ESERCIZI SU INTEGRAZIONE MULTIPLA

1 L'intersezione di algebre (σ-algebre) è un'algebra (σ-algebra).

Soluzione: Ricordiamo che $\mathcal{A} \subseteq \mathcal{P}(X)$ è un'algebra (σ-algebra) di parti di X se:

1) $\emptyset \in \mathcal{A}$
2) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ $\left(\forall (A_n)_{n \in \mathbb{N}}, A_n \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}\right)$
3) $A \in \mathcal{A} \Rightarrow X \setminus A \ (= A^c) \in \mathcal{A}$.

Siano allora $\mathcal{A}_i$, algebre (σ-algebre) $\forall i$, con $i \in I$ insieme qualsiasi di indici.

1) $\emptyset \in \mathcal{A}_i \ \forall i$, poiché $\mathcal{A}_i$ algebra (σ-algebra) $\Rightarrow \emptyset \in \bigcap_{i \in I} \mathcal{A}_i$

2) $A, B \in \bigcap_{i \in I} \mathcal{A}_i \Rightarrow A, B \in \mathcal{A}_i \ \forall i \in I$. Ma poiché $\mathcal{A}_i$ algebra, $A \cup B \in \mathcal{A}_i \ \forall i \Rightarrow A \cup B \in \bigcap_{i \in I} \mathcal{A}_i$. Analogamente si mostra che se $(A_n)_{n \in \mathbb{N}} \in \bigcap_{i \in I} \mathcal{A}_i \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \bigcap_{i \in I} \mathcal{A}_i$.

3) $A \in \bigcap_{i \in I} \mathcal{A}_i \Rightarrow A \in \mathcal{A}_i \ \forall i \in I \Rightarrow (\mathcal{A}_i$ algebra$)$ $A^c \in \mathcal{A}_i \ \forall i \Rightarrow$ $A^c \in \bigcap_{i \in I} \mathcal{A}_i$.

Questo conclude la prova.

$\square$

2 Siano $A, B \subseteq \mathcal{P}(X)$ due insiemi disgiunti $(A \cap B = \emptyset)$.

Qual è $\sigma(\{A,B\})$?

<u>Soluzione</u>: Ricordiamo che data $\mathcal{F} \subseteq \mathcal{P}(X)$ una famiglia di sottoinsiemi di X, esiste la <u>minima</u> algebra di parti di X contenente $\mathcal{F}$: essa è data da $\bigcap_{\substack{\mathcal{F} \subseteq \mathcal{a},\\ \mathcal{a} \text{ algebra}}} \mathcal{a}$ (l'intersezione di tutte le algebre contenenti $\mathcal{F}$).

Consideriamo prima, come esempio, $\sigma(\{A\})$.

$\sigma(\{A\})$ deve contenere <u>almeno</u>: $\emptyset$, A; le loro unioni $\emptyset \cup A = A$; i complementari: $\emptyset^c = X$; A^c; Quindi

$\sigma(\{A\}) \supseteq \{\emptyset, X, A, A^c\}$. È vero "$\subseteq$", cioè che è proprio questa la (σ-) algebra generata da A? Se fosse una (σ-) algebra, sì, poiché siccome $\sigma(\{A\})$ deve essere l'intersezione di tutte le algebre contenenti A, quindi deve valere "$\subseteq$". Che sia un'algebra è una verifica immediata.

Consideriamo ora $\sigma(\{A,B\})$. Procediamo come prima. $\sigma(\{A,B\})$ <u>deve</u> contenere almeno $\{\emptyset, X, A, B, A^c, B^c, A\cup B, (A\cup B)^c\}$. Se questa è un'algebra, allora è proprio $\sigma(\{A,B\})$. Il fatto che sia un'algebra deriva dalle def. i), ii), iii) di algebra e dal fatto $A \cap B = \emptyset$ (fare tutte le possibili unioni e complementari degli elementi $\{\emptyset, X, A, B, A^c, B^c, A\cup B, (A\cup B)^c\}$ e notare che questo insieme è chiuso per queste operazioni).

INTEGRALI ITERATI SU RETTANGOLI (In tutti gli esercizi supponiamo che la formula di riduzione valga sempre)

(ES. 6)

a) $\iint_D xye^y\,dx\,dy$, $R=\{(x,y): 0\le x\le 1, 0\le y\le 2\}$

$$\left(\int_0^1 x\,dx\right)\left(\int_0^2 y e^y dy\right) = \left(\frac{x^2}{2}\Big|_0^1\right)\left(e^y y\Big|_0^2 - \int_0^2 e^y dy\right) = \frac{1}{2}\left(2e^2 - \left(e^y\Big|_0^2\right)\right) = \frac{1}{2}\left(2e^2 - e^2 + 1\right) = \boxed{\frac{e^2+1}{2}}$$

b) $\iint_D (x^3y^2 - 4x^2y)\,dx\,dy$, $R=\{(x,y): 0\le y\le 3, 0\le x\le 1\}$

$$\left(\int_0^1 x^3 dx\right)\left(\int_0^3 y^2 dy\right) - 4\left(\int_0^1 x^2 dx\right)\left(\int_0^3 y\,dy\right) = \left(\frac{x^4}{4}\Big|_0^1\right)\left(\frac{y^3}{3}\Big|_0^3\right) - 4\left(\frac{x^3}{3}\Big|_0^1\right)\left(\frac{y^2}{2}\Big|_0^3\right)$$

$$= \frac{1}{4}\cdot 9 - 4\cdot\frac{1}{3}\cdot\frac{9}{2} = \frac{9}{4} - 6 = \boxed{\frac{-15}{4}}$$

c) $\iint_D \sin(x+3y)\,dx\,dy$, $R=\{(x,y): 0\le y\le \frac{\pi}{3}, 0\le x\le \pi\}$

$$\int_0^{\frac{\pi}{3}}\left(\int_0^{\pi}\sin(x+3y)\,dx\right)dy = \int_0^{\frac{\pi}{3}}\left(-\cos(x+3y)\Big|_{x=0}^{x=\pi}\right)dy = \int_0^{\frac{\pi}{3}} \cos(3y) - \cos(\pi+3y)\,dy =$$

$$= \frac{\sin(3y)}{3} - \frac{\sin(3y+\pi)}{3}\Big|_{y=0}^{y=\frac{\pi}{3}} = \frac{\sin(\pi)}{3} - \frac{\sin(2\pi)}{3} - \frac{\sin(0)}{3} + \frac{\sin(\pi)}{3} = \boxed{0}$$

d) $\iint_D y\cos(xy)\,dx\,dy$, $R=\{[-\frac{\pi}{4},\frac{\pi}{4}]\times[0,1]\}$

$$\int_0^1 y\left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos(xy)\,dx\right)dy = \int_0^1 y\left(\frac{\sin(xy)}{y}\Big|_{x=-\frac{\pi}{4}}^{x=\frac{\pi}{4}}\right)dy = \int_0^1 \sin\left(\frac{\pi}{4}y\right) - \sin\left(-\frac{\pi}{4}y\right)dy =$$

$$= 2\int_0^1 \sin\left(\frac{\pi}{4}y\right)dy = -\frac{2\cdot 4}{\pi}\cos\left(\frac{\pi}{4}y\right)\Big|_{y=0}^{y=1} = -\frac{8}{\pi}\left(\frac{\sqrt{2}}{2} - 1\right) = \boxed{\frac{4(2-\sqrt{2})}{\pi}}$$

e) $$\int_0^1\int_2^4 e^{2x+y}\,dx\,dy = \left(\int_0^1 e^{y}\,dy\right)\left(\int_2^4 e^{2x}\,dx\right) = \left(e^{y}\Big|_{y=0}^{y=1}\right)\left(\frac{e^{2x}}{2}\Big|_{x=2}^{x=4}\right) = (e-1)\left(\frac{e^{8}-e^{4}}{2}\right)$$

f) $$\int_{-1}^1\int_0^{\sqrt{3}} \frac{1+x^2}{1+y^2}\,dy\,dx = \left(\int_{-1}^1 (1+x^2)\,dx\right)\left(\int_0^{\sqrt{3}} \frac{1}{1+y^2}\,dy\right) = \left(x+\frac{x^3}{3}\Big|_{-1}^{1}\right)\left(\arctan y\Big|_0^{\sqrt{3}}\right) =$$

$$\overset{!}{=} \left(1+\frac{1}{3}+1+\frac{1}{3}\right)\left(\arctan(\sqrt{3})-\arctan(0)\right) =$$

$$\overset{!}{=} \left(\frac{8}{3}\right)\left(\frac{\pi}{3}\right) = \boxed{\frac{8\pi}{9}} =$$

g) $$\int_0^2\int_0^2 \sqrt{x+y}\,dx\,dy = \int_0^2\left(\int_0^2 \sqrt{x+y}\,dx\right)dy = \int_0^2\left(\frac{2}{3}(x+y)^{3/2}\Big|_{x=0}^{x=2}\right)dy =$$

$$= \frac{2}{3}\int_0^2\left[(y+2)^{\frac{3}{2}} - y^{3/2}\right]dy = \frac{2}{3}\cdot\frac{2}{5}\left[(y+2)^{\frac{5}{2}}-y^{\frac{5}{2}}\right]\Big|_{y=0}^{y=2} = \boxed{\frac{4}{15}\left[4^{5/2}-2^{5/2}\right]}$$

h) $$\int_{-2}^2\int_{-1}^1 (x^2+y^2)\,dx\,dy = \int_{-2}^2\left(\int_{-1}^1 (x^2+y^2)\,dx\right)dy = \int_{-2}^2\left(\frac{x^3}{3}+xy^2\Big|_{x=-1}^{x=1}\right)dy =$$

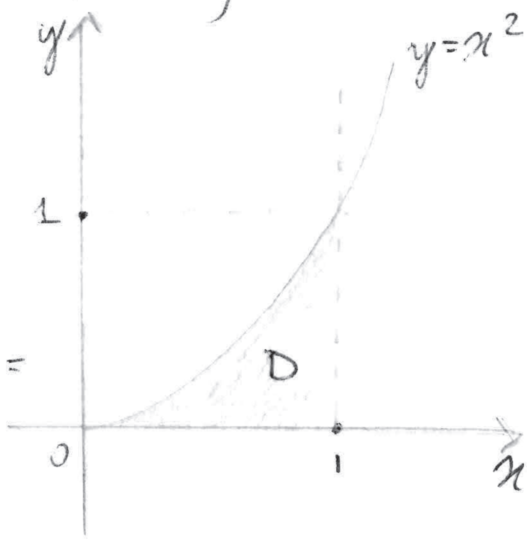
$$= \int_{-2}^2\left(\frac{1}{3}+y^2+\frac{1}{3}+y^2\right)dy = \int_{-2}^2\left(\frac{2}{3}+2y^2\right)dy = \frac{2}{3}y+\frac{2}{3}y^3\Big|_{y=-2}^{y=2} =$$

$$= \frac{2}{3}(y+y^3)\Big|_{y=-2}^{y=2} = \frac{2}{3}(10+10) = \boxed{\frac{40}{3}}$$

INTEGRALI SU DOMINI SEMPLICI

(ES. 8)

a) $\iint_D x\cos y \, dx\, dy$, $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x^2\}$

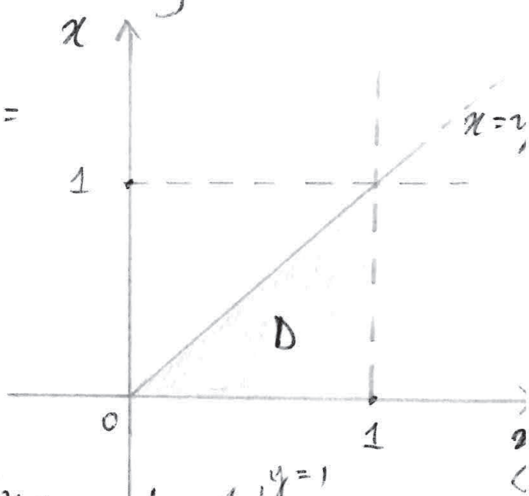


$$\int_0^1 x\left(\int_0^{x^2} \cos y\, dy\right)dx = \int_0^1 x\left(\sin y \Big|_{y=0}^{y=x^2}\right)dx =$$

$$= \int_0^1 x \sin x^2\, dx = \int_0^1 \frac{\sin(x^2)}{2}\cdot 2x\, dx \underset{\substack{\uparrow \\ f = x^2 \\ df = 2x\,dx}}{=} \int_{f(0)=0}^{f(1)=1} \frac{\sin(f)}{2}\, df =$$

$$= -\frac{\cos f}{2}\Big|_{f=0}^{f=1} = \frac{-\cos(1)+\cos(0)}{2} = \boxed{\frac{1-\cos(1)}{2}}$$

b) $\iint_D x\sqrt{y^2-x^2}\, dx\, dy$, $D = \{(x,y) : 0 \le y \le 1,\ 0 \le x \le y\}$



$$\int_0^1\left(\int_0^y x\sqrt{y^2-x^2}\, dx\right)dy \underset{\uparrow}{=} \int_0^1\left(-\frac{1}{2}\int_{f(0)=y^2}^{f(y)=0} \sqrt{f}\, df\right)dy =$$

$y^2 - x^2 = f$

$df = -2x\,dx$

(y è parametro! stiamo integrando in x!)

$$= -\frac{1}{2}\int_0^1 \cdot \frac{2}{3} f^{3/2}\Big|_{f=y^2}^{f=0} = -\frac{1}{3}\int_0^1 (-y^3)\, dy = \frac{1}{3}\int_0^1 y^3\, dy = \frac{1}{3}\frac{y^4}{4}\Big|_{y=0}^{y=1} =$$

$$= \boxed{\frac{1}{12}}$$

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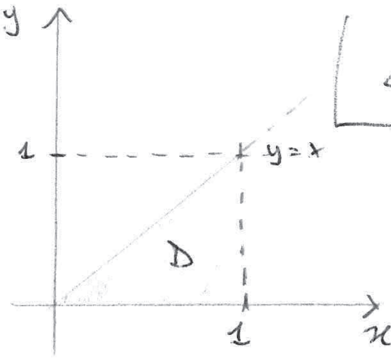
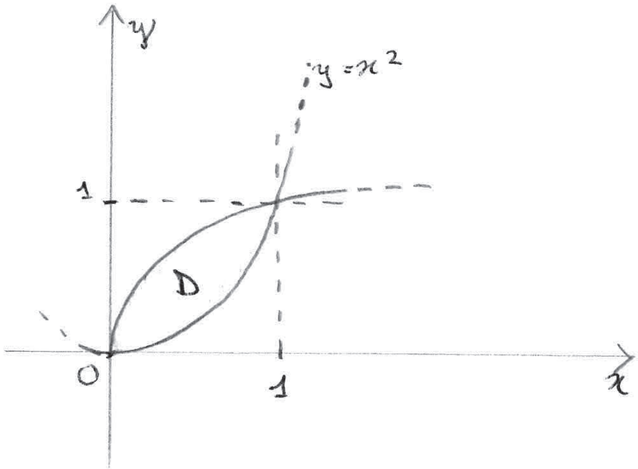
c) $\iint_D e^{x^2}\,dx\,dy \qquad D=\{(x,y) : 0\le x\le 1,\ 0\le y\le x\}$



$$\int_0^1\left(\int_0^x e^{x^2}\,dy\right)dx=\int_0^1 x e^{x^2}\,dx \underset{\substack{x^2=f \\ df=2x\,dx}}{=} \int_{f(0)}^{f(1)}\frac{e^f}{2}=\frac{e^f}{2}\Bigg|_{f(0)=0}^{f(1)=1}=\boxed{\frac{e-1}{2}}$$

d) $\iint_D (x+y)\,dx\,dy$, D delimitato da $y=x^2$ e $y=\sqrt{x}$



$\sqrt{x}$ è definita per $x\geq 0$.

per $0\le x\le 1$, $\sqrt{x}\geq x^2$. Quindi la regione D è descritta da

$D=\{(x,y) : 0\le x\le 1 \quad x^2\le y\le \sqrt{x}\}$

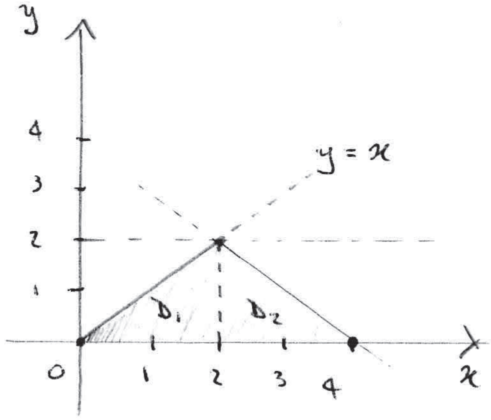
per cui

$$\iint_D (x+y)\,dx\,dy=\int_0^1\left(\int_{x^2}^{\sqrt{x}}(x+y)\,dy\right)dx=\int_0^1\left(xy+\frac{y^2}{2}\right)\Bigg|_{y=x^2}^{y=\sqrt{x}}dx=$$

$$=\int_0^1\left(x\sqrt{x}+\frac{x}{2}-x^3-\frac{x^4}{2}\right)dx=\int_0^1\left(x^{\frac{3}{2}}+\frac{x}{2}-x^3-\frac{x^4}{2}\right)dx=\left(\frac{2}{5}x^{\frac{5}{2}}+\frac{x^2}{4}-\frac{x^4}{4}-\frac{x^5}{10}\right)\Bigg|_{x=0}^{x=1}$$

$$=\frac{2}{5}+\frac{1}{4}-\frac{1}{4}-\frac{1}{10}=\frac{6}{20}=\boxed{\frac{3}{10}}$$

e) $\iint_D x e^y \, dx\, dy$, D è il triangolo di vertici $(0,0)$, $(4,0)$, $(2,2)$



qui dobbiamo descrivere opportunamente il dominio. Innanzitutto, scriviamo le equazioni delle rette per $(0,0)$, $(2,2)$ e $(2,2)$, $(0,4)$.

la retta per $(0,0)$ e $(2,2)$ è $y = x$

la retta per $(2,2)$ e $(0,4)$ è $y = -x + 4$

Suddividiamo D in $D_1 \cup D_2$, (con $D_1 \cap D_2 = \phi$). Per l'additività dell'integrale, $\iint_D = \iint_{D_1} + \iint_{D_2}$.

$D_1 := \{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq x\}$

$D_2 = \{(x,y) : 2 \leq x \leq 4, 0 \leq y \leq -x+4\}$.

$$\iint_{D_1} x e^y = \int_0^2 \left(\int_0^x x e^y dy \right) dx = \int_0^2 x \left(e^y \Big|_{y=0}^{y=x} \right) dx = \int_0^2 x (e^x - 1)\, dx =$$

$$= \int_0^2 x e^x - \int_0^2 x = \left(x e^x \Big|_{x=0}^{x=2} \right) - \int_0^2 e^x dx - \left(\frac{x^2}{2} \Big|_0^2 \right) = 2e^2 - \left(e^x \Big|_0^2 \right) - 2$$

$$= 2e^2 - e^2 + 1 - 2 = (e^2 - 1)$$

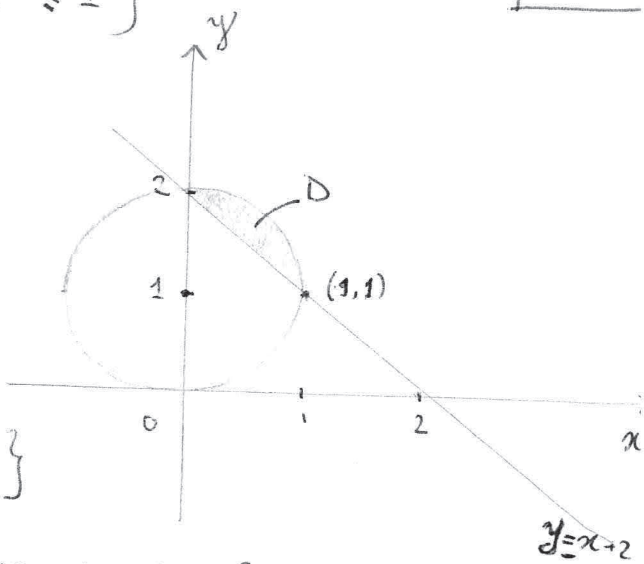
$$\iint_{D_2} x e^y = \int_2^4 x \left(\int_0^{4-x} e^y dy \right) dx = \int_2^4 x \left(e^{4-x} - 1 \right) dx = \int_2^4 x e^4 e^{-x} dx - \int_2^4 x\, dx =$$

$$= e^4 \left[-x e^{-x} \Big|_2^4 + \int_2^4 e^{-x} \right] - \left[\left(\frac{x^2}{2} \right) \Big|_2^4 \right] = e^4 \left(-4e^{-4} + 2e^{-2} - e^{-4} + e^{-2} \right) - 8 + 2$$

$$= e^4 \left(-5e^{-4} + 3e^{-2} \right) - 6 = \boxed{-11 + 3e^2}$$

<u>Finally</u> $\iint_D x e^y = 4e^2 - 12$

f) $\iint_D x\,dx\,dy$, $D=\{(x,y): y \geq -x+2,\ x^2+(y-1)^2 \leq 1\}$



$x^2+(y-1)^2=1$ è la circonferenza di centro $(0,1)$ e raggio 1.

Quindi possiamo descrivere D come:

$D=\{(x,y): 0 \leq x \leq 1,\ -x+2 \leq y \leq 1+\sqrt{1-x^2}\}$

questo perché $(y-1)^2 \leq 1-x^2$, ma per $0 \leq x \leq 1$, $(1-x^2) \geq 0$ e $(y-1) \geq 0$.

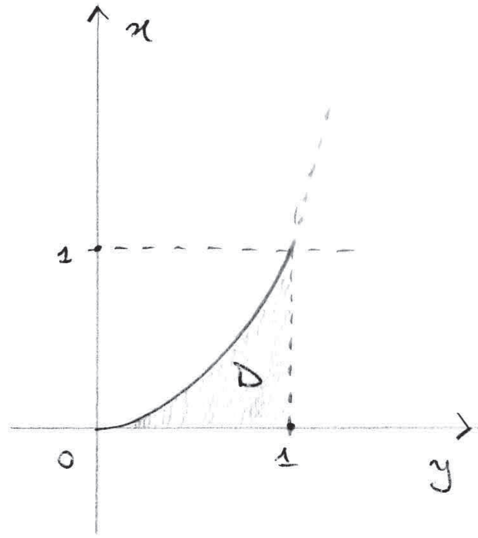
$$\int_0^1 x\left(\int_{-x+2}^{1+\sqrt{1-x^2}} dy\right) dx = \int_0^1 x\left(1+\sqrt{1-x^2}+x-2\right) dx =$$

$$\int_0^1 \left(x + x\sqrt{1-x^2}+x^2-2x\right) dx = \int_0^1 x^2 dx - \int_0^1 x\,dx + \int_0^1 x\sqrt{1-x^2}\,dx = \frac{x^3}{3}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 + \int_0^1 x\sqrt{1-x^2}\,dx$$

$$= \frac{1}{3}-\frac{1}{2} - \frac{1}{2}\int_{f(0)}^{f(1)} \sqrt{f}\,df \quad \text{dove } f(x)=1-x^2 \quad = -\frac{1}{6} - \frac{1}{2}\int_1^0 \sqrt{f}\,df = -\frac{1}{6}+\frac{1}{2}\int_0^1 \sqrt{f}\,df =$$

$$= -\frac{1}{6}+\frac{1}{2}\cdot\frac{2}{3}\, f^{3/2}\Big|_0^1 = -\frac{1}{6}+\frac{1}{3} = \boxed{\frac{1}{6}}$$

9) $\iint_D e^{\frac{x}{y}}\,dx\,dy \quad , \quad D=\{(x,y): 0\le y\le 1,\ 0\le x\le y^2\}$

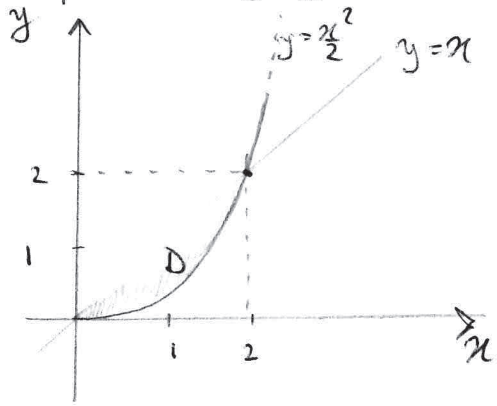


$$\int_0^1\left(\int_0^{y^2} e^{\frac{x}{y}}\,dx\right)dy = \int_0^1\left(y e^{\frac{x}{y}}\Big|_{x=0}^{x=y^2}\right)dy =$$

$$= \int_0^1 (y e^{y} - y)\,dy = -\frac{y^2}{2}\Big|_0^1 + y e^{y}\Big|_0^1 - \int_0^1 e^{y}\,dy =$$

$$= -\frac{1}{2} + e - e + 1 = \boxed{\frac{1}{2}}$$

i) $\iint_D \frac{x}{x^2+y^2}\,dx\,dy$, D regione compresa tra la parabola $y=\frac{x^2}{2}$ e la retta $y=x$



$D=\{(x,y):\ 0\le x\le 2,\ \frac{x^2}{2}\le y\le x\}$

$$\int_0^2 x\left(\int_{\frac{x^2}{2}}^{x}\frac{1}{x^2+y^2}\,dy\right)dx = \int_0^2 x\left(\frac{1}{x}\arctan\left(\frac{y}{x}\right)\Big|_{y=\frac{x^2}{2}}^{y=x}\right)dx$$

$$= \int_0^2 \arctan(1) - \arctan\left(\frac{x}{2}\right)dx = \int_0^2\left(\frac{\pi}{4} - \arctan\left(\frac{x}{2}\right)\right)dx =$$

$$\frac{\pi}{2} - \int_0^2 \arctan\left(\frac{x}{2}\right) = \frac{\pi}{2} - x\arctan\left(\frac{x}{2}\right)\Big|_{x=0}^{x=2} + \int_0^2 \frac{1}{2}\cdot\frac{x}{1+\frac{x^2}{4}}\,dx =$$

$$= \frac{\pi}{2} - 2\frac{\pi}{4} + 0 + \int_0^2 \frac{2x}{4+x^2}\,dx = \int_0^2 \frac{2x}{x^2+4}\,dx = \int_{f(0)=4}^{f(2)=8}\frac{df}{f} = \ln f\Big|_4^8 = \ln(8) - \ln(4)$$

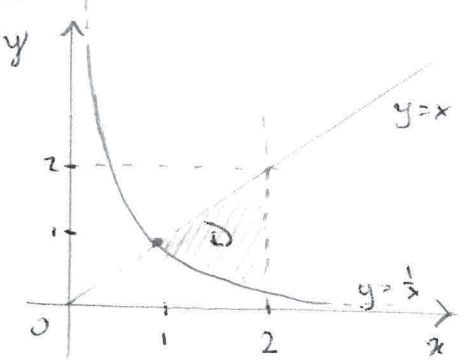
$f = x^2+4$

$df = 2x\,dx$

$$= \ln\left(\frac{8}{4}\right) = \boxed{\ln(2)}$$

i) $\iint_D \frac{x^2}{y^2}\,dx\,dy \quad , D=\{(x,y): 1\le x\le 2,\ \frac{1}{x}\le y\le x\}$

$$\int_1^2 x^2\left(\int_{\frac{1}{x}}^{x}\frac{1}{y^2}dy\right)dx=\int_1^2 x^2\left(-\frac{1}{y}\Big|_{y=\frac{1}{x}}^{y=x}\right)dx =$$



$$= \int_1^2 x^2\left(-\frac{1}{x}+x\right)dx=\int_1^2 -x+x^3 = \left(-\frac{x^2}{2}+\frac{x^4}{4}\right)\Big|_1^2 =$$

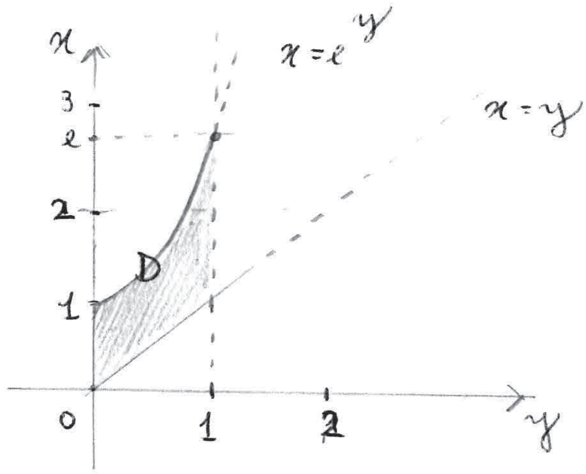
$$= -\frac{4}{2}+\frac{16}{4}+\frac{1}{2}-\frac{1}{4} = -2+4+\frac{1}{2}-\frac{1}{4}=\boxed{\frac{9}{4}}$$

INTEGRAZIONE SU DOMINI SEMPLICI

Qui si disegni il dominio di integrazione e si inverta l'ordine di integrazione qualora necessario.

(ES. 18)

a) $\int_0^1\left(\int_y^{e^y}\sqrt{x}\,dx\right)dy$

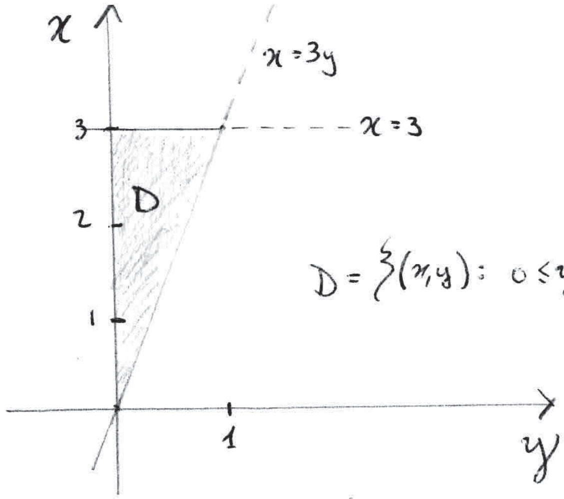


$$\frac{2}{3}\int_0^1\left(x^{3/2}\Big|_{x=y}^{x=e^y}\right)dy=\frac{2}{3}\int_0^1\left(e^{\frac{3}{2}y}-y^{3/2}\right)dy=$$

$$=\frac{2}{3}\cdot\frac{2}{3}\cdot e^{\frac{3}{2}y}\Big|_0^1 -\frac{2}{3}\cdot\frac{2}{5}\,y^{\frac{5}{2}}\Big|_0^1 =$$

$$=\frac{4}{9}\left(e^{\frac{3}{2}}-1\right)-\frac{4}{15}=\boxed{\frac{4}{9}e^{\frac{3}{2}}-\frac{32}{45}}$$

b) $\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy$



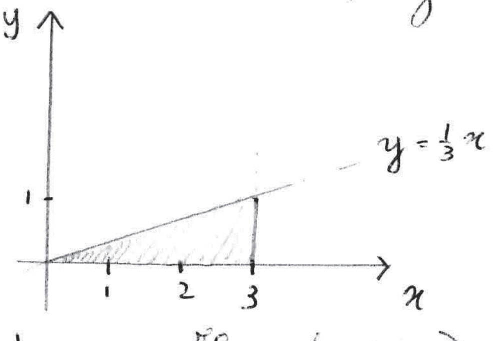
$D=\{(x,y):\ 0\leq y\leq 1,\ 3y\leq x\leq 3\}$

Qui notiamo che $\int_{3y}^{3} e^{x^2}dx$ non riusciamo ad esprimerlo!

L'altra strada da percorrere è scambiare l'ordine di integrazione. Però dobbiamo riscrivere il dominio in modo opportuno. Notiamo che il dominio D ci viene dato da $D=\{(x,y):\ 0\leq y\leq 1,\ 3y\leq x\leq 3\}^{(*)}$, ovvero nella forma $a\leq y\leq b$, $\alpha(y)\leq x\leq \beta(y)$. Dobbiamo cercare di riscriverlo nella forma

$$c\leq x\leq d,\quad \gamma(x)\leq y\leq \delta(x).$$

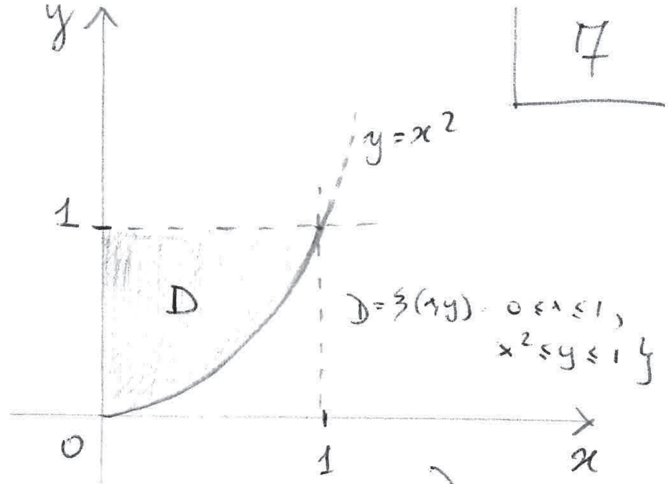
Dal grafico si vede che $0\leq x\leq 3$ ($\Rightarrow c=0,\ d=3$). In prima cosa allora, come regola generale, si individua la proiezione dell'insieme bidimensionale D sull'asse coordinato (x in questo caso). Per scrivere $\gamma(x)\leq y\leq\delta(x)$, graficamente dobbiamo "invertire" gli assi:



Sarà quindi $0\leq y\leq f(x)$, e $f(x)$, se guardiamo (*) $\boxed{3y\leq x}\Rightarrow y\leq\frac{1}{3}x$ è data da $f(x)=\frac{1}{3}x$. Quindi $0\leq y\leq\frac{1}{3}x$. Il dominio D allora si può scrivere: $D=\{(x,y):\ 0\leq x\leq 3,\ 0\leq y\leq \frac{1}{3}x\}$.

$$\int_0^3\left(\int_0^{\frac{1}{3}x} e^{x^2}dy\right)dx=\int_0^3\frac{1}{3}x\,e^{x^2}dx\underset{\substack{t=x^2\\ dt=2x\,dx}}{=}\int_{t(0)=0}^{t(3)=9}\frac{e^t}{6}dt=\frac{e^t}{6}\Big|_{t=0}^{t=9}=\boxed{\frac{e^9-1}{6}}$$

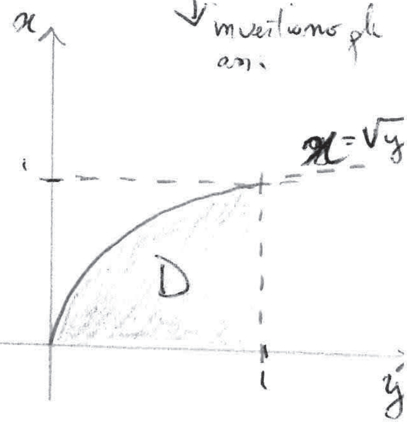
e) $\displaystyle\int_0^1\int_{x^2}^1 x^3\sin(y^3)\,dy\,dx$



l'integrale definito $\displaystyle\int_{x^2}^1 \sin(y^3)\,dy$ non lo si può direttamente scrivere. Allora proviamo ad invertire l'ordine di integrazione:

$0\le y\le 1, \quad 0\le x\le\sqrt{y}$ quindi

invertiamo gli assi.



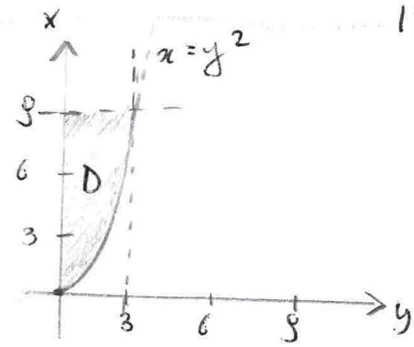
$$\int_0^1\left(\int_0^{\sqrt{y}} x^3\sin(y^3)\,dx\right)dy=\int_0^1 \sin(y^3)\left(\frac{x^4}{4}\Bigg|_{x=0}^{x=\sqrt{y}}\right)dy=$$

$$=\int_0^1\frac{\sin(y^3)\,y^2}{4}\,dy \underset{\uparrow}{=} \int_{f(0)=0}^{f(1)=1}\frac{\sin(f)}{12}\,df=-\frac{\cos(f)}{12}\Bigg|_{f=0}^{f=1}=$$

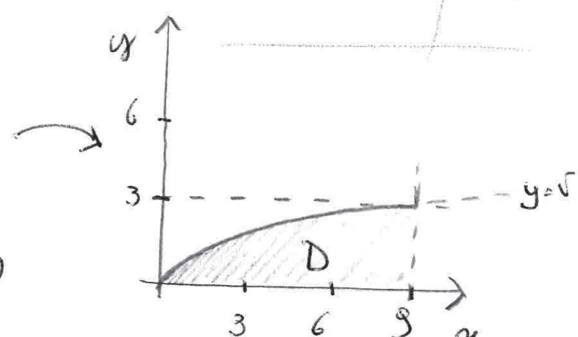
$f=y^3$
$df=3y^2dy$

$$=\frac{-\cos(1)+\cos(0)}{12}=\boxed{\frac{1-\cos(1)}{12}}$$

d) $\displaystyle\int_0^3\int_{y^2}^9 y\cos(x^2)\,dx\,dy$



$D=\{(x,y):\ 0\le y\le 3,\ y^2\le x\le 9\}$



$D=\{(x,y):\ 0\le x\le 9,\ 0\le y\le\sqrt{x}\}$

Anche qui, $\cos(x^2)$ non ha una forma nota per l'integrale definito.

Riscriviamo il dominio: $D=\{(x,y):\ 0\le x\le 9,\ 0\le y\le\sqrt{x}\}$

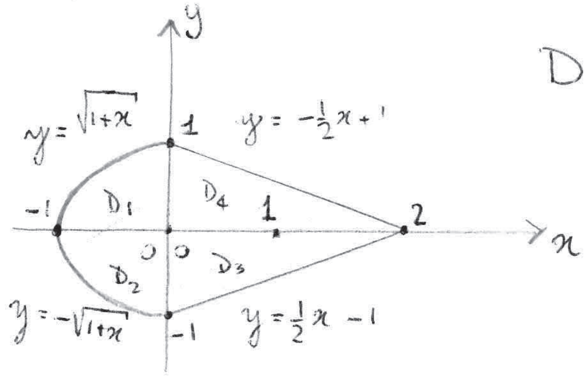
$$\int_0^9\cos(x^2)\left(\int_0^{\sqrt{x}} y\,dy\right)dx=\int_0^9\left(\frac{y^2}{2}\Bigg|_0^{\sqrt{x}}\right)\cos(x^2)\,dx=\int_0^9\frac{x}{2}\cos(x^2) \underset{\uparrow}{=} \int_{f(0)=0}^{f(9)=81}\frac{\cos(f)}{4}\,df=$$

$f=x^2$
$df=2x\,dx$

$$=\frac{\sin(f)}{4}\Bigg|_{f=0}^{f=81}=\boxed{\frac{\sin(81)}{4}}$$

Calcolare l'integrale $\iint_D xy \, dx\, dy$ nei seguenti quattro casi

a)



D:

Scomponiamo il dominio D come $D = D_1 \cup D_2 \cup D_3 \cup D_4$, con $D_i \cap D_j = \phi$ $i,j=1,\dots,4$.

$D_1 = \{(x,y) : -1 \le x \le 0,\ 0 \le y \le \sqrt{1+x}\}$; $D_4 = \{(x,y) : 0 \le x \le 2,\ 0 \le y \le -\frac{1}{2}x+1\}$

$D_2 = \{(x,y) : -1 \le x \le 0,\ -\sqrt{1+x} \le y \le 0\}$; $D_3 = \{(x,y) : 0 \le x \le 2,\ \frac{1}{2}x-1 \le y \le 0\}$

$$\iint_{D_1} xy = \int_{-1}^{0} x \left(\int_0^{\sqrt{1+x}} y\, dy\right) dx = \int_{-1}^{0} \frac{x(1+x)}{2}\, dx = \int_{-1}^{0} \left(\frac{x}{2}+\frac{x^2}{2}\right) dx = \left(\frac{x^2}{4}+\frac{x^3}{6}\right)\Bigg|_{-1}^{0} = -\frac{1}{4}+\frac{1}{6} = \frac{-1}{12}$$

$$\iint_{D_2} xy = \int_{-1}^{0} x \left(\int_{-\sqrt{1+x}}^{0} y\, dy\right) dx = \int_{-1}^{0} -\frac{x(1+x)}{2}\, dx = \frac{1}{12}$$

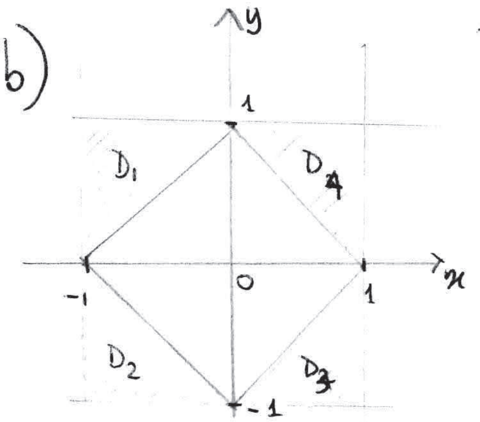
$$\iint_{D_3} xy = \int_0^2 x \left(\int_{\frac{1}{2}x-1}^{0} y\, dy\right) dx = \int_0^2 -\frac{x}{2}\left(\frac{1}{2}x-1\right)^2 dx = -\int_0^2 \frac{x}{2}\left(\frac{1}{4}x^2+1-x\right) dx = \int_0^2 -\frac{x^3}{8}-\frac{x}{2}+\frac{x^2}{2}\, dx =$$

$$= -\frac{x^4}{32}-\frac{x^2}{4}+\frac{x^3}{6}\Bigg|_0^2 = -\frac{1}{2}-1+\frac{4}{3} = \frac{-1}{6}$$

$$\iint_{D_4} xy = \int_0^2 x \left(\int_0^{1-\frac{1}{2}x} y\, dy\right) dx = \int_0^2 \frac{x}{2}\left(1-\frac{1}{2}x\right)^2 dx = \frac{1}{6}.$$

Quindi, sommando i pezzi, $\boxed{\iint_D xy\, dx dy = 0}$

Si poteva concludere immediatamente poiché $\boxed{\int_{D_1} xy = -\int_{D_2} xy \quad e \quad \int_{D_4} xy = -\int_{D_3} xy.}$

b) 

$D = D_1 \cup D_2 \cup D_3 \cup D_4.$

$D_1 = \{(x,y) : -1 \le x \le 0,\ x+1 \le y \le 1\}$

$D_2 : \{(x,y) : -1 \le x \le 0,\ -1 \le y \le -x-1\}$

$D_3 = \{(x,y) : 0 \le x \le 1,\ -1 \le y \le x-1\}$

$D_4 = \{(x,y) : 0 \le x \le 1,\ -x+1 \le y \le 1\}$

$$\iint_{D_1} xy\,dx\,dy = \int_{-1}^{0} x\left(\int_{x+1}^{1} y\,dy\right)dx = \int_{-1}^{0} x\left(\frac{y^2}{2}\Big|_{y=x+1}^{y=1}\right)dx = \int_{-1}^{0} x\left(\frac{1-(x+1)^2}{2}\right)dx =$$

$$= \int_{-1}^{0} \frac{x(-x^2-2x)}{2}\,dx = -\frac{1}{2}\int_{-1}^{0}(x^3+2x^2)\,dx = -\frac{1}{2}\left(\frac{x^4}{4}+\frac{2x^3}{3}\right)\Big|_{-1}^{0} = \frac{1}{2}\left(\frac{1}{4}-\frac{2}{3}\right) = \frac{-5}{24}$$

$$\iint_{D_2} xy\,dx\,dy = \int_{-1}^{0} x\left(\int_{-1}^{-x-1} y\,dy\right)dx = \int_{-1}^{0} x\left(\frac{y^2}{2}\Big|_{y=-1}^{y=-x-1}\right)dx = \int_{-1}^{0} x\left(\frac{(-x-1)^2-1}{2}\right)dx = \frac{5}{24}$$

$$\iint_{D_3} xy\,dx\,dy = \int_{0}^{1} x\left(\int_{-1}^{x-1} y\,dy\right)dx = \int_{0}^{1} x\left(\frac{y^2}{2}\Big|_{y=-1}^{y=x-1}\right)dx = \int_{0}^{1}\frac{x}{2}\left((x-1)^2-1\right)dx = \frac{1}{2}\int_{0}^{1}(x^3-2x^2)\,dx =$$

$$= \frac{1}{2}\int_{0}^{1}(x^3-2x^2)\,dx = \frac{1}{2}\left(\frac{x^4}{4}-\frac{2x^3}{3}\right)\Big|_{0}^{1} = \frac{1}{2}\left(\frac{1}{4}-\frac{2}{3}\right) = \frac{-5}{24}$$ e

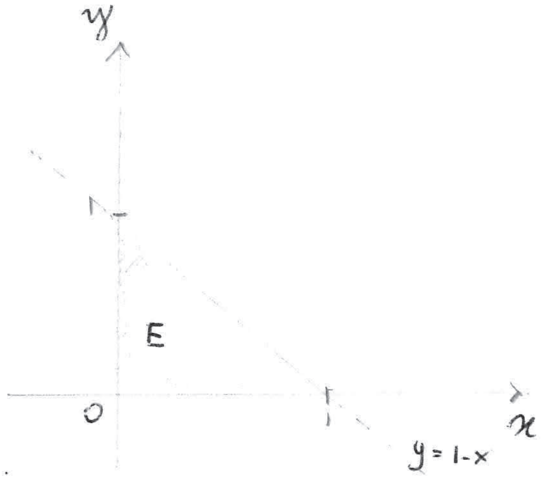
$$\iint_{D_4} xy\,dx\,dy = \frac{5}{24}, \quad \text{per cui} \quad \left|\ \iint_{D} xy\,dx\,dy = 0\ \right|$$

Anche in questo caso si poteva direttamente concludere.

8.2 Sia $f(x,y) = x^2 + \sin y$ e sia

$$E = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x+y \leq 1\}$$

la terza disuguaglianza dice $y \leq 1-x$; per cui E può essere scritto come



$$E = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \ 0 \leq y \leq 1-x\}.$$

$$\int_0^1 \left(\int_0^{1-x} (x^2 + \sin y)\, dy \right) dx = \int_0^1 \left(x^2 y - \cos y \Big|_{y=0}^{y=1-x} \right) dx = \int_0^1 \left(x^2(1-x) - \cos(1-x) + 1 \right) dx =$$

$$= \int_0^1 (x^2 - x^3 - \cos(1-x) + 1)\, dx = \left(\frac{x^3}{3} - \frac{x^4}{4} + x + \sin(1-x) \right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} + 1 + \sin(0) - \sin(1) =$$

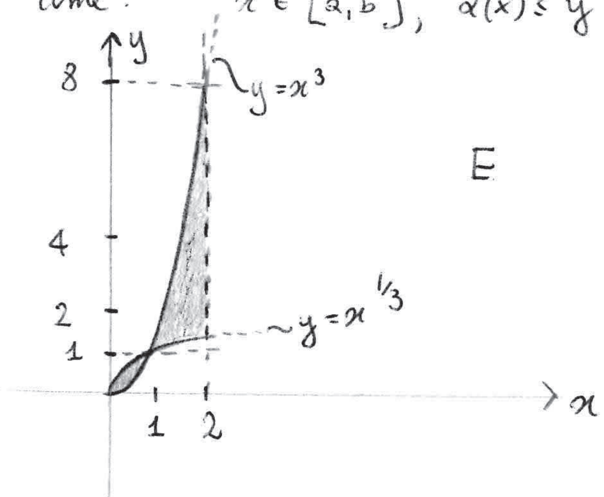
$$\boxed{\frac{13}{12} - \sin(1)}$$

8.3 Sia $E = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, \ \alpha(x) \leq y \leq \beta(x)\}$,

dove $\alpha(x) = \min\{x^3, x^{1/3}\}$, $\beta(x) = \max\{x^3, x^{1/3}\}$. Disegnare E.

Calcolare $\int_E x^5 \, dx\, dy$ come dominio normale rispetto all'asse x

(Vuol dire, descritto come: $x \in [a,b]$, $\alpha(x) \leq y \leq \beta(x)$)



Scriviamo $E = E_1 \cup E_2$

$$E_1 = \left\{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1, \ x^3 \le y \le x^{1/3} \right\}$$

$$E_2 = \left\{(x,y) \in \mathbb{R}^2 : \ 1 \le x \le 2, \ x^{1/3} \le y \le x^3 \right\}$$

$$\iint_{E_1} x^5 \, dx dy = \int_0^1 \left(\int_{x^3}^{x^{1/3}} x^5 dy \right) dx = \int_0^1 x^5 \left(y \Big|_{y=x^3}^{y=x^{1/3}} \right) dx = \int_0^1 x^5 (x^{1/3} - x^3) dx$$

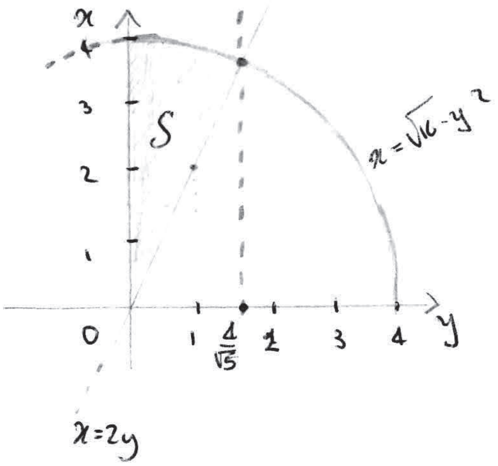
$$= \int_0^1 (x^{\frac{16}{3}} - x^8) dx = \frac{3}{19} x^{\frac{19}{3}} - \frac{x^9}{9} \Big|_0^1 = \frac{3}{19} - \frac{1}{9}$$

$$\iint_{E_2} x^5 \, dx dy = \int_1^2 \left(\int_{x^{1/3}}^{x^3} x^5 dy \right) dx = \int_1^2 (-x^{\frac{16}{3}} + x^8) dx = -\frac{3}{19} x^{\frac{19}{3}} + \frac{x^9}{9} \Big|_1^2 = -\frac{3}{19} \cdot 2^{\frac{19}{3}} + \frac{2^9}{9} + \frac{3}{19} - \frac{1}{9}.$$

La somma $\displaystyle\iint_E x^5 \, dx dy = -\frac{3}{19} \cdot 2^{\frac{19}{3}} + \frac{2^9}{9} + \frac{6}{19} - \frac{2}{9}$

$\boxed{8.4}$ Interpretare l'integrale iterato $\displaystyle\int_{y=0}^{y=4/\sqrt{5}} dy \int_{x=2y}^{x=\sqrt{16-y^2}} x^2 y \, dx$.

come integrale doppio su un sottoinsieme S di $\mathbb{R}^2$. Invertire l'ordine di integrazione e calcolarlo;



$x = \sqrt{16 - y^2}$ è il semicerchio superiore $\{x^2 + y^2 = 16, \ x \ge 0\}$.

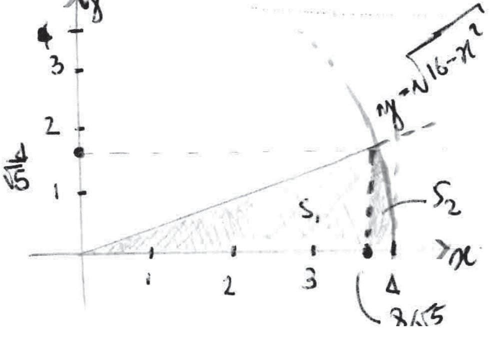
Calcoliamo l'intersezione della retta $x = 2y$ e della circonferenza $x = \sqrt{16-y^2}$:

$2y = \sqrt{16 - y^2}$, $4y^2 = 16 - y^2 \Rightarrow y = \pm \frac{4}{\sqrt{5}}$

(a noi interessa solo $y = \frac{4}{\sqrt{5}}$)

S è la regione in figura. Invertiamo gli assi:

$0 \le x \le 4$ e $0 \le y \le \sqrt{16 - x^2}$, e $0 \le y \le \frac{x}{2}$, cioè



$S = S_1 \cup S_2$, dove

$S_1 = \{(x,y) : \ 0 \le x \le \frac{8}{\sqrt{5}}, \ 0 \le y \le \frac{x}{2}\}$

$S_2 = \{(x,y) : \ \frac{8}{\sqrt{5}} \le x \le 4, \ 0 \le y \le \sqrt{16 - x^2}\}$.

Per cui $\iint_S x^2 y \, dx dy = \iint_{S_1} x^2 y + \iint_{S_2} x^2 y$;

$$\iint_{S_1} x^2 y = \int_0^{\frac{8}{\sqrt5}} \left(\int_0^{\frac{x}{2}} x^2 y \, dy \right) dx = \int_0^{8/\sqrt5} x^2 \left(\frac{y^2}{2} \Big|_{y=0}^{y=\frac{x}{2}} \right) dx = \int_0^{8/\sqrt5} x^2 \cdot \frac{x^2}{8} = \frac{1}{8} \frac{x^5}{5} \Big|_0^{8/\sqrt5} = \boxed{\frac{1}{40}\left(\frac{8}{\sqrt5}\right)^5}$$

$$\iint_{S_2} x^2 y = \int_{\frac{8}{\sqrt5}}^4 x^2 \left(\frac{y^2}{2} \Big|_{y=0}^{y=\sqrt{16-x^2}} \right) dx = \int_{\frac{8}{\sqrt5}}^4 \frac{x^2(16-x^2)}{2} \, dx = \int_{\frac{8}{\sqrt5}}^4 \left(8x^2 - \frac{x^4}{2} \right) dx =$$

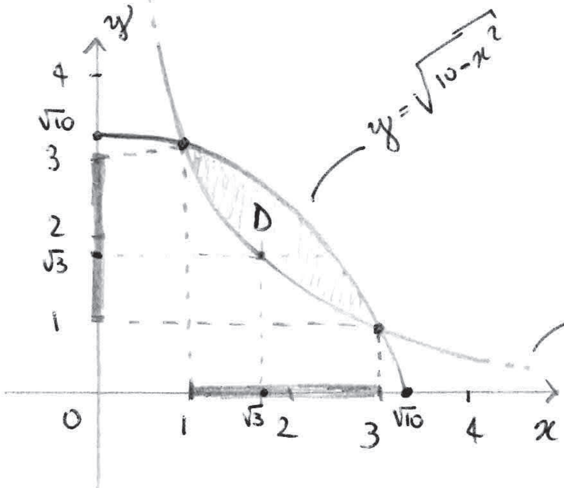
$$= \frac{8x^3}{3} \Big|_{\frac{8}{\sqrt5}}^4 - \frac{x^5}{10} \Big|_{\frac{8}{\sqrt5}}^4 = \boxed{\frac{8 \cdot 4^3}{3} - \frac{8}{3}\left(\frac{8}{\sqrt5}\right)^3 - \frac{4^5}{10} + \frac{1}{10}\left(\frac{8}{\sqrt5}\right)^5}.$$

L'integrale sarà dato dalla somma dei due pezzi

□

$\boxed{8.5}$ Calcolare $\int_D \frac{2xy}{(1+y^2)^2} \, dx dy$, dove

D è il sottoinsieme del primo quadrante delimitato dalle curve di equazioni $x^2+y^2=10$, $xy=3$



- $x^2+y^2=10$ è la circonferenza di centro $(0,0)$ e raggio $\sqrt{10}$
- $xy=3$ è l'iperbole equilatera per $(\sqrt3, \sqrt3)$

Calcoliamo le proiezioni sugli assi coordinati

$$\begin{cases} xy=3 \\ x^2+y^2=10 \end{cases}, \; x \geq 0, y \geq 0 \quad ; \quad x^2 + \frac{9}{x^2} = 10 \quad , \quad x^4 - 10x^2 + 9 = 0 \; , \; x^2_{1/2} = \frac{10 \pm 8}{2} \begin{matrix} 9 \\ 1 \end{matrix}$$

$x_1 = 1, \; x_2 = 3$

$y_1 = 3, \; y_2 = 1$

Quindi $[1,3]$ e $[1,3]$ sono le proiezioni di D sugli assi coordinati.
Scegliamo di descrivere D , ad esempio, come

$$D = \left\{ (x,y) \in \mathbb{R}^2 : \; 1 \leq x \leq 3, \; \frac{3}{x} \leq y \leq \sqrt{10-x^2} \right\}$$

l'integrale diventa

$$\int_1^3 \left(\int_{\frac{3}{x}}^{\sqrt{10-x^2}} \frac{2xy}{(1+y^2)^2} dy \right) dx = \int_1^3 x \left(\int_{\frac{3}{x}}^{\sqrt{10-x^2}} \frac{2y\,dy}{(1+y^2)^2} \right) dx \underset{\substack{1+y^2=f \\ df = 2y\,dy}}{=} \int_1^3 x \left(\int_{f(\frac{3}{x}) = 1+\frac{9}{x^2}}^{f(\sqrt{10-x^2}) = 1+(10-x^2)} \frac{df}{f^2} \right) dx$$

$$= -\int_1^3 x \cdot \left(\frac{1}{f} \Big|_{1+\frac{9}{x^2}}^{11-x^2} \right) dx = -\int_1^3 x \left(\frac{1}{11-x^2} - \frac{1}{1+\frac{9}{x^2}} \right) dx =$$

$$= -\int_1^3 \frac{x}{11-x^2} dx + \int_1^3 \frac{x^3}{9+x^2} dx = \int_1^3 \frac{-x}{11-x^2} dx + \int_1^3 \frac{x \cdot x^2}{9+x^2} dx =$$

$$f = 11-x^2, \quad df = -2x\,dx \qquad g = 9+x^2, \quad dg = 2x\,dx$$

$$\int_{f(1)}^{f(3)} \frac{df}{2f} + \int_{g(1)}^{g(3)} \frac{g-9}{2g} dg =$$

$$= \int_{10}^{2} \frac{df}{2f} + \int_{10}^{18} \left(\frac{1}{2} - \frac{9}{2g} \right) dg = -\int_2^{10} \frac{df}{2f} + \int_{10}^{18} \frac{1}{2} dg - \frac{9}{2} \int_{10}^{18} \frac{1}{g} dg =$$

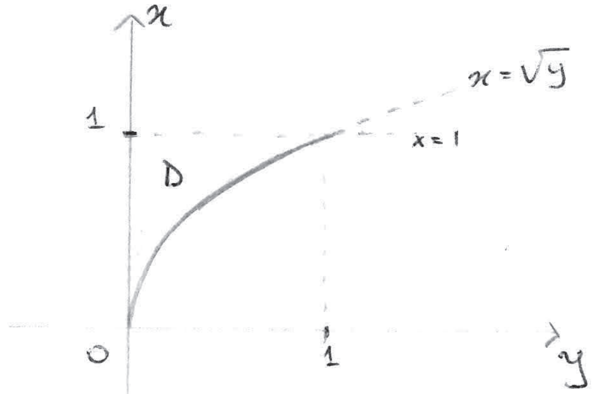
$$= -\frac{1}{2} \ln(f) \Big|_{f=2}^{f=10} + \frac{g}{2} \Big|_{g=10}^{g=18} - \frac{9}{2} \ln g \Big|_{g=10}^{g=18} =$$

$$= -\frac{1}{2} \ln(5) + 4 - \frac{9}{2} \ln\left(\frac{9}{5}\right) = -\frac{1}{2} \ln(5) + 4 - \frac{9}{2} \ln(9) + \frac{9}{2} \ln(5) = \boxed{4 + 4\ln(5) - 9\ln(3)}$$

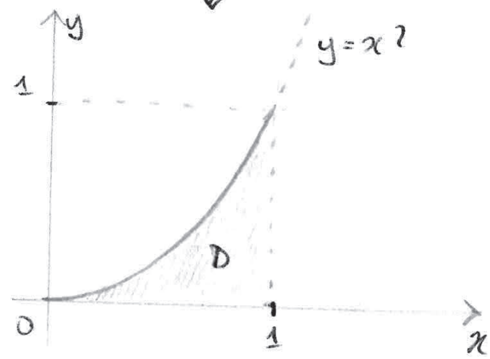
8.7 Calcolare l'integrale

$$\int_D x^3 \cos(xy)\,dx\,dy \qquad D = \{(x,y) : y \geq 0, \sqrt{y} \leq x \leq 1\}.$$

$$\int_0^1 \int_{\sqrt{y}}^1 x^3 \cos(xy)\,dx\,dy.$$ Ci serve cambiare l'ordine di integrazione.



$$D = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq x^2\}$$



$$\int_0^1 \left(\int_0^{x^2} x^3 \cos(xy)\,dy \right) dx = \int_0^1 x^3 \left(\int_0^{x^2} \cos(xy)\,dy \right) dx =$$

$$= \int_0^1 x^3 \left(\frac{\sin(xy)}{x} \Big|_{y=0}^{y=x^2} \right) dx =$$

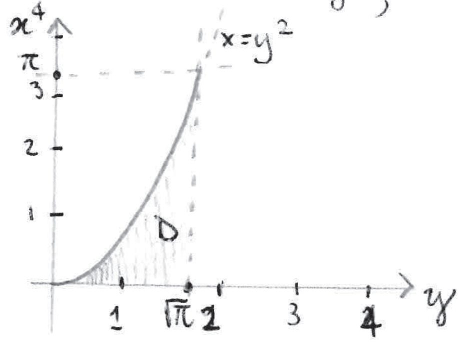
$$= \int_0^1 x^2 \left(\sin(x^3) - 0 \right) dx = \int_0^1 x^2 \sin(x^3)\,dx =$$

$f = x^3$, $df = 3x^2 dx$

$$= \int_{f(0)}^{f(1)} \frac{\sin(f)}{3}\,df = \int_0^1 \frac{\sin(f)}{3}\,df = -\frac{\cos(f)}{3}\Big|_{f=0}^{f=1} = \frac{-\cos(1) + 1}{3} = \boxed{\frac{1 - \cos(1)}{3}}$$

8.8 Siano $f(x,y) = \dfrac{\sin(y^2)}{y}$ e $D = \{(x,y) : 0 \leq x \leq y^2,\ 0 \leq y \leq \sqrt{\pi}\}$.

Calcolare $\int_D f(x,y)\,dx\,dy$.



$$\int_0^{\sqrt{\pi}} \left(\int_0^{y^2} \frac{\sin(y^2)}{y}\,dx \right) dy = \int_0^{\sqrt{\pi}} \frac{\sin(y^2)}{y} \cdot y^2\,dy =$$

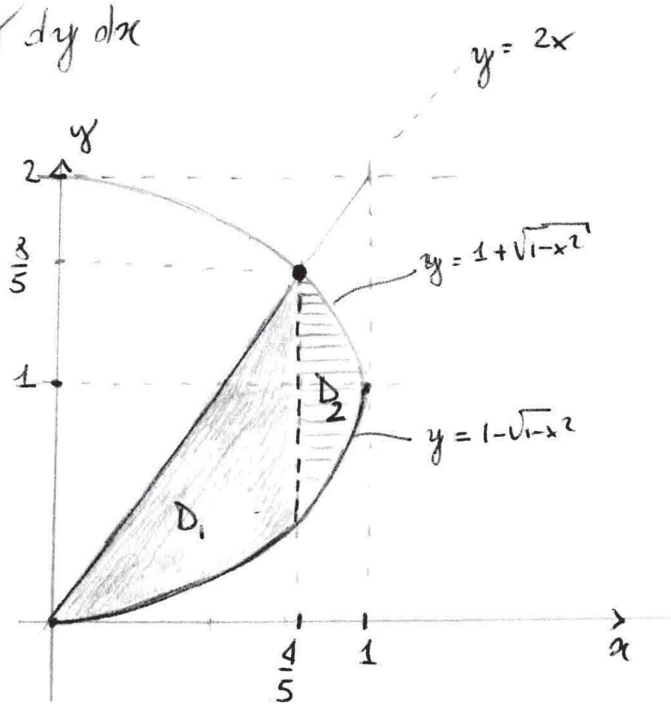
$$= \int_0^{\sqrt{\pi}} y \sin(y^2)\,dy = \int_{f(0)=0}^{f(\sqrt{\pi})=\pi} \frac{\sin(f)}{2}\,df = -\frac{\cos f}{2}\Big|_0^{\pi} = \frac{1+1}{2} = \boxed{1}$$

$y^2 = f$

$2y\,dy = df$

8.9 Considerare la seguente somma di integrali riduttivi come l'integrale di una funzione su un opportuno dominio e calcolarlo:

$$\int_0^{4/5}\int_{1-\sqrt{1-x^2}}^{2x} \frac{x\sin y}{y}\,dy\,dx + \int_{4/5}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} \frac{x\sin y}{y}\,dy\,dx$$



Il primo integrale è sulla regione

$D_1 = \{(x,y) : 0 \le x \le \frac{4}{5},\ 1-\sqrt{1-x^2} \le y \le 2x\}$:

Il secondo è sulla regione:

$D_2 = \{(x,y) : \frac{4}{5} \le x \le 1,\ 1-\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2}\}$:

- $y = 1-\sqrt{1-x^2} \iff (1-y) = \sqrt{1-x^2}$
  è la semicirconferenza "$y<1$", di centro $(0,1)$ e raggio 1 $\left((y-1)^2+x^2=1 \right)$
- $y = 1+\sqrt{1-x^2}$ è la semicirconferenza "$y>1$" di centro $(0,1)$ e raggio 1
- $\begin{cases} y = 1+\sqrt{1-x^2} \\ y=2x \\ y>1 \end{cases} \iff \ldots \quad (2x-1)^2 = 1-x^2 \iff 4x^2-4x+1+x^2-1=0 \quad x(5x-4)=0$
  $\iff x = \frac{4}{5} \ \left(\Rightarrow y = \frac{8}{5}\right)$

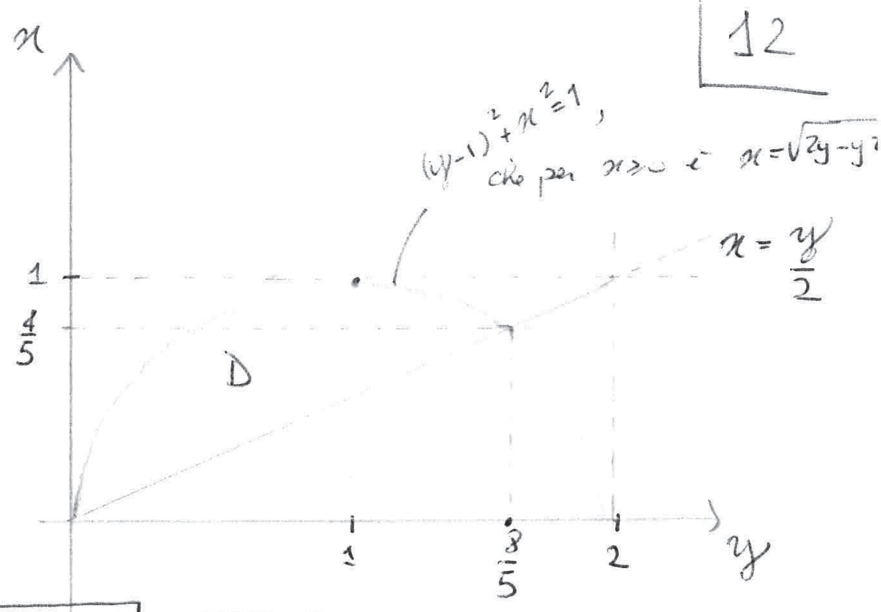
$D = D_1 \cup D_2$ è data da: $\{ y \le 2x,\ x^2+(y-1)^2 \le 1 \}$ intersezione del disco $x^2+(y-1)^2 \le 1$ e del semipiano $y \le 2x$.

Siccome compare $\int \frac{\sin y}{y}$ di cui non sappiamo trovare una formula per l'integrale definito, conviene operare un'inversione.

Riscriviamo il dominio:

Per quanto riguarda la proiezione di D sull'asse y, si ha

$$0 \le y \le \frac{8}{5}$$



siccome poi $x \geq 0$, consideriamo la parte $x \geq 0$ delle circonferenza

$(y-1)^2+x^2=1$, ovvero $x=\sqrt{1-(y-1)^2}=\sqrt{2y-y^2}$.

Dunque $D=\{(x,y)\in\mathbb{R}^2 : 0\le y\le \frac{8}{5},\ \frac{y}{2}\le x\le\sqrt{2y-y^2}\}$.

Calcoliamo

$$\iint_D \frac{x\sin y}{y}\,dy\,dx = \int_0^{\frac{8}{5}}\left(\int_{\frac{y}{2}}^{\sqrt{2y-y^2}}\frac{x\sin y}{y}\,dx\right)dy = \int_0^{\frac{8}{5}}\frac{\sin y}{y}\cdot\left(\frac{x^2}{2}\Big|_{x=\frac{y}{2}}^{x=\sqrt{2y-y^2}}\right)dy =$$

$$= \int_0^{\frac{8}{5}}\frac{\sin y}{y}\left(\frac{2y-y^2-\frac{y^2}{4}}{2}\right)dy = \int_0^{\frac{8}{5}}\frac{\sin y}{y}\left(y-\frac{5}{8}y^2\right)dy = \int_0^{\frac{8}{5}}\left(\sin y-\frac{5}{8}y\sin y\right)dy =$$

$$= -\cos y\Big|_0^{\frac{8}{5}} - \frac{5}{8}\int_0^{\frac{8}{5}} y\sin y\,dy = 1-\cos\left(\frac{8}{5}\right)-\frac{5}{8}\left(-y\cos y\Big|_0^{\frac{8}{5}}\right)-\frac{5}{8}\int_0^{\frac{8}{5}}\cos y\,dy =$$

$$= 1-\cos\left(\frac{8}{5}\right)+\cos\left(\frac{8}{5}\right)-\frac{5}{8}\sin y\Big|_0^{\frac{8}{5}} = \boxed{1-\frac{5}{8}\sin\left(\frac{8}{5}\right)}$$