z}$ such that the following implication holds :

$$\left\{ \begin{array}{lcl} \pi_{n-1}(x_{n-1} \,|\, y^{n-1}) & \equiv & \pi_{\alpha_{n-1},z_{n-1}}(x_{n-1}) \\ \\ p_n(x_n \,|\, x_{n-1}; y^{n-1}) & \equiv & p_{\alpha_{n-1},z_{n-1}}(x_n \,|\, x_{n-1})\} \end{array} \right. \tag{12}$$

for some $\alpha_{n-1},\ z_{n-1},$ implies

$$\pi_n(x_n \,|\, y^n) \in \mathcal{P}_{A,Z} \tag{13}$$

We shall show that the search of such a family $\{p_{\alpha,z}(x \,|\, x')\}$ can be reduced to the following

Problem :
Find $\{p_{\alpha,z}(x \,|\, x')\}_{\alpha\in A, z\in Z}$ such that

$$\int_{\mathcal{X}} a^{\alpha}(x')\exp\{\langle z, q(x')\rangle\} p_{\alpha,z}(x \,|\, x')\,dx' \propto a^{\phi(\alpha,z)}(x)\exp\{\langle \psi(\alpha,z), q(x)\rangle\} \tag{14}$$

for some functions $\phi(\alpha, z) \in A,\ \psi(\alpha, z) \in Z$.

As we shall also discuss, if $p_{\alpha,z}(x \,|\, x')$ and $\phi(\alpha, z)$ do not depend on z, equation (14) can be solved in p_α by a method based on inverting the Laplace transform.

In the next section 2 we describe the structure of the proposed method, deriving the basic equation (14); we shall also discuss the choice of the particular conjugate family of the type (11). In section 3 we discuss the method of the inverse Laplace transform and derive conditions that allow to verify whether a given function can be considered as Laplace transform. In section 4 we present various examples showing how the method of the inverse Laplace transform can be applied to solve our problem. Section 5 contains concluding remarks.

In addition to allowing to derive new (discrete time) filtering models that admit finite dimensional filters, our method allows also to study in more detail classes of filtering models $\{f(y_n|\,x_n),\, \{p_n(x_n|\,x_{n-1};y^{n-1})\}\}$ that are known to admit a finite-dimensional filter, namely the Gaussian and the Poisson-Gamma (or Gamma-Gamma) classes. Although it is known that finite-dimensional filters exist in these classes (for the Poisson- Gamma class, studied in section 4.4 below, the first such results appear to be in [3] and [16]), the method allows to identify the possible variants within these classes thereby sheding light on why in particular they admit finite-dimensional filters. Due mainly to the fact that we restrict ourselves to exponential families of the form (11), the method will not identify the totality of models (satisfying regularity assumptions) that admit a finite dimensional filter, the remaining models being those for which the filtering and prediction distributions belong to different exponential classes (see end of section 2 below).

2 The fundamental equation

In line with our problem statement given in the Introduction, let the observation density be given in the exponential form (10) and consider f-conjugate families $\mathcal{P}_{A,Z}$ of the special exponential form (11). This particular choice of the family $\mathcal{P}_{A,Z}$ is motivated by the fact that it has a number of simplifying properties in our problem setting, namely :

- *under appropriate conditions on the sets A, Z, the filter densities belong to $\mathcal{P}_{A,Z}$, if and only if also the prediction densities belong to $\mathcal{P}_{A,Z}$ (see Proposition 2.2 below);*

- *$\mathcal{P}_{A,Z}$ is conjugate to $\{f(y\,|\,x)\}$ in the classical Bayesian sense of Definition 1.1, provided A and Z are such that*

$$\alpha+1\in A, \quad z+d(y)\in Z; \qquad \forall\alpha\in A, \forall z\in Z, \forall y\in\mathcal{Y}; \tag{15}$$

- *the family $\mathcal{P}_{A,Z}$ can also be rewritten as*

$$\pi_{\alpha,z}(x)\propto\exp\{\alpha\,\log a(x)+\langle z,q(x)\rangle\}, \qquad \alpha\in A,\, z\in Z \tag{16}$$

This implies on one hand that it is actually an exponential family of order $\dim\mathcal{X}+1$, which allows to better model the evolution of the shape of the filter distribution. On the other hand it will be exploited later to *show that, whenever dim $q = 1$, for each problem admitting a finite-dimensional filter there is a dual problem with the same property. In fact, in (16) the parameters α and z play a dual role.*

In view of Proposition 2.2 below, it is convenient to introduce the following further

Definition 2.1 *A family of densities $\{\pi_\theta(x)\}_{\theta\in\Theta}$ is* **transition conjugate** *(t-conjugate) for the family of kernels $\{p_n(x_n \mid x_{n-1}; y^{n-1})\}$ if, $\forall\theta \in \Theta$, $n \in \mathbb{N}, y^{n-1} \in \mathcal{Y}^{n-1}$, there is $\hat{\theta} \in \Theta$, such that*

$$\int_{\mathcal{X}} p_n(x_n \mid x_{n-1}; y^{n-1})\pi_\theta(x_{n-1})dx_{n-1} \propto \pi_{\hat{\theta}}(x_n), \qquad \forall x_n \in \mathcal{X} \tag{17}$$

Note now that, for the special case of the family $\mathcal{P}_{A,Z}$, letting

$$\lambda(\alpha, z) := \left[\int_{\mathcal{X}} a^\alpha(x) \exp\{\langle z, q(x)\rangle\}\, dx\right]^{-1} \tag{18}$$

equation (17) becomes of the type of equation (14); namely, with the normalization factor included, we have $\forall x_n \in \mathcal{X}$

$$\begin{aligned}&\textstyle\int_{\mathcal{X}} a^{\alpha_{n-1}}(x_{n-1}) \exp\{\langle z_{n-1}, q(x_{n-1})\rangle\}\, p_n(x_n \mid x_{n-1}; y^{n-1})\, dx_{n-1}\\ &= \tfrac{\lambda[\phi(\alpha_{n-1},z_{n-1}),\psi(\alpha_{n-1},z_{n-1})]}{\lambda(\alpha_{n-1},z_{n-1})} a^{\phi(\alpha_{n-1},z_{n-1})}(x_n) \exp\{\langle\psi(\alpha_{n-1}, z_{n-1}), q(x_n)\rangle\}\end{aligned} \tag{19}$$

for suitable functions $\phi(\alpha, z) : A \times Z \to A$, $\psi(\alpha, z) : A \times Z \to Z$ that, following [3], we shall occasionally call *connecting functions*.

Analogously, and always for the family $\mathcal{P}_{A,Z}$, the condition of being f-conjugate becomes

$$\begin{aligned}&a(x_n)b(y_n) \exp\{\langle d(y_n), q(x_n)\rangle\}\\ &\qquad \textstyle\int_{\mathcal{X}} a^{\alpha_{n-1}}(x_{n-1}) \exp\{\langle z_{n-1}, q(x_{n-1})\rangle\}p_n(x_n \mid x_{n-1}; y^{n-1})\, dx_{n-1} \qquad (20)\\ &\propto a^{\mu(\alpha_{n-1},z_{n-1})}(x_n) \exp\{\langle\nu(\alpha_{n-1}, z_{n-1}, y_n), q(x_n)\rangle\}\,, \quad \forall x_n \in \mathcal{X}\end{aligned}$$

for suitable functions $\mu(\alpha, z) : A \times Z \to A$, $\nu(\alpha, z, y) : A \times Z \times \mathcal{Y} \to Z$.

¿From the above we immediately have

Proposition 2.2 *If*

- *$\mathcal{P}_{A,Z}$ is t-conjugate for $\{p_n(x_n \mid x_{n-1}; y^{n-1})\}$;*
- *$\alpha + 1 \in A$, $\quad z + d(y) \in Z$, $\quad \forall\alpha \in A$, $z \in Z$, $y \in \mathcal{Y}$*

then $\mathcal{P}_{A,Z}$ *is* f*-conjugate for* $\{p_n(x_n \mid x_{n-1}; y^{n-1})\}$.
If

- $\mathcal{P}_{A,Z}$ *is* f*-conjugate for* $\{p_n(x_n \mid x_{n-1}; y^{n-1})\}$*;*
- $\mu(\alpha, z) - 1 \in A, \quad \nu(\alpha, z, y) - d(y) \in Z, \quad \forall \alpha \in A,\ z \in Z,\ y \in \mathcal{Y}$

then $\mathcal{P}_{A,Z}$ *is* t*-conjugate for* $\{p_n(x_n \mid x_{n-1}; y^{n-1})\}$.

As immediate consequence of the foregoing we have that, if a given family of densities $\mathcal{P}_{A,Z}$ is t-conjugate for the family of transition kernels $\{p_n(x_n \mid x_{n-1}; y^{n-1})\}$, then the following holds : letting $(\alpha_0, z_0) \in A \times Z$ be such that $\pi_0(x) = \pi_{\alpha_0, z_0}(x)$ $(\pi_0(x) \in \mathcal{P}_{A,Z})$, and letting the functions μ, ν be defined by

$$\mu(\alpha, z) = \phi(\alpha, z) + 1, \qquad \nu(\alpha, z, y) = \psi(\alpha, z) + d(y) \tag{21}$$

with ϕ and ψ as in (19) then, under the conditions (15), we have that

$$\pi_n(x \mid y^n) = \pi_{\alpha_n, z_n}(x) \quad \text{and} \quad p_n(x \mid y^{n-1}) = \pi_{\phi(\alpha_{n-1}, z_{n-1}), \psi(\alpha_{n-1}, z_{n-1})}(x) \tag{22}$$

where $\{(\alpha_n, z_n)\}$ satisfy the recursive relations

$$\alpha_n = \mu(\alpha_{n-1}, z_{n-1}), \qquad z_n = \nu(\alpha_{n-1}, z_{n-1}, y_n) \tag{23}$$

Following standard terminology (see e.g. [17]) we shall call $\{(\alpha_n, z_n)\}$ the *filter process*.

Notice now that, under conditions to be discussed below, the left hand side of (19) can be seen formally as Laplace transform of $a^{\alpha_{n-1}}(x_{n-1}) \cdot p_n(x_n | x_{n-1}; y^{n-1})$, considered as function of x_{n-1}, parametrized by x_n, y^{n-1}, and α_{n-1}. Provided the right hand side of (19) is indeed a Laplace transform, we can thus use the inverse Laplace transform to determine $p_n(x_n | x_{n-1}; y^{n-1})$. Since the right hand side of (19) is parametrized by α_{n-1} and z_{n-1}, we have to expect that, in general, a solution $p_n(x_n | x_{n-1}; y^{n-1})$, if it exists, should depend on y^{n-1} only by way of α_{n-1} and z_{n-1}. However, since z_{n-1} is the parameter of the Laplace transform, an explicit dependence of $p_n(x_n | x_{n-1}; y^{n-1})$ on z_{n-1} would not allow the left hand side of (19) to be interpreted as Laplace transform with parameter z_{n-1}. We shall therefore impose the condition

- $p_n(x_n | x_{n-1}; y^{n-1})$ *depends on* n *and* y^{n-1} *at most through* α_{n-1}.

(in the sequel we shall thus denote the kernel generically by $p_\alpha(x \mid x')$). Furthermore, in view of actually applying the inverse Laplace transform, it will be convenient (see (25) below) to transfer the factor $a^{\varphi(\alpha, z)}$ from the right hand side of (19) to the left hand side. For this reason we shall require also $\phi(\alpha, z)$

to depend only on α. From (23) it is clear that $\{(\alpha_n, z_n)\}$ are adapted to the filtration generated by $\{y_n\}$.

For further convenience we put

$$\tau_{\varphi,\psi}(\alpha, z) := \frac{\lambda(\varphi(\alpha), \psi(\alpha, z))}{\lambda(\alpha, z)} \tag{24}$$

dropping the pedices φ, ψ when not strictly needed.

Our main remaining task is now to show existence and obtain state/signal transition kernels $p_\alpha(x|x')$ satisfying (19). This will be accomplished in the rest of the paper by using the inverse Laplace transform. For this purpose it is convenient to rewrite (19) in a suitable equivalent form.

Proposition 2.3 *Let $\varphi(\alpha)$ and $\psi(\alpha, z)$ be two given functions. Assume that we can find a family of functions $K_x^{(\alpha)} : \mathcal{X} \to R_+$ ($x \in \mathcal{X}, \alpha \in A$) such that, for $\alpha \in A, z \in Z, x \in \mathcal{X}$,*

$$\int_{\mathcal{X}} K_x^{(\alpha)}(x') \exp\{\langle z, q(x')\rangle\} dx' = \tau_{\phi,\psi}(\alpha, z) \cdot \exp\{\langle \psi(\alpha, z), q(x)\rangle\} \tag{25}$$

Then
(i) $p_\alpha(x|x') \equiv \frac{[a(x)]^{\varphi(\alpha)}}{[a(x')]^\alpha} \cdot K_x^{(\alpha)}(x')$ *is a transition kernel,* $\forall \alpha \in A$.
(ii) $(p_\alpha(x|x'), \varphi(\alpha), \psi(\alpha, z))$ *is a solution of (19).*

Proof :

(i). We must show that $K_x^{(\alpha)}(x')$ is such that

$$\int_{\mathcal{X}} [a(x)]^{\varphi(\alpha)} K_x^{(\alpha)}(x') dx = [a(x')]^\alpha . \tag{26}$$

Indeed, from (26) and from the very definition of $p_\alpha(x|x')$ above, it will obviously follow that

$$\int_{\mathcal{X}} p_\alpha(x|x') dx = 1, \quad \forall \alpha \in A, \quad x' \in \mathcal{X}.$$

Let us consider the Laplace transform, generalized in the sense that $\mathcal{X}$ need not be the positive half line, of the integral in the left hand side of (26):

$$\int_{\mathcal{X}} \left[\int_{\mathcal{X}} [a(x)]^{\varphi(\alpha)} K_x^{(\alpha)}(x') dx\right] \exp\{\langle z, q(x')\rangle\} dx' =$$

$$\int_{\mathcal{X}} [a(x)]^{\varphi(\alpha)} \left[\int_{\mathcal{X}} K_x^{(\alpha)}(x') \exp\{\langle z, q(x')\rangle\} dx'\right] dx =$$

$$\int_{\mathcal{X}} [a(x)]^{\varphi(\alpha)}\,\tau(\alpha,z)\cdot\exp\left\{\langle\psi(\alpha,z),q(x\}\rangle\right\}dx =$$

$$= \frac{1}{\lambda(\alpha,z)} = \int_{\mathcal{X}} [a(x)]^{\alpha}\exp\left\{\langle z,q(x)\rangle\right\}dx$$

where we have used (25) and the definitions of $\lambda(\varphi(\alpha),\psi(\alpha,z))$ and $\tau_{\varphi,\psi}(\alpha,z)$ in (18) and (24) respectively. Then (26) follows from the unicity of inverse of the (generalized) Laplace transform.

(ii) immediately follows from (25) and from the definition of $p_\alpha(x|x')$.

On the other hand it is obvious that, if $(p_\alpha(x|x'),\varphi(\alpha),\psi(\alpha,z))$ is a solution of (19), then $\left(K_x^{(\alpha)}(x'),\varphi(\alpha),\psi(\alpha,z)\right)$is a solution of (25).

In the next Section, we shall concentrate attention on sufficient conditions for the existence of solutions to (25) and also on methods to determine them. Examples of actual application of these methods follow in section 4.

Remark 2.4 So far we assumed that, at any stage n, the observation distribution is specified by

$$f(y_n|\,x_n) = a(x_n)b(y_n)\exp\{\langle d(y_n),q(x_n)\rangle\} \tag{27}$$

However, with minor changes, the same arguments as above can be used to deal with the case

$$f_n(y_n|\,x_n) = a^{\beta_n}(x_n)b(y_n)\exp\{\langle d_n(y_n),q(x_n)\rangle\} \tag{28}$$

Indeed, $\mathcal{P}_{A,Z}$ being t-conjugate for the family of kernels $\{p_n(x_n|\,x_{n-1};y^{n-1})\}$ remains a sufficient condition for it to be f-conjugate (the generalization of the notion of f-conjugate to this nonstationary case being obvious). We only have to take into account that, in this case, the filter dynamics (21) generalize to

$$\mu_n(\alpha_n,z_n) = \phi_n(\alpha_n,z_n)+\beta_n\ ,\quad \nu_n(\alpha_n,z_n,y_n) = \psi_n(\alpha_n,z_n)+d_n(y_n) \tag{29}$$

It is implicit that the conditions $\mu_n\in A,\nu_n\in Z$ must hold.

We end this section by pointing out that condition (11) on the form of the exponential family $\{\pi_\theta(x)\}_{\theta\in\Theta}$ is by no means necessary for it to be filter conjugate (see examples in [3], [18]). In such cases one has however

- $\{\pi_\theta(x)\}_{\theta\in\Theta}$ *need not be conjugate to* $\{f(y\,|\,x)\}$ *in the sense of Definition 1.1;*
- *the families* $\{\pi_n(x\,|\,y^n)\}$ *and* $\{p_n(x\,|\,y^{n-1})\}$ *have in general different analytic forms (even if* π_0 *is chosen within the family* $\{\pi_\theta(x)\}_{\theta\in\Theta}$*), which would not allow to apply our method.*

3 Existence of solutions of the fundamental equation (inverse Laplace transform method)

In this section we study the problem of existence of a solution to the fundamental equation (19) in its equivalent form (25). More specifically, we shall study the existence of a solution of (25) that can be obtained by applying the inverse Laplace transform to its right hand side. Since, given a triple $(\mathcal{X}, a(x), q(x))$ corresponding to a chosen observation model, the right hand side in (25) depends on the choice of the connecting functions $\varphi(\alpha), \psi(\alpha, z)$, the problem reduces to finding conditions on these latter functions so that the right hand side in (25) is indeed a Laplace transform. For convenience and simplicity of presentation, in what follows we shall assume that

- *The exponential family of observation densities in (10) is of order 1, i.e.*

$$q : \mathcal{X} \subset \mathbb{R} \to q(\mathcal{X}) \subset \mathbb{R} \tag{30}$$

implying that $z \in Z \subset \mathbb{R}$. In addition we assume $q(\cdot)$ to be an invertible function of class $\mathcal{C}^1$ so that, modulo a change of variables, it is not restrictive, for the theoretical purposes of this section, to assume that $q(x) \equiv x$.

If $q(x) = x$, the left hand side in (25) becomes indeed a Laplace transform of $K_x^{(\alpha)}(x')$, generalized in the sense that $x' \in \mathcal{X}$ which need not be the positive halfline and so, provided integrability is guaranteed, it is also not important to have a minus sign in the argument of $\exp(z \cdot x')$ as in the ordinary Laplace transform over the positive halfline.

First we recall some known results concerning the ordinary Laplace transform, that are taken from Ch.XIII of [10] and that we state in the form of the Lemmas 3.3 and 3.4 below.

Definition 3.1 *A function $\rho : [0, \infty) \to (0, \infty)$ is said to be* **completely monotone** *if it possesses derivatives $\rho^{(n)}(x)$ of all orders and*

$$(-1)^n \rho^{(n)}(x) \geq 0 \,; \quad \forall n = 1, 2, \cdots, \forall x \geq 0 \tag{31}$$

Definition 3.2 *A function $\rho : [0, \infty) \to (0, \infty)$ is said to satisfy* **condition (C)***, if it possesses derivatives $\rho^{(n)}(x)$ of all orders and*

$$0 \leq \frac{(-x)^n \cdot \rho^{(n)}(x)}{n!} \leq C \tag{32}$$

for some $C > 0$

Lemma 3.3 (Ch.XIII [10])

1. *A function $\rho : [0,\infty) \to (0,\infty)$ is the Laplace transform of a nonnegative measure M on $[0,\infty)$, i.e.*

$$\rho(x) = \int_0^\infty \exp\{-x \cdot \xi\} \, M(d\xi),$$

if and only if it is completely monotone.

2. *If $\rho_1(x), \rho_2(x)$ are two completely monotone functions, so is their product $\rho_1(x) \cdot \rho_2(x)$ (and, trivially, $\rho_1(x) \cdot \rho_2(x) = \int_0^\infty \exp(-x \cdot \xi) M_1 \star M_2(d\xi)$, where $\star$ denotes the convolution operator).*

3. *If $\rho(x)$ is completely monotone and $\sigma(x)$ is a positive function with a completely monotone derivative, then $\rho(\sigma(x))$ is completely monotone. In particular, $\exp\{-\sigma(x)\}$ is completely monotone and, if $\sigma(0) = 0$, $\exp\{-\sigma(x)\}$ is the Laplace transform of an infinitely divisible probability distribution P; one can then associate with $\exp\{-\sigma(x)\}$ an entire (convolution) semigroup of probability distributions: $\forall z > 0, \exp\{-z \cdot \sigma(x)\}$ is the Laplace trasform of a probability distribution $P_z \equiv (P)^{z\star}$.*

Lemma 3.4 (Ch.XIII [10]) *A function $\rho : [0,\infty) \to (0,\infty)$ is the ordinary Laplace transform of a nonnegative bounded function g, i.e.*

$$\rho(x) = \int_0^\infty g(\xi) \exp(-x \cdot \xi) d\xi \tag{33}$$

if and only if it satisfies condition (C) in (32). In this case $g(x) \leq C$, $\forall x \geq 0$.

We shall distinguish between four cases according to whether or not $\mathcal{X} = (0,\infty)$ and whether $\psi(\alpha, z)$ is linear in z. More precisely

Case I	$\mathcal{X} = (0,\infty)$,	$\psi(\alpha, z) = c(\alpha) \cdot z$
Case II	$\mathcal{X} = (0,\infty)$,	$\psi(\alpha, z)$ non-linear in z
Case III	$\mathcal{X} \cap (-\infty, 0) \neq \emptyset$,	$\psi(\alpha, z) = c(\alpha) \cdot z$
Case IV	$\mathcal{X} \cap (-\infty, 0) \neq \emptyset$,	$\psi(\alpha, z)$ non-linear in z

Remark 3.5 Before studying the four cases let us remark that the inverse Laplace transform of the right hand side in (25) can be obtained by inverting separately the two factors and then obtaining $K_x^{(\alpha)}(x')$ as the convolution

product of the two inverses. From this point of view, the fact of having $\psi(\alpha, z)$ linear of the form

$$\psi(\alpha, z) = c(\alpha) \cdot z \tag{34}$$

is particularly significant because then the second factor in the right of (25) becomes $\exp\left(c(\alpha)z \cdot x\right)$ which is the Laplace transform of a measure, degenerate at the point $c(\alpha)x$.

Remark 3.6 In cases I and II, i.e. when $\mathcal{X} = (0, \infty)$, in view of the integrability requirement in (11), it is natural to assume that $Z = (-\infty, 0)$. Since for $\mathcal{X} = (0, \infty)$ we may refer to the classical Laplace transform (with a minus sign in the exponent), it is convenient to transform variables by putting $t = -z$ and letting

$$\tilde{\psi}(\alpha, t) := -\psi(\alpha, -t) \tag{35}$$

Correspondingly, (24) becomes

$$\tilde{\tau}_{\varphi,\psi}(\alpha, t) = \frac{\int_0^\infty [a(x)]^\alpha \exp\{-t \cdot x\}\, dx}{\int_0^\infty [a(x)]^{\varphi(\alpha)} \exp\left\{-\tilde{\psi}(\alpha, t) \cdot x\right\} dx} \tag{36}$$

We consider now **case I** for which (25) becomes

$$\int_0^\infty K_x^{(\alpha)}(x') \exp\{-t \cdot x'\}\, dx' = \tilde{\tau}(\alpha, t) \cdot \exp\{-c(\alpha) \cdot t \cdot x\} \tag{37}$$

Proposition 3.7 *Let $\varphi(\alpha)$ and $c(\alpha)$ be such that $\tilde{\tau}(\alpha, t)$ is completely monotone and the measure M_α, given by the inverse Laplace transform of $\tilde{\tau}(\alpha, t)$ (point 1. of lemma 3.3) has a density $g_\alpha(x) : (0, \infty) \to (0, \infty)$ such that*

$$\int_0^\infty g_\alpha(x) \exp(-tx) dx = \tilde{\tau}(\alpha, t), \tag{38}$$

then $K_x^{(\alpha)}(x') := g_\alpha(x' - c(\alpha)x)$ satisfies (25) for all $\alpha \in A$ and with x, x' such that $x' - c(\alpha)x \geq 0$. If, furthermore, $\tilde{\tau}(\alpha, t)$ satisfies condition (C) in (32), then a $g_\alpha(x)$ can be found that is bounded by C.

Proof : The proof is an easy consequence of lemma 3.3, of lemma 3.4, and remark 3.5. A simple direct proof is as follows :

Since $g_\alpha(x)$ satisfies (38), letting $u = x' - c(\alpha) \cdot x \geq 0$, one has

$$\int_0^\infty g_\alpha(x' - c(\alpha) \cdot x) \exp\{-t \cdot x'\}\, dx'$$

$$= \int_0^\infty g_\alpha(u) \exp\{-t \cdot (u + c(\alpha) \cdot x)\}\, du \quad = \quad \tilde{\tau}(\alpha, t) \cdot \exp\{-t \cdot c(\alpha) \cdot x\} \tag{39}$$

Using the same kind of arguments, for **case II** we obtain

Proposition 3.8 *Let $\varphi(\alpha), \widetilde{\psi}(\alpha, t)$ be such that the*

a) $\widetilde{\psi}(\alpha, t)$ has a completely monotone partial derivative with respect to $t, \forall \alpha \in A$;

b) $\widetilde{\tau}(\alpha, t)$ satisfies condition (C) in (32).

Then, $\forall \alpha \in A, x \in X$, there exists a positive measure $N_{\alpha,x}$ over $(0, +\infty)$ and a bounded function g_α, such that $K_x^{(\alpha)}(x') = \int_0^\infty g_\alpha(x' - u) N_{\alpha,x}(du)$ satisfies (25).

If $\widetilde{\tau}(\alpha, t)$ is only completely monotone then, as in Proposition 3.7, the measure M_α, given by the inverse Laplace transform of $\widetilde{\tau}(\alpha, t)$, has to have a density for which (38) holds; the above result then carries over to this $g_\alpha(x)$, provided the integral on the left in (25) exists.

Proof : Again an easy consequence of lemma 3.3, lemma 3.4 and remark 3.5 noticing that, by condition a), there exists (see part 3. of lemma 3.3) a positive measure $N_{\alpha,x}$ satisfying

$$\int_0^\infty \exp\{-t \cdot u\} N_{\alpha,x}(du) = \exp\left\{-x \cdot \widetilde{\psi}(\alpha, t)\right\} \tag{40}$$

Remark 3.9 Notice that, by part 3. of lemma 3.3, the measure $N_{\alpha,x}$ is infinitely divisible.

For **case III** the arguments of case I can be extended rather straightforwardly making use of the following

Lemma 3.10 (see [18]) *Fix $\beta \in \mathbb{R}$ and let $g : \mathbb{R} \to (0, +\infty)$ be such that for its support set $\Gamma_g := \{x \,|\, g(x) > 0\}$ one has $\Gamma_g + \beta \subset \mathcal{X}$. Then, for any x such that $\int_{\Gamma_g} g(x') \exp\{x \cdot x'\} dx' < \infty$, we have*

$$\int_{\mathcal{X}} g(x' - \beta) \exp\{x \cdot x'\} dx' = \exp\{x \cdot \beta\} \int_{\Gamma_g} g(x') \exp\{x \cdot x'\} dx' \tag{41}$$

Proof (following the proof of Lemma 2.2 in [18]) :
We have $g(x' - \beta) > 0$ if and only if $x' \in \Gamma_g + \beta$ and so, considering the transformation $w = x' - \beta$, we have

$$\begin{aligned} \textstyle\int_{\mathcal{X}} g(x' - \beta) \exp\{x \cdot x'\} dx' &= \textstyle\int_{\mathcal{X} \cap \{\Gamma_g + \beta\}} g(x' - \beta) \exp\{x \cdot x'\} dx' \\ &= \textstyle\int_{\{\mathcal{X} \cap \{\Gamma_g + \beta\} - \beta\}} g(w) \exp\{x \cdot (w + \beta)\} dw \end{aligned} \tag{42}$$

The assumption $\Gamma_g + \beta \subset \mathcal{X}$ implies that $\{\mathcal{X} \cap \{\Gamma_g + \beta\} - \beta\} \equiv \Gamma_g$ and so from (42) one obtains (41).

Recalling that, for case III, there is no reason to transform z into t so that (25) remains in its given form (with $q(x) = x$), from lemma 3.10 and putting $\beta = c(\alpha) \cdot x$, we immediately obtain

Proposition 3.11 *Assume that, $\forall \alpha \in A$, one can find $g_\alpha : \mathbb{R} \to (0, +\infty)$ with support set Γ_{g_α} such that*

i) $\Gamma_{g_\alpha} + c(\alpha) \cdot \mathcal{X} \subset \mathcal{X}$

ii) $\int_{\Gamma_{g_\alpha}} g_\alpha(x') \exp\{z \cdot x'\}\, dx' = \tau_{\varphi,\psi}(\alpha, z)$.

Then, $\forall \alpha \in A$, $K_x^{(\alpha)}(x') = g_\alpha(x' - c(\alpha) \cdot x)$ satisfies (25).

Notice that condition i) certainly holds for $\mathcal{X} = \mathbb{R}$.

Case IV is more complex in that equation (25) has to be approached directly and this is a possible object of further research.

Remark 3.12 In addition to remark 3.5 we want to point out that the choice of a linear $\psi(\alpha, z)$ as in (34), besides being convenient for the calculations, has also a relevant interpretation in terms of the dependence structure between x_{n-1} and x_n. Notice in fact that, in both cases I and III, we get (see propositions 3.7 and 3.11) a solution of (25) of the form

$$K_x^{(\alpha)}(x') = g_\alpha(x' - c(\alpha) \cdot x) \tag{43}$$

implying that

$$p_\alpha(x|x') = \frac{(a(x))^{\phi(x)}}{(a(x'))^\alpha}\; g_\alpha(x' - c(\alpha) \cdot x) \tag{44}$$

This is, in particular, the case if for the dynamics of the process $\{X_n\}$ one has either an additive model of the form $X_{n+1} = \gamma_1(\alpha)X_n + \gamma_2(\alpha)V_n$ as in the examples of section 4.1 below, or a multiplicative model $X_{n+1} = \gamma(\alpha)X_n \cdot V_n$ as in the example of subsection 4.2.1.

Remark 3.13 A final remark concerns the fact that we use the inverse Laplace transform to obtain a solution to (25) and for this purpose we invert separately the two factors in the right hand side of (25). This, as we shall also see in the next section of examples, greatly simplifies our task. In general one may still try a global inversion of the right hand side of (25). This is the approach taken in [6] and may have the advantage of giving useful indications on how to choose the connecting functions $\varphi(\alpha)$ and $\psi(\alpha, z)$. This method, in synthesized form, is used in the example of section 4.4 below.

4 Examples

In this section we apply the results of the previous section 3 to explicitly determine transition kernels $p_n(x_n | x_{n-1}; y^{n-1})$ such that, given an observation density $f(y \,|\, x)$ belonging to the exponential class of the form in (10), the system $\{f(y_n | x_n),\ \{p_n(x_n | x_{n-1}; y^{n-1})\}\ ,\ n = 0, 1, 2, \cdots\}$ admits a finite-dimensional filter of the form in (11). More precisely, given such an $f(y \,|\, x)$, on the basis of the discussion following proposition 2.2 we shall determine triples $(p_\alpha(x \,|\, x'), \varphi(\alpha), \psi(\alpha, z))$ that satisfy the fundamental equation (19) in its equivalent form (25) and where $\varphi(\alpha), \psi(\alpha, z)$ can in principle be chosen arbitrarily, provided the inverse Laplace transformation can be applied to obtain a solution to (25).

We recall that the particular choice of an observation model $f(y \,|\, x)$ affects (19) only through the triple $(\mathcal{X}, a(x), q(x))$. Although, therefore, we shall consider specific observation models, the results concerning $p_\alpha(x \,|\, x')$ are valid for all alternative models that lead to the same triple $(\mathcal{X}, a(x), q(x))$. On the other hand, for a fixed $(\mathcal{X}, a(x), q(x))$, one can find various triples $(p_\alpha(x \,|\, x'), \varphi(\alpha), \psi(\alpha, z))$ that satisfy (19) and they depend on the choice of $\varphi(\alpha), \psi(\alpha, z)$. For each of these triples one gets different dynamics, both for the state/signal process $\{X_n\}$ and for the filter process; the filter dynamics depend, in addition, also on the specific observation model $f(y \,|\, x)$ via the function $d(y)$ (see (21)). However, $(\mathcal{X}, a(x), q(x))$ and the choice of the parameter sets A, Z (satisfying (15)) uniquely determine the filter-conjugate family in (11).

We furthermore recall that, by (11), we implicitly always have $dim(a(x)) = 1$ while, more generally than in (30), one may have $dim\, q(x) \geq 1$. This implies that, while $dim\ A = 1$, it may be that $dim\ Z \geq 1$ so that there is in general no symmetry between the parameters $\alpha \in A$ and $z \in Z$. When $dim\ q(x) = 1$, the roles of α and z become symmetric. In fact, recalling (see (16)) that the right hand side in (11) can also be rewritten as $\exp\{\alpha \log a(x) + \langle z, q(x) \rangle\}$, it is easily seen that a same filter-conjugate family can be originated by two triples $(\mathcal{X}, a(x), q(x))$ and $(\mathcal{X}, \tilde{a}(x), \tilde{q}(x))$ where

$$\tilde{a}(x) = \exp\{q(x)\} \quad , \quad \tilde{q}(x) = \log\{a(x)\} \tag{45}$$

In such a case one obtains a pair of dual models, where the roles of α and z are interchanged so that, in one model, $p(x \,|\, x')$ is parametrized by α, in the other by z and for this latter model the function φ has to be a function only of z instead of of α.

In subsection 4.1 we start with the classical Gaussian observation model, for which $\mathcal{X} = \mathbb{R}$ and $q(x) = x$. We show how one can obtain different results even if one considers only linear functions $\psi(\alpha, z) = c(\alpha) \cdot z$ (case III of section 3). In subsection 4.2 we then consider an observation model of the Gamma-type and derive two solutions, one by taking $\psi(\alpha, z)$ to be linear (case I of section 3), and one for $\psi(\alpha, z)$ nonlinear (case II of section 3). In this same subsection we then illustrate how a different observation model can lead to the same triple $(\mathcal{X}, a(x), q(x))$ and thus to the same state/signal transition kernels as for the Gamma-type observations. For the triple $(\mathcal{X}, a(x), q(x))$ corresponding to the Gamma-type observation model we present in subsection 4.3 a dual model with Poisson-type observations that corresponds to the triple $(\mathcal{X}, \tilde{a}(x), \tilde{q}(x))$ with $\tilde{a}(x) = \exp\{q(x)\}$, $\tilde{q}(x) = \log\{a(x)\}$. If the Poisson-observation model is treated directly, and not as dual to the Gamma-type observation model, then an example results where $q(x)$ is not directly given as the identity function. A change of variable is thus needed and this is shown in section 4.4 where also a global inverse Laplace transform is applied to the right hand side of equation (25).

4.1 Examples for case III (Gaussian models)

Consider an observation model of the form

$$Y_n = X_n + W_n \tag{46}$$

with $\{W_n\}$ i.i.d. standard Gaussian, so that

$$f(y \mid x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \exp\left\{-\frac{y^2}{2}\right\} \exp\{x \cdot y\} \tag{47}$$

It leads to a triple $(\mathcal{X}, a(x), q(x))$ with

$$\mathcal{X} = \mathbb{R}, \quad a(x) = \exp\left\{-\frac{x^2}{2}\right\}, \quad q(x) = x \tag{48}$$

and all that follows, except for the dynamics of the parameters of the filter distribution (see (53) below), depends on (47) only through (48). Notice also that the form of $a(x)$ guarantees integrability in (11) if we take $A = (0, \infty)$. Taking furthermore $Z = \mathbb{R}$, we have that the conditions (15) are fulfilled.

For what follows it is convenient to recall that :

a) if $f(x)$ is the density of a Gaussian r.v. $\mathcal{N}(x; \mu, \sigma^2)$, it can be written as

$$f(x) \propto \left[\exp\left\{-\frac{x^2}{2}\right\}\right]^\alpha \exp\{x \cdot z\} \tag{49}$$

with $\alpha = \sigma^{-2}$, $z = \mu\sigma^{-2}$;

b) the following equality holds

$$\int_{\mathbb{R}} \left[\exp\left\{-\frac{x^2}{2}\right\}\right]^\alpha \exp\{x \cdot z)\}dx = \sqrt{\frac{2\pi}{\alpha}} \exp\left[\frac{z^2}{2\alpha}\right] \tag{50}$$

¿From (48) and (49) and the choice of A and Z one then has that a filter-conjugate family of the form (11) is given by the family of Gaussian random variables $\mathcal{N}\left(x; \frac{z}{\alpha}, \frac{1}{\alpha}\right)$.

4.1.1 General linear variants

Let us choose $\varphi(\alpha) > 0$ and

$$\psi(\alpha, z) = c(\alpha) \cdot z \tag{51}$$

with $c(\alpha)$ a real function of $\alpha \in A$, so that we are in the situation of case III as discussed in the previous section 3. The functions $\varphi(\alpha) > 0$ and $c(\alpha) \in \mathbb{R}$ may be arbitrary, provided they satisfy the condition

$$|\, c(\alpha) \,| < \sqrt{\frac{\varphi(\alpha)}{\alpha}} \tag{52}$$

(see (56) below). From (21)-(23) and the fact that in (47) we have $d(y) = y$, it follows that, if in the generic period n the filter distribution is $\mathcal{N}\left(x_n; \frac{z_n}{\alpha_n}, \frac{1}{\alpha_n}\right)$ and one has observed y_{n+1}, then in period $n+1$ it is $\mathcal{N}\left(x_{n+1}; \frac{z_{n+1}}{\alpha_{n+1}}, \frac{1}{\alpha_{n+1}}\right)$ where

$$\begin{cases} \alpha_{n+1} &= \varphi(\alpha_n) + 1 \\ \\ z_{n+1} &= c(\alpha_n) z_n + y_{n+1} \end{cases} \tag{53}$$

Next we determine a transition kernel $p_\alpha(x \,|\, x')$ for which $\pi_{\alpha,z}(x) \propto \mathcal{N}\left(x; \frac{z}{\alpha}, \frac{1}{\alpha}\right)$ is a filter-conjugate family with transformation of the parameters given by (53). Using (50) we obtain

$$\lambda(\alpha, z) := \left[\int_{\mathbb{R}} a^\alpha(x) \exp\{x \cdot z\} dx\right]^{-1} = \sqrt{\frac{\alpha}{2\pi}} \exp\left[-\frac{z^2}{2\alpha}\right] \tag{54}$$

and so

$$\tau_{\varphi,\psi}(\alpha, z) := \frac{\lambda\left(\varphi(\alpha), \psi(\alpha, z)\right)}{\lambda(\alpha, z)} = \sqrt{\frac{\varphi(\alpha)}{\alpha}} \exp\left(\frac{1}{2} Q(\alpha) z^2\right) \tag{55}$$

having put

$$Q(\alpha) := \frac{\varphi(\alpha) - \alpha c^2(\alpha)}{\alpha \varphi(\alpha)} \tag{56}$$

which, by (52), is positive.

According to proposition 3.11 it suffices to find a nonnegative function $g_\alpha(x')$ such that

$$\int_{\Gamma_{g_\alpha}} g_\alpha(x') \exp\{z \cdot x'\}\, dx' = \tau_{\varphi,\psi}(\alpha, z) \tag{57}$$

Using again (50), it is easily seen that a solution of (57), with $\tau_{\varphi,\psi}(\alpha, z)$ given by (55), is

$$g_\alpha(x') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\varphi(\alpha)}{\alpha Q(\alpha)}}\exp\left\{-\frac{(x')^2}{2Q(\alpha)}\right\} \tag{58}$$

with $\Gamma_{g_\alpha} = \mathbb{R}$. (Notice the importance of having $Q(\alpha) > 0$, i.e. that (52) holds). Again by proposition 3.11 one then has

$$K_x^{(\alpha)}(x') := g_\alpha(x' - c(\alpha)x) = \sqrt{\frac{\varphi(\alpha)}{2\pi\alpha Q(\alpha)}}\exp\left\{-\frac{(x' - c(\alpha)x)^2}{2Q(\alpha)}\right\} \tag{59}$$

and

$$\begin{aligned} p_\alpha(x \mid x') &:= \tfrac{(a(x))^{\varphi(\alpha)}}{(a(x'))^{\alpha}} K_x^{(\alpha)}(x') \\ &= \sqrt{\tfrac{\varphi(\alpha)}{2\pi\alpha Q(\alpha)}}\exp\left\{-\tfrac{x^2}{2}\left(\varphi(\alpha) + \tfrac{c^2(\alpha)}{Q(\alpha)}\right)\right\}\exp\left\{-\tfrac{(x')^2}{2}\left(\tfrac{1}{Q(\alpha)} - \alpha\right)\right\}\exp\left\{\tfrac{x\cdot x'\cdot c(\alpha)}{Q(\alpha)}\right\} \end{aligned} \tag{60}$$

Recalling (56), we have on one hand

$$\varphi(\alpha) + \frac{c^2(\alpha)}{Q(\alpha)} = \frac{\varphi^2(\alpha)}{\varphi(\alpha) - \alpha(c(\alpha))^2} = \frac{\varphi(\alpha)}{\alpha Q(\alpha)} := T(\alpha) > 0 \tag{61}$$

on the other

$$\frac{1}{Q(\alpha)} - \alpha = \frac{\alpha c^2(\alpha)}{\varphi(\alpha)Q(\alpha)} \tag{62}$$

so that we may write

$$\begin{aligned} p_\alpha(x \mid x') &= \sqrt{\tfrac{T(\alpha)}{2\pi}}\exp\left\{-\tfrac{T(\alpha)}{2}x^2 + \tfrac{c(\alpha)xx'}{Q(\alpha)} - \tfrac{\alpha c^2(\alpha)}{2\varphi(\alpha)Q(\alpha)}(x')^2\right\} \\ &= \sqrt{\tfrac{T(\alpha)}{2\pi}}\exp\left\{-\tfrac{T(\alpha)}{2}\left[x - \tfrac{c(\alpha)}{Q(\alpha)T(\alpha)}x'\right]^2\right\}\exp\left\{\left[\tfrac{T(\alpha)c^2(\alpha)}{2(T(\alpha)Q(\alpha))^2} - \tfrac{\alpha c^2(\alpha)}{2\varphi(\alpha)Q(\alpha)}\right](x')^2\right\} \end{aligned} \tag{63}$$

from where, by the definition of $T(\alpha)$ in (61) and considering $p_\alpha(x \mid x')$ as function of x for x' given,

$$p_\alpha(x \mid x') \sim \mathcal{N}\left(x; \frac{\alpha c(\alpha)}{\varphi(\alpha)}x', \frac{1}{T(\alpha)}\right) \tag{64}$$

which corresponds to a state evolution model of the form

$$X_{n+1} = \frac{\alpha_n c(\alpha_n)}{\varphi(\alpha_n)}X_n + \frac{1}{\sqrt{T(\alpha_n)}}V_{n+1} \tag{65}$$

where $\{V_n\}$ are i.i.d. standard Gaussian.

Concluding this first linear variant, we have that the filtering model

$$\left\{\begin{array}{lcl} X_{n+1} & = & \frac{\alpha_n c(\alpha_n)}{\varphi(\alpha_n)} X_n + \frac{\sqrt{\varphi(\alpha_n) - \alpha_n c^2(\alpha_n)}}{\varphi(\alpha_n)} V_{n+1} \\ & & \\ Y_n & = & X_n + W_n \end{array}\right. \tag{66}$$

with $\{V_n\}$ and $\{W_n\}$ i.i.d. standard Gaussian, independent from each other, and where $\varphi(\alpha) > 0$ and $c(\alpha)$ need only to satisfy (52), admits a filter-conjugate distribution that is a Gaussian $\mathcal{N}\left(x_n; \frac{z_n}{\alpha_n}, \frac{1}{\alpha_n}\right)$ with the parameters $(\alpha_n, z_n) \in (\mathbb{R}^+ \times \mathbb{R})$ evolving according to (53).

4.1.2 A particular linear variant (Conditionally Gaussian models)

Let the observation model be of the form

$$Y_n = \gamma_n(Y^n) X_n + \delta_n(Y^n) W_n \tag{67}$$

again with $\{W_n\}$ i.i.d. standard Gaussian. It is easily seen that also for (67) we have the same triple $(\mathcal{X}, a(x), q(x))$ as in (48) and also the parameter sets A and Z remain the same, so that the procedure of the previous subsection remains valid in the same form provided $\varphi(\alpha)$ and $\psi(\alpha, z)$ are chosen appropriately (notice also that, instead of $d(y) = y$, here we now have $d_n(y_n) = \frac{\gamma_n(y^n)}{\delta_n^2(y^n)} y_n$).

For $\varphi(\alpha)$ and $\psi(\alpha, z)$ we make now the following particular choice

$$\varphi(\alpha_n) = \alpha_n \rho_n(y^n) \quad , \quad \psi(\alpha, z) = z \tag{68}$$

where, due to (52), we must have

$$\rho_n(y^n) > 1 \tag{69}$$

otherwise $\rho_n(y^n)$ can be arbitrary (the choice for ψ corresponds to (51) with $c(\alpha) = 1$). With this choice of $\varphi(\alpha)$ and $\psi(\alpha, z)$ we have that (see (66)) the following filtering model

$$\left\{\begin{array}{lcl} X_{n+1} & = & a_n(Y^n) x_n + b_n(Y^n) V_{n+1} \\ & & \\ Y_n & = & \gamma_n(Y^n) X_n + \delta_n(Y^n) W_n \end{array}\right. \tag{70}$$

where $\gamma_n(y^n)$ and $\delta_n(y^n)$ are arbitrary functions of y^n and

$$\left\{\begin{array}{lcl} a_n(y^n) & = & \rho_n^{-1}(y^n) < 1 \\ & & \\ b_n(y^n) & = & \sqrt{\frac{\rho_n(y^n) - 1}{\alpha_n \rho_n^2(y^n)}} \end{array}\right. \tag{71}$$

admits a filter distribution that is Gaussian $\mathcal{N}\left(x_n; \frac{z_n}{\alpha_n}, \frac{1}{\alpha_n}\right)$, where (α_n, z_n) evolve according to

$$\begin{cases} \alpha_{n+1} = \alpha_n \rho_n(y^n) + 1 \\ \\ z_{n+1} = z_n + \frac{\gamma_n(y^{n+1})}{\delta_n^2(y^{n+1})}\, y_{n+1} \end{cases} \tag{72}$$

Models of the type of (70) are known from the literature as *conditionally Gaussian* models (see [13]).

4.2 Examples for cases I and II

Let us now consider observation models that, for a given positive real $c > 0$, lead to a triple $(\mathcal{X}, a(x), q(x))$ with

$$\mathcal{X} = (0, \infty) \quad , \quad a(x) = x^c \quad , \quad q(x) = x \tag{73}$$

as could e.g. be a model of the form (46) where $\{W_n\}$ are i.i.d. of the Gamma-type, with each W_n independent of X_n, so that we have

$$f(y \,|\, x) \propto 1_{\{y>0\}} \frac{x^c}{\Gamma(c)} y^{c-1} \exp\{-y \cdot x\}\,; \quad x > 0\,, \quad c > 0 \tag{74}$$

In this case we take $A = (0, \infty)$ and, to guarantee integrability in (11), it is natural to let $Z = (-\infty, 0)$. Notice that, since $d(y) = -y$ with $y > 0$, the so defined parameter sets A and Z satisfy the conditions (15). From (73) and the form of A and Z it follows that a filter-conjugate family of the form (11) is given by the family of Gamma-type distributions, i.e., putting for convenience $t := -z > 0$,

$$\pi_{\alpha,t}(x) \propto x^{c\alpha} \exp\{-t \cdot x\}\, 1_{\{x>0\}} \tag{75}$$

4.2.1 Linear variant (case I)

As in subsection 4.1 let us choose $\varphi(\alpha) > 0$ and $\psi(\alpha, z) = c(\alpha) \cdot z$ implying (see remark 3.6) $\tilde{\psi}(\alpha, t) = c(\alpha)t$. The real function $c(\alpha)$ may be arbitrary, provided it takes on positive values, while for $\varphi(\alpha)$ we require

$$0 < \varphi(\alpha) < \alpha \tag{76}$$

(see comment after (78) below). From (21)-(23) and the fact that in (74) we have $d(y) = -y$, it follows that, if in the generic period n the filter distribution is $\Gamma(x_n; \alpha_n, t_n)$ and one has observed y_{n+1}, then in period $n+1$ it is $\Gamma(x_{n+1}; \alpha_{n+1}, t_{n+1})$ where

$$\begin{cases} \alpha_{n+1} &= \varphi(\alpha_n) + 1 \\ \\ t_{n+1} &= c(\alpha_n) t_n + y_{n+1} \end{cases} \tag{77}$$

Again we determine a transition kernel $p_\alpha(x \mid x')$ for which $\pi_{\alpha,t}(x) \propto \Gamma(x;\alpha,t)$ is a filter-conjugate family with transformation of the parameters given by (77).

We easily have (see (36))

$$\tilde{\tau}_{\varphi,\tilde{\psi}}(\alpha,t) = \frac{\Gamma(c\alpha+1)\tilde{\psi}(\alpha,t)^{c\varphi(\alpha)+1}}{\Gamma(c\varphi(\alpha)+1)t^{c\alpha+1}} = \frac{\Gamma(c\alpha+1)}{\Gamma(c\varphi(\alpha)+1)}\,(c\cdot\alpha)^{c\varphi(\alpha)+1}\,t^{c(\varphi(\alpha)-\alpha)} \tag{78}$$

As a function of t, $\tilde{\tau}_{\varphi,\tilde{\psi}}(\alpha,t)$ is by (76) completely monotone and therefore (see 1. of lemma 3.3) it is the Laplace transform of a nonnegative measure on $(0,\infty)$. In fact, using the definition of Euler's Gamma function, it can directly be seen that $\tilde{\tau}_{\varphi,\tilde{\psi}}(\alpha,t)$ is, under assumption (76), the Laplace transform of a measure having the (unbounded) density

$$g_\alpha(\xi) = \frac{\Gamma(c\alpha+1)}{\Gamma(c\varphi(\alpha)+1)\Gamma(c(\alpha-\varphi(\alpha))}\,(c(\alpha))^{c\varphi(\alpha)+1}\,\xi^{c(\alpha-\varphi(\alpha))-1}1_{\{\xi>0\}} \tag{79}$$

so that by part a) of proposition 3.7

$$\begin{aligned} &K_x^{(\alpha)}(x') = g_\alpha(x'-c(\alpha)x) \\ &= \tfrac{\Gamma(c\alpha+1)}{\Gamma(c\varphi(\alpha)+1)\Gamma(c(\alpha-\varphi(\alpha))}\,(c(\alpha))^{c\varphi(\alpha)+1}\,(x'-c(\alpha)x)^{c(\alpha-\varphi(\alpha))-1}1_{\{0\le\frac{c(\alpha)x}{x'}\le1\}} \end{aligned} \tag{80}$$

By proposition 2.3 we obtain from (80) the following transition kernel

$$\begin{aligned} p_\alpha(x \mid x') = &\tfrac{\Gamma(c\alpha+1)}{\Gamma(c\varphi(\alpha)+1)\Gamma(c(\alpha-\varphi(\alpha))}\tfrac{c(\alpha)}{x'}\cdot \\ &\cdot\left[\tfrac{c(\alpha)x}{x'}\right]^{c\varphi(\alpha)}\left[1-\tfrac{c(\alpha)x}{x'}\right]^{c(\alpha-\varphi(\alpha))-1}1_{\{0\le\frac{c(\alpha)x}{x'}\le1\}} \end{aligned} \tag{81}$$

i.e., given $x_n = x'$, the r.v. $\frac{c(\alpha)x_{n+1}}{x_n}$ has a Beta-type distribution with parameters $(c\varphi(\alpha)+1, c(\alpha-\varphi(\alpha)))$.

Concluding this subsection, we have constructed a Beta-Gamma filtering model that admits as filter-conjugate distribution a Gamma-type distribution $\Gamma(x_n;\alpha_n,t_n)$, where the parameters $(\alpha_n,t_n) \in \mathbb{R}^+ \times \mathbb{R}^+$ evolve according to (77) with $c(\alpha)$ an arbitrary positive function and $\varphi(\alpha)$ satisfying (76).

4.2.2 Nonlinear variant (Case II)

We take the same setting as in the previous subsection 4.2.1, but shall specify a different (nonlinear) function $\tilde{\psi}(\alpha,t)$, choosing also $\varphi(\alpha) > 0$ in a specific way and, to keep formulas simple, we let $c = 1$. From the proof of proposition 3.8 it follows that it suffices to choose $\varphi(\alpha)$ and $\tilde{\psi}(\alpha,t)$ such that a solution to (38) and (40) can be found. For this example we thus recall the following equation known from mathematical physics (see e.g. pg.103 in [9])

$$\int_0^{+\infty} \frac{x}{\xi^{\frac{3}{2}}} \cdot \exp\left\{-\frac{x^2}{2\xi}\right\} \cdot \exp\{-t\cdot\xi\}\, d\xi = \sqrt{2\pi} \cdot \exp\left\{-(2t)^{\frac{1}{2}} \cdot x\right\}, \quad \forall x > 0 \tag{82}$$

In the language of infinitely divisible distributions this amounts to stating that the density

$$h(\xi) = \frac{1}{\sqrt{2\pi}} \frac{1}{\xi^{\frac{3}{2}}} \cdot \exp\left\{-\frac{1}{2\xi}\right\} \quad , \qquad \xi > 0 \tag{83}$$

is stable with characteristic exponent 1/2; notice also that the corresponding distribution function is

$$\Psi(\xi) = 2\left[1 - \Phi(\frac{1}{\sqrt{\xi}})\right] 1_{(\xi>0)} \tag{84}$$

where Φ is the distribution function of the standard gaussian (see [10], pg.173). From our point of view, (82) is a special case of (40) with $\tilde{\psi}(\alpha,t) = (2t)^{\frac{1}{2}}$ so that the solution is

$$N_{\alpha,x}(d\xi) = \frac{1}{\sqrt{2\pi}} \frac{x}{\xi^{\frac{3}{2}}} \cdot \exp\left\{-\frac{x^2}{2\xi}\right\} d\xi \tag{85}$$

We therefore choose

$$\tilde{\psi}(\alpha,t) = (2t)^{\frac{1}{2}} \tag{86}$$

as a nonlinear function of t and this leads to (see (78))

$$\tilde{\tau}_{\varphi,\tilde{\psi}}(\alpha,t) = \frac{\Gamma(\alpha+1)}{\Gamma(\varphi(\alpha)+1)} 2^{\frac{\varphi(\alpha)+1}{2}} t^{\frac{\varphi(\alpha)}{2}-\alpha-\frac{1}{2}} \tag{87}$$

It remains to find a solution $g_\alpha(\xi)$ to (38), namely for

$$\int_0^{\infty} g_\alpha(x) \exp(-tx) dx = \tilde{\tau}_{\varphi,\tilde{\tau}}(\alpha,t) \tag{88}$$

¿From the identity

$$\int_0^{\infty} \exp\{-t\cdot\xi\}\, d\xi = t^{-1}, \qquad \forall t > 0 \tag{89}$$

it is easily seen that, by taking

$$\varphi(\alpha) = 2\alpha - 1 \tag{90}$$

i.e. letting $\frac{\varphi(\alpha)}{2} - \alpha - \frac{1}{2} = -1$, a (bounded) solution to (88) is given by

$$g_\alpha(\xi) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha)} \cdot 2^\alpha = \text{constant}. \tag{91}$$

¿From the proof of proposition 3.8 and (84) it then follows that

$$\begin{aligned} K_x^{(\alpha)}(x') &= \int_0^\infty g_\alpha(x'-\xi) N_{\alpha,x}(d\xi) \\ &= \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha)} \cdot 2^\alpha \int_0^{x'} \tfrac{1}{\sqrt{2\pi}} \tfrac{x}{\xi^{\frac{3}{2}}} \cdot \exp\left\{-\tfrac{x^2}{2\xi}\right\} d\xi = \tfrac{\Gamma(\alpha+1)}{\Gamma(2\alpha)} 2^{\alpha+1} \left[1-\Phi(\sqrt{\tfrac{x}{x'}})\right] \end{aligned} \tag{92}$$

and thus, by proposition 2.3,

$$p_\alpha(x \mid x') = \frac{x^{2\alpha-1}}{(x')^\alpha} K_x^{(\alpha)}(x') = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha)} \frac{2^{\alpha+1} x^{2\alpha-1}}{(x')^\alpha} \left[1-\Phi(\sqrt{\frac{x}{x'}})\right] \tag{93}$$

is, $\forall x, x' > 0$, the transition kernel we looked for.

Concluding this subsection, we have again constructed a filtering model that admits as filter-conjugate distribution a Gamma-type distribution $\Gamma(x_n; \alpha_n, t_n)$, where the parameters $(\alpha_n, t_n) \in \mathbb{R}^+ \times \mathbb{R}^+$ evolve according to

$$\left\{ \begin{array}{lcl} \alpha_{n+1} & = & 2\,\alpha_n \\ \\ t_{n+1} & = & \sqrt{2t_n} + y_{n+1} \end{array} \right. \tag{94}$$

Remark 4.1 The triple $(\mathcal{X}, a(x), q(x))$ used in this subsection 4.2 and defined in (73), has been seen to arise from an observation model of the Gamma type, i.e. corresponding to (74). For the sake of illustration, we now show how a completely different observation model can lead to the same triple and thus to the same filter- conjugate distribution and the same transition kernels as before; only the dynamic transformation of the filter parameters may change due to a different function $d(y)$.

Instead of (46) consider in fact the observation model

$$Y_n = \frac{1}{\sqrt{X_n}}\; W_n \quad ; \quad W_n \sim \text{i.i.d.} \sim \mathcal{N}(0,1) \tag{95}$$

so that

$$f(y \mid x) = \frac{\sqrt{x}}{\sqrt{2\pi}} \exp\left\{-\frac{x}{2} y\right\} \tag{96}$$

which leads to the same triple (73) with $c = 1/2$. Since $d(y) = -(y/2)$, the only change with respect to subsection 4.2.1 is in (77), where in the second equation one adds $(y_{n+1}/2)$ instead of y_{n+1}.

4.3 A dual model

Consider an observation model where, given x_n, the observation Y_n is a Poisson random variable with parameter x_n, i.e.

$$f(y \mid x) = \frac{1}{y!} x^y e^{-x} 1_{\{\mathbb{N}\}}(y) = 1_{\{\mathbb{N}\}}(y) \frac{1}{y!} e^{-x} \exp\{y \log x\} \tag{97}$$

For this observation model we can consider a triple of the form

$$\mathcal{X} = \mathbb{R}^+ \quad ; \quad \tilde{a}(x) = e^x \quad ; \quad \tilde{q}(x) = \log x \tag{98}$$

that is dual in the sense of (45) to that in (73) if we choose $c = 1$. Notice that in the duality represented by (45) we have to invert the roles of α and z so that, with respect to the situation of subsection 4.2, we now have $A = (-\infty, 0)$, $Z = (0, \infty)$; this again guarantees integrability in (11) and a filter-conjugate family of the form (11) for (97), (98) is then

$$\pi_{z,\alpha}(x) \propto \tilde{a}^{\alpha}(x) \exp\{\langle z, \tilde{q}(x)\rangle\} = \exp(\alpha \cdot x) \cdot x^z \, 1_{\{x>0\}} \tag{99}$$

which, putting $t = -\alpha > 0$, becomes

$$\pi_{z,t}(x) \propto \exp(-t \cdot x) \cdot x^z \, 1_{\{x>0\}} \tag{100}$$

and is the same Gamma-type distribution (75) used in subsection 4.2, except that the role of the parameter α is now played by z and, viceversa, $t = -\alpha$ becomes the parameter of the Laplace transform. All the considerations of section 4.2 can now be repeated with inverted roles for α and z, paying attention to the form of the function $d(y)$ when determining the dynamics for the parameters of the filter distribution.

For the linear variant of subsection 4.2.1 one then takes a $\varphi(z)$ with $0 < \varphi(z) < z$ and $\tilde{\psi}(z,t) = c(z) \cdot t$ obtaining as filter distribution in the generic period n a $\Gamma(x_n; z_n, t_n)$ where (see (77), the parameters are updated according to

$$\begin{cases} z_{n+1} &= \varphi(z_n) + 1 \\ \\ t_{n+1} &= c(z_n)t_n - y_{n+1} \end{cases} \tag{101}$$

Notice in fact that, contrary to section 4.2, now we have $d(y) = y$. The transition kernel is the same as in (81) with α replaced by z and $c = 1$. Analogously one can adjust the results of subsection 4.2.2.

4.4 Global Laplace inversion (An example with $q(x) \neq x$)

We consider again the Poisson observation model (97) and associate with it the more convenient triple

$$\mathcal{X} = \mathbb{R}^+ \quad ; \quad a(x) = e^{-x} \quad ; \quad q(x) = \log x \tag{102}$$

By choosing $A = (0, \infty)$, $Z = (-1, +\infty)$, a filter-conjugate family of the form (11) is again a family of Gamma-type distributions as in (100), i.e.

$$\pi_{\alpha,z}(x) \propto e^{-\alpha x} e^{z \log x} 1_{\{x>0\}} = x^z e^{-\alpha x} 1_{\{x>0\}} \tag{103}$$

This time however we do not exploit the duality with respect to the model of section 4.2, as done in section 4.3, and thus we do not invert the roles of α and z, which implies that we cannot carry over the results of section 4.2.

In line with the general procedure we therefore start computing

$$\lambda(\alpha, z) = \left[\int_0^\infty e^{-\alpha x} e^{z \log x}\, dx\right]^{-1} = \left[\int_0^\infty e^{-\alpha x} x^z dx\right]^{-1} = \frac{\alpha^{z+1}}{\Gamma(z+1)} \tag{104}$$

so that equation (25) becomes

$$\int_0^\infty K_x^{(\alpha)}(x') \exp\left(z \cdot \log x'\right)\, dx' = \frac{\varphi(\alpha)^{\psi(\alpha,z)+1}\Gamma(z+1)}{\alpha^{z+1}\Gamma(\psi(\alpha,z)+1)} e^{\psi(\alpha,z)\log x} \tag{105}$$

In order to be able to consider the left hand side in (105) as a Laplace transform and thus use the method of the inverse Laplace transform, we perform the change of variables

$$u = \log x' \tag{106}$$

whereby (105) becomes

$$\int_{-\infty}^{+\infty} K_x^{(\alpha)}(e^u) e^u\, e^{zu} du = \frac{\varphi(\alpha)^{\psi(\alpha,z)+1}\Gamma(z+1)}{\alpha^{z+1}\Gamma(\psi(\alpha,z)+1)} e^{\psi(\alpha,z)\log x} \tag{107}$$

On the left hand side of (107) we have now a (generalized) Laplace transform of the function $K_x^{(\alpha)}(e^u)e^u$ with $u \in \mathbb{R}$. Such Laplace transforms can be inverted on the basis of the inverse Fourier transform (details are in [6]).

As first step one has to choose the functions $\varphi(\alpha)$ and $\psi(\alpha, z)$, that determine the filter dynamics, and this will be done in a way that the Laplace transform can be actually inverted. To be able to deal with the ratio of the two Gamma functions in the right side of (107), it is convenient to let

$$\psi(\alpha, z) = z + k \tag{108}$$

for a positive integer k. Choosing, furthermore, $\varphi(\alpha) = \alpha$ and taking, to be specific, $k = 2$, the right hand side in (107) becomes $\frac{\alpha^2 x^{z+2}}{(z+1)(z+2)}$. Using the inverse Laplace transform, based on the inverse Fourier transform, one obtains $K_x^{(\alpha)}(e^u)e^u$ as (for details see [6])

$$\begin{aligned} K_x^{(\alpha)}(e^u)e^u &= \tfrac{\alpha^2}{2\pi} \int_{-\infty}^{+\infty} \tfrac{x^{(x_0+i\xi+2)} e^{-(x_0+i\xi)u}}{(x_0+i\xi+1)(x_0+i\xi+2)} d\xi \\ &= \tfrac{\alpha^2}{2\pi \cdot i^2} \int_{-\infty}^{+\infty} \tfrac{x^{(x_0+i\xi+2)} e^{-(x_0+i\xi)u}}{[\xi - i(x_0+1)]\,[\xi - i(x_0+2)]} d\xi \end{aligned} \tag{109}$$

which does not depend on the choice of the value for x_0. The integrand is a holomorphic function on $\mathbb{C}$ with two poles of order 1 at $i(x_0+1)$ and $i(x_0+2)$. Using complex integration theory (method of residuals) one obtains (see [6])

$$K_x^{(\alpha)}(e^u)e^u = \alpha^2 e^u [x - e^u] \tag{110}$$

Applying the inverse change of variables with respect to (106), namely $x' = e^u$, it follows that $K_x^{(\alpha)}(x') = \alpha^2(x - x')$ and, finally, by proposition 2.3 we have

$$p_\alpha(x \mid x') = \alpha^2 e^{-\alpha(x-x')}(x - x') \tag{111}$$

which corresponds to a Gamma-type distribution with parameters $(2, \alpha)$, provided $x > x'$.

Concluding this section we have that the filtering model

$$\begin{cases} X_{n+1} & = & X_n + W_n \, ; \quad x_0 = w_0 \, , \quad W_n \text{ indep. } \sim \Gamma(2, \alpha_n) \\ \\ Y_n & : & \text{Poisson with parameter } X_n \end{cases} \tag{112}$$

admits a filter distribution that is a Gamma $\Gamma(x_n; z_n + 1, \alpha_n)$ where the parameters evolve according to

$$\begin{cases} \alpha_{n+1} & = & \alpha_n + 1 \\ \\ z_{n+1} & = & z_n + 2 + y_{n+1} \end{cases} \tag{113}$$

5 Conclusions

We have presented a procedure, based on the inverse Laplace transform, to construct discrete-time filtering models that admit a filter-conjugate distribution of exponential class of finite order.

In a classical filtering problem one is given a state/signal as well as an observation model and one tries to find the filter distribution and its dynamical evolution. Our problem is in a sense the inverse to this problem : given an observation model and having assigned a class of filter distributions and their dynamic evolution, we determine a state/signal model so that the filtering model, formed by this state/signal and the given observations, admits as filter-conjugate distribution a distribution in the preassigned class together with its dynamics.

We have presented theoretical results giving conditions on the filter dynamics in order that the inverse Laplace transform can be applied to obtain the state/signal dynamics. In various examples we have illustrated possible ways to actually apply this procedure. Analogs to this procedure have to some extent already been dealt with in the literature, namely in [3] and [18] by exlpoiting properties of the exponential class and of infinitely divisible distributions respectively. The peculiar features of our contribution are :

1. Based on the use of the inverse Laplace transform, we develop a workable methodology;

2. considering filter-conjugate families of the special form (11), our methodology gains in flexibility;

3. we allow nonhomogeneous state-transition kernels $p(x \mid x')$ which depend on the current filter distribution through one of the parameters, namely the parameter α, that generally depends only on time but in some cases also on the actual observations.

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