On the spectral behavior of a biharmonic Steklov problem

Luigi Provenzano

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Introduction and motivations

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Let Ω be a domain in $\mathbb{R}^N$. We start by recalling the classical Steklov eigenvalue problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega. \end{cases}$$

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If Ω is a bounded domain with Lipschitz boundary, then the spectrum is discrete and the eigenvalues form a sequence

$$0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots \nearrow +\infty.$$

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- Steklov eigenvalues can be interpreted as the eigenvalues of the Dirichlet to Neumann operator $\mathcal{D} : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$, which is relevant in applications (e.g., in medical and geophysical imaging).

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- Shape optimization 2: Weinstock's inequality states that "*among all simply connected planar domains with fixed perimeter,* λ_2 *is maximized by a disk*". The general case is an open problem.

Remark

The Steklov problem can be formulated in a more general setting:

- Ω *is a compact Riemannian manifold of dimension* $N \geq 2$ *with boundary* $\partial\Omega$*;*
- Δ *is the Laplace-Beltrami operator on functions on* Ω*;*
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A. Girouard and I. Polterovich. Spectral geometry of the Steklov problem. *arXiv:1411.6567*, 2014.

I. Chavel. Eigenvalues in Riemannian geometry. *Academic Press, INC.*, 1984.

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What about Steklov boundary conditions for the biharmonic operator?

In the recent literature, the following Steklov problem for the biharmonic operator has been discussed

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- The first eigenvalue is the *sharp constant* of a priori estimates for solutions of the Laplace equation.
- If Ω is the unit ball in $\mathbb{R}^N$ and $\lambda = N - 1$, we have the hinged plate problem for the associated Poisson problem $\Delta^2 u = f$.
- Shape optimization: the behavior is completely different from the second order case and not well understood.

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- Consider shape optimization problems, in particular, the problem of characterizing the optimal sets (maximizers or minimizers) for the first positive eigenvalue (usually called the fundamental tone).
- Consider the issue of the stability of the optimal shapes and quantify it.

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Let Ω be a bounded domain of class C^1. Let $\tau \geq 0$ be a fixed constant and $\rho : \Omega \to \mathbb{R}^+$ a positive function. We consider the Neumann problem

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda \rho u, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = \tau \frac{\partial u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u \cdot \nu\right) - \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

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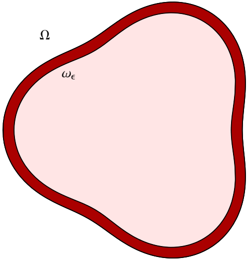

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Theorem

For all $j \in \mathbb{N}$, $\lim_{\varepsilon\to 0}\lambda_j(\varepsilon) = \lambda_j$, where λ_j are the eigenvalues of

$$\begin{cases} \Delta^2 u - \tau\Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial\nu^2} = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial u}{\partial\nu} - \mathrm{div}_{\partial\Omega}\left(D^2 u\cdot\nu\right) - \frac{\partial\Delta u}{\partial\nu} = \frac{M}{|\partial\Omega|}\lambda u, & \text{on } \partial\Omega. \end{cases}$$

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and denote by $\lambda_j(\varepsilon)$, λ_j the corresponding sequences of eigenvalues, then for all $j \in \mathbb{N}$,

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The proof of the convergence of the eigenvalues consists in showing that the resolvent operators of the Neumann problems with ρ_ε compactly converge to the resolvent operator of the limiting problem.

A biharmonic Steklov problem

From now on we consider $\tau > 0$. The biharmonic Steklov problem reads

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The space $\Phi(\Omega)$ is a linear space, so we can make Differential Calculus on it.

Let $F \subset \mathbb{N}$ be fixed. We introduce the following quantity

$$\mathcal{A}_\Omega[F] = \left\{\phi \in \Phi(\Omega) : \lambda_l[\phi] \neq \lambda_j[\phi] \quad \forall j \in F,\ \forall l \in \mathbb{N} \setminus F\right\}$$

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Such functions turn out to be important objects of study in shape optimization problems.

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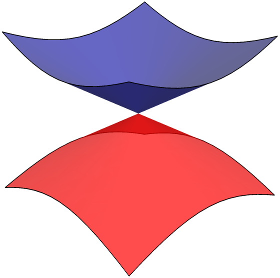

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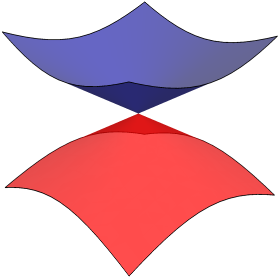

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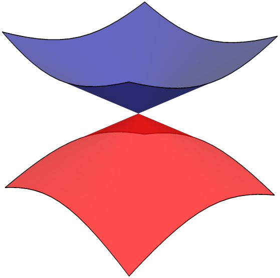

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Symmetric functions of $\lambda_1[\alpha_1, \alpha_2], \lambda_2[\alpha_1, \alpha_2]$ are even analytic.

Theorem (Analyticity)

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a finite non-empty subset of $\mathbb{N}$. Then

i) *The set $\mathcal{A}_\Omega[F]$ is open in $\Phi(\Omega)$.*

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ii) *The function $\Lambda_{F,s}[\phi]$ from $\mathcal{A}_\Omega[F]$ to $\mathbb{R}$ is real analytic.*

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Theorem (Derivatives)

Let $\tilde{\phi} \in \mathcal{A}_\Omega[F]$ be such that $\lambda_j[\tilde{\phi}] = \lambda_F[\tilde{\phi}]$ for all $j \in F$ and such that $\tilde{\phi}(\Omega)$ is of class C^4. Let $v_1, ..., v_{|F|}$ be a orthonormal basis of the eigenspace associated with $\lambda_F[\tilde{\phi}]$. Then

$$d|_{\phi=\tilde{\phi}}\left(\Lambda_{F,s}\right)[\psi] = -\lambda_F^{s-1}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{j=1}^{|F|}\int_{\partial\tilde{\phi}(\Omega)}\Big(\lambda_F[\tilde{\phi}]Kv_j^2 + \lambda_F[\tilde{\phi}]\frac{\partial(v_j^2)}{\partial\nu} - \tau|\nabla v_j|^2 - |D^2 v_j|^2\Big)\psi\circ\tilde{\phi}^{(-1)}\cdot\nu d\sigma,$$

for all $\psi \in \left(C^2(\bar{\Omega})\right)^N$, where K denotes the mean curvature of $\partial\tilde{\phi}(\Omega)$.

Critical domains

Now we turn our attention to extremum problems of the type

$$\min_{\mathcal{V}(\phi)=\text{const.}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{\mathcal{V}(\phi)=\text{const.}} \Lambda_{F,s}[\phi],$$

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Hence, *balls are critical domains for all simple eigenvalues and for all the symmetric functions of all multiple eigenvalues under measure constraint*.

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- (Classical) isoperimetric inequality for weighted perimeters.

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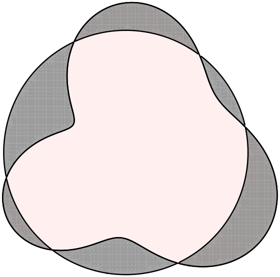

Fraenkel Asymmetry measures the "distance" in the L^1 sense of a generic set from the "family" of balls.

We have the following

Theorem

For every domain Ω in $\mathbb{R}^N$ of class C^1 the following estimate holds:

$$\lambda_2(\Omega) \leq \lambda_2(\Omega^*)\left(1 - c_N \mathcal{A}(\Omega)^2\right), \tag{2}$$

where c_N is a suitable constant and Ω^* is a ball with the same measure as Ω.

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This is the isoperimetric inequality in quantitative form.

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To do so we shall exhibit a family $\{\Omega_\varepsilon\}$ of sets approaching the unit ball B such that

$$\mathcal{A}(\Omega_\varepsilon) \simeq \frac{|\Omega_\varepsilon \triangle B|}{|\Omega_\varepsilon|} \simeq \varepsilon \quad \text{and} \quad \lambda_2(B) - \lambda_2(\Omega_\varepsilon) \simeq \varepsilon^2 , \quad \varepsilon \ll 1.$$

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N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2) 168* (2008), no. 3, 941–980.

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$$\mu_1(\Omega) \geq \mu_1(\Omega^*),$$

where Ω^* is a ball with the same measure as Ω. Also in this case we have a quantitative version of the isoperimetric inequality

$$\mu_1(\Omega) \geq \mu_1(\Omega^*)\left(1 + c_N \mathcal{A}(\Omega)^2\right).$$

L. Brasco, G. De Philippis, and B. Velichkov. Faber-Krahn inequalities in sharp quantitative form. *Duke Math. J. 164* (2015), no. 9, 1777–1831.

Sharpness of the exponent

In both the case of the classical isoperimetric inequality and of the Faber-Krahn inequality, nearly spherical ellipsoids converge with the sharp convergence rate 2

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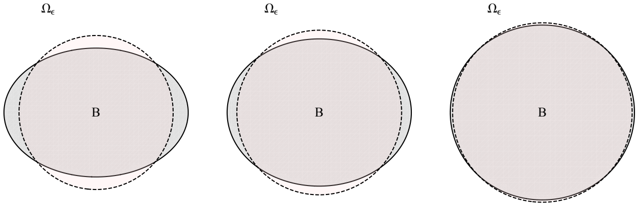

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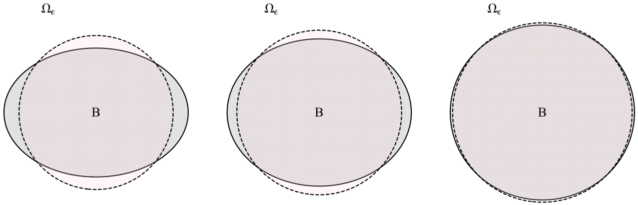

This means that $\mathcal{A}(\Omega_\varepsilon) \simeq \frac{|\Omega_\varepsilon \triangle B|}{|\Omega_\varepsilon|} \simeq \varepsilon$ and moreover

$$P(\Omega_\varepsilon) - P(B) \simeq \varepsilon^2 \quad \text{and} \quad \mu_1(\Omega_\varepsilon) - \mu_1(B) \simeq \varepsilon^2 \quad \varepsilon \ll 1.$$

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In the case of the first positive eigenvalue of our biharmonic Steklov problem, we have that

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The answer relies not only on the geometry of the critical set (the ball), but also on the features of the problem itself (that is, on the properties of the eigenfunctions).

Some observations:

- An eigenvalue does not have a straightforward geometrical meaning like in the case of the perimeter, for example. Thus, it is not straightforward to understand how deformations of an optimal shape affect the eigenvalues.

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- If an eigenvalue is "shape differentiable" (e.g., if it is always simple as in the case of $\mu_1(\Omega)$), then its "shape derivative" at the "extremal point" would be zero (this is rather heuristic).

- In the case of $\mu_1(\Omega)$ (first Dirichlet eigenvalue), any perturbation $\Omega_\varepsilon = (\mathrm{Id} + \varepsilon V)B$, for some smooth vector field V, should provide an expansion of the form

$$\mu_1(\Omega_\varepsilon) \simeq \mu_1(B) + O(\varepsilon^2)\,, \quad \varepsilon \ll 1.$$

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To prove that the exponent is sharp, we have to exclude that this happens for every “direction”.

We define a family $\{\Omega_\varepsilon\}$ in this way

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^N \; : \; |x| < 1 + \varepsilon\psi(x/|x|) \right\},$$

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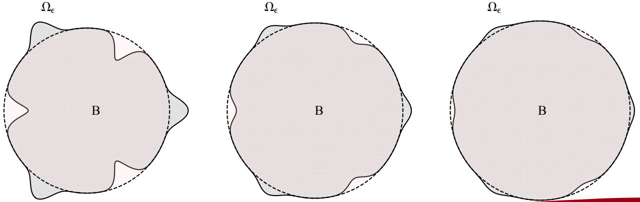

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$$|\xi|^2_{|\partial B} = (a \cdot x)^2 \quad \text{and} \quad |D^2\xi|^2 + \tau |D\xi|^2_{|\partial B} = (b \cdot x)^2,$$

for some $a, b \in \mathbb{R}^N$ and such relations are crucial in proving $\lambda_2(B) - \lambda_2(\Omega_\varepsilon) \simeq \varepsilon^2$ for $\varepsilon \ll 1$.

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Conditions 1 and 2 (which are the purely geometrical assumptions for the approximation of the domains) are satisfied also by the nearly spherical ellipsoids.

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Along the "ellipsoid" direction, $\lambda_2(\Omega)$ *has a non-trivial super-differential*

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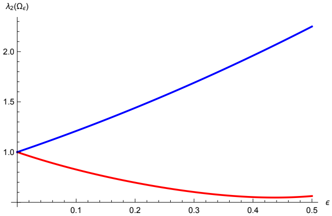
First two eigenvalues of nearly spherical ellipsoids Ω_ε and $\tau = 1$.



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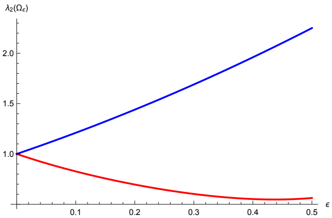
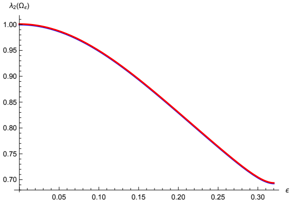


First two (multiple) eigenvalues of the "flower domains" Ω_ε and $\tau = 1$.



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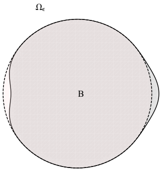

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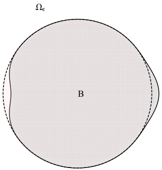

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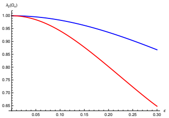

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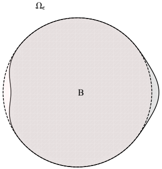

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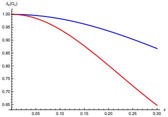

Note that also in this case the eigenvalue is not differentiable at the “maximum point” but converges with the sharp exponent 2.

Essential bibliography

- C. Bandle,
  Isoperimetric inequalities and applications,
  Pitman advanced publishing program, monographs and studies in mathematics, vol. 7, 1980.
- L. Brasco, G. De Philippis, and B. Ruffini,
  Spectral optimization for the Stekloff-Laplacian: the stability issue,
  J. Funct. Anal., 262(11):4675–4710, 2012.
- L. Brasco, G. De Philippis, and B. Velichkov,
  Faber-Krahn inequalities in sharp quantitative form,
  Duke Math. J., 164(9):1777–1831, 2015.
- L. Brasco and A. Pratelli,
  Sharp stability of some spectral inequalities,
  Geom. Funct. Anal., 22(1):107–135, 2012.

Essential bibliography

D. Bucur, A. Ferrero, and F. Gazzola,,
On the first eigenvalue of a fourth order Steklov problem,
Calc. Var. Partial Differential Equations, 35:103–131, 2009.

D. Buoso, L. M. Chasman, and L. Provenzano,
On the stability of some isoperimetric inequalities for the fundamental tones of free plates,
Preprint, 2015

D. Buoso and P.D. Lamberti,
Eigenvalues of polyharmonic operators on variable domains,
ESAIM Control Optim. Calc. Var., 19:1225–1235, 2013.

D. Buoso and L. Provenzano,
A few shape optimization results for a biharmonic Steklov problem,
J. Differential Equations, 259(5):1778–1818, 2015.

Essential bibliography

- L. M. Chasman,
  An isoperimetric inequality for the fundamental tones of free plates,
  Comm. Math. Phys., 303(2):421–449, 2011.
- F. Gazzola, H.-C. Grunau, and G. Sweers,
  Polyharmonic Boundary Value Problems. Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains
  Lecture Notes in Math., Springer-Verlag, Berlin, 2010.
- A. Girouard and I. Polterovich,
  Spectral geometry of the Steklov problem,
  arXiv:1411.6567, 2014
- A. Henrot,
  Extremum Problems for Eigenvalues of Elliptic Operators
  Frontiers in Math., Birkhäuser Verlag, Basel, 2006.

THANK YOU