Neumann to Steklov eigenvalues: an asymptotic analysis

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UNIVERSITÀ DEGLI STUDI DI PADOVA

The Neumann problem

Let Ω be a bounded domain in $\mathbb{R}^2$ of class C^2 and $M > 0$ be fixed. We consider

$$\begin{cases} -\Delta u_\varepsilon = \lambda(\varepsilon)\rho_\varepsilon u_\varepsilon & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

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where

$$\rho_\varepsilon = \begin{cases} \varepsilon & \text{in } \Omega \setminus \overline{\omega}_\varepsilon, \\ \frac{M - \varepsilon|\Omega \setminus \overline{\omega}_\varepsilon|}{|\omega_\varepsilon|} & \text{in } \omega_\varepsilon \end{cases} \quad \text{and} \quad \omega_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}.$$

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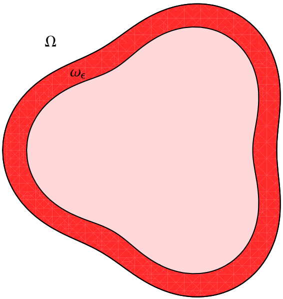
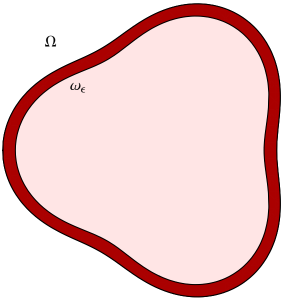
For all $\varepsilon > 0$

$$0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \cdots \leq \lambda_j(\varepsilon) \leq \cdots.$$

The Neumann problem





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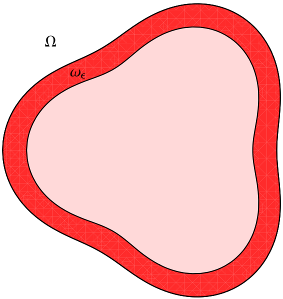

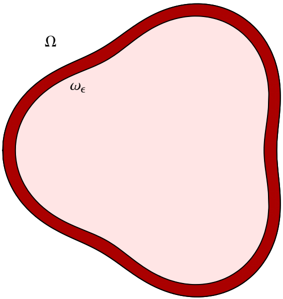

$$\int_\Omega \rho_\varepsilon = M \ \ \forall \varepsilon > 0.$$

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Consider the Steklov eigenvalue problem on Ω

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Spectrum

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_j \leq \cdots .$$

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Theorem

For all $j \in \mathbb{N}$,

$$\lim_{\varepsilon \to 0} \lambda_j(\varepsilon) = \mu_j.$$

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Answers via asymptotic analysis

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Answers via asymptotic analysis for simple eigenvalues

Asymptotic expansions

Let μ be a simple Steklov eigenvalue, $\lambda(\varepsilon)$ for all $\varepsilon > 0$ small enough, be a simple Neumann eigenvalue such that $\lambda(\varepsilon) \to \mu$ as $\varepsilon \to 0$.

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- $v_\varepsilon, v_\varepsilon^1$ depend on ε explicitly and are supported on ω_ε.
- u^1, μ^1 solve an auxiliary boundary value problem which does not depend on ε.
- μ^1 is the *topological derivative* of $\lambda(\varepsilon)$ at $\varepsilon = 0$.

Asymptotic expansions

Strategy:

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- Justifying the expansions up to the desired order

Postulating the expansions

Main tools:

- The map $\psi_\varepsilon : [0, |\partial\Omega|) \times (0, 1) \to \omega_\varepsilon$

$$\psi_\varepsilon(s, \xi) = \gamma(s) - \varepsilon\xi\nu(\gamma(s)),$$

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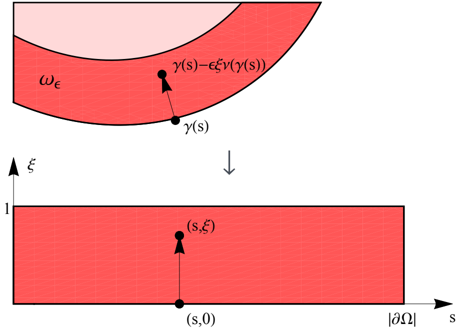

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- Laplacian in coordinates (s, ξ)

$$\Delta = \frac{1}{\varepsilon^2}\partial_\xi^2 - \frac{1}{\varepsilon}\kappa(s)\partial_\xi - \kappa(s)^2\xi\partial_\xi + \partial_s^2 + \cdots$$

Postulating the expansions

In the strip ω_ε:

- Expansion of u:

$$(u \circ \psi_\varepsilon)(s,\xi) = (u \circ \psi_\varepsilon)(s,0) - \varepsilon\xi((\partial_\nu u) \circ \psi_\varepsilon)(s,0) + O(\varepsilon^2).$$

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$$(u^1 \circ \psi_\varepsilon)(s,\xi) = (u^1 \circ \psi_\varepsilon)(s,0) + O(\varepsilon)$$

- We look for $v_\varepsilon, v^1_\varepsilon$ supported on ω_ε of the form

$$w = v_\varepsilon \circ \psi_\varepsilon\,, \quad w^1 = v^1_\varepsilon \circ \psi_\varepsilon,$$

where $w(s,\xi), w^1(s,\xi)$ are functions on $[0,|\partial\Omega|) \times (0,1)$.

Postulating the expansions

Plug the asymptotic expansions for u_ε and $\lambda(\varepsilon)$ in the equation

$$-\Delta(u+\varepsilon u^1+\varepsilon v_\varepsilon+\varepsilon^2 v_\varepsilon^1) = \left(\varepsilon + \frac{1}{\varepsilon}\tilde{\rho}_\varepsilon \chi_{\omega_\varepsilon}\right)(\mu+\varepsilon\mu^1)(u+\varepsilon u^1+\varepsilon v_\varepsilon+\varepsilon^2 v_\varepsilon^1).$$

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We obtain four problems, for $u, \mu, u^1, \mu^1, w, w^1$.

Postulating the expansions

Problems in Ω:

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- $$\begin{cases} -\Delta u^1 = \mu u & \text{in } \Omega, \\ \frac{\partial u^1}{\partial \nu} = \left(\frac{M\mu}{2|\partial\Omega|^2}(K - |\partial\Omega|\kappa) - \frac{2M^2\mu^2}{3|\partial\Omega|^2} + \frac{M\mu^1}{|\partial\Omega| - \frac{\mu|\Omega|}{|\partial\Omega|}} \right) u + \frac{M\mu}{|\partial\Omega|} u^1 & \text{on } \partial\Omega. \end{cases}$$

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Problems in $[0, |\partial\Omega|) \times (0, 1)$:

- For each $s \in [0, |\partial\Omega|)$, $w(s, \xi), w^1(s, \xi)$ solve explicit second order ODEs in the variable $\xi \in (0, 1)$ with suitable initial conditions.

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- Once we know u, u^1, the solutions w, w^1 and therefore $v_\varepsilon, v_\varepsilon^1$ are explicitly determined.

Justifying the expansions

Main tool:

Lemma (Oleinik's Lemma)

Let $A : H \to H$ be a linear, self-adjoint, positive and compact operator from a separable Hilbert space H to itself. Let $V \in H$ with $\|V\|_H = 1$. Let $\eta, r > 0$ be such that $\|AV - \eta V\|_H \leq r$. Then there exists an eigenvalue η_i of the operator A which satisfy the inequality $|\eta - \eta_i| \leq r$. Moreover, for any $r^* > r$ there exist $V^* \in H$ with $\|V^*\|_H = 1$, V^* belonging to the space generated by all the eigenspaces associated with an eigenvalue of the operator A lying on the segment $[\eta - r^*, \eta + r^*]$ and such that

$$\|V - V^*\|_H \leq \frac{2r}{r^*}.$$

Justifying the expansions

Setting:

- Hilbert space $\mathcal{H}_\varepsilon(\Omega)$ of $H^1(\Omega)$ functions and scalar product

$$\langle u, v\rangle_\varepsilon := \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega \rho_\varepsilon u v dx\,,\ \forall u, v \in \mathcal{H}_\varepsilon(\Omega);$$

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- The operator $\mathcal{A}_\varepsilon$ from $\mathcal{H}_\varepsilon(\Omega)$ to itself defined by

$$\mathcal{A}_\varepsilon f = u \iff \int_\Omega \nabla u \cdot \nabla \varphi dx + \int_\Omega \rho_\varepsilon u \varphi dx = \int_\Omega \rho_\varepsilon f \varphi dx \,, \ \forall \varphi \in \mathcal{H}_\varepsilon(\Omega).$$

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$\lambda(\varepsilon)$ Neumann eigenvalue $\iff \frac{1}{1+\lambda(\varepsilon)}$ eigenvalue of $\mathcal{A}_\varepsilon$.

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- u, u^1 and u_ε normalized in a suitable way. In particular: $\int_{\partial\Omega} u^2 d\sigma = 1$, $\int_\Omega \rho_\varepsilon u_\varepsilon^2 = \frac{M}{|\partial\Omega|}$.

Justifying the expansions

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The expansions are correct up to the first order terms.

Topological derivative

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Consider the problem for u^1, μ^1

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\begin{cases}
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$$\downarrow$$

$$\mu^1 = \frac{\mu}{M}\left(|\Omega| - |\partial\Omega| \int_\Omega u^2 dx\right) + \frac{2M\mu^2}{3|\partial\Omega|} + \frac{\mu}{2|\partial\Omega|} \int_{\partial\Omega} \left(|\partial\Omega| u^2 - 1\right) \kappa d\sigma.$$

The ball

Which is the sign of this derivative? Case of the unit ball in $\mathbb{R}^2$

The ball

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Asymptotic expansion of Neumann eigenvalues:

$$\begin{aligned}\lambda_{2j-1}(\varepsilon) &= \mu_{2j-1} + \left(\frac{2j\mu_{2j-1}}{3} + \frac{{\mu_{2j-1}}^2}{2(j+1)}\right)\varepsilon + O(\varepsilon^2) \\ &= \frac{2\pi j}{M} + \frac{2\pi j^2}{M}\left(\frac{2}{3} + \frac{\pi}{M(1+j)}\right)\varepsilon + O(\varepsilon^2), \text{ as } \varepsilon \to 0.\end{aligned}$$

The ball

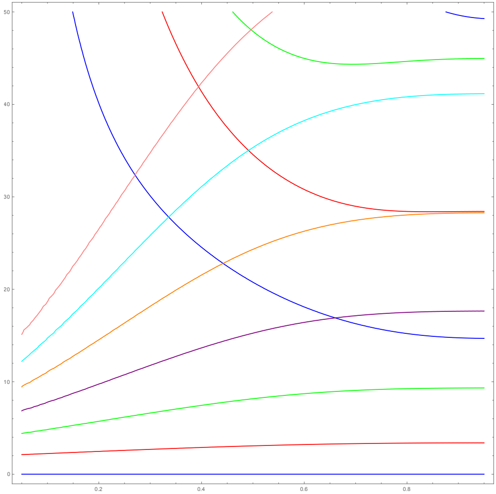

Figure: $\lambda_{2j-1} = \lambda_{2j}$ with $M = \pi$ in the range $(\varepsilon, \lambda) \in (0, 1) \times (0, 50)$.

To be done

Remark: for the ball in $\mathbb{R}^N$, $N \geq 2$, it is possible to obtain the derivative μ^1 explicitly by means of another technique (write the solutions in terms of ultraspherical Bessel functions, find implicit equation for the eigenvalues, use suitable Taylor's expansions, estimates for the remainders,...)

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- Is μ^1 always positive?
- What about a generic Ω in $\mathbb{R}^N$?

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THANK YOU

Postulating the expansions

Problems in $[0, |\partial\Omega|) \times (0, 1)$:

- $$\begin{cases} -\partial_\xi^2 w(s,\xi) = \frac{M\mu}{|\partial\Omega|}(u \circ \psi_\varepsilon)(s,0) & \text{on } [0, |\partial\Omega|) \times (0,1), \\ \partial_\xi w(s,0) = \frac{M\mu}{|\partial\Omega|}(u \circ \psi_\varepsilon)(s,0) & s \in [0, |\partial\Omega|), \\ \partial_\xi w(s,1) = w(s,1) = 0 & s \in [0, |\partial\Omega|); \end{cases}$$

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Once we know u, u^1, the solutions w, w^1 and therefore $v_\varepsilon, v_\varepsilon^1$ are explicitly determined.