On the eigenvalues of Steklov-type problems

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Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Consider the classical **Steklov eigenvalue problem**

$$\begin{cases}\Delta u = 0, & \text{in } \Omega\\ \frac{\partial u}{\partial \nu} = \lambda u, & \text{on } \partial\Omega,\end{cases} \tag{1}$$

in the unknowns λ, u. For $N=2$ this problem models the transverse vibrations of a free thin elastic membrane the mass of which is concentrated at the boundary.

This concentration phenomenon can be described as follows (see [6]). Let $\varepsilon \in]0, \varepsilon_0[$ and consider the following **Neumann eigenvalue problem**

$$\begin{cases}-\Delta u = \lambda \rho_\varepsilon u, & \text{in } \Omega\\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega.\end{cases} \tag{2}$$

For $N=2$ this problems models the transverse vibrations of a free membrane the mass of which is displaced on the whole of Ω with density

$$\rho_\varepsilon = \begin{cases}\varepsilon, & \text{in } \Omega_\varepsilon\\ \frac{\mathcal{H}^{N-1}(\partial\Omega) - \varepsilon|\Omega_\varepsilon|}{|\Omega\setminus\overline{\Omega}_\varepsilon|}, & \text{in } \Omega\setminus\overline{\Omega}_\varepsilon,\end{cases}$$

where $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. Note that the mass is concentrated in a **neighborhood of $\partial\Omega$**.

Problems (1) and (2) admit an increasing sequence of non-negative eigenvalues of finite multiplicity diverging to $+\infty$, respectively

$$0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots$$
$$0 = \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \le \cdots \le \lambda_j(\varepsilon) \le \cdots.$$

Let T_0, T_ε be the resolvent operators associated with problems (1) and (2) respectively.

Theorem. $T_\varepsilon \to T_0$ as $\varepsilon \to 0$, the convergence being in norm. In particular, for all $j \in \mathbb{N}$, $\lambda_j(\varepsilon) \to \lambda_j$ as $\varepsilon \to 0$.

The biharmonic operator

Consider the classical **biharmonic Neumann problem** (see [5]) with tension parameter $\tau > 0$ and mass density ρ_ε, namely

$$\begin{cases}\Delta^2 u - \tau\Delta u = \lambda\rho_\varepsilon u, & \text{in } \Omega\\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\\ \tau\frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega}(D^2u.\nu) - \frac{\partial\Delta u}{\partial\nu} = 0, & \text{on } \partial\Omega.\end{cases} \tag{3}$$

This problem models the transverse vibrations of a free thin **plate** with density ρ_ε. The spectrum is made up of eigenvalues of finite multiplicity increasing to $+\infty$

$$0 = \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \le \cdots \le \lambda_j(\varepsilon) \cdots.$$

We perform the **same analysis** of the second order case. The eigenvalues of (3) converge to those of an appropriate **limiting problem**.

We refer to the limiting problem obtained in this way as to the **biharmonic Steklov problem**.

The Bh Steklov problem

The classical formulation of this **new fourth-order Steklov problem** is: find λ, u such that

$$\begin{cases}\Delta^2 u - \tau\Delta u = 0, & \text{in } \Omega\\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\\ \tau\frac{\partial u}{\partial \nu} - \operatorname{div}_{\partial\Omega}(D^2u.\nu) - \frac{\partial\Delta u}{\partial\nu} = \lambda u, & \text{on } \partial\Omega.\end{cases} \tag{4}$$

We aim at:

- studying the dependence $\Omega \mapsto \lambda[\Omega]$;
- characterizing the **critical domains** for this map under the constraint $|\Omega|$ **fixed**;
- finding **global maxima** for this map under the constraint $|\Omega|$ **fixed**.

Shape derivatives

Fix Ω of class C^1 and consider the family of **perturbations**

$$\Phi(\Omega) := \left\{\phi \in \left(C^2(\overline{\Omega})\right)^N, \phi \text{ inj.}, \inf_{\Omega}|\det D\phi| > 0\right\}.$$

We want to study the map $\phi \mapsto \lambda_j[\phi] := \lambda_j[\phi(\Omega)]$. To avoid the occurrence of bifurcation phenomena, we consider the symmetric functions of the eigenvalues (see e.g., [2]). We have the following (see [3, 4])

Theorem. Let $F \subset \mathbb{N}$ finite and non-empty. Let

$$\mathcal{A}_\Omega[F] := \{\phi \in \Phi(\Omega) : \lambda_l[\phi] \notin \{\lambda_j[\phi] : j \in F\},\ \forall l \in \mathbb{N}\setminus F\}.$$

The set $\mathcal{A}_\Omega[F]$ is open in $\Phi(\Omega)$ and the maps from $\mathcal{A}_\Omega[F]$ to $\mathbb{R}$ defined by

$$\Lambda_{F,s}[\phi] := \sum_{j_1<\cdots<j_s \in F} \lambda_{j_1}[\phi]\cdots\lambda_{j_s}[\phi],$$

for $s \in \{1, ...|F|\}$ are **real analytic**. Moreover, let $\tilde\phi \in \mathcal{A}_\Omega[\phi]$ be such that all the eigenvalues with indexes in F have a common value $\lambda_F[\tilde\phi]$ and $\partial\tilde\phi(\Omega) \in C^4$. Then, for all $\psi \in \left(C^2(\overline{\Omega})\right)^N$,

$$d_{|_{\phi=\tilde\phi}}\Lambda_{F,s}[\psi] = -\lambda_F^s[\tilde\phi]\binom{|F|-1}{s-1} \cdot \sum_{l=1}^{|F|}\int_{\partial\tilde\phi(\Omega)}\left(\lambda_F[\tilde\phi]\kappa v_l^2 + \lambda_F[\tilde\phi]\frac{\partial(v_l^2)}{\partial\nu} - \tau|\nabla v_l|^2 - |D^2v_l|^2\right)\left(\psi\circ\tilde\phi^{(-1)}\right)\cdot\nu d\sigma, \tag{5}$$

where κ is the mean curvature on $\partial\tilde\phi(\Omega)$ and $\{v_1, ..., v_l\}$ is a orthonormal basis of the eigenspace associated with $\lambda_F[\tilde\phi]$.

Critical points

We consider now volume preserving perturbations, i.e., $|\phi(\Omega)| = |\Omega|$. Using (5) and the Lagrange Multipliers Theorem, we can prove the following (see [3, 4])

Theorem. Under the assumptions of the previous theorems, $\tilde\phi$ is a critical point for $\Lambda_{F,s}$ if and only if

$$\sum_{l=1}^{|F|}\left(\lambda_F[\tilde\phi]\left(\kappa v_l^2 + \frac{\partial(v_l^2)}{\partial\nu}\right) - \tau|\nabla v_l|^2 + |D^2v_l|^2\right)$$

is constant on $\partial\tilde\phi(\Omega)$.

It is known that **balls** play an important role in the frame of shape optimization for the eigenvalues of the Laplacian and biharmonic operator. We have the following result for problem (4) (see [3, 4]).

Theorem. Let $\tilde\phi$ such that $\tilde\phi(\Omega)$ is a ball. Then $\tilde\phi$ is a critical point for $\Lambda_{F,s}$ for any $s = 1, ..., |F|$.
Remark. The analogue of the previous result holds for problem (1). Moreover it is known that for problem (1) the ball is the **unique maximizer** for the first positive eigenvalue.

Isoperimetric inequality

The first positive eigenvalue of the Laplacian or the biharmonic operator is called the **fundamental tone**. In the case of problem (4) the fundamental tone is $\lambda_2(\Omega)$.

We introduce the notion of **Fraenkel asymmetry** $\mathcal{F}(\Omega)$

$$\mathcal{F}(\Omega) := \inf_{\substack{B \text{ ball},\\ |B|=|\Omega|}} \frac{\|\chi_\Omega - \chi_B\|_{L^1(\mathbb{R}^N)}}{|\Omega|}.$$

Note that $\mathcal{F}(\Omega)$ is the distance in the $L^1(\mathbb{R}^N)$ of Ω from the set of all balls of measure $|\Omega|$.

It turns out that the ball is the **unique maximizer** for $\lambda_2(\Omega)$ (see [3, 4]; cfr [1]).

Theorem. For every bounded domain Ω in $\mathbb{R}^N$ of class C^1 it holds

$$\lambda_2(\Omega) \le \lambda_2(\Omega^*)\left(1 - \delta_N\mathcal{F}(\Omega)^2\right),$$

where δ_N is a dimensional constant and Ω^* is the ball with the same measure as Ω. In particular,

$$\lambda_2(\Omega) \le \lambda_2(\Omega^*),$$

with equality if and only if Ω is a ball.

References

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