A filtering approach to pricing in multifactor term structure models

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Abstract

We present an approach for pricing of illiquid bonds (and bond derivatives) in an arbitrage-free way and consistently with observed prices of liquid bonds. The basic model is a multifactor term structure model with abstract latent factors. The approach is based on stochastic filtering techniques, leading to a continuous update of the distribution of the latent factors on the basis of the information coming from the observations. This allows our model to continuously "track" the real market.

1 Introduction

The main purpose of this paper is to derive an approach for pricing of illiquid bonds, as well as of bond derivatives, in an arbitrage-free way and consistently with the observed prices of liquid bonds.

The basic model is a multifactor term structure model with abstract factors, that may be viewed as describing the evolution of market fundamentals, but need not have a specific interpretation. They are considered as latent variables, that are not directly observable, but can be filtered from observations of bond prices or of the yield curve (to this effect see also the comment in Section 4 of [10] and the Introduction in [1]; an analogous situation, although in a different context, appears in [4]). This will allow our term structure model to continuously "track" the actual market situation.

In section 2 we shall first define a theoretical arbitrage-free term structure model, in which prices and rates are functions of the latent factors. The prices of the liquid bonds, that can actually be observed on the market, are then modeled as their theoretical values corrupted by noise. This setup, described in section 3, can be justified as e.g. in [2], [3], [13] on the basis of model misspecification, errors of observation (spread), mispricing and thin trading. A further, general argument in favour of considering real observed prices as differing from the theoretical arbitrage-free values is that (see e.g. [8] and some of the references therein) the constraints imposed by absence of arbitrage are often too strong to reproduce faithfully real observed prices. Finally, since the term structure has to be viewed as infinite-dimensional, for any number of explanatory factors there will always be some residual (in this context see also Sections 5 and 6 in [11]).

The values according to our pricing approach are obtained (see section 3) by computing the conditional expectation of suitably perturbed values of the theoretical prices, given the actual observations of the liquid bond prices. This leads to an arbitrage-free pricing system, where (see Proposition 3.4) the discounted values of the computed prices are martingales with respect to the filtration generated by the observed prices, i.e. with respect to the filtration that represents the actually available information. In addition, this pricing system is consistent with the observed prices in the sense that, for the maturities corresponding to the observed liquid bonds, the prices computed by our pricing system are equal to the actually observed ones.

Since the theoretical prices are functions of the latent factors, the required conditional expectation can be computed on the basis of the conditional distribution of the latent factors, given the observations of the liquid bond prices. This conditional distribution corresponds to the solution of what is called a stochastic filtering problem, where the latent factors form the unobserved state or signal process, whose distribution is continuously updated on the basis of the incoming information due to the observations. This makes stochastic filtering a very powerful tool also for financial applications. In fact, provided a market model is made flexible by parametrizing it with hidden/latent random quantities as are here the abstract factors, the market model can be continuously adjusted to the current information. In section 4 we show how in specific cases the filtering problem admits an explicit analytic solution that in turn leads to an explicit solution of our pricing approach.

While in the past there were only few financial applications of filtering techniques, researchers are becoming increasingly aware of the potentialities of these techniques.

Here we just mention [4], [6], [13],[16],[17], [18] and references therein, as well as [1],[2],[3], [9],[21] for a more specific application to term structure models. The main goal of these latter papers differs however from our's.

2 The theoretical arbitrage-free factor model

Given an underlying filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where the filtration $\{\mathcal{F}_t\}$ is generated by a standard scalar Brownian motion, let $x_t \in \mathbb{R}^k$ be a Markov process describing the evolution of the abstract factors in our model. Denote by $f(t,T)$, $t \leq T$, the instantaneous forward rate with maturity T, contracted at t. More specifically, we shall consider a model of the form

$$\begin{aligned} dx_t &= F\,x_t dt + D\,dw_t \quad , \qquad t \geq 0 \quad , \qquad x_0 = 0 \\ f(t,T) &= a(t,T) + b'(t,T)x_t + x_t' c(t,T)x_t \end{aligned} \tag{1}$$

where $'$ denotes transposition, F and D are matrices of appropriate dimensions to be considered as parameters in the model, and $a(t,T), b(t,T), c(t,T)$ are scalar, vector- and matrix-valued functions respectively, differentiable with respect to t and with $c(t,T)$ symmetric. In (1) we consider an exponentially quadratic term structure model as instance of a non-affine model (see also [12]). By imposing the conditions of absence of arbitrage as in Proposition 2.1 below, it can be shown (see [15]) that, for our linear-Gaussian factor model, general exponentially polynomial term structure models with degree larger than 2 reduce to the quadratic model, i.e. the coefficients of the powers of x larger than 2 have to be equal to zero. On the other hand, for the explicit calculations (see section 4 below) we shall restrict ourselves to affine models. Notice, furthermore, that the term $a(t,T)$ in (1) implicitly includes the observed forward rate curve $f^*(0,T)$ for an initial time $t=0$ (for an explicit representation of $a(t,T)$ in the case when $c(t,T) \equiv 0$, see (58) below). Our model (1) is thus a linear factor - nonlinear term structure model. Dually one could also consider nonlinear factor - affine models and there is also some equivalence between the two possible settings.

Model (1) for the forward rates implies for the short rate r_t and the zero-coupon bond prices $p(t,T)$ the representations

$$\begin{aligned} r_t &= a(t) + b'(t)x_t + x_t' c(t)x_t \\ p(t,T) &= \exp\left[-A(t,T) - B'(t,T)x_t - x_t' C(t,T)x_t\right] \end{aligned} \tag{2}$$

with $a(t) = a(t,t)$, $A(t,T) = \int_t^T a(t,u)du$ and, analogously, for $b(t), c(t), B(t,T), C(t,T)$.

So far $a(t,T), b(t,T), c(t,T)$ in (1) also appear as parameters in our model and they induce the parametric functions $a(t), A(t,T), \cdots, c(t), C(t,T)$ in (2). However, in order that our model excludes the possibility of arbitrage, they cannot be chosen arbitrarily. We shall therefore impose on them conditions for absence of arbitrage, that can equivalently be imposed on their integrated variants $A(t,T), \cdots, C(t,T)$ in (2). We have

Proposition 2.1 *A sufficient condition for the term structure model (1), (2) to be arbitrage-free is that the coefficients $A(t,T), B(t,T), C(t,T)$ in (2) satisfy the system of differential equations in t*

$$\begin{array}{l} C_t(t,T) + F'C(t,T) + C(t,T)F - 2C(t,T)DD'C(t,T) + c(t) = 0 \\ B_t(t,T) + B(t,T)F - 2B(t,T)DD'C(t,T) + b(t) = 0 \\ A_t(t,T) + tr\,(D'C(t,T)D) - \frac{1}{2}B'(t,T)DD'B(t,T) + a(t) = 0 \end{array} \tag{3}$$

with terminal conditions $A(T,T) = B(T,T) = C(T,T) = 0$, where $A_t(t,T)$, $B_t(t,T)$,$C_t(t,T)$ denote the partial derivatives with respect to t. The functions $b(t), c(t)$ are here to be considered as parameters, while for $a(t)$ we have

$$a(t) = f^*(0,t) + \frac{1}{2}\int_0^t \beta_T(s,t)ds \tag{4}$$

having put

$$\beta(t,T) := B'(t,T)DD'B(t,T) - 2\,tr\,(D'C(t,T)D) \tag{5}$$

and where $f^*(0,t)$ is the observed initial forward rate curve and the subscript T denotes partial differentiation with respect to the second variable.

Proof : The proof of the first statement can be obtained as a corollary to the more general conditions for absence of arbitrage derived in section 4.2 of [14]. Here we present an independent proof for our specific quadratic case, based on the traditional "HJM-drift condition". Applying the Ito differential rule to $f(t,T)$ in (1), we obtain

$$\begin{aligned} df(t,T) &= x_t'c_t(t,T)x_tdt + dx_t'c(t,T)x_t + x_t'c(t,T)dx_t \\ &+ tr\,(D'c(t,T)D)\,dt + b_t(t,T)'x_tdt \\ &+ b(t,T)'Fx_tdt + b(t,T)'Ddw_t + a_t(t,T)dt \\ &= [x_t'c_t(t,T)x_t + x_t'F'c(t,T)x_t + x_t'c(t,T)Fx_t]\,dt \\ &+ dw_t'D'c(t,T)x_t + x_t'c(t,T)Ddw_t \\ &+ tr\,(D'c(t,T)D)\,dt + b_t(t,T)'x_tdt \\ &+ b(t,T)'Fx_tdt + b(t,T)'Ddw_t + a_t(t,T)dt \end{aligned} \tag{6}$$

so that the volatility is given by

$$\sigma(t,T) = 2x_t'c(t,T)D + b(t,T)'D \tag{7}$$

Apply now the well-known Heath-Jarrow-Morton drift condition [19], whereby we obtain absence of arbitrage if the drift, that we denote by $\mu(t,T)$, is given by

$$\mu(t,T) = \sigma'(t,T)\int_t^T \sigma(t,u)du \tag{8}$$

By (7) this leads to

$$\begin{aligned}\mu(t,T) &= 4x_t'c(t,T)DD'\int_t^T c(t,u)du\,x_t + 2x_t'c(t,T)DD'\int_t^T b(t,u)du \qquad (9)\\ &+ 2b(t,T)'DD'\int_t^T c(t,u)du\,x_t + b(t,T)'DD'\int_t^T b(t,u)du\end{aligned}$$

Equating terms quadratic in x_t in (6) and (9), we obtain

$$\begin{aligned}&x_t'[c_t(t,T) + F'c(t,T) + c(t,T)F]\,x_t \qquad (10)\\ &= 2x_t'[c(t,T)DD'\int_t^T c(t,u)du + \int_t^T c(t,u)duDD'c(t,T)]\,x_t\end{aligned}$$

Since this must hold for every x, we can in fact equate the coefficients. Integrating with respect to the $T-$variable we obtain

$$\begin{aligned}&\int_t^T c_t(t,u)du + F'\int_t^T c(t,u)du + \int_t^T c(t,u)du\,F \qquad (11)\\ &= 2\int_t^T c(t,u)duDD'\int_t^T c(t,u)du\end{aligned}$$

which is a differential Riccati equation. This is easier to see observing that $C_t(t,T) = \int_t^T c_t(t,u)du - c(t)$ with (see (2)) $c(t) = c(t,t)$. We can now rewrite (11) as

$$C_t(t,T) + F'C(t,T) + C(t,T)F - 2C(t,T)DD'C(t,T) + c_t = 0 \qquad (12)$$

which corresponds to the first equation in (3).

Equating the linear terms in (6) and (9) leads to

$$\begin{aligned}&2x_t'c(t,T)DD'\int_t^T b(t,u)du + 2b(t,T)'DD'\int_t^T c(t,u)du\,x_t \qquad (13)\\ &= b_t(t,T)'x_t + b(t,T)'Fx_t\end{aligned}$$

or, dropping x and integrating in T,

$$2\int_t^T c(t,u)'duDD'\int_t^T b(t,u)du = \int_t^T b_t(t,u)'du + \int_t^T b(t,u)'Fdu \qquad (14)$$

from which we get

$$B_t(t,T) + B(t,T)F - 2B(t,T)DD'C(t,T) + b(t) = 0 \qquad (15)$$

and which corresponds to the second equation in (3).

Finally, the constant term must satisfy

$$tr\,(D'c(t,T)D) + a_t(t,T) = b(t,T)'DD'\int_t^T b(t,u)du \qquad (16)$$

or, again integrating in T,

$$\int_t^T tr\left(D'c(t,u)D\right) du + \int_t^T a_t(t,u)du = \frac{1}{2}\int_t^T b(t,u)'duDD'\int_t^T b(t,u)du \qquad (17)$$

It follows

$$A_t(t,T) + tr\left(D'C(t,T)D\right) - \frac{1}{2}B'(t,T)DD'B(t,T) + a(t) = 0 \qquad (18)$$

which is the third equation in (3).

For the second statement notice that from (18), with $\beta(t,T)$ as in (5), one obtains

$$A(t,T) = -\frac{1}{2}\int_t^T \beta(s,T)ds + \int_t^T a(s)ds \qquad (19)$$

On the other hand, being $x_0 = 0$, from (1) one has $f^*(0,T) = a(0,T)$ so that

$$A(0,T) = \int_0^T f^*(0,s)\, ds \qquad (20)$$

Combining (19) with (20) results in

$$\int_0^T f^*(0,s)ds + \frac{1}{2}\int_0^T \beta(s,T)ds - \int_0^T a(s)ds = 0 \qquad (21)$$

from which, differentiating (under an implicit regularity assumption) with respect to T, and noticing that $T > 0$ is arbitrary and can thus be replaced by t, we obtain (4). ∎

Remark 2.2 *For $c(t) \equiv const$ and $b(t) = a(t) = 0$, our setup, restricted to the short rate, reduces to that in [24] for the case of multi-factor models. In fact, we make some progress, since we consider a representation for the forward rates rather than just the short rate, like in [23] where however only the scalar case is being treated.*

Combining (2) and (3) and applying Ito's rule, with an immediate proof we obtain our theoretical arbitrage-free model for the evolution of the prices of zero-coupon bonds, namely

Proposition 2.3 *Under conditions of absence of arbitrage, the zero-coupon bond prices as represented in (2) satisfy*

$$dp(t,T) = p(t,T)\left[r_t\, dt + \sigma'(t,T)dw_t\right] \qquad (22)$$

where r_t is the short rate as in (2) with $b(t), c(t)$ to be considered as parameters, and

$$\sigma'(t,T) := -\left[B'(t,T) + 2x_t'C(t,T)\right] D \qquad (23)$$

with $B(t,T), C(t,T)$ satisfying (3) where, again, $b(t), c(t)$ are parameters.

Remark 2.4 *As it follows from (22), our term structure model satisfies the principle of absence of arbitrage also in the sense that the zero-coupon bond prices, discounted with respect to the money market account, are $(P, \mathcal{F}_t)-$martingales.*

3 An arbitrage-free pricing system consistent with real liquid bond price observations

We assume now that we have some extra observation noise in the market data, which we still want to model in an arbitrage free context.

Let $\Gamma(t,T)$ and $\Delta(t,T)$ be, respectively, a scalar and a row n-dimensional vector valued function satisfying the condition:

$$\left(\frac{\partial}{\partial T}\Gamma\right)(t,T) = \left(\frac{\partial}{\partial T}\Delta\right)(t,T)[\Delta'(t,T) - \Delta'(t,t)] \tag{24}$$

This is basically the Heath-Jarrow-Morton condition and it completely determines $\Gamma(t,T)$ given $\Delta(t,T)$. Furthermore, let

$$\tilde{r}_t = r_t + \int_0^t \Gamma_T(s,t)ds + \int_0^t \Delta_T(s,t)dv_s \tag{25}$$

where $v_t = [v_t^{(1)}, ...v_t^{(n)}]$ is a an n-dimensional $(P, \mathcal{F}_t)-$ Wiener process, independent of $\{w_t\}$.

Lemma 3.1 *The arbitrage free prices $\tilde{p}(t,T)$ associated to the short rate $\tilde{r}_t$ can be expressed as*

$$\tilde{p}(t,T) = p(t,T)\exp\left\{\int_0^t -[\Gamma(s,T) - \Gamma(s,t)]ds - \int_0^t [\Delta(s,T) - \Delta(s,t)]dv_s\right\} \tag{26}$$

and thus, discounted with respect to the money market account associated to $\tilde{r}$, are $(P, \mathcal{F}_t)-$martingales satisfying:

$$d\tilde{p}(t,T) = \tilde{p}(t,T)\left\{\tilde{r}(t)dt + \sigma'(t,T)dw_t - [\Delta(t,T) - \Delta(t,t)]dv_t\right\} \tag{27}$$

Proof: differentiation of (26) with respect to t yields (27); in fact

$$\begin{aligned}
d\tilde{p}(t,T) &= \tilde{p}(t,T)[r_t dt + \sigma'(t,T)dw_t \\
&\quad -(\Gamma(t,T) - \Gamma(t,t))dt + \left(\int_0^t \Gamma_T(s,t)ds\right)dt \\
&\quad -(\Delta(t,T) - \Delta(t,t))dv_t + \left(\int_0^t \Delta_T(s,t)dv_s\right)dt \\
&\quad +\frac{1}{2}(\Delta(t,T) - \Delta(t,t))'(\Delta(t,T) - \Delta(t,t))dt] \\
&= \tilde{p}(t,T)[r_t dt + \left(\int_0^t \Gamma_T(s,t)ds + \int_0^t \Delta_T(s,t)dv_s\right)dt \\
&\quad +\sigma'(t,T)dw_t - (\Delta(t,T) - \Delta(t,t))dv_t] \\
&= \tilde{p}(t,T)[\tilde{r}_t dt + \sigma'(t,T))dw_t - (\Delta(t,T) - \Delta(t,t))dv_t]
\end{aligned}$$

where we have used the integrated version of condition (24) in the second equality. ∎

We have thus defined a perturbed price process which still satisfies the no arbitrage assumption when discounted with respect to its own riskless asset.

Assume now that on the market one can observe n liquid bond prices for the maturities $T_1 < T_2 < \cdots < T_n$. According to the considerations made in the introduction, these real observed bond prices may not necessarily correspond to the theoretical arbitrage-free values associated to r_t. Therefore, for the maturities T_i $(i = 1, 2 \cdots, n)$ we assume that, instead of $p(t, T_i)$, one observes $\tilde{p}(t, T_i)$ as given in (26). An example of this setup can be obtained by choosing the volatility in (24) as follows: set

$$\delta_i(t, T) := \chi_{(T_{i-1}, T_i]}(T)$$

where $\chi_{(a,b]}$ is the characteristic function of the interval $(a, b]$ and where, for $t < T_1$, we choose $T_0 = t$. Next, set

$$\delta(t, T) := [\delta_1(t, T), \delta_2(t, T), ..., \delta_n(t, T)]$$

and define

$$\Delta(t, T) := \int_t^T \delta(t, s) ds.$$

A simple computation yields for the i-th component of $\Delta(t, T)$:

$$\begin{aligned} \Delta_i(t, T) &= \begin{cases} 0 & \text{if } T \leq T_{i-1} \\ T - T_{i-1} & \text{if } T_{i-1} \leq T \leq T_i \text{ and } 0 \leq t \leq T_{i-1} \\ T - t & \text{if } T_{i-1} \leq t < T \leq T_i \\ T_i - t & \text{if } T_{i-1} \leq t \leq T_i < T \\ T_i - T_{i-1} & \text{if } t < T_{i-1} \text{ and } T_i < T \end{cases} \\ &= (T_i \wedge T) \vee T_{i-1} - (T_{i-1} \vee t) \wedge T_i \end{aligned} \tag{28}$$

Notice that $\Delta_i(t, t) = 0$ for $i = 1, ..., n$.

Remark 3.2 *With the choice of $\Delta(t, T)$ as in (28), for each different value T_i, in (27) we add a perturbation driven by a Wiener process $v_t^{(i)}$ and this i-th perturbation only affects $\tilde{p}(t, T)$ for $T \leq T_i$; $i = 1, \cdots, n$. The structure implied by (27) and (28) is therefore different from what one would obtain by adding v_t as additional factor, that would affect the bond prices for all maturities T in the same way.*

In addition to observing the liquid bond prices $\tilde{p}(t, T_i)$, we assume that one can observe also the short rate $\tilde{r}_t$ (a proxy thereof). The observation filtration $\tilde{\mathcal{F}}_t$ is therefore specified by

$$\tilde{\mathcal{F}}_t := \sigma \{\tilde{p}(s, T_i) \, ; \; s \leq t \, , \; i \leq n\} \cup \{\tilde{r}_s \, ; \; s \leq t\} \tag{29}$$

We can now define our arbitrage-free pricing system that is consistent with the real liquid bond price observations.

Definition 3.3 *According to our pricing system, we define $\hat{p}(t,T)$ as:*

$$\hat{p}(t,T) := E\left\{\tilde{p}(t,T) \mid \tilde{\mathcal{F}}_t\right\} = E\left\{\exp\left[-\int_t^T \tilde{r}_s ds\right] \mid \tilde{\mathcal{F}}_t\right\} \tag{30}$$

That this pricing system is arbitrage-free and consistent with the observed prices follows from

Proposition 3.4 *The prices $\hat{p}(t,T)$, discounted with respect to the money market account associated to $\tilde{r}_t$, are $P-$martingales, not however with respect to the full filtration $\mathcal{F}_t$, but with respect to $\tilde{\mathcal{F}}_t$ that represents the actually available information. Furthermore, for $T = T_i \quad (i = 1, \cdots, n)$ one has $\hat{p}(t,T_i) = \tilde{p}(t,T_i)$.*

Proof : The first statement follows from Lemma 3.1 and the second is obvious. ∎

Define now

$$z_t := \int_0^t \Delta_T(s,t) dv_s \tag{31}$$

and assume that $\Delta(t,T)$ is such that the process

$$\xi_t := \begin{bmatrix} x_t \\ z_t \end{bmatrix} \tag{32}$$

is Markov (take e.g. $\Delta(t,T)$ as in (28)). We may then write (25) as:

$$\begin{aligned} \tilde{r}_t &= r_t + \int_0^t \Gamma_T(s,t)ds + z_t \\ &= a(t) + \int_0^t \Gamma_T(s,t)ds + b'(t)x_t + x_t'c(t)x_t + z_t \end{aligned} \tag{33}$$

so that $\tilde{r}$ is a function of the Markov process ξ_t that becomes linear for $c(t) \equiv 0$.

Putting

$$\Phi_{t,T}(\xi_t) := E\left\{\exp\left[-\int_t^T \tilde{r}_s ds\right] \mid \xi_t\right\} \tag{34}$$

the pricing formula (30) can be given the representation

$$\hat{p}(t,T) = E\left\{\exp\left[-\int_t^T \tilde{r}_s ds\right] \mid \tilde{\mathcal{F}}_t\right\} = E\left\{\Phi_{t,T}(\xi_t) \mid \tilde{\mathcal{F}}_t\right\} \tag{35}$$

The computation of the prices $\hat{p}(t,T)$ reduces thus to a filtering problem involving the unobservable state/signal process ξ_t and the observations given by $\tilde{p}(t,T_i)$ and $\tilde{r}_t$.

To actually apply the filtering approach, in what follows we shall restrict ourselves to the case of an affine model, i.e. when in (1) and (2) we have $c(t,T) = c(t) = C(t,T) \equiv 0$.

Notice that our factor model is a multi-factor model. This implies that, although we assume that the short rate satisfying (25) is observable, its observation does not fully determine the values of the latent process ξ_t, not even in the affine case and under the assumption that the model parameter $b(t)$ is known. The filtering problem to be defined below remains thus meaningful and does not degenerate.

Notice finally that, the fact of having defined our theoretical factor model in section 2 in terms of the latent abstract factors $\{x_t\}$, allows us through the pricing formula (30) to fully adapt to the actually observed prices on the market. Our model is thus capable of "tracking" the market.

4 Explicit computation of the pricing formulae

As mentioned in the previous section, in what follows we confine ourselves to affine models. The computation of the prices $\hat{p}(t,T)$ according to (35) requires the computation of the conditional expectation of $\Phi_{t,T}(\xi_t)$ given $\tilde{\mathcal{F}}_t$, which in turn can be computed on the basis of the conditional distribution of the latent process ξ_t, given the observations of the processes $\tilde{r}_t$ and $\tilde{p}(t,T_i)$, $(i = 1, \cdots, n)$ that generate $\tilde{\mathcal{F}}_t$. Computing this conditional distribution corresponds now to solving the filtering problem for ξ_t, given the observations of $\tilde{r}_t$ and $\tilde{p}(t,T_i)$. In the next subsection 4.1 we shall describe the form of the corresponding filtering model, that in the ensuing subsection 4.2 will then be used for the explicit computation of $\hat{p}(t,T)$.

4.1 The induced filtering problem

Notice that for $c(t,T) \equiv 0$ $(c(t) \equiv 0, C(t,T) \equiv 0)$ we have from (33) and (23)

$$\begin{aligned} \tilde{r}_t &= a(t) + \int_0^t \Gamma_T(s,t)ds + b'(t)x_t + z_t \\ \sigma'(t,T) &= -B'(t,T)\,D \end{aligned} \tag{36}$$

Notice, furthermore, that observing $\tilde{p}(t,T_i)$ is equivalent to observing the yields $\tilde{y}(t,T_i) := \log \tilde{p}(t,T_i)$ for $i = 1, ...n$.

The filtering model, describing the evolution of the latent state process $\xi_t = \begin{bmatrix} x_t \\ z_t \end{bmatrix}$ and of the observables $\tilde{r}_t$ and $\tilde{y}(t,T_i)$ is now (see (1), (31), (36) and (27))

$$\begin{aligned} dx_t &= Fx_t dt + Ddw_t \\ dz_t &= \left[\int_0^t \Delta_{TT}(s,t)dv_s\right] dt + \Delta_T(t,t)dv_t \\ d\tilde{r}_t &= \Big[\dot{a}(t) + \Gamma_T(t,t) + \int_0^t \Gamma_{TT}(s,t)ds + \left(\dot{b}'(t) + b'(t)F\right) x_t \end{aligned} \tag{37}$$

$$
\begin{aligned}
& + \int_0^t \Delta_{TT}(s,t)dv_s \Big] dt + b'(t)Ddw_t + \Delta_T(t,t)dv_t \\
dy(t,T_i) \quad = \quad & \left(a(t) + \int_0^t \Gamma_T(s,t)ds + b'(t)x_t + z_t\right) dt \\
- \quad & B'(t,T_i)Ddw_t - [\Delta(t,T_i) - \Delta(t,t)]dv_t \\
- \quad & \left(\frac{1}{2}B'(t,T_i)DD'B(t,T_i) - \frac{1}{2}[\Delta(t,T_i) - \Delta(t,t)]\,[\Delta(t,T_i) - \Delta(t,t)]'\right) dt
\end{aligned}
$$

The degree of difficulty of the associated filtering problem clearly depends on the choice of $\Delta(t,T)$. An explicit solution can be obtained by choosing $\Delta(t,T)$ as in (28), and this is what we shall assume from now on. Notice first that, with this choice, the above expressions simplify considerably, since we have $\Delta(t,t) \equiv 0$, $\Delta_{TT}(t,T) \equiv 0$ and $\Gamma_T(t,t) \equiv 0$. Letting, furthermore, $b(t) \equiv b$ yields also $\dot{b}(t) \equiv 0$. The above filtering model then becomes a classical linear-Gaussian filtering model where the matrix $\Sigma(t)$ of the coefficients of the observation noises is given by:

$$
\Sigma(t) = \begin{bmatrix} b'D & \Delta_T(t,t) \\ -B'(t,T_1)D & -\Delta(t,T_1) \\ \vdots & \vdots \\ -B'(t,T_n)D & -\Delta(t,T_n) \end{bmatrix} \tag{38}
$$

It is well known that, if $\Sigma(t)\Sigma(t)'$ is non singular, the conditional distribution $g(\xi_t|\tilde{\mathcal{F}}_t)$ of ξ_t, given the observations of $\tilde{r}_t$ and $\tilde{y}(t,T_i) = \log \tilde{p}(t,T_i)$ for $i = 1,...n$, is Gaussian and thus determined by its conditional mean vector $m_t = \begin{bmatrix} m_t^x \\ m_t^z \end{bmatrix}$ and covariance matrix V_t, i.e.

$$
g(\xi_t|\tilde{\mathcal{F}}_t) \sim \mathcal{N}(\xi_t; m_t, V_t) \tag{39}
$$

The evolution of m_t and V_t is given by the so-called Kalman-Bucy filter (see e.g. [22], section 10.3).

Since $\Sigma(t)$ is a square matrix, it is non singular if and only if $\Sigma(t)\Sigma(t)'$ is non singular. Notice also that, without loss of generality, we can restrict ourselves to proving that $\Sigma(t)$ is non singular for $t < T_1$.

Lemma 4.1 *Let $\Delta(t,T)$ be as in (28). The matrix $\Sigma(t)$ is non singular for $t < T_1$ if and only if*

$$
b'D(T_1 - t) - B'(t,T_1)D \neq 0 \qquad \text{for } t < T_1 \tag{40}
$$

Proof: notice first that, for $t < T_1$, one has

$$
\Sigma(t) = \begin{bmatrix} b'D & 1 & 0 & \dots & 0 \\ -B'(t,T_1)D & -T_1 + t & 0 & \dots & 0 \\ -B'(t,T_2)D & -T_1 + t & -T_2 + T_1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -B'(t,T_n)D & -T_1 + t & -T_2 + T_1 & \dots & -T_n + T_{n-1} \end{bmatrix} \tag{41}
$$

implying that

$$det(\Sigma(t)) = [b'D(-T_1+t) + B'(t,T_1)D] \prod_{i=2}^{n} (-T_i + T_{i-1})$$

from which the conclusion follows. ∎

In our affine model setting, the functions $B(t,T_i)$, that have to satisfy the second equation in (3), admit an explicit solution described in the following

Lemma 4.2 *Under the assumptions of this section, that is $C(t,T) \equiv 0$ in addition to $b(t) \equiv b$ and the invertibility of F in (1), the second equation in (3), i.e.*

$$B_t(t,T) + FB(t,T) + b = 0 \;\; ; \quad B(T,T) = 0 \tag{42}$$

has the explicit solution

$$B(t,T) = F^{-1}\left(e^{F(T-t)} - I\right)\, b \tag{43}$$

Proof : From (42) it follows that

$$B(t,T) = e^{-Ft}\left[B(0,T) - \int_0^t e^{Fs} b\, ds\right] \tag{44}$$

The terminal condition $B(T,T) = 0$ implies

$$B(0,T) = \int_0^T e^{Fs} b\, ds \tag{45}$$

from which the conclusion follows. ∎

Notice that, in the context of Lemma 4.2, condition (40) takes the form

$$b'\left[(T_1 - t)I - (F')^{-1}(e^{F'(T_1-t)} - I)\right] D \neq 0 \qquad \text{for } t < T_1 \tag{46}$$

In view of the fact that for $t = T_1$ the first term in (46) is 0, a sufficient condition for (46) to be satisfied for $t < T_1$ is that its derivative does not vanish, i.e.

$$b'(I - e^{F'(T_1-t)})D \neq 0$$

For the computation of the pricing formula (35) we still need an explicit expression for $\Phi_{t,T}(\xi_t)$ defined in (34). This will be accomplished in the next subsection.

4.2 The explicit pricing formula

We shall derive an explicit expression for $\hat{p}(t,T)$ for the case when $\Delta(t,T)$ is chosen according to (28) and $b(t) \equiv b$. Recall also from (36) that then $\tilde{r}_t$ becomes:

$$\tilde{r}_t = a(t) + \int_0^t \Gamma_T(s,t)ds + b'x_t + z_t$$

We first have

Lemma 4.3 *Under the assumptions of this section, the function $\Phi_{t,T}(\xi_t)$ in (34) has the explicit representation*

$$\Phi_{t,T}(\xi_t) = \phi(t,T) \exp\left[-b'\Psi(t,T)x_t + (T-t)z_t\right] \tag{47}$$

where

$$\phi(t,T) := \exp\left[-\int_t^T a(s)ds - \int_t^T \int_0^s \Gamma_T(u,s)duds + \frac{1}{2}b'\rho^x(t,T)b + \frac{1}{2}\rho^z(t,T)\right] \tag{48}$$

with the $(k \times k)-$matrix $\rho^x(t,T)$ and the scalar $\rho^z(t,T)$ defined as

$$\begin{aligned} \rho^x(t,T) &:= \int_t^T F^{-1}\left[e^{F(T-u)} - I\right] DD' \left[e^{F'(T-u)} - I\right] (F')^{-1} du & (49) \\ \rho^z(t,T) &:= \frac{(T-t)^3}{3} & (50) \end{aligned}$$

respectively and where the $(k \times k)-$matrix $\Psi(t,T)$ is

$$\Psi(t,T) = e^{-Ft} \int_t^T e^{Fs} ds \tag{51}$$

Proof : From the dynamic equations (1) we have $x_t = \int_0^t e^{F(t-u)} D dw_u$, so that

$$\begin{aligned} \int_t^T x_s ds &= \int_t^T \int_0^s e^{F(s-u)} D dw_u ds & (52) \\ &= \int_t^T \int_0^t e^{F(s-u)} D dw_u ds + \int_t^T \left(\int_t^s e^{F(s-u)} D dw_u \right) ds \\ &= \left[\int_t^T e^{F(s-t)} ds\right] x_t + \int_t^T \int_u^T e^{F(s-u)} D ds dw_u \end{aligned}$$

where the last term on the right is a $k-$variate Gaussian vector with mean zero and whose $(k \times k)$ covariance matrix is easily seen to be

$$\begin{aligned} \rho^x(t,T) &= \int_t^T \left[\int_u^T e^{F(s-u)} D ds\right] \left[\int_u^T D' e^{F'(s-u)} ds\right] du \\ &= \int_t^T F^{-1}\left[e^{F(T-u)} - I\right] DD' \left[e^{F'(T-u)} - I\right] (F')^{-1} du \end{aligned}$$

which is (49). Together with (51) this gives

$$\int_t^T x_s ds = \Psi(t,T)x_t + \mathcal{N}(0, \rho^x(t,T))$$

On the other hand, from (37) with $\Delta(t,T)$ according to (28), we have

$$dz_t = \Delta_T(t,t)dv_t$$

from which we obtain

$$\begin{aligned}
\int_t^T z_s ds &= \int_t^T \int_0^s \Delta_T(u,u)dv_u ds \qquad (53)\\
&= \int_t^T \int_0^t \Delta_T(u,u)dv_u ds + \int_t^T \int_t^s \Delta_T(u,u)dv_u ds\\
&= (T-t)z_t + \int_t^T \int_u^T \Delta_T(u,u)ds dv_u
\end{aligned}$$

As before, the last term on the right is Gaussian random variable with mean zero covariance given by

$$\begin{aligned}
\rho^z(t,T) &= \int_t^T (T-u)^2 \Delta_T(u,u)\Delta_T(u,u)' du\\
&= \int_t^T (T-u)^2 du = \frac{(T-t)^3}{3}
\end{aligned}$$

which is (50). This gives

$$\int_t^T z_s ds = (T-t)z_t + \mathcal{N}(0, \rho^z(t,T))$$

The conclusion now follows immediately recalling the expression of the moment generating function of a Gaussian random vector and variable, respectively, by which

$$\begin{aligned}
E\left\{\exp\left[-b'\mathcal{N}(0, \rho^x(t,T))\right]\right\} &= \exp\left[\frac{1}{2}b'\rho^x(t,T)b\right]\\
E\left\{\exp\left[\mathcal{N}(0, \rho^z(t,T))\right]\right\} &= \exp\left[\frac{1}{2}\rho^z(t,T)\right]
\end{aligned}$$

∎

Notice that the expression of the variance in (50) coincides with that of the integral of a single Brownian motion, and this may lead to think that our construction coincides with that obtained by simply adding another factor. However, as was already pointed out in Remark 3.2, the difference becomes apparent if one looks at the yields volatilities, as seen in the expression (41) of $\Sigma(t)$.

The explicit pricing formula is now given in the following

Proposition 4.4 *Under the assumptions of this section, we have*

$$\begin{aligned} \hat{p}(t,T) &= \left\{ \phi(t,T) \exp\left[\frac{1}{2} \left[b'\Psi(t,T), (T-t) \right] V_t \left[b'\Psi(t,T), (T-t) \right]' \right] \right\} \\ &\cdot \exp\left[-b'\Psi(t,T) m_t^x + (T-t) m_t^z \right] \end{aligned} \quad (54)$$

with $\phi(t,T)$, $\Psi(t,T)$ as defined in (48) and (51) respectively and where $m_t = \begin{bmatrix} m_t^x \\ m_t^z \end{bmatrix}$ and V_t are the conditional mean vector and covariance matrix respectively of (39).

Proof : By formula (35) and the expression (47) for $\Phi_{t,T}(\xi_t)$, we only need to compute the moment generating function

$$E\left\{ \exp\left[-b'\Psi(t,T)x_t + (T-t)z_t \right] \mid \tilde{\mathcal{F}}_t \right\} \quad (55)$$

where, see (39), the distribution of ξ_t, conditional on $\tilde{\mathcal{F}}_t$, is Gaussian with mean vector m_t and covariance matrix V_t. This immediately leads to (54). ∎

Remark 4.5 *The prices given by (54) correspond to an exponentially affine structure, where the factors are now given by m_t, a vector of dimension $k+1$, i.e. one more than that of the original factors x_t. The values of m_t and of V_t can be computed by means of the Kalman filter whereby V_t can be precomputed off-line.*

We conclude this section with an additional result, which shows that the number of parameters in our model can be considerably reduced without affecting the main results (see e.g. [20] for details).

Proposition 4.6 *In the case of $c(t) \equiv 0$ and $b(t) \equiv b$, all pricing formulas remain the same by taking F and D in (1) of the form*

$$F = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ \vdots & & & & 1 \\ q_1 & q_2 & q_3 & \dots & q_k \end{pmatrix} \qquad D = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (56)$$

i.e. the number of parameters in our model can be reduced to $2k$ constants, namely the k parameters $[q_1, \dots, q_k]$ and the k components of the vector $b = [b_1, \dots, b_k]$.

Proof : Under the given assumptions, from (1) it follows that in the unperturbed model

$$f(t,T) = a(t,T) + b'(t,T)x_t \quad (57)$$

where, see [5],

$$\begin{aligned} a(t,T) &= f^*(0,T) + \frac{1}{2}\Big\{b'F^{-1}e^{FT}DD'e^{F'T}F'^{-1}b - b'F^{-1}e^{F(T-t)}DD'e^{F'(T-t)}F'^{-1}b \\ &+ b'F^{-1}e^{F(T-t)}\Big[e^{Ft}-I\Big]DD'F'^{-1}b \\ &+ b'F^{-1}DD'\Big[e^{F't}-I\Big]e^{F(T-t)}F'^{-1}b\Big\} \end{aligned} \tag{58}$$

while from (1) and (43)

$$b'(t,T) = b'\,e^{F'(T-t)} \tag{59}$$

∎

It follows that, in the expression for $f(t,T)$ and, as can easily be seen, in all pricing formulas, the parameters b, F, D appear always grouped in expressions of the form $b'e^{Fs}D$. From systems theory it is now well known (see e.g.[7]) that this corresponds to an impulse response, which is invariant under the actions of an invertible matrix. More precisely, if $M \in GL(n)$ and one defines $b'_1 = b'M$, $F_1 = M^{-1}F$ and $D_1 = MD$, then

$$b'e^{Fs}D = b'MM^{-1}e^{Fs}MM^{-1}D = b'_1e^{F_1s}D_1 \tag{60}$$

The choice of F and D according to (56) corresponds now to the choice of a particular triple, called standard controllable realization, as an atlas to parametrize all impulse responses.

5 Concluding remarks and possible extensions

Starting from the (background) theoretical arbitrage-free term structure model described in section 2, in section 3 we have defined an arbitrage-free pricing system, consistent with the actually observed liquid bond prices, that may deviate from their theoretical arbitrage-free values and are modeled as their theoretical values corrupted by noise. It is an arbitrage-free pricing system in the sense that the prices, discounted with respect to the money market, are martingales with respect to the sub-filtration representing the actually available information. In addition, for the maturities corresponding to liquid bonds, the prices obtained through our pricing system correspond to the actually observed ones. Like more traditional approaches such as those based on calibrating an arbitrage-free model to market data, also our system defines prices consistently with the market data; the way our procedure works is however different from how those more traditional approaches work.

According to our procedure, prices of illiquid bonds (and bond derivatives) are derived from suitably perturbed values of their theoretical arbitrage-free prices by projecting the latter onto the subfiltration generated by the observed prices. This projection corresponds to the solution of a stochastic filtering problem, that in section 4 is shown to admit an explicit analytic solution in the case of an affine model, for which one obtains a linear filtering problem.

The filtering setup for our model with the latent abstract factors allows to continuously adjust the model to the currently available information; in other words, it allows the model to "track" the market. To obtain a filtering model, we have modeled the dynamics of the real prices by the dynamics of their theoretical values plus additive Gaussian noise. We could also have added a jump-noise component or modeled the factors themselves as jump-diffusion processes. This would allow for greater flexibility in exchange however for greater model complexity.

We have not discussed the identification of the parameters in our model, for which we shall briefly mention here three possibilities (in this context see also Section 3 in [6]). Taking the Bayesian point of view, a first possibility corresponds to considering the parameters as random variables, whose distribution can be updated jointly with that of the latent factors. A second possibility corresponds to the classical approaches of model calibration : choose the parameters in a way that the computed values for the prices match as closely as possible their observed values. Contrary to the first alternative, this latter identification approach does not allow for an evaluation of the accuracy of the parameter estimates. A third possibility is finally given by likelihood-based methods (In this context see e.g. also [2], [3], where the likelihood approach is combined with the filtering setup).

To test the effectiveness of the proposed pricing approach, numerical investigations with real market data are needed and this is the subject of ongoing research.

Acknowledgments : It is a great pleasure to acknowledge the extremely valuable help of Professor Tomas Björk during the early stages of this investigation. We are also grateful to Professor Nicole El Karoui, Professor Vincent Lacoste as well as Martino Grasselli for helpful comments.

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