Adaptive and robust control procedures for risk minimization under uncertainty

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Abstract

We consider the problem of *model risk*, namely the risk due to model uncertainty. The context is that of *shortfall risk minimization*, that is widely used as hedging criterion in incomplete markets. We discuss the applicability of specific adaptive and robust control procedures. Given that trading takes place in discrete time, the setting is mainly in discrete time.

1 Introduction

For the solution of an applied problem, specifically of a financial problem, one needs first to build a model. The solution may be rather sensitive to this model. On the other hand, one has usually only partial knowledge about the true model that, furthermore, may be considerably complex and has thus to be approximated by tractable models. All this leads to deal with the issue of model uncertainty, which in financial applications corresponds to what is called *model risk*.

The particular financial application that we shall be dealing with in this paper is *shortfall risk minimization* that has recently attracted considerable interest as hedging criterion in incomplete markets (see e.g. [2], [3], [6],[7],[9]) and that we shall review in section 2. We point out that also hedging in formally complete markets may require criteria for incomplete markets if e.g. the initial capital is insufficient to achieve a perfect hedge or if the market model depends on additional random quantities.

One way to deal with model uncertainties is to consider parametrized families of models and to update successively the knowledge about the parameter values on the basis of observed quantities such as the prices of traded assets. Using the Bayesian point of view, whereby an unknown quantity is considered as realization of a random variable, updating the knowledge of a parameter value corresponds to updating its distribution. This leads to adaptive procedures, in particular *Bayesian Dynamic Programming (DP)* (see e.g. [8], [12]), as a methodology for adaptive optimization. In section 3 we shall discuss Bayesian DP as an adaptive control procedure for shortfall risk minimization.

An alternative way to deal with model uncertainty is to investigate the sensitivity of a solution to the model, which is also called *robustness* of a solution. One may in fact ask how reliable is the solution obtained for a hypothetical model, when applied to the real problem. This is the subject of the last section 4 where, in particular, we shall recall from [5] a robustness result for general discrete time stochastic control problems and mention its applicability (see [4]) to shortfall risk minimization in a binomial price evolution model.

2 The problem of shortfall risk minimization

Consider a market, in which there are $M+1$ assets with prices at the generic time t given by

$$S_0(t), S_1(t), \cdots\cdots, S_M(t) \tag{1}$$

where $S_0(t)$ corresponds to the non-risky asset and $S_i(t),\ 1 \leq i \leq M$ are the prices of the (locally) risky assets. The processes $S_i(t),\ 1 \leq i \leq M$ are defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Consider then an agent, who trades in this market according to a trading strategy

$$\phi(t) = (\phi_0(t), \cdots\cdots, \phi_M(t)) \tag{2}$$

where $\phi_i(t)$ denotes the amount in units of the $i-$th asset that are kept in the portfolio at time t. The processes $\phi_i(t)$ are supposed to be predictable on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Given $\phi(t)$, the value of the corresponding portfolio is then

$$V_\phi(t) = \sum_{i=0}^{M} \phi_i(t)\, S_i(t) \tag{3}$$

The portfolio corresponding to $\phi(t)$ is said to be *self-financing*, if it possesses an Ito differential given by

$$dV_\phi(t) = \sum_{i=0}^{M} \phi_i(t)\, dS_i(t) \tag{4}$$

In integral form this becomes

$$V_\phi(t) = V_\phi(0) + \sum_{i=0}^{M} \int_0^t \phi_i(s)\, dS_i(s) \tag{5}$$

expressing the current value $V_\phi(t)$ of the portfolio as its initial value $V_\phi(0)$ (initial endowment/capital) plus the total gain from trading up to time t given by $G_\phi(t) := \sum_{i=0}^{M} \int_0^t \phi_i(s)\, dS_i(s)$. To simplify the expressions, we shall consider the prices $S_i(t)$ ($i \geq 1$) discounted with respect to the non-risky asset $S_0(t)$; we then have $dS_0(t) \equiv 0$ and in (4),(5) the sum extends from $i = 1$ to M.

There are various reasons for which agents may trade in a market. One of them is to *hedge a given liability* at a given maturity T. We shall asssume that the liability (or claim) H_T is such that $H_T \in \mathcal{F}_T^S$, where $\mathcal{F}_T^S$ is the filtration generated by $S(t) = (S_1(t), \cdots, S_M(t))$ up to time $t = T$. In particular, one may consider simple claims of the form $H_T = H(S(T))$, among them European call options on one (the first) of the risky assets, namely $H(S(T)) = (S_1(T) - K)^+$.

In the ideal case of a complete market one has that, given any claim H_T, there exists a self-financing strategy $\phi(t)$ and an initial endowment $V_\phi(0)$ such that $V_\phi(T) = H_T$ a.s. In this case the *hedging problem* consists in determining $\phi(t)$ and $V_\phi(0)$ for a given H_T. It is known (see [1]) that, in continuous time, a market without friction is complete if the number of risky assets available for hedging equals the number of independent sources of randomness.

If, as is the case in realistic situations, the market is incomplete, then a perfect hedging and self financing strategy does not exist and one has to consider alternative criteria for hedging. A natural extension of the criterion of perfect hedging is

Superhedging : *Determine a self financing $\phi(t)$ with smallest $V_\phi(0)$ such that*

$$V_\phi(T) \geq H_T \qquad \text{a.s.} \tag{6}$$

A superhedging strategy eliminates completely the risk related to the liability H_T, but requires in general a large initial capital $V_\phi(0)$. An alternative criterion is to minimize a downside-type risk, also called

Shortfall risk : *Given $S(0)$ and $V(0)$, determine*

$$J\left(S(0), V(0)\right) := \inf_{\phi \in A} E\left\{ (H_T - V_\phi(T))^+ \right\} \tag{7}$$

as well as a minimizing strategy ϕ^*, where A is some admissible set of strategies satisfying the property of self financing in addition to possible further requirements such as shortselling prohibition, etc. and $(Z)^+$ is the positive part of Z.

It is immediately seen that the smallest value of $V(0)$ for which, given $S(0)$, one has $J\left(S(0), V(0)\right) = 0$, corresponds to the initial capital required for superhedging.

2.1 A discrete time setting

Since trading takes place in discrete time, for problems concerning the determination of an investment strategy as is the shortfall risk minimization problem, it makes sense to consider a discrete time setting. With $n \in \{0, \cdots, N\}$, a general model for the evolution of the prices of the risky assets is a multinomial model, namely

$$S_i(n+1) = S_i(n) \cdot \omega_i(n) \quad \text{with} \quad i = 1, \cdots, M \quad \text{and} \quad \omega_i(n) \in \left\{\omega^1, \cdots, \omega^K\right\} \tag{8}$$

for some integer $K > 1$. For $K = 2$ we have the classical binomial model. The multinomial model is completely specified by specifying the possible values $\left\{\omega^1, \cdots, \omega^K\right\}$ for the random stock returns $\omega_i(n)$ as well as the corresponding probabilities $p_i^k(n) := P\left\{\omega_i(n) = \omega^k\right\}$. In what follows we shall for simplicity consider the case of a single risky asset, i.e. $M = 1$, thereby writing $S(n)$ for $S_1(n)$ and $\omega(n)$ for $\omega_1(n)$ and assume time-homogeneous probabilities

$$p^k := P\left\{\omega(n) = \omega^k\right\} \; ; \quad k = 1, \cdots, K \tag{9}$$

Recalling that we consider the prices discounted with respect to the non risky asset and that the strategies are assumed to be predictable, the condition of self financing (see (4), (5)) can, in the discrete time single risky asset model, be expressed as

$$\begin{aligned} V_\phi(n+1) &= V_\phi(n) + \phi_1(n+1)\left(S(n+1) - S(n)\right) \\ &= V_\phi(n) + \phi_1(n+1)S(n)\ (\omega(n) - 1) \end{aligned} \tag{10}$$

or, equivalently, as $\phi_0(n+1) + \phi_1(n+1)\, S(n) = \phi_0(n) + \phi_1(n)\, S(n)$. If proportional transaction costs with cost rate $\gamma > 0$ are present, then condition (10) generalizes to

$$V_\phi(n+1) = V_\phi(n) + \phi_1(n+1)S(n)\ (\omega(n) - 1) - \gamma S(n)\left|\phi_1(n+1) - \phi_1(n)\right| \tag{11}$$

Notice that, for $M = 1$, by the self financing condition the value of $\phi_0(n+1)$ follows once $\phi_1(n+1)$ is chosen and so $\phi_1(n+1)$ is the only decision variable in period n (recall that ϕ is predictable).

Notice also that, for the problem of superhedging, all price evolution models, that induce equivalent probability measures, lead to the same solution. For the multinomial model this means that the solution does not depend on the particular choice of the values of the probabilities p^k, provided they are all strictly positive. The situation is different for the problem of shortfall risk minimization, as can be seen in the following Figure I. In this Figure we consider an underlying binomial price evolution model and plot, for a given value of $S(0)$, the values of $J\left(S(0), V(0)\right)$ against $V(0)$ for each of the three values of $p^1 = 0.1, 0.5, 0.9$ for the probability of an "up-movement". The liability corresponds to an European call option, i.e. $H_N = (S(N) - K)^+$, with $S(0) = K = 100$, the stock return values are $\omega^1 = 1.3, \omega^2 = 0.9$, the horizon is $N = 5$, the short rate has the constant value $r = 0.3$, and proportional transaction costs are also taken into account with cost rate $\gamma = 0.4$.

Fig. I

Shortfall risk against initial capital in a binomial model ($p^1 = 0.1, 0.5, 0.9;\ N = 5$)

As can be seen from this Figure, the initial capital required for superhedging is the same for all three values of $p^1 = 0.1, 0.5, 0.9$ and is given by $V(0) = 28.2$; the shortfall risk however varies considerably with the choice of the value for p^1.

If, as is quite realistic to assume, there is uncertainty about the actual value of p^1, one faces the problem of how reliable is the shortfall risk minimizing strategy computed for a hypothetical value of p^1, that may differ from its actual value. In the next two sections we mention two approaches to this problem : an adaptive control approach and a robust control approach.

3 An adaptive control approach

Recall that, for the multinomial single risky asset model of subsection 2.1, the shortfall risk minimization problem under uncertainty and with proportional transaction costs can be described as the stochastic adaptive terminal cost control problem

$$\begin{aligned} S(n+1) &= S(n) \cdot \omega(n) \\ V_\phi(n+1) &= V_\phi(n) + \phi_1(n+1)S(n)\ (\omega(n)-1) - \gamma S(n)\,|\,\phi_1(n+1) - \phi_1(n)\,| \\ J\left(S(0), V(0)\right) &= \min_{\phi_1 \in A} E_{S(0),V(0)} \left\{ (H_N - V_\phi(N))^+ \right\} \end{aligned} \tag{12}$$

The solution to (12) can, in principle, be obtained by the method of (backwards) Dynamic Programming (DP). Denote by $J_n\left(S(n), V_\phi(n), \phi_1(n)\right)$ the minimal value of the shortfall risk from period n onwards, given the current values of $S(n), V_\phi(n), \phi_1(n)$. Assuming for a moment that the values of p^k are known, the generic $n-$th step of DP consists in computing

$$J_n\left(S(n), V_\phi(n), \phi_1(n)\right) = \inf_{\phi_1(n+1)\in\Gamma_{n+1}(S(n),V_\phi(n),\phi_1(n))} \sum_{k=1}^{K} p^k J_{n+1}\left(S(n)\omega^k, V_\phi(n+1;\phi_1(n+1),\omega^k), \phi_1(n+1)\right) \tag{13}$$

where, considering only self financing constraints in addition to $\phi_0(n) \geq 0$, the set $\Gamma_{n+1}(\cdot,\cdot,\cdot)$ of admissible values for $\phi_1(n+1)$ is given by (see (11),

$$\begin{aligned}\Gamma_{n+1}&\left(S(n), V_\phi(n), \phi_1(n)\right)\\ &= \{\phi_1(n+1) \geq 0 \,|\, \phi_1(n+1)\cdot S(n) \leq V_\phi(n) - \gamma S(n) \,|\, \phi_1(n+1) - \phi_1(n)\,|\}\end{aligned} \tag{14}$$

and the portfolio value $V_\phi(n+1)$ in period $n+1$ depends on $\phi_1(n+1)$ and ω^k according to (see again (11))

$$V_\phi\left(n+1;\phi_1(n+1),\omega^k\right) = V_\phi(n)+\phi_1(n+1)S(n)\,\left(\omega^k-1\right)+\gamma S(n)\,|\phi_1(n+1)-\phi_1(n)| \tag{15}$$

Since we assume that there is uncertainty about p^k, we have to infer their values on the basis of the observations $S(m)$, $m \leq n$ of the risky asset's prices. Taking the Bayesian point of view, the unknown parameter vector $\bar{p} = (p^1,\cdots,p^K)$ is then considered as a random vector with distribution $\pi_n(\bar{p})$ in the generic period n, that can be updated on the basis of Bayes' formula for each new observation of the price of the risky asset. Since, given $\bar{p}$, the distribution of $\omega(n)$ is multinomial, a conjugate family of distributions for $\bar{p}$ is the family of Dirichlet distributions (see e.g. [8]) which implies that : if in the generic period n we have $\pi_n(\bar{p}) \propto (p^1)^{\alpha_n^1}\cdots(p^k)^{\alpha_n^k}$ for some $\alpha_n^1,\cdots,\alpha_n^k > -1$ with $\propto$ denoting "proportional to" and the new observation $S(n+1)$ is such that $S(n+1)/S(n) = \omega^k$, then the updated distribution of $\bar{p}$ is

$$\pi_{n+1}(\bar{p}) \propto \left(p^1\right)^{\alpha_{n+1}^1}\cdots\left(p^k\right)^{\alpha_{n+1}^k} \quad \text{with} \quad \begin{cases}\alpha_{n+1}^i &= \alpha_n^i \ \text{ for } i \neq k\\ \alpha_{n+1}^k &= \alpha_n^k + 1\end{cases} \tag{16}$$

Notice now that the unknown values of $\bar{p} = (p^1,\cdots,p^K)$ enter the DP relations (13) only linearly. This implies that, if in period n the information is restricted to $\mathcal{F}_n^S$ in addition to the knowledge of $V_\phi(n)$ and $\phi_1(n)$, then, by the smoothing property of conditional expectations, the recursions (13) remain the same also for unknown $\bar{p}$, provided the values of p^k are replaced by their conditional expectations $E\left\{p^k|\,\mathcal{F}_n^S\right\}$ that, in the case of the Dirichlet distribution, become $E\left\{p^k\,|\,\mathcal{F}_n^S\right\} = \alpha_n^k/\sum_{i=1}^K \alpha_n^i$.

Also in view of Fig.I it is clear that, if there is uncertainty in the value of $\bar{p}$, it is advisable to compute the solution to the shortfall risk minimizing problem according to the just outlined Bayesian Dynamic programming adaptive procedure. The effect of using an adaptive procedure can be seen from Figures 4.1 and 4.2 of [11] where, for the same problem data as in Fig.I except for a horizon of $N = 6$, the shortfall risk (as well as other risk values) is plotted against initial capital for the case when $p^1 = 0.5$ and when it is assigned a uniform prior respectively. The adaptive approach allows one to be on the safe side : the actual risk may be considerably lower than its computed value. On the other hand, if the risk minimizing strategy is computed for the wrong hypothetical value of p^1, then the actual risk may be much larger than its computed value.

As already mentioned, the DP approach allows one to compute the solution "in principle". For larger horizons the numerical solution becomes infeasible, unless one introduces appropriate approximations (see e.g. [11]). It is thus of interest to look for cases, where one can obtain an explicit analytic solution. For a binomial model, without transaction costs, with the self financing requirement as the only constraint on the trading strategy and with an initial capital that is insufficient for a perfect hedge, this has been accomplished in [10]. The explicit analytic solution in [10] turns out to be a rather intuitive extension of the classical Cox-Ross-Rubinstein solution.

4 A robust control approach

When there is model uncertainty (uncertainty in the parameters of the model), an alternative to an adaptive approach is a robust control approach. In this section we mention a robustness result for a general discrete time stochastic control problem, taken from [5] and that has been applied in [4] to the shortfall risk minimization problem in the binomial case. The application to the binomial case has also led (see always [4]) to an interesting alternative to the Bayesian DP approach of the previous section.

Consider the following generic discrete time, finite horizon stochastic control problem, of which (12) is a particular case when $H_N = H(S(N))$,

$$
\begin{aligned}
&x(n+1) = F_n\left(x(n), \phi(n+1), \omega^P(n)\right) \\
&J^P\left(x(0), \phi^*\right) = \min_{\phi \in A} J^P\left(x(0), \phi\right) := \min_{\phi \in A} E^P_{x(0)}\left\{\sum_{n=0}^{N-1} c\left(x(n), \phi(n+1)\right) + b\left(x(N)\right)\right\}
\end{aligned}
\tag{17}
$$

$x(n)$ is the controlled state in period n and $\omega^P(n)$ are the random *disturbances*. In (12) the parameters $\bar{p} = (p^1, \cdots, p^K)$ determine the distribution of the disturbances $\omega(n)$; in other words, they determine the underlying probability measure P. Correspondingly, in (17) we use the notation $\omega^P(n)$ as well as J^P and E^P. A change in the parameters from $\bar{p}$ to $\bar{q}$ (with all components positive) can thus be seen as an equivalent change of measure from P to Q.

Suppose then that P is the real-world probability measure and (17) is the real problem, while we solve the analogous problem

$$x^Q(n+1) = F_n\left(x^Q(n), \phi(n+1), \omega^Q(n)\right) \tag{18}$$

$$J^Q\left(x(0), \phi^Q\right) = \min_{\phi\in A} E^Q_{x(0)}\left\{\sum_{n=0}^{N-1} c\left(x^Q(n), \phi(n+1)\right) + b\left(x^Q(N)\right)\right\} \tag{19}$$

for a hypothetical measure $Q \sim P$ leading to a minimizing control $\phi^Q \neq \phi^*$.

Question : *Can we find a (possibly uniform in $x(0)$) bound on*

$$J^P\left(x(0), \phi^Q\right) - J^P\left(x(0), \phi^*\right) \geq 0 \tag{20}$$

in terms of the difference between $\omega^P = \{\omega^P(n)\}$ and $\omega^Q = \{\omega^Q(n)\}$ (P and Q) ?

Notice that $J^P\left(x(0), \phi^Q\right)$ is the value of the real problem when one applies the control ϕ^Q that is optimal in the hypothetical problem.

Dropping the argument $x(0)$, that we may assume as given, we have

$$\begin{array}{l} J^P(\phi^Q) - J^P(\phi^*) \\ = \left(J^P(\phi^Q) - J^Q(\phi^Q)\right) + \left(J^Q(\phi^Q) - J^Q(\phi^*)\right) + \left(J^Q(\phi^*) - J^P(\phi^*)\right) \\ \leq 2\sup_{\phi\in A} \mid J^P(\phi) - J^Q(\phi) \mid \end{array} \tag{21}$$

so that it suffices to find a bound on the right hand side in (21). Let $\delta > 0$ and $C > 0$ be such that

$$\begin{array}{l} \sup_\omega \left|1 - \frac{dQ(\omega)}{dP(\omega)}\right| \leq \delta \\ \sup_{\omega,\,\phi\in A} \left|\sum_{n=0}^{N-1} c\left(x(n), \phi(n+1)\right) + b\left(x(N)\right)\right| \leq C \end{array} \tag{22}$$

Proposition 4.1 *(See [5]) In the above setting we have*

1. $\mid J^P(\phi) - J^Q(\phi) \mid \leq \delta\, C$;
2. *If $\delta < 1$, the bound in 1. can be improved by replacing $\delta\, C$ with* $\frac{\delta(1+\delta)}{1-\delta} \min\limits_{\phi\in A} J^Q(\phi)$

Proof : 1. is a consequence of the following immediate calculations, where we put $C(\phi, \omega) := \sum_{n=0}^{N-1} c\left(x(n), \phi(n+1)\right) + b\left(x(N)\right)$

$$\mid J^P(\phi) - J^Q(\phi) \mid \left|\int C(\phi, \omega)\left[1 - \frac{dQ(\omega)}{dP(\omega)}\right] dP(\omega)\right| \leq \delta\, C \tag{23}$$

For the proof of 2. we refer to [5].

In [5] one can also find an additional bound on the right hand side in (21) that is alternative to the one in the above Proposition. Depending on the situation, one may prefer one or the other of these bounds.

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