On the first positive eigenvalue of fourth-order Steklov and Neumann problems

Luigi Provenzano
based on joint works with D. Buoso and L.M. Chasman

Napoli, February 11, 2016

UNIVERSITÀ DEGLI STUDI DI PADOVA

Introduction

Let Ω be a domain in $\mathbb{R}^N$. We consider the Neumann eigenvalue problem problem for the biharmonic operator

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u \cdot \nu\right) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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D. Buoso, L. P.. A few shape optimization results for a biharmonic Steklov problem. *J. Differential Equations (259)*, 1778–1818, 2015.

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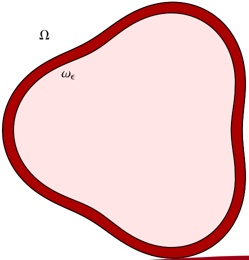

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For each fixed ε we have an increasing sequence

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Theorem (Buoso, P.)

For all $j\in\mathbb{N}$, $\lim_{\varepsilon\to 0}\lambda_j(\varepsilon) = \lambda_j$, where λ_j are the eigenvalues of

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Critical points?

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The space $\Phi(\Omega)$ is a linear space.

Analyticity and derivatives

Let $F \subset \mathbb{N}$ be fixed. We introduce the following quantity

$$\mathcal{A}_\Omega[F] = \left\{\phi \in \Phi(\Omega) : \lambda_l[\phi] \neq \lambda_j[\phi] \quad \forall j \in F,\ \forall l \in \mathbb{N} \setminus F\right\}$$

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Then we consider the symmetric functions of the eigenvalues, for $s \in \{1, ..., |F|\}$

$$\Lambda_{F,s}[\phi] = \sum_{j_1 < \cdots < j_s \in F} \lambda_{j_1}[\phi] \cdots \lambda_{j_s}[\phi]$$

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Such functions turn out to be important objects of study in shape differentiability and shape optimization problems.

Theorem (Buoso, P. - 'Analyticity')

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N}$. Then

i) *The set $\mathcal{A}_\Omega[F]$ is open in $\Phi(\Omega)$.*

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i) *The set $\mathcal{A}_\Omega[F]$ is open in $\Phi(\Omega)$.*

ii) *The function $\Lambda_{F,s}[\phi]$ from $\mathcal{A}_\Omega[F]$ to $\mathbb{R}$ is real analytic.*

Theorem (Buoso, P. - ‘Derivatives, Neumann’)

Let $\tilde{\phi} \in \mathcal{A}_\Omega[F]$ be such that $\lambda_j[\tilde{\phi}] = \lambda_F[\tilde{\phi}]$ for all $j \in F$ and such that $\tilde{\phi}(\Omega)$ is of class C^4. Let $v_1, ..., v_{|F|}$ be a orthonormal basis of the eigenspace associated with $\lambda_F[\tilde{\phi}]$. Then

$$d|_{\phi=\tilde{\phi}}\left(\Lambda_{F,s}\right)[\psi] = -\lambda_F^{s-1}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{j=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Big(\lambda_F[\tilde{\phi}]v_j^2 - \tau|\nabla v_j|^2 - |D^2 v_j|^2\Big)\psi \circ \tilde{\phi}^{(-1)} \cdot \nu d\sigma,$$

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for all $\psi \in \left(C^2(\bar{\Omega})\right)^N$, where K denotes the mean curvature of $\partial\tilde{\phi}(\Omega)$.

Critical domains

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Hence, *balls are critical domains for all simple eigenvalues and for all the symmetric functions of all multiple eigenvalues under measure constraint*.

Can we say more on the critical nature of balls for the Steklov eigenvalues?

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Theorem (Chasman; Buoso, P.)

Among all bounded domains of class C^1 with fixed measure, the ball is the unique maximizer of the first non-negative eigenvalue of problem (1.1) *(Chasman) and of problem* (1.2) *(Buoso, P.), that is*

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*),$$

where Ω^* is a ball with the same measure as Ω

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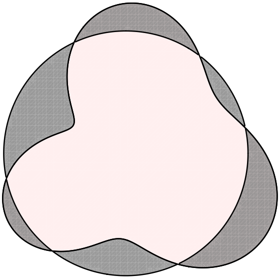

Fraenkel Asymmetry measures the “distance” in the L^1 sense of a generic set from the “family” of balls.

We have the following

Theorem (Buoso, Chasman, P., 'Neumann'; Buoso, P., 'Steklov')

For every domain Ω in $\mathbb{R}^N$ of class C^1 the following estimate holds:

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*)\left(1 - c_N \mathcal{A}(\Omega)^2\right), \tag{2}$$

where c_N is a suitable constant and Ω^* is a ball with the same measure as Ω.

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This is the isoperimetric inequality in quantitative form.

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To do so we shall exhibit a family $\{\Omega_\varepsilon\}$ of sets approaching the unit ball B such that

$$\mathcal{A}(\Omega_\varepsilon) \simeq \frac{|\Omega_\varepsilon \triangle B|}{|\Omega_\varepsilon|} \simeq \varepsilon \quad \text{and} \quad \lambda_2(B) - \lambda_2(\Omega_\varepsilon) \simeq \varepsilon^2 \,, \quad \varepsilon \ll 1.$$

Sharpness of the exponent

We define a family $\{\Omega_\varepsilon\}$ in this way

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^N \;:\; |x| < 1 + \varepsilon\psi(x/|x|) \right\},$$

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where $\psi \in C^\infty(\partial B)$ and satisfies

1. $\int_{\partial B} \psi d\sigma = 0$;
2. $\int_{\partial B} (a \cdot x)\psi d\sigma = 0$ for all $a \in \mathbb{R}^N$;
3. $\int_{\partial B} (a \cdot x)^2 \psi d\sigma = 0$ for all $a \in \mathbb{R}^N$.

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This family of sets is such that $\mathcal{A}(\Omega_\varepsilon) \simeq \varepsilon$ and $\lambda_2(B) - \lambda_2(\Omega_\varepsilon) \simeq \varepsilon^2$, proving that the exponent 2 is sharp.

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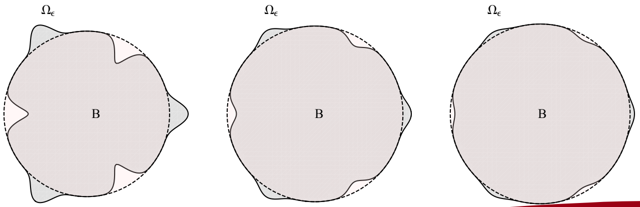

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