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Eigenvalues of harmonic and poly-harmonic operators subject to mass density perturbations

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INTRODUCTION

In this dissertation we consider different classes of eigenvalue problems for poly-harmonic operators subject to homogeneous boundary conditions on open sets in $\mathbb{R}^N$. First of all we consider the eigenvalue problem for the poly-harmonic operator with Dirichlet boundary conditions

$$\begin{cases} (-\Delta)^n u = \lambda \rho u, & \text{in } \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{n-1} u}{\partial \nu^{n-1}} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{0.0.1}$$

where Ω is a domain (i.e., a connected open set) of finite measure in $\mathbb{R}^N$, $\rho \in L^\infty(\Omega)$. Along with the Dirichlet case, we consider a class of eigenvalue problems for poly-harmonic operators subject to intermediate boundary conditions

$$\begin{cases} (-\Delta)^n u = \lambda \rho u, & \text{in } \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0, & \text{on } \partial\Omega\,, \\ B_m(x;D)u = B_{m+1}(x;D)u = \dots = B_{n-1}(x;D)u = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{0.0.2}$$

with $\frac{n}{2} \leq m \leq n$ if n is even, $\frac{n+1}{2} \leq m \leq n$ if n is odd, where $B_j(x;D)$, $j = m, m+1, \dots, n-1$, are suitable linear differential operators of order m_j (the conditions $B_j(x;D)u = 0$, for $j = m, \dots, n-1$, are called complementing conditions). The limiting case $m = n$ corresponds to the case of Dirichlet boundary conditions (0.0.1). From a physical point of view, one may think of the open set Ω as a vibrating N-dimensional membrane with mass density ρ and total mass $M = \int_\Omega \rho dx$. The cases $n = 1$ and $n = 2$ model concrete problems of physical interest. For $n = 1$ problem (0.0.1) is reduced to the problem of the Laplace operator with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{0.0.3}$$

which arises in the study of a vibrating membrane of mass density ρ with a fixed frame. For $n = 2$ we have two problems for the biharmonic operator:

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{0.0.4}$$

and

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega\,, \\ u = \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\,. \end{cases} \tag{0.0.5}$$

Problem (0.0.4) models a vibrating thin clamped plate with mass density ρ, while problem (0.0.5) models a vibrating simply supported thin plate with mass density ρ. For a detailed discussion of the physical interpretation of problems

(0.0.3),(0.0.4) and (0.0.5) we refer to [7].

As is well-known the eigenvalues λ of problem (0.0.2) have finite multiplicity and form a strictly positive and increasing sequence $\{\lambda_j\}_{j\in\mathbb{N}}$, with $\lambda_j \to +\infty$ as $j \to \infty$.

We are interested in the problem of the dependence of λ_j on ρ. More precisely, we investigate the analyticity of the dependence of the eigenvalues, or their functions, upon variations of the density ρ in a suitable subspace of $L^\infty(\Omega)$. We consider densities ρ in the open subset of $L^\infty(\Omega)$ of those functions ρ such that $\operatorname{ess\,inf}_\Omega \rho > 0$. Then we show that the symmetric functions of multiple eigenvalues depend real analytically on ρ and we compute formulas for the derivatives of such functions. Note that, in general, multiple eigenvalues are even not differentiable with respect to ρ.

Then we consider the problem of maximizing or minimizing the eigenvalues with respect to the variable ρ under the constraint $\int_\Omega \rho dx = \text{const}$. There are some results in this direction, see e.g., [16, 8, 9]. The case $n = 1, N = 1$ has been completely solved in [18, Krein], under the assumption that the admissible densities satisfy the condition

$$\alpha \leq \rho \leq \beta, \tag{0.0.6}$$

where α, β are fixed positive constants. Namely, for each index $j \in \mathbb{N}$ the densities $\check{\rho}_j$ and $\hat{\rho}_j$ which minimize and maximize the eigenvalue λ_j are explicitly constructed. It turns out that they are extreme points of the convex set of densities defined by (0.0.6), i.e., $\check{\rho}_j, \hat{\rho}_j$ are 'bang-bang' controls (see [18] for details). Moreover, in the case of the first eigenvalue λ_1, the result of Krein has been generalized in [8, 9, Cox-McLaughlin] to arbitrary dimensions under certain regularity assumptions of the boundary of Ω. Here we will generalize the results obtained in [19, Lamberti] for the Laplace operator with Dirichlet boundary conditions to the class of problems (0.0.2).

We shall also consider mixed boundary conditions in which case Dirichlet or intermediate conditions are considered on a part Γ_1 of $\partial\Omega$ and Neumann boundary conditions are considered on the remaining part $\partial\Omega \setminus \Gamma_1$ of $\partial\Omega$.

Keeping this in mind, after proving that elementary symmetric functions of the eigenvalues depend real analytically on ρ, we compute formulas for their derivatives and thanks to the Lagrange multiplier theorem, we show that there are no critical mass densities under the sole fixed mass constraint. We show then that eigenvalues are continuous with respect to the weak* topology of $L^\infty(\Omega)$; it immediately follows that the restriction of the symmetric functions of the eigenvalues to weakly* compact sets of $L^\infty(\Omega)$ (the set defined by (0.0.6) is of this kind) admit points of maximum and minimum and such points belong to the boundary of such sets.

The second part of this dissertation is devoted to the eigenvalue problem for the Laplace operator with Neumann boundary conditions

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{0.0.7}$$

which models a free vibrating membrane of mass density ρ. Here we assume that Ω is of class C^1, which guarantees the existence of a sequence of positive eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ increasing to $+\infty$. Following the scheme of [19], we are able to prove that the symmetric functions of multiple eigenvalues depend real analytically on ρ and compute formulas for their derivatives. The fact that there are no critical mass densities under fixed mass constraint is not immediate. In fact we prove it in a few special cases. Then we prove the continuity of the eigenvalues with respect to the weak* topology of $L^\infty(\Omega)$.

Finally, we consider the following eigenvalue problem for the Laplace operator with Steklov boundary conditions

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = \lambda \rho u, & \text{on } \partial\Omega\,, \end{cases} \tag{0.0.8}$$

which is related to the study of a vibrating membrane whose mass is concentrated at the boundary. Problem (0.0.8) can be considered as a limiting case for problem (0.0.7). In fact, we are able to construct a sequence of densities $\rho_n \in L^\infty(\Omega)$ with fixed total mass $\int_\Omega \rho_n dx = M$ such that the spectrum of problems (0.0.7) with density ρ_n converges pointwise to the spectrum of problem (0.0.8) with a suitable constant surface density. This suggests us to look for critical mass densities for Neumann problem among the wider class of problems including the Steklov problem. Finally, we exploit the same procedure used for problems (0.0.2) and (0.0.7) in order to prove the real analyticity of symmetric functions of eigenvalues and compute their derivatives. Then we show that for the ball B, the constant surface density is a critical point for certain symmetric functions of the first eigenvalue of problem (0.0.8).

This thesis is organized as follows. In Chapter 1, we introduce some preliminaries. We recall basic results of Sobolev Spaces theory and general results of perturbation theory for compact selfadjoint operators in Hilbert Spaces. In Chapter 2, we study the case of the biharmonic operator with Dirichlet and intermediate boundary conditions, we characterize the spectra, and prove real analyticity of the symmetric functions of eigenvalues. Then we compute explicit formulas for their differentials and we prove that there are no critical mass densities under fixed mass constraint. Moreover, we generalize the results of [8, 9]. Then we extend these results to poly-harmonic operators. Finally, we prove that these results hold also for poly-harmonic operators subject to mixed boundary conditions where Dirichlet or intermediate conditions are imposed on a part of the boundary, and Neumann boundary conditions are imposed on the remaining part. In Chapter 3, we study the eigenvalue problem for the Laplace operator with Neumann boundary conditions, we prove real analiticity of symmetric functions of the eigenvalues, then compute explicit formulas for their derivatives. Then we give partial results on the non-existence of critical mass densities. In Chapter 4, we construct a sequence of densities for which we are able to prove the pointwise convergence of the spectrum of the appropriate problem (0.0.7) to the spectrum of problem (0.0.8) with a suitable constant density. Then we study the eigenvalue problem for the Laplace operator with Steklov boundary conditions and show that the constant density is a critical point for certain functions of the first eigenvalue for the ball.

Introduzione

In questa dissertazione ci proponiamo di studiare problemi agli autovalori per operatori armonici e poliarmonici con condizioni al contorno omogenee su aperti connessi Ω di $\mathbb{R}^N$ di misura finita (eventualmente limitati, di classe C^1), sui quali consideriamo densità di massa $\rho \in L^\infty(\Omega)$ che soddisfano la condizione $\operatorname{ess\,inf}_\Omega \rho > 0$. Inizialmente consideriamo la classe di problemi (0.0.2) per gli operatori poliarmonici $(-\Delta)^n$ (il caso $m = n$ è il caso di condizioni al bordo di Dirichlet) e ci interessiamo alla dipendenza analitica delle funzioni elementari simmetriche degli autovalori λ_j (che sappiamo formare una successione crescente e strettamente positiva), dalla densità di massa ρ (in generale invece i singoli autovalori non sono funzioni neppure derivabili di ρ). Lo scopo in questo caso è generalizzare a questa classe di problemi i risultati ottenuti in [8, 9, Cox] e [19, Lamberti] per l'operatore di Laplace con condizioni al bordo di Dirichlet. Mostriamo infatti che le funzioni simmetriche elementari degli autovalori non ammettono densità di massa critiche sotto la sola condizione che la massa totale dell'aperto, data da $\int_\Omega \rho dx$, sia fissata. Inoltre proviamo la continuità rispetto alla topologia debole* degli autovalori, e quindi mostriamo che su insiemi debolmente* compatti di $L^\infty(\Omega)$ le funzioni elementari simmetriche degli autovalori ammettono massimi e minimi, e pertanto tali punti devono trovarsi al bordo, generalizzando così i risultati di [18, Krein] e [8, 9, Cox-Mc.Laughlin]. In seguito si generalizzeranno tali risultati al caso in cui le condizioni al contorno precedentemente considerate sono imposte su una parte del bordo, imponendo sulla restante parte condizioni di Neumann.

In seguito si tenterà di estendere i risultati trovati in [19] al caso del laplaciano con condizioni al bordo di Neumann. Anche in questo caso si proverà la reale analiticità delle funzioni simmetriche degli autovalori, e si daranno parziali risultati sulla non esistenza di densità di massa critiche sotto la sola condizione che la massa sia fissata. In seguito ci si interesserà al problema di Steklov per il laplaciano. Le motivazioni che ci spingono a studiare tale problema sono dovute al fatto che in un certo senso il problema (0.0.8) può essere visto come caso limite di problemi del tipo (0.0.7); infatti possiamo costruire una successione di densità ρ_n per le quali lo spettro dei problemi di Neumann con densità ρ_n converge puntualmente allo spettro del problema di Steklov con densità costante C sul bordo (dove ρ_n e C forniscono la stessa massa su Ω). Dunque questo ci suggerisce che possiamo cercare densità di massa critiche per il problema di Neumann, nella classe più ampia costituita dall'aggiunta del problema di Steklov. Infine proviamo che le funzioni simmetriche elementari degli autovalori del problema (0.0.8) dipendono analiticamente dalla densità superficiale ρ, e mostriamo che nel caso della palla la densità superficiale costante è critica per il primo autovalore.

CONTENTS

1. PRELIMINARIES

We present here a number of results widely used in the following chapters.

1.1 Sobolev Spaces

Let's introduce some definitions and basic results on Sobolev Spaces. For all proofs we refer to [6].

For any set Ω in $\mathbb{R}^N$ and $\rho > 0$ we denote by Ω_ρ the set $\{x \in \Omega : \ d(x, \partial\Omega) > \rho\}$. Moreover, by a cuboid we mean any roto-translation of a rectangular parallelepiped in $\mathbb{R}^N$.

Definition 1.1.1. *Let $\rho > 0$, $s, s' \in \mathbb{N}$, $s' \leq s$ and $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of isometries in $\mathbb{R}^N$.*

We say that $\mathcal{A} = (\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ is an atlas in $\mathbb{R}^N$ with the parameters $\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$, briefly an atlas in $\mathbb{R}^N$.

We denote by $C(\mathcal{A})$ the family of all open sets Ω in $\mathbb{R}^N$ satisfying the following properties:

(i) $\Omega \subset \bigcup_{j=1}^{s} (V_j)_\rho$ *and* $(V_j)_\rho \cap \Omega \neq \emptyset$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ *for* $j = 1, \dots s'$, $V_j \cap \partial\Omega = \emptyset$ *for* $s' < j \leq s$;

(iii) for $j = 1, ..., s$

$$r_j(V_j) = \{\, x \in \mathbb{R}^N : \ a_{ij} < x_i < b_{ij},\ i = 1, ..., N\}$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^N : \ a_{Nj} < x_N < g_j(\bar{x}),\ \ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, ..., x_{N-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{N-1} : \ a_{ij} < x_i < b_{ij},\ i = 1, ..., N-1\}$ and g_j is a continuous function defined on $\overline{W}_j$ (it is meant that if $s' < j \leq s$ then $g_j(\bar{x}) = b_{Nj}$ for all $\bar{x} \in \overline{W}_j$); moreover for $j = 1, \dots, s'$

$$a_{Nj} + \rho \leq g_j(\bar{x}) \leq b_{Nj} - \rho,$$

for all $\bar{x} \in \overline{W}_j$.

We say that an open set Ω in $\mathbb{R}^N$ is an open set with a continuous boundary if Ω is of class $C(\mathcal{A})$ for some atlas $\mathcal{A}$.

Let $m \in \mathbb{N}, M > 0$. We say that an open set Ω is of class $C^m_M(\mathcal{A})$ if Ω is of class $C(\mathcal{A})$ and all the functions g_j in (iii) are of class $C^m(\overline{W}_j)$ with

$$|g_j|_{c^m(\overline{W}_j)} = \sum_{1 \leq |\alpha| \leq m} \|D^\alpha g_j\|_{L^\infty(\overline{W}_j)} \leq M.$$

We say that an open set Ω in $\mathbb{R}^N$ is an open set of class C^m if Ω is of class $C^m_M(\mathcal{A})$ for some atlas $\mathcal{A}$, $m \in \mathbb{N}$ and $M > 0$.

Let $C_c^\infty(\Omega)$ be the space of $C^\infty(\Omega)$ functions compactly supported in Ω (test functions). Then we have the following

Definition 1.1.2. *Let Ω be an open set in $\mathbb{R}^N$, $u, v \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}_0^N$ a multiindex. We say that v is the α^{th}-weak partial derivative of u and we write $D^\alpha u = v$ if*

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v \phi dx,$$

for all $\phi \in C_c^\infty(\Omega)$.

Definition 1.1.3. *Let Ω be an open set in $\mathbb{R}^N$. The Sobolev Space $W^{k,p}(\Omega)$ consists of all functions u in $L^p(\Omega)$ such that for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$.*

Definition 1.1.4. *If $u \in W^{k,p}(\Omega)$, we set*

$$\|u\|_{W^{k,p}_\Omega} := \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)} \right)^{\frac{1}{p}}, \quad \text{if p} \neq \infty,$$
$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} .$$

Definition 1.1.5. *We denote with $W_0^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.*

For $p = 2$, we write $H^k(\Omega) = W^{k,2}(\Omega)$, $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

Theorem 1.1.6. *For each $1 \leq k \leq \infty$, $W^{k,p}(\Omega)$ is a Banach space.*

We will need some approximation results.

Theorem 1.1.7. (Global approximation by smooth functions). *Let Ω be an open set in $\mathbb{R}^N$. Let $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C^\infty(\Omega) \cap W^{k,p}(\Omega)$ converging to u in $W^{k,p}(\Omega)$.*

Theorem 1.1.8. (Global approximation by smooth functions up to the boundary). *Assume that Ω is a bounded open set of class C^1. Let $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists a sequence $\{u_m\}_{m\in\mathbb{N}} \subset C^\infty(\bar{\Omega})$ converging to u in $W^{k,p}(\Omega)$.*

As a consequence of Theorem 1.1.7 we have the following

Theorem 1.1.9. *Let Ω be an open set in $\mathbb{R}^N$, $1 \leq p < \infty$, $u \in L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$ if and only if u coincides almos everywhere with a function $\tilde{u}$ such that for almos all lines l parallel to the coordinate axis, $u_{|_l}$ is locally absolutely continuous, and the classic derivatives $\frac{\partial \tilde{u}}{\partial x_1}, ..., \frac{\partial \tilde{u}}{\partial x_N}$, which exist almost everywhere, belong to $L^p(\Omega)$.*

Under suitable regularity conditions of $\partial\Omega$ it makes sense to define the trace of a function $u \in W^{k,p}(\Omega)$.

Theorem 1.1.10. (Trace Theorem). *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. Then there exists a bounded linear operator* Tr *from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$ such that:*

i) $\operatorname{Tr}[u] = u_{|\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$*;*

ii) $\|\operatorname{Tr}[u]\|_{L^p(\partial\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}$, $\forall u \in W^{1,p}(\Omega)$, *the constant* C *depending only on* p *and* Ω*.*

$\operatorname{Tr}[u]$ *is called the trace of* u *on* $\partial\Omega$*.*

Theorem 1.1.11. *Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *of class* C^1*. Then* $u \in W_0^{1,p}(\Omega)$ *if and only in* $\operatorname{Tr}[u] = 0$*.*

Next results concern embeddings of Sobolev Spaces.

Theorem 1.1.12. (Gagliardo-Nirenberg-Sobolev inequality)*. For* $1 \leq p < N$ *let the Sobolev conjugate* p^* *be defined by* $p^* := \frac{Np}{N-p}$*. Then there exists* $C > 0$*, depending only on* p *and* N*, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^N)},$$

for all $u \in C_c^1(\mathbb{R}^N)$*.*

Lemma 1.1.13. (Poincaré inequality)*. Let* Ω *be an open set in* $\mathbb{R}^N$ *of finite measure,* $1 \leq p < \infty$*. Then there exists* $C > 0$*, depending only on* p*,* N *and* Ω *such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

for all $u \in W_0^{1,p}(\Omega)$*.*

Theorem 1.1.14. (Rellich-Kondravhov compactness Theorem)*. Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *of class* C^1*,* $1 \leq p < N$*. Then* $W^{1,p}(\Omega)$ *is compactly embedded into* $L^q(\Omega)$ *for all* $1 \leq q < p^*$*.*

Corollary 1.1.15. *If* Ω *is an open set of finite measure, then for all* $1 \leq p < \infty$*,* $W_0^{1,p}(\Omega)$ *is compactly embedded into* $L^p(\Omega)$*. If* Ω *is a bounded open set of class* C^1*, then for all* $1 \leq p < \infty$*,* $W^{1,p}(\Omega)$ *is compactly embedded into* $L^p(\Omega)$*.*

Theorem 1.1.16. (Poincaré-Wirtinger inequality)*. Let* Ω *be a bounded open set in* $\mathbb{R}^N$ *of class* C^1*,* $1 \leq p < \infty$*. Then there exists* $C > 0$*, depending only on* p*,* N *and* Ω *such that*

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)},$$

where $(u)_\Omega = \frac{\int_\Omega u}{|\Omega|}$*.*

Thanks to Rellich-Kondrachov Theorem (1.1.14), we are able to prove the following

Theorem 1.1.17. (Compact trace Theorem)*. Let* Ω *be an open bounded set in* $\mathbb{R}^N$ *of class* C^1*,* $1 \leq p < \infty$*. Then the trace operator* $\operatorname{Tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *is compact.*

Proof. We consider only the case $p > 1$. For the case $p = 1$ we refer to [22]. By Theorem 1.1.8, it suffices to prove that if $\{v_n\}_{n\in\mathbb{N}}$ is a sequence in $C^\infty(\bar{\Omega})$, there exists a subsequence $\{v_{n_m}\}_{m\in\mathbb{N}}$ such that $\{v_{n_m|\partial\Omega}\}_{m\in\mathbb{N}}$ is convergent in $L^p(\partial\Omega)$. Moreover, $\Omega \in C(\mathcal{A})$, for a certain $\mathcal{A}(\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$. By [23, 1.2.4, p. 27] there exist functions $\phi_j \in C_c^\infty(V_j)$, $j = 1, ..., s$, such that $\sum_{j=1}^s \phi_j(x) = 1$ for

$x \in \Omega$, and that $\sum_{j=1}^{s'} \phi_j(x) = 1$ for $x \in \partial\Omega$. Then we can directly assume that v_n has support within $V_j \cap \Omega$ for some $j = 1, ..., s'$. We denote the set $V_j \cap \Omega$ by U_j and the set $V_j \cap \partial\Omega$ by Λ_j. Then for $\bar{x} \in W_j$ we have

$$\begin{aligned} |v_n(\bar{x}, g(\bar{x})) - v_m(\bar{x}, g(\bar{x}))|^p &\leq |v_n(\bar{x}, a_N) - v_m(\bar{x}, a_N)|^p \\ &+ \int_{a_N}^{g(\bar{x})} \frac{\partial\, |v_n(\bar{x}, x_N) - v_m(\bar{x}, x_N)|^p}{\partial x_N} dx_N. \end{aligned}$$

Then it follows, integrating on W_j, that

$$\begin{aligned} \|v_n - v_m\|^p_{L^p(\Lambda_j)} &\leq C\, \|v_n - v_m\|^p_{L^p(U_j)} \\ &+ Cp \int_{U_j} |v_n(x) - v_m(x)|^{p-1} \left| \frac{\partial(v_n(x) - v_m(x))}{\partial x_N} \right| dx \\ &\leq C \left(\|v_n - v_m\|^p_{L^p(U_j)} + p\, \|v_n - v_m\|^{p-1}_{L^p(U_j)} \|v_n - v_m\|_{W^{1,p}(U_j)} \right). \end{aligned}$$

Now by Theorem 1.1.14 we can extract a subsequence, again denoted by $\{v_n\}_{n\in\mathbb{N}}$, that converges strongly in $L^p(U_j)$. Then v_n is a Cauchy sequence in $L^p(\Lambda_j)$. This concludes the proof. □

Remark 1.1.18. *We observe that the Rellich-Kondrachov Theorem and compact trace theorem hold true even under lower regularity assumptions of the boundary. In fact in [23, Thm. 6.1, p. 106] is proved that Rellich Theorem holds true for* Ω *of class* $C^{0,1}$.

1.2 Symmetric functions of the eigenvalues

In this section we present some techniques and results developed by P.D. Lamberti and M. Lanza de Cristoforis in [20] on the analytic dependence of symmetric functions of eigenvalues of a compact selfadjoint operator on a Hilbert space equipped with a variable scalar product depending on the operator. For all proofs and details we refer to [20].

Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be real Banach spaces. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the Banach space of bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$, endowed with the usual norm $\|A\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} := \sup_{\|x\|_{\mathcal{X}}=1} \|Ax\|_{\mathcal{Y}}$. Let $\mathcal{B}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$ the space of bilinear continuous maps from $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Z}$, endowed with the usual norm of the uniform convergence on the product of the unit ball of $\mathcal{X}$ and the one of $\mathcal{Y}$. Let $(H, < \cdot, \cdot >)$ be a real Hilbert space, $\|\cdot\|$ the norm associated with the scalar product $< \cdot, \cdot >$ of H. We denote by H_Q the vector space H endowed with the scalar product $Q = Q(\cdot, \cdot)$, and $\|\cdot\|_Q$ the associated norm. We denote by $\mathcal{K}(H, H)$ the subspace of $\mathcal{L}(H, H)$ of compact operators, which is closed in $\mathcal{L}(H, H)$. We denote by $\mathcal{K}_{\mathcal{S}}(H_Q, H_Q)$ the closed subspace of $\mathcal{K}(H_Q, H_Q)$ of those T such that $Q(Tu, u) = Q(u, Tu)$ for all $u, v \in H_Q$. Let T be a compact selfadjoint operator on H, $\sigma(T)$ the spectrum of T, that is well-known to be a finite or countable subset of $\mathbb{R}$. The elements of $\sigma(T) \setminus \{0\}$ are the eigenvalues of T, and 0 is the only possible accumulation point for $\sigma(T)$ (for the proof of the characterization of the spectrum of a compact selfadjoint operator we refer to [5]). We denote by $j^+(T)$ the number of positive eigenvalues of T, each counted according to its multiplicity, and by $j^-(T)$ the number of negative eigenvalues of T, each counted according to its multiplicity. We set

$$\begin{aligned} J^+(T) &:= \{j \in \mathbb{Z} : 1 \le j \le j^+(T)\}, \\ J^-(T) &:= \{j \in \mathbb{Z} : -j^-(T) \le j \le -1\}. \end{aligned}$$

Then there exists a unique function $j \to \mu_j(T)$ of $J(T) := J^+(T) \cup J^-(T)$ to $\mathbb{R}$, which is decreasing on $J^-(T)$ and on $J^+(T)$, with

$$\sigma(T) \setminus \{0\} = \{\mu_j(T) : j \in J(T)\},$$

such that each eigenvalue is repeated according to its multiplicity. We set

$$\mathcal{B}_{\mathcal{S}}(H^2, \mathbb{R}) := \{B \in \mathcal{B}(H^2, \mathbb{R}) \ : \ B(u_1, u_2) = B(u_2, u_1) \ \text{ for all } u_1, u_2 \in H\},$$

a closed subspace of $\mathcal{B}(H^2, \mathbb{R})$, and

$$\mathcal{Q}(H^2, \mathbb{R}) := \{B \in \mathcal{B}_{\mathcal{S}}(H^2, \mathbb{R}) \ : \ \eta[B] > 0\},$$

where

$$\eta[B] := \inf \left\{ \frac{B(u, u)}{\|u\|^2} \ : \ u \in H \setminus \{0\} \right\}.$$

The set $\mathcal{Q}(H)$ is the set of those scalar products on H coercive with respect to the fixed one $< \cdot, \cdot >$. We observe that Q is a coercive scalar product if and only if the embedding of H_Q in H is a homeomorphism. Now we set

$$\mathcal{M} := \{(Q, T) \in \mathcal{B}_{\mathcal{S}}(H^2, \mathbb{R}) \times \mathcal{K}(H, H) \ : \ Q(Tu, v) = Q(u, Tv) \ \text{ for all } u, v \in H\}.$$

The set $\mathcal{M}$ is closed in $\mathcal{B}_{\mathcal{S}}(H^2,\mathbb{R})\times\mathcal{K}(H,H)$. Moreover we set

$$\begin{aligned}\mathcal{O} := \mathcal{M}\cap(\mathcal{Q}(H^2,\mathbb{R}) &\times \mathcal{K}(H,H))\\ &= \{(Q,T)\in\mathcal{Q}(H^2,\mathbb{R})\times\mathcal{K}(H,H)\,:\,T\in\mathcal{K}_{\mathcal{S}}(H_Q,H_Q)\}.\end{aligned}$$

The set $\mathcal{O}$ is open in $\mathcal{M}$. Now we have the following

Theorem 1.2.1. *Let H be a real Hilbert space, $j\in\mathbb{Z}\setminus\{0\}$. Then the set*

$$\mathcal{A}_j := \{(Q,T)\in\mathcal{O}\,:\,j\in J(T)\}$$

is open in $\mathcal{M}$, and the function $\mu_j[\cdot]$ which takes $(Q,T)\in\mathcal{A}_j$ to $\mu_j[T]$ is continuous.

We now consider a fixed finite subset F of $\mathbb{Z}\setminus\{0\}$, and set

$$\mathcal{A}[F] := \{(Q,T)\in\mathcal{O}\,:\,j\in J(T)\,\forall j\in F\,,\ \mu_l[T]\notin\{\mu_j[T]\,:\,j\in F\}\,\forall l\in J(T)\backslash F\}. \tag{1.2.2}$$

By Theorem 1.2.1 it follows that $\mathcal{A}[F]$ is open in $\mathcal{M}$ and $\mu_j[\cdot]$ are continuous on $\mathcal{A}[F]$. Finally we denote the ortogonal projection $P_F[Q,T]$ of H_Q on the subsapce $E[T,F]$ generated by

$$\{u\in H_Q\,:\,Tu=\mu u,\ \exists\ \mu\in\{\mu_j[T]\,:\,j\in F\}\}.$$

We can state the following

Theorem 1.2.3. *Let H be a real Hilbert space and F a finite subset of $\mathbb{Z}\setminus\{0\}$. Then $E[T,F]$ has dimension equal to the cardinality of F, and it is an invariant subspace of H for T.*

Then we have [17, Kato] the following

Theorem 1.2.4. *Let H a real Hilbert space, F a finite subset of $\mathbb{Z}\setminus\{0\}$. Then the map P_F which takes $(Q,T)\in\mathcal{A}[F]$ to $P_F[Q,T]\in\mathcal{L}(H,H)$ is continuous.*

It is shown in [20] that $P_F[Q,T]$ depends analytically on (Q,T), in the sense of the following Theorem

Theorem 1.2.5. *Let H be a real Hilbert space, F a finite nonempty subset of $\mathbb{Z}\setminus\{0\}$ and $(\tilde{Q},\tilde{T})\in\mathcal{A}[F]$. Then there exists an open neighbourhood $\widetilde{\mathcal{W}}$ of $(\tilde{Q},\tilde{T})$ in $\mathcal{Q}(H^2,\mathbb{R})\times\mathcal{L}(H,H)$, and a real analytic operator $P_F^\sharp$ of $\widetilde{\mathcal{W}}$ to $\mathcal{L}(H,H)$ such that $P_F^\sharp[Q,T]=P_F[Q,T]$ for all $(Q,T)\in\widetilde{\mathcal{W}}\cap\mathcal{A}[F]$.*

We can choose an orthonormal basis of $E[T,F]$ which depends analitically on (Q,T).

Proposition 1.2.6. *Let H be a real Hilbert space, F a finite subset of $\mathbb{Z}\setminus\{0\}$ and $(\tilde{Q},\tilde{T})\in\mathcal{A}[F]$. Let $\{\tilde{u}_j\,:\,j\in F\}$ be an othonormal basis for $E[\tilde{T},F]$ in $H_{\tilde{Q}}$. Then there exists an open neighbourhood $\mathcal{W}_0$ of $(\tilde{Q},\tilde{T})$ in $\mathcal{Q}(H^2,\mathbb{R})\times\mathcal{L}(H,H)$ which is contained in the neighbourhood $\tilde{\mathcal{W}}$ of Theorem 1.2.5 , and $|F|$ real analytic operators $u_j[\cdot,\cdot]$, $j\in F$, of $\mathcal{W}_0$ to H such that:*

i) $\{u_j[Q,T]\,:\,j\in F\}$ is an orthonormal set in H_Q, for all $(Q,T)\in\mathcal{W}_0$, *ii)* $\{u_j[Q,T] : j \in F\}$ *is an orthonormal basis for the range of* $P^\sharp_F[Q,T]$*, which coincide with* $E[T,F]$*, in* H_Q*, for all* $(Q,T) \in \mathcal{W}_0 \cap \mathcal{A}[F]$ *,*

iii) $u_j[\tilde{Q},\tilde{T}] = \tilde{u}_j$ *for all* $j \in F$*.*

The problem is now reduced to a finite-dimensional one.

Proposition 1.2.7. *Let* H *be a real Hilbert space,* F *a finite subset of* $\mathbb{Z} \setminus \{0\}$ *and* $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$*. Let* $\{\tilde{u}_1, ..., \tilde{u_{|F|}}\}$ *be an orthonormal basis of* $E[\tilde{T},F]$ *in* $H_{\tilde{Q}}$*, and* $\{u_j[Q,T] : j = 1, ..., |F|\}$ *as in the previous proposition and* $\mathcal{S}$ *the map of* $\mathcal{W}_0$ *to the set* $M_{|F|}(\mathbb{R})$ *of* $|F| \times |F|$ *matrices with real coefficients, defined by*

$$\mathcal{S}[Q,T] = (\mathcal{S}_{hk}[Q,T])_{h,k=1,...,|F|} := (Q(Tu_k[Q,T], u_h[Q,T]))_{h,k=1,...,|F|}\,,$$

for all $(Q,T) \in \mathcal{W}_0$*. Then* $\mathcal{S}[\cdot,\cdot]$ *is real analytic and* $\mathcal{S}[Q,T]$ *is symmetric for all* $(Q,T) \in \mathcal{W}_0 \cap \mathcal{A}[F]$*. Moreover, if* $(Q,T) \in \mathcal{W}_0 \cap \mathcal{A}[F]$*, then* $\{\mu_j[T]\}_{j\in F}$ *are the eigenvalues of* $\mathcal{S}[Q,T]$ *repeated according to their multiplicity. Finally, if we assume that* $\mu_j[\tilde{T}]$ *assume a common value* $\tilde{\mu}_j$ *for all* $j \in F$*, then the differential of* $\mathcal{S}[\cdot,\cdot]$ *in* $(\tilde{Q},\tilde{T})$ *is given by the formula*

$$d\mathcal{S}[\tilde{Q},\tilde{T}](\dot{Q},\dot{T}) = \Big(\tilde{Q}(\dot{T}\tilde{u}_k, \tilde{u}_h)\Big)_{h,k=1,...,|F|}, \quad \text{for all } (\dot{Q},\dot{T}) \in \mathcal{B}_{\mathcal{S}}(H^2,\mathbb{R}) \times \mathcal{L}(H,H)\,.$$

Finally, we have

Theorem 1.2.8. *Let* H *be a real Hilbert space and* F *a finite nonempty subset of* $\mathbb{Z} \setminus \{0\}$*. Let*

$$M_{F,s}[T] = \sum_{\substack{j_1,...,j_s \in F \\ j_1<\cdots<j_s}} \mu_{j_1}[T] \cdots \mu_{j_s}[T], \quad \forall\, s \in \{1, \dots, |F|\},$$

for all $(Q,T) \in \mathcal{A}[F]$*, be the elementary symmetric functions of the eigenvalues* $\mu_j[T]$ *with indices* $j \in F$*. Let* $(\tilde{Q},\tilde{T}) \in \mathcal{A}[F]$*. Then there exists an open neighbourhood* $\widetilde{\mathcal{W}}$ *of* $(\tilde{Q},\tilde{T})$ *in* $\mathcal{Q}(H^2,\mathbb{R}) \times \mathcal{L}(H,H)$*, and real analytic functions* $M^\sharp_{F,s}[\cdot,\cdot]$*, for* $s = 1, ..., |F|$*, of* $\widetilde{\mathcal{W}}$ *in* $\mathbb{R}$ *such that*

$$M^\sharp_{F,s}[Q,T] = M_{F,s}[T],$$

for all $(Q,T) \in \widetilde{\mathcal{W}} \cap \mathcal{A}[F]$*, and for all* $s = 1, ..., |F|$*. If we further assume that there exists* $\tilde{\mu} \in \mathbb{R}$ *such that* $\tilde{\mu} = \mu_j[T]$ *for all* $j \in F$*, and if* $\{\tilde{u}_1, ..., \tilde{u}_{|F|}\}$ *is an orthonormal basis for* $E[\tilde{T},F]$ *in* $H_{\tilde{Q}}$*, then the partial derivative of* $M^\sharp_{F,s}$ *with respect to the variable* T *at* $(\tilde{Q},\tilde{T})$ *is given by the formula*

$$d_T M^\sharp_{F,s}[\tilde{Q},\tilde{T}](\dot{T}) = \binom{|F|-1}{s-1} \tilde{\mu}^{s-1} \sum_{l=1}^{|F|} \tilde{Q}(\dot{T}\tilde{u}_l, \tilde{u}_l), \tag{1.2.9}$$

for all $\dot{T} \in \mathcal{K}_{\mathcal{S}}(H_{\tilde{Q}}, H_{\tilde{Q}})$*, and for all* $s = 1, ..., |F|$*.*

2. DIRICHLET, INTERMEDIATE AND MIXED BOUNDARY CONDITIONS FOR POLY-HARMONIC OPERATORS

In this chapter we study the eigenvalue problems for the poly-harmonic operators subject to Dirichlet or intermediate boundary conditions. We start considering the biharmonic operator, which models a vibrating clamped plate in the case of Dirichlet boundary conditions, and a vibrating simply supported plate in the case of intermediate boundary conditions. Then we generalize the results to poly-harmonic operators. Finally, we study the case of mixed Neumann-Dirichlet and Neumann-Intermediate boundary conditions.

2.1 The bilaplacian

Throughout this chapter Ω is a domain, i.e., a connected open set in $\mathbb{R}^N$, of finite measure $|\Omega|$.
Let $\mathcal{R}$ be the set of those $\rho \in L^\infty(\Omega)$ such that ess $\inf_\Omega \rho > 0$. Note that $\mathcal{R}$ is open in $L^\infty(\Omega)$. We start considering the classical formulation of Dirichlet problem for the bilaplacian, which models the bending of a clamped plate:

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{2.1.1}$$

where ν denotes the outer unit normal to $\partial\Omega$, $u \in C^4(\Omega) \cap C^1(\bar{\Omega})$ and $\lambda \in \mathbb{R}$. We consider the weak formulation of the problem (2.1.1), that is

$$\int_\Omega \Delta u \Delta \phi \, dx = \lambda \int_\Omega \rho u \phi \, dx \;, \quad \forall\, \phi \in H_0^2(\Omega) \tag{2.1.2}$$

in the unknows $u \in H_0^2(\Omega)$, $\lambda \in \mathbb{R}$.

We reduce the study of problem (2.1.2) to the study of the spectrum of a compact selfadjoint operator in a suitable Hilbert space. We start considering the operator Δ^2 as a map from $H_0^2(\Omega)$ to its dual $(H_0^2(\Omega))'$ defined by

$$\Delta^2[u][\phi] = \int_\Omega \Delta u \Delta \phi dx \;, \quad \forall\, u, \phi \in H_0^2(\Omega).$$

The hypothesis of the Lax-Milgram Theorem (cfr. [27]) are fulfilled by Δ^2, in fact:

1. Δ^2 is bounded:

$$\begin{aligned} \left|\Delta^2[u][\phi]\right| &= \left|\int_\Omega \Delta u \Delta \phi \, dx\right| \leq \int_\Omega |\Delta u| \, |\Delta \phi| \; dx \leq \|\Delta u\|_{L^2(\Omega)} \|\Delta \phi\|_{L^2(\Omega)} \\ &\leq C \|u\|_{H_0^2(\Omega)} \|\phi\|_{H_0^2(\Omega)} \;, \quad \forall u, \phi \in H_0^2(\Omega). \end{aligned}$$

2. Δ^2 is coercive. In fact by the Poincaré inequality (Lemma 1.1.13),

$$\|u\|^2_{H^2_0(\Omega)} \leq C(\Omega, N)\, \|\Delta u\|^2_{L^2(\Omega)} = C(\Omega, N)\Delta^2[u][u]\ , \quad \forall\ u \in H^2_0(\Omega)\,,$$
with $C(\Omega, N) > 0$.

Then by the Lax-Milgram Theorem we have that Δ^2 is a linear homeomorphism between $H^2_0(\Omega)$ and its dual.

We denote by i the canonical (compact) embedding of $H^2_0(\Omega)$ into $L^2(\Omega)$ and by J the canonical (continuous) embedding of $L^2(\Omega)$ into $(H^2_0(\Omega))'$ defined by

$$J[u][\phi] = \int_\Omega u\phi dx\ , \quad \forall u \in L^2(\Omega), \phi \in H^2_0(\Omega).$$

For all $\rho \in \mathcal{R}$ we denote by M_ρ the map which takes $u \in L^2(\Omega)$ to $\rho u \in L^2(\Omega)$, and by J_ρ the map $J \circ M_\rho$, which is a continuous embedding of $L^2(\Omega)$ into $(H^2_0(\Omega))'$:

$$J_\rho[u][\phi] = \int_\Omega u\phi\rho dx\ , \quad \forall u \in L^2(\Omega), \phi \in H^2_0(\Omega).$$

It is now easy to see that problem (2.1.2) is equivalent to the following one:

$$(\Delta^2)^{-1} \circ J_\rho \circ iu = \lambda^{-1} u, \tag{2.1.3}$$

in the unknows $u \in H^2_0(\Omega)$, $\lambda \in \mathbb{R}$. By the Poincaré inequality (1.1.13) it is easy to see that the bilinear form

$$< u, v >= \int_\Omega \Delta u \Delta v dx, \quad \forall u, v \in H^2_0(\Omega),$$

defines on $H^2_0(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. We will denote $\mathcal{H}^2_0(\Omega)$ the space $H^2_0(\Omega)$ endowed with this scalar product. We can now state the following

Lemma 2.1.4. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure, $\rho \in \mathcal{R}$. The operator $T_\rho := (\Delta^2)^{-1} \circ J_\rho \circ i$ is a compact selfadjoint operator in $\mathcal{H}^2_0(\Omega)$, whose eigenvalues coincide with the reciprocals of the eigenvalues of problem (2.1.2) for all $j \in \mathbb{N}$.*

Proof. The compactness of T_ρ follows immediately from the compactness of i and the continuity of $(\Delta^2)^{-1}$ and J_ρ. We observe now that

$$\begin{aligned} < T_\rho u, v >_{\mathcal{H}^2_0(\Omega)} &= < (\Delta^2)^{-1} \circ J_\rho \circ iu, v >_{\mathcal{H}^2_0(\Omega)} = \Delta^2[(\Delta^2)^{-1} \circ J_\rho \circ iu][v] \\ &= J_\rho[iu][v] = J_\rho[iv][u], \end{aligned}$$

for all $u, v \in \mathcal{H}^2_0(\Omega)$. This proves the selfadjointness. The proof of the other statements is straightforward. □

Then we have the following

Theorem 2.1.5. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and $\rho \in \mathcal{R}$. Then the set Σ of the eigenvalues of (2.1.2) is contained in $]0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity. The operator Δ^2 has a Hilbert basis in $\mathcal{H}^2_0(\Omega)$ which consists of eigenfunctions.*

Proof. Let $\lambda \in \Sigma$, $u \in \mathcal{H}_0^2(\Omega) \setminus \{0\}$ be such that $\Delta^2 u = \lambda \rho u$ in the weak sense. Thus

$$\int_\Omega (\Delta u)^2 \, dx = \lambda \int_\Omega \rho u^2 \, dx \,,$$

hence $\lambda \geq 0$. By the Poincaré inequality it follows that if $u \neq 0$, then $\int_\Omega (\Delta u)^2 dx > 0$, hence 0 is not an eigenvalue and $\Sigma \in]0, +\infty[$. Moreover, T_ρ is injective: indeed $(\Delta^2)^{-1} \circ J_\rho u = 0$ implies $J_\rho u = 0$ and hence $u = 0$. Then $\mathrm{Ker} T = \{0\}$ and it is well-known that the eigenvalues of T_ρ consist of the image of a decreasing sequence $\{\mu_j\}$, such that $\mu_j > 0$ for all $j \in \mathbb{N}$ and $\lim_{j \to \infty} = 0$. Since $\Sigma = \left\{ \frac{1}{\mu_j} : j \in \mathbb{N} \right\}$ the remaining statements follow immediately. □

We represent the set Σ of the eigenvalues of (2.1.2) by means of an increasing sequence

$$\lambda_1[\rho] \,,\ \lambda_2[\rho] \,,\ \lambda_3[\rho] \,,\ \dots ,\ \lambda_n[\rho] \,,\ \dots$$

where each eigenvalue is repeated according to its multiplicity. The first eigenvalue in general is not simple, cfr. [14] (for the laplacian $-\Delta$ the first eigenvalue is always simple, cfr. [19]).

We have the following variational representation of the eigenvalues.

Theorem 2.1.6. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and $\rho \in \mathcal{R}$. Then we have*

i)

$$\lambda_1[\rho] = \inf_{\substack{u \in \mathcal{H}_0^2(\Omega) \\ u \neq 0}} \frac{\int_\Omega (\Delta u)^2 \, dx}{\int_\Omega u^2 \rho \, dx} \,. \tag{2.1.7}$$

The eigenfunctions corresponding to $\lambda_1[\rho]$ are exactly the minimizers in (2.1.7).

ii) For all $j \in \mathbb{N}$

$$\lambda_j[\rho] = \sup \left\{ \frac{\int_\Omega (\Delta u)^2 \, dx}{\int_\Omega u^2 \rho \, dx} \ : \ u \in \langle u_1, ..., u_j \rangle \right\} ,$$

where $u_1, ..., u_j$ are the linearly independent eigenfunctions corresponding to $\lambda_1[\rho], ..., \lambda_j[\rho]$.

iii) Let

$$\Lambda(E) := \sup \left\{ \frac{\int_\Omega (\Delta v)^2 \, dx}{\int_\Omega v^2 \rho \, dx} \ : \ 0 \neq v \in E \leq \mathcal{H}_0^2(\Omega) \right\} ,$$

for all $E \leq \mathcal{H}_0^2(\Omega)$. Then

$$\lambda_j[\rho] = \inf_{\substack{E \leq \mathcal{H}_0^2(\Omega) \\ \mathrm{dim} E = j}} \Lambda(E). \tag{2.1.8}$$

By (2.1.8) it immediately follows that the map $\rho \to \lambda_j[\rho]$ is locally Lipschitz-continuous in $\rho \in \mathcal{R}$. In fact it is easy to see that

$$\left| \frac{\int_\Omega (\Delta u)^2 \, dx}{\int_\Omega u^2 \rho_1 dx} - \frac{\int_\Omega (\Delta u)^2 \, dx}{\int_\Omega u^2 \rho_2 \, dx} \right| \leq \frac{1}{\alpha} \frac{\int_\Omega (\Delta u)^2 \, dx}{\int_\Omega u^2 \rho_1 \, dx} \left\| \rho_2 - \rho_1 \right\|_\infty$$

for all $\rho_1, \rho_2 \in \mathcal{R}$, where

$$\alpha = \min\{\operatorname{ess\,inf}_\Omega \rho_1, \operatorname{ess\,inf}_\Omega \rho_2\} > 0.$$

Then

$$\begin{aligned} \frac{\int_\Omega (\Delta u)^2\, dx}{\int_\Omega u^2 \rho_1\, dx}\left(1 - \frac{1}{\alpha}\|\rho_2 - \rho_1\|_\infty\right) &\leq \frac{\int_\Omega (\Delta u)^2\, dx}{\int_\Omega u^2 \rho_2\, dx} \\ &\leq \frac{\int_\Omega (\Delta u)^2\, dx}{\int_\Omega u^2 \rho_1\, dx}\left(1 + \frac{1}{\alpha}\|\rho_2 - \rho_1\|_\infty\right). \end{aligned} \tag{2.1.9}$$

If ρ_1, ρ_2 satisfy

$$\|\rho_2 - \rho_1\|_\infty < \alpha\,,$$

then taking the infimum and the supremum in (2.1.9) yields

$$|\lambda_j[\rho_2] - \lambda_j[\rho_1]| \leq \lambda_j[\rho_1]\frac{1}{\alpha}\|\rho_2 - \rho_1\|_\infty\ ,$$

hence the local Lipschitz-continuity of $\lambda_j[\cdot]$ is proved.

We now study the eigenvalue problem for the operator Δ^2 subject to intermediate boundary conditions. The classical formulation of the problem is

$$\begin{cases} \Delta^2 u = \lambda \rho u, & \text{in } \Omega\,, \\ u = 0, & \text{on } \partial\Omega\,, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{2.1.10}$$

in the unknown $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$, $\lambda \in \mathbb{R}$. We introduce now some basic elements of tangential calculus in order to state the weak formulation of problem (2.1.10). Let $b_\Omega(x) = d(x,\Omega) - d(x,\Omega^c)$, $S_h(\partial\Omega) = \{x \in \mathbb{R}^N : |b_\Omega(x)| < h\}$, where $d(x, A)$ denotes the euclidean distance of x to a set A in $\mathbb{R}^N$. If Ω is of class C^2 then $b_\Omega \in C^2(S_h(\partial\Omega))$, cfr.[11, Ch.4].

Definition 2.1.11. *Let Ω be an open set in $\mathbb{R}^N$ of class C^3. Let $F \in C^1(S_h(\partial\Omega))$, $V \in C^1(S_h(\partial\Omega))^N$. We set $f := F_{|\partial\Omega}$, $v := V_{|\partial\Omega}$, $v_\nu := V \cdot \nu$, $v_{\partial\Omega} := v - v_\nu$. Then the tangential gradient and divergence on $\partial\Omega$ are defined as follows:*

$$\begin{aligned} &\nabla_{\partial\Omega} f := \nabla F - \frac{\partial F}{\partial \nu}\nu, \\ &\operatorname{div}_{\partial\Omega} v := \operatorname{div} V - DV\nu \cdot \nu, \end{aligned}$$

where D denotes the Jacobian. Moreover, if $F \in C^2(S_h(\partial\Omega))$, it is defined the operator $\Delta_{\partial\Omega}$ (Laplace-Beltrami operator) as

$$\Delta_{\partial\Omega} f := \operatorname{div}_{\partial\Omega}(\nabla_{\partial\Omega} f)\,.$$

Under the assumptions of Definition 2.1.11 we can state the tangential Green formula:

$$\int_{\partial\Omega} f \operatorname{div}_{\partial\Omega} v + \nabla_{\partial\Omega} f \cdot v\, d\sigma = \int_{\partial\Omega} H f v \cdot \nu\, d\sigma, \tag{2.1.12}$$

where $H = (N-1)\bar{H}$, and $\bar{H}$ is the mean curvature of $\partial\Omega$. For all proofs and details concerning tangential calculus, we refer to [11].

Consider the equation $\Delta^2 u = \lambda\rho u$. We multiply both members for a test function $\phi \in C_c^\infty(\Omega)$ and integrate by parts twice. Then we obtain

$$\begin{aligned}\int_\Omega \Delta^2 u\phi dx &= \sum_{i,j=1}^N \int_\Omega \partial^4_{iijj} u\phi dx = \sum_{i,j=1}^N \left(\int_{\partial\Omega} \partial^3_{iij} u\phi\nu_j d\sigma - \int_\Omega \partial^3_{iij} u\partial_j\phi dx\right) \\ &= \sum_{i,j=1}^N \left(\int_{\partial\Omega} \partial^3_{iij} u\phi\nu_j - \partial^2_{ij} u\partial_j\phi\nu_i d\sigma + \int_\Omega \partial^2_{ij} u\partial^2_{ij}\phi dx\right). \quad (2.1.13)\end{aligned}$$

Since boundary integrals are all equal to 0, we obtain

$$\int_\Omega \Delta^2 u\phi dx = \int_\Omega \sum_{i,j=1}^N \partial^2_{ij} u\partial^2_{ij}\phi dx \quad \forall \phi \in C_c^\infty(\Omega). \quad (2.1.14)$$

We relax our conditions and assume that u and ϕ are in a suitable subspace of $H^2(\Omega)$. Let's consider the first two terms of (2.1.13). We have

$$\begin{aligned}\int_{\partial\Omega} \sum_{i,j=1}^N \partial^3_{iij} u\phi\nu_j d\sigma &= \int_{\partial\Omega} \frac{\partial\Delta u}{\partial\nu}\phi d\sigma, \\ \int_{\partial\Omega} \sum_{i,j=1}^N \partial^2_{ij} u\partial_j\phi\nu_i d\sigma &= \int_{\partial\Omega} \left(D^2 u.\nu\right)\cdot\nabla\phi d\sigma, \quad (2.1.15)\end{aligned}$$

where $D^2 u$ denotes the Hessian of u. The term (2.1.15) can be written as

$$\begin{aligned}\int_{\partial\Omega} \left(D^2 u.\nu\right)\cdot\nabla\phi d\sigma &= \int_{\partial\Omega} \left(D^2 u.\nu\right)\cdot\nabla_{\partial\Omega}\phi d\sigma + \int_{\partial\Omega} \left(D^2 u.\nu\right)\cdot\frac{\partial\phi}{\partial\nu}\nu d\sigma \\ &= \int_{\partial\Omega} \left(D^2 u.\nu\right)\cdot\nabla_{\partial\Omega}\phi d\sigma + \int_{\partial\Omega} \frac{\partial^2 u}{\partial\nu^2}\frac{\partial\phi}{\partial\nu} d\sigma \\ &= \int_{\partial\Omega} \left(-\mathrm{div}_{\partial\Omega}\left(D^2 u.\nu\right) + H\frac{\partial^2 u}{\partial\nu^2}\right)\phi d\sigma + \int_{\partial\Omega} \frac{\partial^2 u}{\partial\nu^2}\frac{\partial\phi}{\partial\nu} d\sigma,\end{aligned}$$

where we have used the fact that $\frac{\partial^2 u}{\partial\nu^2} = \left(D^2 u.\nu\right)\cdot\nu$. The natural choice of the subspace of $H^2(\Omega)$ in order to state the weak formulation of the problem (2.1.10) is then $H^2(\Omega)\cap H^1_0(\Omega)$. The weak formulation is

$$\int_\Omega \sum_{i,j=1}^N \partial^2_{ij} u\partial^2_{ij}\phi dx = \lambda\int_\Omega \rho u\phi dx, \quad \forall\phi \in H^2(\Omega)\cap H^1_0(\Omega), \quad (2.1.16)$$

in the unknowns $u \in H^2(\Omega)\cap H^1_0(\Omega)$, $\lambda \in \mathbb{R}$. Problem (2.1.16) makes sense under less restrictive boundary regularity assumptions. As in the case of Dirichlet boundary conditions, we study problem (2.1.16) in an open subset Ω of $\mathbb{R}^N$ of finite measure.

Remark 2.1.17. *Having in mind the canonical decomposition of Δ on the boundary $\partial\Omega$ of a bounded open set of class C^3, i.e., $\Delta u_{|\partial\Omega} = \Delta_{\partial\Omega} u + H\frac{\partial u}{\partial\nu} + \frac{\partial^2 u}{\partial\nu^2}$, since $u = 0$ on $\partial\Omega$, hence $\Delta_{\partial\Omega} = 0$, the boundary condition $\frac{\partial^2 u}{\partial\nu^2} = 0$ on $\partial\Omega$ in (2.1.10) is equivalent to the condition $\Delta u - H\frac{\partial u}{\partial\nu} = 0$ on $\partial\Omega$.*

We now reduce problem (2.1.16) to an eigenvalue problem for a compact selfadjoint operator on a Hilbert space. We denote by $V(\Omega)$ the space $H^2(\Omega)\cap H^1_0(\Omega)$. We consider the operator Δ^2 as the map from $V(\Omega)$ to its dual $(V(\Omega))'$ defined by

$$\Delta^2[u][\phi]=\int_\Omega\sum_{i,j=1}^N\partial^2_{ij}u\partial^2_{ij}\phi dx,\quad\forall\phi\in V(\Omega).\tag{2.1.18}$$

The operator Δ^2 is a linear homeomorphism between $V(\Omega)$ and $(V(\Omega))'$. This follows immediately by observing that there exists $C>0$ such that

$$\|u\|_{L^2(\Omega)}\le C\left\|D^2u\right\|_{L^2(\Omega)},\quad\forall u\in V(\Omega).\tag{2.1.19}$$

In fact we observe that for any $u\in V(\Omega)$

$$\int_\Omega\partial^2_{ii}uudx=\int_{\partial\Omega}\partial_iuu\nu_id\sigma-\int_\Omega(\partial_iu)^2\,dx=-\int_\Omega(\partial_iu)^2\,dx\,,$$

hence,

$$\int_\Omega(\partial_iu)^2\,dx=\left|\int_\Omega\partial^2_{ii}uudx\right|\le\|u\|_{L^2(\Omega)}\left\|\partial^2_{ii}u\right\|_{L^2(\Omega)}.$$

We sum over index i, obtaining

$$\begin{aligned}\|\nabla u\|^2_{L^2(\Omega)}&\le\|u\|_{L^2(\Omega)}\sum_{i=1}^N\left\|\partial^2_{ii}u\right\|_{L^2(\Omega)}\\&\le\|u\|_{L^2(\Omega)}\sum_{i,j=1}^N\left\|\partial^2_{ij}u\right\|_{L^2(\Omega)}\le C(N)\,\|u\|_{L^2(\Omega)}\left\|D^2u\right\|_{L^2(\Omega)}\end{aligned}$$

Since $u\in H^1_0(\Omega)$, there exists $C'>0$ such that $\|u\|_{L^2(\Omega)}\le C'\,\|\nabla u\|_{L^2(\Omega)}$. It immediately follows then

$$\|u\|_{L^2(\Omega)}\le C(N)C'\left\|D^2u\right\|_{L^2(\Omega)},$$

hence the coercivity of Δ^2. The proof of the continuity of Δ^2 is straightforward. Next we denote by i the canonical (compact) embedding of $V(\Omega)$ into $L^2(\Omega)$, and by J_ρ the (continuous) embedding of $L^2(\Omega)$ into $(V(\Omega))'$, defined by

$$J_\rho[u][\phi]=\int_\Omega\rho u\phi dx\quad\forall u\in L^2(\Omega),\phi\in V(\Omega).$$

Let T_ρ be the operator from $V(\Omega)$ to itself defined by $T_\rho:=(\Delta^2)^{-1}\circ J_\rho\circ i$. Problem (2.1.16) is then equivalent to

$$T_\rho u=\lambda^{-1}u,\tag{2.1.20}$$

in the unknows $u\in V(\Omega)$, $\lambda\in\mathbb{R}$. We now consider the space $V(\Omega)$ endowed with the bilinear form

$$<u,v>=\int_\Omega\sum_{i,j=1}^N\partial^2_{ij}u\partial^2_{ij}vdx,\quad\forall u,v\in V(\Omega).\tag{2.1.21}$$

This is a scalar product on $V(\Omega)$ whose induced norm is equivalent to the standard one. We denote by $\mathcal{V}(\Omega)$ the space $V(\Omega)$ endowed with the scalar product defined by (2.1.21). Then we can state the following

Lemma 2.1.22. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and $\rho \in \mathcal{R}$. The operator $T_\rho := (\Delta^2)^{-1} \circ J_\rho \circ i$ is a compact selfadjoint operator in $\mathcal{V}(\Omega)$, whose eigenvalues coincide with the reciprocals of the eigenvalues of problem (2.1.16) for all $j \in \mathbb{N}$.*

The proof of Lemma 2.1.22 is very similar to the proof of Lemma 2.1.4, hence we omit it.

Theorem 2.1.23. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure, $\rho \in \mathcal{R}$. Then the set Σ of the eigenvalues of (2.1.16) is contained in $]0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity. The operator Δ^2 has a Hilbert basis in $\mathcal{V}(\Omega)$ which consists of eigenfunctions.*

The proof of Theorem 2.1.23 is very similar to the proof of Theorem 2.1.5.

We represent the set Σ of the eigenvalues of (2.1.16) by means of an increasing sequence

$$\lambda_1[\rho]\,,\ \lambda_2[\rho]\,,\ \lambda_3[\rho]\,,\ \dots,\ \lambda_n[\rho]\,,\ \dots$$

where each eigenvalue is repeated accordingly its multiplicity.

We have the following variational representation of the eigenvalues.

Theorem 2.1.24. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and $\rho \in \mathcal{R}$. Then we have*

i)

$$\lambda_1[\rho] = \inf_{\substack{u \in \mathcal{V}(\Omega) \\ u \neq 0}} \frac{\int_\Omega \left\|D^2 u\right\|^2 dx}{\int_\Omega u^2 \rho\, dx}\,. \tag{2.1.25}$$

The eigenfunctions corresponding to $\lambda_1[\rho]$ are exactly the minimizers in (2.1.25).

ii) For all $j \in \mathbb{N}$

$$\lambda_j[\rho] = \inf_{\substack{E \leq \mathcal{V}(\Omega) \\ \dim E = j}} \sup_{0 \neq u \in E} \frac{\int_\Omega \left\|D^2 u\right\|^2 dx}{\int_\Omega u^2 \rho\, dx}\,. \tag{2.1.26}$$

Exactly as in the Dirichlet boundary conditions case, by this representation we deduce the local Lipschitz-continuity of $\lambda_j[\rho]$.

In the sequel we will denote by $\lambda_j^D[\rho]$ and by $\lambda_j^I[\rho]$ the eigenvalues of the problems (2.1.2) and (2.1.16) respectively, by T_ρ^D and T_ρ^I the respective resolvent operators and by $\mathcal{H}_D(\Omega)$ and $\mathcal{H}_I(\Omega)$ the Hilbert spaces $\mathcal{H}_0^2(\Omega)$ and $\mathcal{V}(\Omega)$ respectively.

Theorem 2.1.27. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Let*

$$\mathcal{R}^{(\cdot)}[F] := \{\rho \in \mathcal{R} \,:\, \lambda_j^{(\cdot)}[\rho] \neq \lambda_l^{(\cdot)}[\rho]\,,\ \forall j \in F, l \in \mathbb{N} \setminus F\}\,,$$
$$\Theta^{(\cdot)}[F] := \{\rho \in \mathcal{R}[F] \,:\, \lambda_{j_1}^{(\cdot)}[\rho] = \lambda_{j_2}^{(\cdot)}[\rho]\,,\ \forall j_1, j_2 \in F\},$$

where $(\cdot)$ stands for D or I. Then $\mathcal{R}^{(\cdot)}[F]$ is an open subset of $L^\infty(\Omega)$ and the symmetric functions of eigenvalues

$$\Lambda^{(\cdot)}_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h \in F\\ j_1<\dots j_h}} \lambda^{(\cdot)}_{j_1}[\rho]\cdots\lambda^{(\cdot)}_{j_h}[\rho]\,, \quad h = 1,\,\dots,|F|\,, \tag{2.1.28}$$

are real analytic in $\mathcal{R}^{(\cdot)}[F]$. Moreover, if $\rho \in \Theta^{(\cdot)}[F]$ and the eigenvalues $\lambda^{(\cdot)}_j[\rho]$ assume the common value $\lambda^{(\cdot)}_F[\rho]$ for all $j \in F$, then the differential of $\Lambda^{(\cdot)}_{F,h}$ at ρ is given by

$$d\Lambda^{(\cdot)}_{F,h}[\rho][\dot\rho] = \left(-\lambda^{(\cdot)}_F[\rho]\right)^{h+1}\binom{|F|-1}{h-1}\sum_{l\in F}\int_\Omega \left(u^{(\cdot)}_l\right)^2\dot\rho\,dx\,, \tag{2.1.29}$$

for all $\dot\rho \in L^\infty(\Omega)$, where $\{u^{(\cdot)}_l\}$ is an orthonormal basis for $\lambda^{(\cdot)}_F[\rho]$ in $\mathcal{H}_{(\cdot)}(\Omega)$.

Proof. We write the proof in the case of Dirichlet boundary conditions. The proof of the other case is essentially the same. First, we observe that the map which takes $\rho\in\mathcal{R}$ to $T^D_\rho \in \mathcal{K}(\mathcal{H}^2_0(\Omega),\mathcal{H}^2_0(\Omega))$ is a bounded linear map, hence real analytic. Then the map which takes $\rho\in\mathcal{R}$ to $(\langle\cdot,\cdot\rangle_{\mathcal{H}^2_0(\Omega)}, T^D_\rho) \in \mathcal{Q}(\mathcal{H}^2_0(\Omega)^2,\mathbb{R}) \times \mathcal{K}(\mathcal{H}^2_0(\Omega),\mathcal{H}^2_0(\Omega))$ is real analytic. The operator T^D_ρ is a compact selfadjoint operator with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}^2_0(\Omega)}$, and its eigenvalues, denoted by $\mu^D_j[\rho]$, coincide with the reciprocals of $\lambda^D_j[\rho]$. Then the set $\mathcal{R}^D[F]$ coincides with the set $\{\rho\in\mathcal{R} \,:\, \mu^D_j[\rho] \neq \mu^D_l[\rho]\,,\ \forall j \in F, l \in \mathbb{N}\setminus F\}$. The function $\rho \mapsto (\langle\cdot,\cdot\rangle_{\mathcal{H}^2_0(\Omega)}, T^D_\rho)$ is an analytic map from $\mathcal{R}$ to

$$\begin{aligned}\mathcal{O}^D_\Omega := \big\{(Q,T) \in \mathcal{Q}(\mathcal{H}^2_0(\Omega)^2,\mathbb{R}) \times \mathcal{K}(\mathcal{H}^2_0(\Omega),\mathcal{H}^2_0(\Omega)) \,:\, \\ Q(Tu,v) = Q(u,Tv) \ \text{for all}\ u,v \in \mathcal{H}^2_0(\Omega)\big\},\end{aligned}$$

and the set $\mathcal{R}^D[F]$ coincides with the set

$$\{\rho\in\mathcal{R} \,:\, (\langle\cdot,\cdot\rangle_{\mathcal{H}^2_0(\Omega)}, T^D_\rho) \in \mathcal{A}^D[F]\},$$

where $\mathcal{A}^D[F]$ is defined in (1.2.2), with $H = \mathcal{H}^2_0(\Omega)$. Since $\mathcal{A}^D[F]$ is open in $\mathcal{O}^D_\Omega$ (Theorem 1.2.1) and $\rho \mapsto (\langle\cdot,\cdot\rangle_{\mathcal{H}^2_0(\Omega)}, T^D_\rho)$ is a continuous map of $\mathcal{R}$ into $\mathcal{O}^D_\Omega$, it follows that $\mathcal{R}^D[F]$ is open in $L^\infty(\Omega)$. By Theorem 1.2.8 it follows that the maps which take $\rho\in\mathcal{R}^D[F]$ to

$$\Gamma^D_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h \in F\\ j_1<\dots<j_h}} \mu^D_{j_1}[\rho]\cdots\mu^D_{j_h}[\rho]\,, \tag{2.1.30}$$

are real analytic for all $h = 1,...,|F|$. Now one can easily see that

$$\Lambda^D_{F,h}[\rho] = \frac{\Gamma^D_{F,|F|-h}[\rho]}{\Gamma^D_{F,|F|}[\rho]}\,, \tag{2.1.31}$$

for all $h = 1,...,|F|$, where we have set $\Gamma^D_{F,0} := 1$. Then the symmetric functions of eigenvalues $\Lambda^D_{F,h}[\rho]$ are real analytic.
We now show formula (2.1.29). The function $\Gamma^D_{F,h}[\rho]$ is given by the composition of $M_{F,h}[T]$ defined in Theorem 1.2.8 with the map which takes $\rho\in\mathcal{R}^D[F]$ to T^D_ρ. By standard calculus and Theorem 1.2.8 it follows

$$d\Gamma^D_{F,h}[\rho][\dot\rho] = \binom{|F|-1}{h-1}\left(\lambda^D_F[\,\rho]\right)^{1-h}\sum_{l=1}^{|F|}\langle dT^D_\rho[\dot\rho][u^D_l], u^D_l\rangle_{\mathcal{H}^2_0(\Omega)}\,, \tag{2.1.32}$$

for all $\rho \in \mathcal{R}^{\mathcal{D}}[\mathcal{F}], \dot{\rho} \in L^\infty(\Omega)$.
We have

$$\begin{aligned} \langle dT_\rho^D[\dot{\rho}][u_l^D], u_l^D\rangle_{\mathcal{H}_0^2(\Omega)} &= \Delta^2[dT_\rho^D[\dot{\rho}][u_l^D]][u_l^D] = \Delta^2[(\Delta^2)^{-1}dJ_\rho[\dot{\rho}][u_l^D]][u_l^D] \\ &= dJ_\rho[\dot{\rho}][u_l^D][u_l^D] = \int_\Omega \left(u_l^D\right)^2 \dot{\rho}\, dx\,, \end{aligned} \tag{2.1.33}$$

for all $\dot{\rho} \in L^\infty(\Omega), l \in F$. Now by (2.1.31), (2.1.32) and (2.1.33), and by standard calculus, it follows

$$\begin{aligned} d\Lambda_{F,h}^D[\rho][\dot{\rho}] &= \frac{d\Gamma_{F,|F|-h}^D[\rho][\dot{\rho}]\Gamma_{F,|F|}^D[\rho] - \Gamma_{F,|F|-h}^D[\rho]d\Gamma_{F,|F|}^D[\rho][\dot{\rho}]}{\Gamma_{F,|F|^2}^D[\rho]} \\ &= \left\{\binom{|F|-1}{|F|-h-1}\left(\lambda_F^D[\rho]\right)^{1-2|F|+h} - \binom{|F|}{h}\left(\lambda_F^D[\rho]\right)^{h+1-2|F|}\right\} \\ &\quad \cdot \left(\lambda_F^D[\rho]\right)^{2|F|}\sum_{l=1}^{|F|}\langle dT_\rho^D[\dot{\rho}][u_l^D], u_l^D\rangle_{\mathcal{H}_0^2(\Omega)} \\ &= -\left(\lambda_F^D[\rho]\right)^{h+1}\binom{|F|-1}{h-1}\sum_{l=1}^{|F|}\int_\Omega \left(u_l^D\right)^2\dot{\rho}\, dx\,. \end{aligned}$$

This concludes the proof. □

Remark 2.1.34. *We observe that if $j \in F$, then the restriction of $\lambda_j^{(\cdot)}[\rho]$ to $\Theta^{(\cdot)}[F]$ is a real analytic function, in fact $\lambda_j^{(\cdot)}[\cdot]$ coincides on $\Theta^{(\cdot)}[F]$ with the real analytic function $\frac{\Lambda_{F,1}^{(\cdot)}[\cdot]}{|F|}$.*

2.2 Critical mass densities

In this section we show that there are no critical mass densities for the symmetric functions of eigenvalues under fixed mass constraint.

We recall that the total mass of the set Ω with density $\rho \in \mathcal{R}$ is given by

$$M[\rho] = \int_\Omega \rho dx.$$

Definition 2.2.1. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a differentiable real valued function defined on an open subset U of $L^\infty(\Omega)$. We say that $\rho \in U$ is a critical mass density for F under the constraint*

$$M[\rho] = C \tag{2.2.2}$$

provided that

$$\operatorname{Ker} dM[\rho] \subseteq \operatorname{Ker} dF[\rho]. \tag{2.2.3}$$

We observe that

$$d|_{\rho=\tilde{\rho}}M[\rho][\dot{\rho}] = \int_\Omega \dot{\rho}dx.$$

Moreover, for $M \in]0, +\infty[$ fixed the set $\mathcal{R}[M] := \{\rho \in \mathcal{R} : M[\rho] = M\}$ is a Banach manifold of codimension 1, since $dM[\rho]$ is surjective.

As in the previous section, the symbol $(\cdot)$ stands for both D and I. By using the Lagrange multipliers Theorem we can state the following

Theorem 2.2.4. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Then for all $h = 1, ..., |F|$ there are no critical mass densities for the map which takes $\rho \in \mathcal{R}^{(\cdot)}[F]$ to $\Lambda^{(\cdot)}_{F,h}[\rho]$ under the constraint (2.2.2).*

Proof. Let $\tilde{\rho} \in \mathcal{R}^{(\cdot)}[F]$ be fixed. There exist an integer $n \in \mathbb{N}$ and a partition $\{F_1, ..., F_n\}$ of F such that $\tilde{\rho} \in \cap_{k=1}^n \Theta^{(\cdot)}[F_k]$. The restrictions of the functions $\lambda^{(\cdot)}_k[\cdot]$ to $\Theta^{(\cdot)}[F_k]$ are real analytic. Thus there exists an open neighbourhood $\mathcal{W}$ of $\tilde{\rho}$ in $\mathcal{R}^{(\cdot)}[F]$ such that $\mathcal{W} \subset \cap_{k=1}^n \mathcal{R}^{(\cdot)}[F_k]$. Let $h \in \{1, ..., |F|\}$. We write the function $\Lambda^{(\cdot)}_{F,h}$ in a more convenient way:

$$\Lambda^{(\cdot)}_{F,h}[\rho] = \sum_{\substack{0\le h_1\le|F_1|,...,0\le h_n\le|F_n| \\ h_1+\cdots+h_n=h}} \prod_{k=1}^n \Lambda^{(\cdot)}_{F_k,h_k}[\rho]\,, \tag{2.2.5}$$

for all $\rho \in \mathcal{W}$. Let's compute the differential of (2.2.5) at $\tilde{\rho}$. Thanks to formula (2.1.29) we can write the differential for each $\Lambda^{(\cdot)}_{F_k,h_k}$. We obtain

$$d\Lambda^{(\cdot)}_{F,h}[\tilde{\rho}][\dot{\rho}] = \sum_{\substack{0\le h_1\le|F_1|,...,0\le h_n\le|F_n| \\ h_1+\cdots+h_n=h}} \left(\sum_{k=1}^n d\Lambda^{(\cdot)}_{F_k,h_k}[\tilde{\rho}][\dot{\rho}] \prod_{\substack{j=1 \\ j\ne k}}^n \Lambda^{(\cdot)}_{F_j,h_j}[\tilde{\rho}] \right)$$

$$= \sum_{\substack{0\le h_1\le|F_1|,...,0\le h_n\le|F_n| \\ h_1+\cdots+h_n=h}} \left(\sum_{k=1}^n b_{h_k} \left(-\left(\lambda^{(\cdot)}_{F_k}[\tilde{\rho}]\right)^{h_k+1} \right) \binom{|F_k|-1}{h_k-1} \sum_{l\in F_k} \int_\Omega \left(u^{(\cdot)}_l\right)^2 \dot{\rho}\, dx \right),$$

where $b_{h_k} = \prod_{\substack{j=1 \\ j\ne k}}^n \Lambda^{(\cdot)}_{F_j,h_j}[\tilde{\rho}]$, and $\{u^{(\cdot)}_l\}_{l\in F_k}$ is an orthonormal basis in $\mathcal{H}_{(\cdot)}$ of the eigenspace corresponding to the eigenvalue $\lambda^{(\cdot)}_{F_k}[\tilde{\rho}]$ and $\lambda^{(\cdot)}_{F_k}[\tilde{\rho}]$ is the common value of all eigenvalues in $\lambda^{(\cdot)}_j[\tilde{\rho}]$ with $j \in F_k$. It follows that

$$d\Lambda^{(\cdot)}_{F,h}[\tilde{\rho}][\dot{\rho}] = -\sum_{k=1}^n c_k \int_\Omega \sum_{l\in F_k} \left(u^{(\cdot)}_l\right)^2 \dot{\rho}\, dx = -\int_\Omega \left(\sum_{k=1}^n c_k \sum_{l\in F_k} \left(u^{(\cdot)}_l\right)^2 \right) \dot{\rho}\, dx\,, \tag{2.2.6}$$

for all $\dot{\rho} \in L^\infty(\Omega)$ and suitable positive constants $c_k \in \mathbb{R}$, $k = 1, ..., n$. Suppose now that $\tilde{\rho}$ is a critical mass density for $\Lambda^{(\cdot)}_{F,h}$ under the constraint (2.2.2). This implies the existence of a Lagrange multiplier, i.e., there exists $c \in \mathbb{R}$ such that $d\Lambda^{(\cdot)}_{F,h}[\tilde{\rho}] = -cdM[\tilde{\rho}]$, that is

$$\int_\Omega \left(\sum_{k=1}^n c_k \sum_{l\in F_k} \left(u^{(\cdot)}_l\right)^2 \right) \dot{\rho}\, dx = c \int_\Omega \dot{\rho}\, dx\,,$$

for all $\dot\rho \in L^\infty(\Omega)$. Since $\dot\rho$ is arbitrary, it follows

$$\left(\sum_{k=1}^{n} c_k \sum_{l\in F_k}\left(u_l^{(\cdot)}\right)^2\right) = c\,, \quad \text{a.e. in } \Omega\,.$$

This equality implies $c \geq 0$. Now if $u_l^{(\cdot)} \in \mathcal{H}_{(\cdot)}(\Omega)$, it is shown by a standard approximation argument (cfr. [27]) that $u^{(\cdot)} := (\sum_{k=1}^{n}\sum_{l\in F_k}(\sqrt{c_k}u_l^{(\cdot)})^2)^{1/2}$ is in $\mathcal{H}_{(\cdot)}(\Omega)$ and is equal a.e. on Ω to $\sqrt{c}$. Then $\nabla u^{(\cdot)} = 0$ a.e. on Ω, and by Poincaré inequality we get $c = 0$, hence $u_l^{(\cdot)} = 0$ for all $l \in F$. This is a contradiction. $\square$

Corollary 2.2.7. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a finite nonempty subset of $\mathbb{N}\setminus\{0\}$. Let $M > 0$ and $L_M := \{\rho \in L^\infty(\Omega) : \int_\Omega \rho dx = M\}$. Then for all $h = 1, ..., |F|$ the real valued function which takes $\rho \in \mathcal{R}^{(\cdot)}[F]\cap L_M$ to $\Lambda_{F,h}^{(\cdot)}[\rho]$ has no local maxima or minima.*

We now show a continuity result for the eigenvalues with respect to the weak* topology of $L^\infty(\Omega)$. The proof is based on the argument of [8].

Proposition 2.2.8. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and $C \subset L^\infty(\Omega)$ a weakly* compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho\in C}\operatorname{ess\,inf}_{x\in\Omega}\rho(x) > 0$. Then the functions which take $\rho \in C$ to $\lambda_j^{(\cdot)}[\rho]$ are continuous in the weak* topology of $L^\infty(\Omega)$.*

Proof. It is enough to prove that if $\{\rho_n\}$ is a sequence in C converging to ρ in the weak* topology of $L^\infty(\Omega)$, then the sequence $\{\lambda_j^{(\cdot)}[\rho_n]\}$ converges to $\lambda_j^{(\cdot)}[\rho]$. By Banach-Steinhaus Theorem it follows that C is bounded in $L^\infty(\Omega)$. Let $\alpha := \inf_{\rho\in C}\operatorname{ess\,inf}_{x\in\Omega}\rho(x) > 0$ and $\beta := \sup_{\rho\in C}\operatorname{ess\,sup}_{x\in\Omega}\rho(x) < +\infty$. Thus, for all $\rho \in C$ we have $\alpha \leq \rho(x) \leq \beta$ a.e. in Ω. We denote $\lambda_j^{(\cdot)}[\rho_n] = \lambda_j^{(\cdot),n}$ and by $u_j^{(\cdot),n}$ the respective eigenfunctions, normalized by $\int_\Omega \rho_n u_i^{(\cdot),n}u_j^{(\cdot),n}dx = \delta_{ij}$. By the min-max principles (2.1.8) and (2.1.26) it follows that $\lambda_j^{(\cdot)}[\beta] \leq \lambda_j^{(\cdot),n} \leq \lambda_j^{(\cdot)}[\alpha]$ for all $n \in \mathbb{N}$. Since $\left\|\Delta^2 u_j^{D,n}\right\|_{L^2(\Omega)}^2 = \lambda_j^{D,n}$ and $\left\|D^2 u_j^{I,n}\right\|_{L^2(\Omega)}^2 = \lambda_j^{I,n}$, we get that the sequence $\{u_j^{(\cdot),n}\}$ is bounded in $h_{(\cdot)}(\Omega)$ for all $j \in \mathbb{N}$. By possibly passing to subsequences, we can directly assume that there exist $\bar\lambda_j^{(\cdot)} \in \mathbb{R}$ and $\bar u_j^{(\cdot)} \in \mathcal{H}_{(\cdot)}$ such that for all $j \in \mathbb{N}$ the sequence $\{\lambda_j^{(\cdot),n}\}_{n\in\mathbb{N}}$ converges to $\bar\lambda_j^{(\cdot)}$, and the sequence $\{u_j^{(\cdot),n}\}_{n\in\mathbb{N}}$ weakly converges to $\bar u_j^{(\cdot)}$. Since $\mathcal{H}_{(\cdot)}(\Omega) \subset H_0^1(\Omega)$ the sequence $\{u_j^{(\cdot),n}\}_{n\in\mathbb{N}}$ is strongly convergent in $L^2(\Omega)$. By the fact that $0 < \lambda_1^{(\cdot),n} \leq \lambda_2^{(\cdot),n} \leq \cdots$, we get $0 < \bar\lambda_1^{(\cdot)} \leq \bar\lambda_2^{(\cdot)} \leq \cdots$. It is now easy to see that

$$\lim_{n\to\infty}\int_\Omega \Delta u_j^{D,n}\Delta\phi - \lambda_j^{D,n}\rho_n u_j^{D,n}\phi dx = \int_\Omega \Delta\bar u_j^D\Delta\phi - \bar\lambda_j^D\rho\bar u_j^D\phi dx, \tag{2.2.9}$$

for all $\phi \in \mathcal{H}_D(\Omega)$, and

$$\lim_{n\to\infty}\int_\Omega \sum_{l,m=1}^{N}\partial_{lm}^2 u_j^{I,n}\partial_{lm}^2\phi - \lambda_j^{I,n}\rho_n u_j^{I,n}\phi dx = \int_\Omega \sum_{l,m=1}^{N}\partial_{lm}^2\bar u_j^I\partial_{lm}^2\phi - \bar\lambda_j^I\rho\bar u_j^I\phi dx, \tag{2.2.10}$$

for all $\phi \in \mathcal{H}_I(\Omega)$. Moreover it can be easily seen that

$$\lim_{n\to\infty} \int_\Omega \rho_n u_i^{(\cdot),n} u_j^{(\cdot),n} dx = \int_\Omega \rho \bar{u}_i^{(\cdot)} \bar{u}_j^{(\cdot)} dx = \delta_{ij}. \tag{2.2.11}$$

By (2.2.9), (2.2.10) and (2.2.11) it follows that $\{\bar{\lambda}_j^{(\cdot)}\} \subseteq \{\lambda_j^{(\cdot)}[\rho]\}$. Next we observe that

$$\lim_{n\to\infty} \left\| \Delta u_j^{D,n} \right\|_{L^2(\Omega)} = \lim_{n\to\infty} (\lambda_j^{D,n})^{\frac{1}{2}} = (\bar{\lambda}_j^{D,n})^{\frac{1}{2}} = \left\| \Delta \bar{u}_j^D \right\|_{L^2(\Omega)}$$

and

$$\lim_{n\to\infty} \left\| D^2 u_j^{I,n} \right\|_{L^2(\Omega)} = \lim_{n\to\infty} (\lambda_j^{I,n})^{\frac{1}{2}} = (\bar{\lambda}_j^{I,n})^{\frac{1}{2}} = \left\| D^2 \bar{u}_j^I \right\|_{L^2(\Omega)}.$$

We need to show that $\{\lambda_j^{(\cdot)}[\rho]\} \subseteq \{\bar{\lambda}_j^{(\cdot)}\}$. Suppose that there exists $\bar{\lambda}^{(\cdot)} \in \{\lambda_j^{(\cdot)}[\rho]\} \setminus \{\bar{\lambda}_j^{(\cdot)}\}$, and $\bar{u}^{(\cdot)}$ a non trivial element of the eigenspace associated with $\bar{\lambda}^{(\cdot)}$. Then for all $j \in \mathbb{N}$ we have $\int_\Omega \rho \bar{u}^{(\cdot)} \bar{u}_j^{(\cdot)} dx = 0$. Let $\bar{u}^{(\cdot)}$ be normalized by $\left(\int_\Omega \rho (\bar{u}^{(\cdot)})^2 dx\right)^{\frac{1}{2}} = (\bar{\lambda}^{(\cdot)})^{-1}$. For $\rho \in \mathcal{R}$ we define $\mathcal{A}_j^{(\cdot)}(\rho, u)$ by

$$\mathcal{A}_j^D(\rho, u) := \frac{1}{2} \left\| \Delta u \right\|_{L^2(\Omega)}^2 - \left(\int_\Omega \rho \left((\mathbb{I} - P_{j-1}[\rho]) \, u \right)^2 dx \right)^{\frac{1}{2}},$$

$$\mathcal{A}_j^I(\rho, u) := \frac{1}{2} \left\| D^2 u \right\|_{L^2(\Omega)}^2 - \left(\int_\Omega \rho \left((\mathbb{I} - P_{j-1}[\rho]) \, u \right)^2 dx \right)^{\frac{1}{2}},$$

for all $j \in \mathbb{N}$, $u \in \mathcal{H}_{(\cdot)}(\Omega)$ where

$$P_j[\rho] u := \sum_{i=1}^{j} \left(\int_\Omega \rho u u_i dx \right) u_i.$$

We need to invoke the following variational representation for the eigenvalues of problems (2.1.2) and (2.1.16):

$$\frac{-1}{2\lambda_j^{(\cdot)}[\rho]} = \inf_{u \in h_{(\cdot)}(\Omega)} \mathcal{A}_j^{(\cdot)}(\rho, u), \tag{2.2.12}$$

see [3, pp.55-71]. By (2.2.12) it follows that

$$\frac{-1}{2\lambda_j^{(\cdot),n}} \leq \mathcal{A}_j^{(\cdot)}(\rho_n, \bar{u}^{(\cdot)}). \tag{2.2.13}$$

Furthermore, we observe that

$$\begin{aligned} \lim_{n\to\infty} P_{j-1}[\rho_n] \bar{u}^{(\cdot)} &= \lim_{n\to\infty} \sum_{k=1}^{j-1} \left(\int_\Omega \rho_n \bar{u}^{(\cdot)} u_k^{(\cdot),n} dx \right) u_k^{(\cdot),n} \\ &= \sum_{k=1}^{j-1} \left(\int_\Omega \rho \bar{u}^{(\cdot)} \bar{u}_k^{(\cdot)} dx \right) u_k^{(\cdot),n} = 0 \end{aligned}$$

in $L^2(\Omega)$. Then an easy computation shows that

$$\lim_{n\to\infty} \mathcal{A}_j^{(\cdot)}(\rho_n, \bar{u}^{(\cdot)}) = \frac{-1}{2\bar{\lambda}^{(\cdot)}}.$$

Now let $n \to \infty$ in (2.2.13). We find

$$\frac{-1}{2\bar{\lambda}_j^{(\cdot)}} \le \frac{-1}{2\bar{\lambda}^{(\cdot)}}$$

for all $j \in \mathbb{N}$. But this is a contradiction since $\{\bar{\lambda}_j^{(\cdot)}\}$ is not bounded from above. Then $\{\bar{\lambda}_j^{(\cdot)}\} = \{\lambda_j^{(\cdot)}[\rho]\}$. This concludes the proof. □

Finally we can prove the following

Theorem 2.2.14. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Let $C \subseteq \mathcal{R}^{(\cdot)}[F]$ a weakly* compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho\in C} \operatorname{ess\,inf}_{x\in\Omega} \rho(x) > 0$. Let $M > 0$ and L_M defined as in Corollary 2.2.7 . Then for all $h = 1, ..., |F|$ the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}^{(\cdot)}[\rho]$ admits points of maximum and minimum in $C \cap L_M$, and such points belong to $\partial C \cap L_M$.*

Proof. Since the functions which take $\rho \in C \cap L_M$ to $\lambda_j^{(\cdot)}[\rho]$ are weakly* continuous in $L^\infty(\Omega)$ and $C \cap L_M$ is weakly* compact, then the functions $\Lambda_{F,h}^{(\cdot)}[\rho]$, which are composed by sums and products of the $\lambda_j^{(\cdot)}[\rho]$, admit points of maximum and minimum in $C \cap L_M$, but by (2.2.7) they cannot be in the interior of C, hence they belong to $\partial C \cap L_M$. □

Our aim is now to extend the results proved in [8, 9, Cox-McLaughlin] to our case. We fix a class of weakly* compact and convex subsets of $L^\infty(\Omega)$ and we show that in certain cases, the minimizers and maximizers of the eigenfunctions (that exist by Theorem 2.2.14) are extreme points of such sets (the so-called 'bang bang' controls). Let then Ω be a domain in $\mathbb{R}^N$ of finite measure $|\Omega|$. Let α, β, M be such that $0 < \alpha < \frac{M}{|\Omega|} < \beta$ (if one of the last two inequality were an equality, the problem would not be of interest). Let $\gamma := \frac{M-|\Omega|\beta}{\alpha-\beta}$.

Definition 2.2.15. *We define the subset ad_γ of $L^\infty(\Omega)$ as*

$$ad_\gamma := \{\rho \in L^\infty(\Omega) : \rho = \alpha\chi + \beta(1-\chi),\ \chi \subset \Omega \text{ measurable},\ |\chi| = \gamma\}$$

Proposition 2.2.16. *The weak* closure of ad_γ is the convex weak* compact set*

$$ad_\gamma^* := \left\{\rho \in L^\infty(\Omega) : \alpha \le \rho(x) \le \beta \text{ a.e. in } \Omega,\ \int_\Omega \rho dx = M\right\}.$$

Recall that $v \in K$ is an extreme point of a convex set K if $K \setminus \{v\}$ is convex.

Proposition 2.2.17. *The set of extreme points of ad_γ^* is exactly ad_γ.*

For proofs of the previous statements we refer to [8].

In general, for both Dirichlet and intermediate boundary conditions the first eigenfunction may change sign. However for the ball we have the following

Theorem 2.2.18. *If $\Omega = B \subset \mathbb{R}^N$, the open unit ball, then the first eigenvalue of (2.1.2) is simple and the correspondig eigenfunction does not change sign in Ω.*

We refer to [14, Thm. 3.7] for the proof. There are other cases in which the structure of the domain yelds positivity of first eigenfuncion, eg., in $\mathbb{R}^2$ ellipses with small eccentricity and annuli with a sufficiently big inner radius (cfr. [14]). In the remaining part of this subsection we treat the case of Dirichlet boundary conditions.

First of all, by Theorem 2.2.4 in the case of the ball, we get the following

Corollary 2.2.19. *Let $\Omega = B \subset \mathbb{R}^N$. Then there are no critical mass densities in $\mathcal{R}$ for the function which takes $\rho \to \lambda_1[\rho]$ under the fixed mass constraint (2.2.2).*

Proof. We set $F = \{1\}$ in the Theorem 2.2.4. It is evident that $\mathcal{R}[F] = \Theta[F] = \mathcal{R}$. Then the proof of the corollary follows immediately by Theorem 2.2.4. □

Thanks to Proposition 2.2.8 we get for $j \in \mathbb{N}$ the existence of $\check{\rho}_j, \hat{\rho}_j$, in ad^*_γ such that

$$\check{\lambda}_j := \lambda_j[\check{\rho}_j] = \inf_{\rho \in ad^*_\gamma} \lambda_j[\rho],$$
$$\hat{\lambda}_j := \lambda_j[\hat{\rho}_j] = \sup_{\rho \in ad^*_\gamma} \lambda_j[\rho].$$

Moreover, we have the following

Proposition 2.2.20. *Let $\Omega = B \subset \mathbb{R}^N$, $\rho \in \mathcal{R}$. Then we have*

i) the minimizer $\check{\rho}_1$ may be chosen from ad_γ;

ii) the maximizer $\hat{\rho}_1$ belongs to ad_γ and it is unique.

The proof of the previous proposition can be carried out by using exactly the same argument of [9, Corol. 6.2 (i), Prop.7.10]

2.3 Extension to poly-harmonic operators. Critical mass densities

As in the previous section, we consider a domain Ω in $\mathbb{R}^N$ of finite measure and a density $\rho \in \mathcal{R}$.

We first consider the case of Dirichlet boudary condition, namely:

$$\begin{cases} (-\Delta)^n u = \lambda \rho u, & \text{in } \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{n-1} u}{\partial \nu^{n-1}} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{2.3.1}$$

for $n > 2$. We recall that the case $n = 1$ has been studied in [19], while the case $n = 2$ has been treated in the previous section. The weak formulation of problem (2.3.1) is:

$$\int_\Omega \mathcal{D}_n u \mathcal{D}_n \phi dx = \lambda \int_\Omega \rho u \phi dx, \quad \forall \phi \in H^n_0(\Omega), \tag{2.3.2}$$

in the unknowns $u \in H^n_0(\Omega)$, $\lambda \in \mathbb{R}$, where

$$\mathcal{D}_n u = \begin{cases} \nabla \Delta^m u, & \text{if } n = 2m+1 \\ \Delta^m u, & \text{if } n = 2m. \end{cases}$$

First we consider the poly-harmonic operator $(-\Delta)^n$ as a map from $H_0^n(\Omega)$ to $(H_0^n(\Omega))'$ defined by

$$(-\Delta)^n[u][\phi] = \int_\Omega \mathcal{D}_n u \mathcal{D}_n \phi dx, \quad \forall \phi \in H_0^n(\Omega). \tag{2.3.3}$$

As in the case of the biharmonic operator with Dirichlet boundary conditions, it is easy to see that $(-\Delta)^n$ is a linear homeomorphism between $H_0^n(\Omega)$ and $(H_0^n(\Omega))'$. We denote by i_n the canonical (continuous) embedding of $H_0^n(\Omega)$ into $L^2(\Omega)$, which is compact by (1.1.14), and by $J_{n,\rho}$ the continuous embedding of $L^2(\Omega)$ into $(H_0^n(\Omega))'$, defined by

$$J_{n,\rho}[u][\phi] = \int_\Omega \rho u \phi dx \quad \forall u \in L^2(\Omega), \phi \in H_0^n(\Omega).$$

In this way, problem (2.3.2) is equivalent to the following one:

$$((-\Delta)^n)^{-1} \circ J_{n,\rho} \circ i_n u = \lambda^{-1} u, \tag{2.3.4}$$

in the unknowns $u \in H_0^n(\Omega)$, $\lambda \in \mathbb{R}$. Finally, we observe that on $H_0^n(\Omega)$ the bilinear form

$$< u, v >= \int_\Omega \mathcal{D}_n u \mathcal{D}_m v dx, \quad \forall u, v \in H_0^n(\Omega), \tag{2.3.5}$$

defines a scalar product whose induced norm is equivalent to the standard one. We denote by $\mathcal{H}_0^n(\Omega)$ the space $H_0^n(\Omega)$ endowed with this norm. Then we can state the following

Theorem 2.3.6. *Let $n > 2$, Ω be a domain in $\mathbb{R}^N$ of finite measure and $\rho \in \mathcal{R}$. Then the following facts hold:*

i) The operator $T_{n,\rho} := ((-\Delta)^n)^{-1} \circ J_{n,\rho} \circ i_n$ is a compact selfadjoint operator in $\mathcal{H}_0^n(\Omega)$, whose eigenvalues coincide with the reciprocals of the eigenvalues of (2.3.2).

ii) The set Σ_n of the eigenvalues of (2.3.2) is contained in $]0, +\infty[$ and it consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity. Moreover, $(-\Delta)^n$ has a Hilbert basis in $\mathcal{H}_0^n(\Omega)$ of eigenfunctions.

Next we treat the case of intermediate boundary conditions. Let $n > 2$ and $\frac{n}{2} \leq k \leq n$ if n is even, $\frac{n+1}{2} \leq k \leq n$ if n is odd. The classical formulation of the problem is:

$$\begin{cases} (-\Delta)^n u = \lambda \rho u, & \text{in } \Omega\,, \\ u = \frac{\partial u}{\partial \nu} = \dots = \frac{\partial^{k-1} u}{\partial \nu^{k-1}} = 0, & \text{on } \partial\Omega\,, \\ B_k(x;D)u = B_{k+1}(x;D)u = \dots = B_{n-1}(x;D)u = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{2.3.7}$$

where $B_j(x;D)$, $j = k, k+1, \dots, n-1$, are suitable linear differential operators of order m_j, corresponding to weak problem

$$\int_\Omega \sum_{i_1,\dots,i_n=1}^{N} \frac{\partial^n u}{\partial x_{i_1} \cdots \partial_{x_{i_n}}} \frac{\partial^n \phi}{\partial x_{i_1} \cdots \partial_{x_{i_n}}} dx = \lambda \int_\Omega \rho u \phi dx, \quad \forall \phi \in H^n(\Omega) \cap H_0^k(\Omega), \tag{2.3.8}$$

in the unknowns $u \in H^n(\Omega) \cap H_0^k(\Omega)$, $\lambda \in \mathbb{R}$.

From now on, we will denote the space $H^n(\Omega) \cap H_0^k(\Omega)$ by $V_n(\Omega)$. We first consider the polyharmonic operator $(-\Delta)^n$ as a map from $V_n(\Omega)$ to $(V_n(\Omega))'$ defined by

$$(-\Delta)^n[u][\phi] = \int_\Omega \sum_{i_1,\dots,i_n=1}^N \frac{\partial^n u}{\partial x_{i_1} \cdots \partial_{x_{i_n}}} \frac{\partial^n \phi}{\partial x_{i_1} \cdots \partial_{x_{i_n}}} dx, \quad \forall \phi \in V_n(\Omega). \tag{2.3.9}$$

This is a linear homeomorphism between $V_n(\Omega)$ and its dual. This is a consequence of the fact that there exists $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C \|\mathcal{D}^n u\|_{L^2(\Omega)}, \tag{2.3.10}$$

where

$$\mathcal{D}^n u := \left(\sum_{i_1,\dots,i_n=1}^N \left(\frac{\partial^n u}{\partial x_{i_1}, \cdots \partial x_{i_n}} \right)^2 \right)^{\frac{1}{2}}.$$

In fact an integration by parts yields

$$\int_\Omega \partial^n_{i,\dots,i} u \, u dx = (-1)^m \int_\Omega \left(\partial^m_{i,\dots,i} u\right)^2 dx, \quad \text{if } n = 2m,$$
$$\int_\Omega \partial^n_{i,\dots,i} u \, \partial_i u dx = (-1)^{m-1} \int_\Omega \left(\partial^m_{i,\dots,i} u\right)^2 dx, \quad \text{if } n = 2m-1,$$

for all $u \in V_n(\Omega)$. Then, taking the modulus of both terms, we gain

$$\int_\Omega \left(\partial^m_{i,\dots,i} u\right)^2 dx \leq \left\|\partial^n_{i,\dots,i} u\right\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}, \quad \text{if } n = 2m, \tag{2.3.11}$$
$$\int_\Omega \left(\partial^m_{i,\dots,i} u\right)^2 dx \leq \left\|\partial^n_{i,\dots,i} u\right\|_{L^2(\Omega)} \|\partial_i u\|_{L^2(\Omega)}, \quad \text{if } n = 2m-1. \tag{2.3.12}$$

Summing over i and by applying the Poincaré inequality to the left hand side of (2.3.11) and (2.3.12) respectively, since $H^n(\Omega) \cap H_0^k(\Omega) \subset H^n(\Omega) \cap H_0^{\frac{n}{2}}(\Omega)$ for $\frac{n}{2} \leq k \leq n$ if n is even, $H^n(\Omega) \cap H_0^k(\Omega) \subset H^n(\Omega) \cap H_0^{\frac{n+1}{2}}(\Omega)$ for $\frac{n+1}{2} \leq k \leq n$ if n odd, we obtain that there exist $C > 0$ such that

$$\|u\|^2_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega)} \|\mathcal{D}^n u\|_{L^2(\Omega)}, \quad \text{if } n = 2m,$$
$$\|\nabla u\|^2_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \|\mathcal{D}^n u\|_{L^2(\Omega)}, \quad \text{if } n = 2m-1,$$

respectively. In the second inequality we divide by $\|\nabla u\|_{L^2(\Omega)}$ and use again Poincaré inequality. Now the proof of (2.3.10) is straightforward. Then we have the coercivity of $(-\Delta)^n$. The continuity is clear.

Next, we denote by i_n the canonical (compact) embedding of $V_n(\Omega)$ into $L^2(\Omega)$, and by $J_{n,\rho}$ the continuous embedding of $L^2(\Omega)$ into $(V_n(\Omega))'$ defined by

$$J_{n,\rho}[u][\phi] = \int_\Omega \rho u \phi dx, \quad \forall \phi \in V_n(\Omega).$$

Finally, we observe that the bilinear form

$$< u, v >= \int_\Omega \sum_{i_1,\dots,i_n=1}^N \frac{\partial^n u}{\partial x_{i_1} \cdots \partial_{x_{i_n}}} \frac{\partial^n v}{\partial x_{i_1} \cdots \partial_{x_{i_n}}} dx \tag{2.3.13}$$

defines on $V_n(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. We will denote by $\mathcal{V}_n(\Omega)$ the space $V_n(\Omega)$ endowed with this norm. Then we can state the following

Theorem 2.3.14. *Let $n > 2$, Ω be a domain in $\mathbb{R}^N$ of finite measure, $\rho \in \mathcal{R}$. Then the following facts hold:*

i) The operator $T_{n,\rho} := ((-\Delta)^n)^{-1} \circ J_{n,\rho} \circ i_n$ is a compact selfadjoint operator in $\mathcal{V}_n(\Omega)$, whose eigenvalues coincide with the reciprocals of the eigenvalues of (2.3.8).

ii) The set Σ_n of the eigenvalues of (2.3.8) is contained in $]0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity. Moreover, $(-\Delta)^n$ has a Hilbert basis in $\mathcal{V}_n(\Omega)$ of eigenfunctions.

We represent the set Σ_n^D of the eigenvalues of problems (2.3.2) by means of a sequence

$$0 < \lambda_{n,1}^D[\rho] \leq \lambda_{n,2}^D[\rho], ..., \lambda_{n,j}^D[\rho], ...$$

and the set Σ_n^I of the eigenvalues of problems (2.3.8) by means of a sequence

$$0 < \lambda_{n,1}^I[\rho] \leq \lambda_{n,2}^I[\rho], ..., \lambda_{n,j}^I[\rho], ...$$

where each eigenvalue is repeated accordingly to its multiplicity. We denote $\mathcal{H}_n^D(\Omega) := \mathcal{H}_0^n(\Omega)$, $\mathcal{H}_n^I(\Omega) := \mathcal{V}_n(\Omega)$, by $(-\Delta)_D^n$ the operator defined in (2.3.3) and by $(-\Delta)_I^n$ the operator defined in (2.3.9). We have the following variational representation of the eigenvalues

Theorem 2.3.15. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and $\rho \in \mathcal{R}$. Then we have*

i)

$$\lambda_{n,1}^{(\cdot)}[\rho] = \inf_{\substack{u \in \mathcal{H}_n^{(\cdot)}(\Omega) \\ u \neq 0}} \frac{(-\Delta)_{(\cdot)}^n[u][u]}{\int_\Omega u^2 \rho \, dx} . \tag{2.3.16}$$

The eigenfunctions corresponding to $\lambda_1^{(\cdot)}[\rho]$ are exactly the minimizers in (2.3.16).

ii) For all $j \in \mathbb{N}$

$$\lambda_{n,j}^{(\cdot)}[\rho] = \inf_{\substack{E \leq \mathcal{H}_n^{(\cdot)}(\Omega) \\ \dim E = j}} \sup_{0 \neq u \in E} \frac{(-\Delta)_{(\cdot)}^n[u][u]}{\int_\Omega u^2 \rho \, dx} , \tag{2.3.17}$$

where $(\cdot)$ stands for D or I.

Exactly as in the case $n = 2$ this representation yields the local Lipschitz continuity of the functions $\rho \to \lambda_{n,j}^{(\cdot)}[\rho]$ in $\rho \in \mathcal{R}$.

Now, in the same way as for the case $n = 2$ we can compute the derivatives of symmetric functions of eigenvalues. In fact by the same arguments used in the proof of Theorem 2.1.27 one can prove the following

Theorem 2.3.18. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure, F a nonempty finite subset of $\mathbb{N}\setminus\{0\}$ and let*

$$\begin{aligned}\mathcal{R}_n^{(\cdot)}[F] &:= \{\rho \in \mathcal{R} \,:\, \lambda_{n,j}^{(\cdot)}[\rho] \neq \lambda_{n,l}^{(\cdot)}[\rho]\,,\ \forall j \in F, l \in \mathbb{N}\setminus F\}\,,\\ \Theta_n^{(\cdot)}[F] &:= \{\rho \in \mathcal{R}_n^{(\cdot)}[F] \,:\, \lambda_{n,j_1}^{(\cdot)}[\rho] = \lambda_{n,j_2}^{(\cdot)}[\rho]\,,\ \forall j_1, j_2 \in F\}.\end{aligned}$$

Then $\mathcal{R}_n^{(\cdot)}[F]$ is open in $L^\infty(\Omega)$ and the symmetric functions of the eigenvalues

$$\Lambda_{n,F,h}^{(\cdot)}[\rho] = \sum_{\substack{j_1,\dots,j_h\in F\\ j_1<\dots j_h}} \lambda_{n,j_1}^{(\cdot)}[\rho]\cdots\lambda_{n,j_h}^{(\cdot)}[\rho]\,,\qquad h = 1,\dots,|F| \tag{2.3.19}$$

are real analytic in $\mathcal{R}_n^{(\cdot)}[F]$. Moreover, if $\rho \in \Theta_n^{(\cdot)}[F]$ and the eigenvalues $\lambda_{n,j}^{(\cdot)}[\rho]$ assume the common value $\lambda_{n,F}^{(\cdot)}[\rho]$ for all $j \in F$, then the differential of $\Lambda_{n,F,h}^{(\cdot)}$ at ρ is given by the formula

$$d\Lambda_{n,F,h}^{(\cdot)}[\rho][\dot\rho] = \left(-\lambda_{n,F}^{(\cdot)}[\rho]\right)^{h+1}\binom{|F|-1}{h-1}\sum_{l\in F}\int_\Omega (u_{n,l}^{(\cdot)})^2\dot\rho\,dx\,, \tag{2.3.20}$$

for all $\dot\rho \in L^\infty(\Omega)$, where $\{u_{n,l}^{(\cdot)}\}$ is an orthonormal basis for $\lambda_{n,F}^{(\cdot)}[\rho]$ in $\mathcal{H}_n^{(\cdot)}(\Omega)$.

We can say now that there aren't critical mass densities for the symmetric functions of the eigenvalues under mass constraint. In fact, since $\mathcal{H}_n^{(\cdot)}(\Omega) \subset H_0^1(\Omega)$, the same argument used in the proof of Theorem 2.2.4 holds. We can state then the following

Theorem 2.3.21. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a nonempty finite subset of $\mathbb{N}\setminus\{0\}$. Then for all $h = 1,...,|F|$ the function which takes $\rho \in \mathcal{R}_n^{(\cdot)}[F]$ to $\Lambda_{n,F,h}^{(\cdot)}[\rho]$ has no critical mass densities in $\mathcal{R}_n^{(\cdot)}[F]$ under the constraint (2.2.2).*

It follows immediately

Corollary 2.3.22. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a nonempty finite subset of $\mathbb{N}\setminus\{0\}$. Let $M > 0$ and $L_M := \{\rho \in L^\infty(\Omega) : \int_\Omega \rho dx = M\}$. Then for all $h = 1,...,|F|$ the function which takes $\rho \in \mathcal{R}_n^{(\cdot)}[F] \cap L_M$ to $\Lambda_{n,F,h}^{(\cdot)}[\rho]$ has no local maxima or minima.*

Now we can state the following

Theorem 2.3.23. *Let Ω be a domain in $\mathbb{R}^N$ of finite measure and F a nonempty finite subset of $\mathbb{N}\setminus\{0\}$. Let $C_n \subseteq \mathcal{R}_n^{(\cdot)}[F]$ a weakly* compact set in $L^\infty(\Omega)$ such that $\inf_{\rho\in C_n}\operatorname{ess\,inf}_{x\in\Omega}\rho(x) > 0$. Let $M > 0$ and L_M defined as in Corollary 2.3.22. Then for all $h = 1,...,|F|$ the function which takes $\rho \in C_n \cap L_M$ to $\Lambda_{n,F,h}^{(\cdot)}$ admits points of maximum and minimum, and such points belong to $\partial C_n \cap L_M$.*

Proof. One can prove by the same arguments of Proposition 2.2.8 the continuity of the eigenvalues with respect to the weak* topology of $L^\infty(\Omega)$. The remaining part of the proof is equal to that of Theorem 2.2.14. □

Finally, see [14, 8, 9], we have

Theorem 2.3.24. *If $\Omega \subset \mathbb{R}^N$ is the unit open ball, then the first eigenvalue of (2.3.2)is simple and the correspondig eigenfunction is of one sign. Moreover, the following statements hold:*

i) there are no critical mass densities in $\mathcal{R}$ for the function $\rho \to \lambda^D_{n,1}[\rho]$ under the fixed mass constraint (2.2.2);

ii) there exist minimizers and maximizers for the functions $\rho \to \lambda^D_{n,1}[\rho]$ in the set ad^*_γ defined in (2.2.16), and such minimizers and maximizers can be chosen in the set ad_γ.

2.4 The mixed Neumann-Dirichlet problem for the laplacian

In this section we extend the results of the previous sections to a mixed Neumann-Dirichlet problem. From now on it is understood that Ω is a bounded domain in $\mathbb{R}^N$ of class C^1. We consider two nonempty open parts of the boundary, namely Γ_0 and Γ_1, which consist of a finite number of connected components and which satisfy

$$\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1, \quad \Gamma_0 \cap \Gamma_1 = \emptyset, \tag{2.4.1}$$

where $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ denote the closure in $\partial\Omega$ of Γ_0 and Γ_1 respectively. We consider then the following problem, for $\rho \in \mathcal{R}$

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ u = 0, & \text{on } \Gamma_0\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_1\,, \end{cases} \tag{2.4.2}$$

in the unknowns $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\lambda \in \mathbb{R}$. This problem models a vibrating membrane which has a fixed part of his frame, while the remaining part is free. As for the case studied in [19], we have the following weak formulation of the problem

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \lambda \int_\Omega \rho u \phi dx, \quad \forall \phi \in H^1_{0,\Gamma_0}(\Omega), \tag{2.4.3}$$

in the unknowns $u \in H^1_{0,\Gamma_0}(\Omega)$, $\lambda \in \mathbb{R}$, where

$$H^1_{0,\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0 \text{ in the sense of traces } \right\}.$$

Remark 2.4.4. *One can show that the space*

$$C^\infty_{c,\Gamma_0}(\bar{\Omega}) = \left\{ u \in C^\infty(\bar{\Omega}) : u = 0 \text{ in a neighbourhood of } \Gamma_0 \right\}$$

is dense in $H^1_{0,\Gamma_0}(\Omega)$, see [4].

Now by a standard argument it is easy to prove the following

Proposition 2.4.5. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let Γ_0 and Γ_1 be defined as in (2.4.1). Then there exists $C > 0$ such that for all $u \in H^1_{0,\Gamma_0}(\Omega)$*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}\,. \tag{2.4.6}$$

Proof. We note that in order to prove (2.4.6) it is sufficient to prove $\|u\|_{H^1(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$. Assume by contradiction that there exists a sequence $\{u_m\}$ such that for all $m \in \mathbb{N}$ we have $\frac{1}{m}\|u_m\|_{H^1(\Omega)} > \|\nabla u_m\|_{L^2(\Omega)}$. Let $v_m = \frac{u_m}{\|u_m\|_{H^1(\Omega)}}$. Then we get $\|v_m\|_{H^1(\Omega)} = 1$ and $\frac{1}{m} > \|\nabla v_m\|_{L^2(\Omega)}$ for all $m \in \mathbb{N}$. The sequence $\{v_m\}$ is bounded in $H^1(\Omega)$, and by the compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$, there exists subsequence, denoted again v_m, converging to a certain v strongly in $L^2(\Omega)$, weakly in $H^1(\Omega)$. Since $\|\nabla v_m\|_{L^2(\Omega)} \to 0$, and $\nabla v_m \rightharpoonup \nabla v$ in the sense of distributions, then by the uniqueness of the limit, $\nabla v = 0$. Then v is constant. Since v vanishes on a part of the boundary, then it is identically zero on Ω. Since $\|v\|_{H^1(\Omega)} = 1$, we have a contradiction. □

We consider the operator $-\Delta$ from $H^1_{0,\Gamma_0}(\Omega)$ to $(H^1_{0,\Gamma_0}(\Omega))'$ defined by

$$-\Delta[u][\phi] = \int_\Omega \nabla u \cdot \nabla \phi, \quad \forall \phi \in H^1_{0,\Gamma_0}(\Omega).$$

The operator $-\Delta$ turns out to be a linear homeomorphism by Proposition 2.4.5. Moreover, we consider the canonical compact embedding i of $H^1_{0,\Gamma_0}(\Omega)$ into $L^2(\Omega)$ and the continuous embedding J_ρ of $L^2(\Omega)$ into $(H^1_{0,\Gamma_0}(\Omega))'$ defined by

$$J_\rho[u][\phi] = \int_\Omega \rho u \phi, \quad \forall \phi \in H^1_{0,\Gamma_0}(\Omega).$$

Then the operator $T_\rho = (-\Delta)^{-1} \circ J_\rho \circ i$ is a compact selfadjoint operator in $\mathcal{H}^1_{0,\Gamma_0}(\Omega)$, which is the space $H^1_{0,\Gamma_0}(\Omega)$ endowed with the equivalent scalar product

$$< u, v >_{\mathcal{H}^1_{0,\Gamma_0}(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall u, v \in H^1_{0,\Gamma_0}(\Omega).$$

The eigenvalues $\mu_j[\rho]$ of T_ρ coincide with the reciprocals of the eigenvalues $\lambda_j[\rho]$ of (2.4.3), which consist of the image of a positive sequence increasing to $+\infty$. As usual, we have the following variational representation of eigenvalues

$$\lambda_j[\rho] = \inf_{\substack{E \leq \mathcal{H}^1_{0,\Gamma_0}(\Omega) \\ \dim E = j}} \sup_{0 \neq u \in E} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 \rho \, dx}, \quad \forall j \in \mathbb{N}. \tag{2.4.7}$$

Exactly as in the case of the Dirichlet Laplacian, the first eigenfunction is simple and does not change sign. In fact, if $v \geq 0$ ($v \leq 0$) is a solution of (2.4.3), then the strong maximum (minimum) principle yields $v > 0$ ($v < 0$) in Ω. If we set $\mathcal{B}(u, v) = \int_\Omega \nabla u \cdot \nabla v dx$ for $u, v \in \mathcal{H}^1_{0,\Gamma_0}(\Omega)$, and we take a first eigenfunction u of (2.4.3) normalized by $\int_\Omega u^2 \rho dx = 1$, then we have $\lambda_1[\rho] = \mathcal{B}(u, u)$. Let $u^+ = \max\{u, 0\}$, $u^- = -\min\{-u, 0\}$, which are still in $\mathcal{H}^1_{0,\Gamma_0}$, and suppose that $u^+, u^- \neq 0$. We get

$$\mathcal{B}(u, u) = \mathcal{B}(u^+ - u^-, u^+ - u^-) = \mathcal{B}(u^+, u^+) + \mathcal{B}(u^-, u^-).$$

By (2.4.7) it follows that

$$\lambda_1[\rho] \leq \frac{\mathcal{B}(u^+, u^+)}{\int_\Omega (u^+)^2 \rho dx}, \qquad \lambda_1[\rho] \leq \frac{\mathcal{B}(u^-, u^-)}{\int_\Omega (u^-)^2 \rho dx}.$$

This implies that $\mathcal{B}(u^+,u^+)+\mathcal{B}(u^-,u^-) = \lambda_1[\rho]\left(\int_\Omega ((u^+)^2+(u^-)^2)\,\rho dx\right)$. But now this implies $|u|>0$ since $|u|$ is a minimizer in the Rayleigh quotient, a contradiction. The simplicity is an immediate consequence of the constancy of the sign.

Again, the variational representation (2.4.7) yields local Lipschitz-continuity of the eigenvalues with respect to the variable ρ. As in the previous sections, for $\emptyset \neq F \subset \mathbb{N}$ finite we set

$$\begin{aligned}
\mathcal{R}[F] &:= \{\rho \in \mathcal{R} \,:\, \lambda_j[\rho] \neq \lambda_l[\rho]\,,\ \forall j \in F, l \in \mathbb{N}\setminus F\}\,,\\
\Theta[F] &:= \{\rho \in \mathcal{R}[F] \,:\, \lambda_{j_1}[\rho] = \lambda_{j_2}[\rho]\,,\ \forall j_1, j_2 \in F\}.
\end{aligned}$$

Let

$$\Lambda_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h\in F\\ j_1<\cdots j_h}} \lambda_{j_1}[\rho]\cdots\lambda_{j_h}[\rho]\,,\quad h=1,\dots,|F|\,.$$

the symmetric functions of eigenvalues. Then we can state the following

Theorem 2.4.8. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, F a nonempty finite subset of $\mathbb{N}\setminus\{0\}$. Then for all $h=1,\dots,|F|$ the function which takes $\rho\in\mathcal{R}[F]$ to $\Lambda_{F,h}[\rho]$ is real analytic in $\mathcal{R}[F]$, and has no critical mass densities in $\mathcal{R}[F]$ under mass constraint (2.2.2).*

Proof. Exactly as in the proof of Theorem 2.1.27 one can show that $\Lambda_{F,h}$ are real analytic in $\mathcal{R}[F]$ and compute formulas for their derivatives. Then, by using these formulas as in the proof of Theorem 2.2.4, one gets that if $\tilde\rho$ is a critical mass density for $\Lambda_{F,h}$, then there exist $n\in\mathbb{N}$, a partition $\{F_1,\dots,F_n\}$ of F and real numbers $c_k, c \geq 0$ such that $\left(\sum_{k=1}^n c_k \sum_{j\in F_k} u_j^2\right) = c$ a.e. in Ω, where $\{u_j\}_{j\in F_k}$ is a certain orthonormal set in $\mathcal{H}^1_{0,\Gamma_0}$ of eigenfunctions of (2.4.3). Since $\tilde u := \left(\sum_{k=1}^n \sum_{j\in F_k} (\sqrt{c_k}u_j)^2\right)^{\frac12} = \sqrt{c}$ and $\tilde u \in \mathcal{H}^1_{0,\Gamma_0}$ then $c=0$, hence $u_j = 0$ in Ω for all j, a contradiction. □

Corollary 2.4.9. *Let Ω a bounded domain in $\mathbb{R}^N$ of class C^1. Then there are no critical mass densities in $\mathcal{R}$ for the function which takes $\rho\in\mathcal{R}$ to $\lambda_1[\rho]$ under the mass constraint (2.2.2).*

As a consequence of Theorem 2.4.8, we have

Theorem 2.4.10. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and F a nonempty finite subset of $\mathbb{N}\setminus\{0\}$. Let $C\subseteq\mathcal{R}[F]$ be a weakly* compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho\in C}\operatorname{ess\,inf}_\Omega \rho > 0$. Let $M>0$ and $L_M = \{\rho\in L^\infty(\Omega) : \int_\Omega \rho = M\}$. Then for all $h=1,\dots,|F|$ the function which takes $\rho\in C\cap L_M$ to $\Lambda_{F,h}[\rho]$ admits points of maximum and minimum, and such points belong to $\partial C\cap L_M$.*

By looking at the structure of this problem, we observed that the results of [8, 9, Cox-Mc.Laughlin] can be immediately extended to our case. Let $M>0$

fixed, ad_γ, ad^*_γ defined as in (2.2.15) and (2.2.16). Since ad^*_γ is convex and weakly* compact in $L^\infty(\Omega)$, we get the existence of $\check\rho_j, \hat\rho_j$ in ad^*_γ such that

$$\check\lambda_j := \lambda_j[\check\rho_j] = \inf_{\rho\in ad^*_\gamma} \lambda_j[\rho],$$
$$\hat\lambda_j := \lambda_j[\hat\rho_j] = \sup_{\rho\in ad^*_\gamma} \lambda_j[\rho].$$

We can give a characterization of such extremizers. We state the following results.

Proposition 2.4.11. *Let Ω a bounded domain in $\mathbb{R}^N$ of class C^1, $\rho \in \mathcal{R}$. Then*

i) $\check\rho_1$ can be chosen from ad_γ;

ii) if Ω is of class C^2, then $\check\rho_1$ is uniquely determined and belongs to ad_γ;

iii) for $j > 1$, if Ω is of class C^2 and there exists u in the eigenspace corresponging to $\check\lambda_j$ with exactly j nodal domains, then $\check\rho_j$ can be chosen from ad_γ;

iv) $\hat\rho_1 \in ad_\gamma$ and is unique.

We don't repeat here the proofs of the previous statements. This case is identical to the one of Dirichlet boundary conditions, and it is studied in detail in [9].

Remark 2.4.12. *It is now straightforward that the considerations made for this problem immediately extend to the case of the biharmonic operator with Dirichlet (intermediate) boundary conditions on Γ_0, and Neumann boundary conditions on Γ_1 and also to poly-harmonic operators $(-\Delta)^n$ with Dirichlet (intermediate) boundary conditions on Γ_0 and Neumann boundary conditions on Γ_1, when Γ_0 and Γ_1 satisfy (2.4.1). The problem to consider is (2.3.2) ((2.3.8)) in the space*

$$H^n_{0,\Gamma_0}(\Omega) = \{u \in H^n(\Omega) : D^\alpha u = 0 \text{ on } \Gamma_0\ \forall\ |\alpha| \le n-1\,,\ \text{in the sense of traces}\}\,,$$

$(H^n(\Omega) \cap H^m_{0,\Gamma_0}(\Omega)$, where $\frac{n+1}{2} \le m \le n$ if n odd, $\frac{n}{2} \le m \le n$ if n even), endowed with the equivalent scalar product (2.3.5) ((2.3.13)). Then, the symmetric functions of eigenvalues of this class of problems have no critical points under the fixed mass constraint (2.2.2), and their restrictions to weakly compact set in $\mathcal{R}$ admit points of maximum and minimum, and such points have to belong to the boundary of such sets.*

3. THE NEUMANN PROBLEM FOR THE LAPLACE OPERATOR

Throughout this chapter Ω is a bounded domain in $\mathbb{R}^N$ of class C^1, $\rho \in \mathcal{R}$. The classic formulation of the eigenvalue problem for the Laplace operator with Neumann boundary conditions is

$$\begin{cases} -\Delta u = \lambda \rho u, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{3.0.1}$$

in the unknowns $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, $\lambda \in \mathbb{R}$. This problem models a free vibrating membrane of mass density ρ. We will consider the weak formulation of problem (3.0.1)

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \lambda \int_\Omega \rho u \phi dx\,, \quad \forall \phi \in H^1(\Omega)\,, \tag{3.0.2}$$

in the unknowns $u \in H^1(\Omega)$, $\lambda \in \mathbb{R}$. Actually, we will obtain a problem in $(H^1(\Omega)/\mathbb{R})$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue $\lambda = 0$. We denote by i the canonical (compact) embedding of $H^1(\Omega)$ into $L^2(\Omega)$. We denote by J_ρ the continuous embedding of $L^2(\Omega)$ into $(H^1(\Omega))'$, defined by

$$J_\rho[u][\phi] := \int_\Omega \rho u \phi dx \quad \forall u \in L^2(\Omega), \phi \in H^1(\Omega). \tag{3.0.3}$$

We set

$$H^{1,0}_\rho(\Omega) := \left\{ u \in H^1(\Omega) : \int_\Omega u \rho dx = 0 \right\}.$$

We consider on $H^1(\Omega)$ the bilinear form

$$< u, v >= \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall u, v \in H^1(\Omega). \tag{3.0.4}$$

We denote by $\mathcal{H}^1(\Omega)$, $\mathcal{H}^{1,0}_\rho(\Omega)$ the spaces $H^1(\Omega)$ and $H^{1,0}_\rho(\Omega)$ endowed with this form. We observe that by simply modifications of the the proof of Poincaré-Wirtinger inequality in Evans, [12, Theorem 1, ch. 5, sec. 5.8], one can prove that that for fixed ρ, there exists $C > 0$ such that

$$\left\| u - \frac{\int_\Omega \rho u dx}{\int_\Omega \rho dx} \right\|_{L^2(\Omega)} \leq C \left\| \nabla u \right\|_{L^2(\Omega)} \quad \forall u \in H^1(\Omega).$$

Then the bilinear form (3.0.4) defines on $\mathcal{H}^{1,0}_\rho(\Omega)$ a scalar product whose induced norm is equivalent to the standard one. We consider on $(\mathcal{H}^1(\Omega)/\mathbb{R})$ the bilinear form

$$< p[u], p[v] >= \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall u, v \in \mathcal{H}^1(\Omega),$$

which is a scalar product on $(\mathcal{H}^1(\Omega)/\mathbb{R})$ that renders $(\mathcal{H}^1(\Omega)/\mathbb{R})$ a Hilbert space. The norm associated with this scalar product generates a quotient topology on $(\mathcal{H}^1(\Omega)/\mathbb{R})$ which is equivalent to the quotient topology of $(H^1(\Omega)/\mathbb{R})$. We denote by π_ρ the map of $\mathcal{H}^1(\Omega)$ to $\mathcal{H}^{1,0}_\rho(\Omega)$ defined by

$$\pi_\rho[u] = u - \frac{\int_\Omega u\rho dx}{\int_\Omega \rho dx},$$

for all $u \in \mathcal{H}^1(\Omega)$. We denote by $\pi_\rho^\sharp$ the map of $(\mathcal{H}^1(\Omega)/\mathbb{R})$ onto $\mathcal{H}^{1,0}_\rho(\Omega)$ defined by the equality $\pi_\rho = \pi_\rho^\sharp \circ p$, where p is the canonical projection of $\mathcal{H}^1(\Omega)$ onto $(\mathcal{H}^1(\Omega)/\mathbb{R})$. We set

$$F(\Omega) := \left\{ G \in (H^1(\Omega))' : G[1] = 0 \right\}.$$

We consider the operator $-\Delta_\rho$ as a map from $\mathcal{H}^{1,0}_\rho(\Omega)$ to $F(\Omega)$ defined by

$$-\Delta_\rho[u][\phi] = \int_\Omega \nabla u \cdot \nabla \phi dx \quad \forall u \in \mathcal{H}^{1,0}_\rho(\Omega), \phi \in \mathcal{H}^1(\Omega). \tag{3.0.5}$$

The operator $-\Delta_\rho$ considered as an operator acting on the whole $\mathcal{H}^1(\Omega)$ is surjective onto $F(\Omega)$. Moreover, it is clear that it is injective and continuous if restricted to $\mathcal{H}^{1,0}_\rho(\Omega)$, and thanks to Poincaré-Wirtinger inequality it turns out that its inverse is also continuous. Then $-\Delta_\rho$ turns out to be a linear homeomorphism of $\mathcal{H}^{1,0}_\rho(\Omega)$ onto $F(\Omega)$.

Moreover, the norm on $F(\Omega)$ defined by

$$\|G\|_{F(\Omega)} := \sup_{0 \neq u \in \mathcal{H}^{1,0}_\rho} \frac{|G(u)|}{\|u\|_{\mathcal{H}^{1,0}_\rho}} \quad \forall G \in F(\Omega)$$

is equivalent to the restriction to $F(\Omega)$ of the standard operator norm of $(H^1(\Omega))'$.

We define the operator $T_\rho := (\pi_\rho^\sharp)^{-1} \circ (-\Delta_\rho)^{-1} \circ J_\rho \circ i \circ \pi_\rho^\sharp$ from $(\mathcal{H}^1(\Omega)/\mathbb{R})$ to itself. It is easy to prove the following

Proposition 3.0.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and $\rho \in \mathcal{R}$. The operator T_ρ is a compact selfadjoint operator in $(\mathcal{H}^1(\Omega)/\mathbb{R})$ and its eigenvalues coincide with the reciprocals of the eigenvalues $\lambda_j[\rho]$ of problem (3.0.2) for all $j \in \mathbb{N}$. Moreover, the set of eigenvalues of problem (3.0.2) is contained in $]0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity.*

Proof. For the self-adjointness, it suffices to observe that

$$\begin{aligned} < p[\pi_\rho^\sharp T_\rho u], p[\pi_\rho^\sharp v] > &= < p[\pi_\rho^\sharp \circ (\pi_\rho^\sharp)^{-1} \circ (-\Delta_\rho)^{-1} \circ J_\rho \circ i \circ \pi_\rho^\sharp u], p[\pi_\rho^\sharp v] > \\ &= -\Delta_\rho[(-\Delta_\rho)^{-1} \circ J_\rho \circ i \circ \pi_\rho^\sharp u][\pi_\rho^\sharp v] \\ &= J_\rho[i \circ \pi_\rho^\sharp u][\pi_\rho^\sharp v], \quad \forall u, v \in (\mathcal{H}^1(\Omega)/\mathbb{R}). \end{aligned}$$

The selfadjointness now follows immediately. The remaining statements are straightforward. □

Remark 3.0.7. *We observe that the pair* (λ, u) *of the set* $\mathbb{R} \times (\mathcal{H}^{1,0}_{\rho}(\Omega) \setminus \{0\})$ *satisfies (3.0.2) if and only if* $\lambda > 0$ *and the pair* $(\lambda^{-1}, p[u])$ *of the set* $\mathbb{R} \times ((\mathcal{H}^1(\Omega)/\mathbb{R}) \setminus \{0\})$ *satisfies the equation*

$$\lambda^{-1} p[u] = T_{\rho} p[u].$$

In order to exploit the procedure used in the previous chapters which allows to prove real analyticity of symmetric functions of eigenvalues and compute their derivatives, we observe that the operator T_{ρ} can be written in a more suitable way. We consider the operator $-\Delta$ from $\mathcal{H}^{1,0}(\Omega)$ onto $F(\Omega)$, where

$$\mathcal{H}^{1,0}(\Omega) := \left\{ u \in \mathcal{H}^1(\Omega) : \int_{\Omega} u dx = 0 \right\},$$

defined by

$$-\Delta[u][\phi] = \int_{\Omega} \nabla u \cdot \nabla \phi dx \quad \forall u, \phi \in \mathcal{H}^{1,0}(\Omega). \tag{3.0.8}$$

Then it is easy to prove that this operator is a linear homeomorphism of $\mathcal{H}^{1,0}(\Omega)$ onto $F(\Omega)$. Let π, $\pi^{\sharp}$ be $\pi_{\rho\equiv 1}$, $\pi^{\sharp}_{\rho\equiv 1}$ respectively. We define the operator $\tilde{T}_{\rho}$ of $(\mathcal{H}^1(\Omega)/\mathbb{R})$ to itself as

$$\tilde{T}_{\rho} u := (-\pi^{\sharp})^{-1} \circ (-\Delta)^{-1} \circ J_{\rho} \circ i \circ \pi^{\sharp}_{\rho} u, \quad \forall u \in (\mathcal{H}^1(\Omega)/\mathbb{R}).$$

Then the following diagram commutes

$$\begin{tikzcd}
(\mathcal{H}^1(\Omega)/\mathbb{R}) \arrow[r, "\pi^{\sharp}_{\rho}"] & \mathcal{H}^{1,0}_{\rho}(\Omega) \arrow[r, "J_{\rho} \circ i"] & F(\Omega) \arrow[r, "(-\Delta_{\rho})^{-1}"] \arrow[dr, "(-\Delta)^{-1}"'] & \mathcal{H}^{1,0}_{\rho}(\Omega) \arrow[r, "(\pi^{\sharp}_{\rho})^{-1}"] & (\mathcal{H}^1(\Omega)/\mathbb{R}) \\
 & & & \mathcal{H}^{1,0}(\Omega) \arrow[ur, "(\pi^{\sharp})^{-1}"'] &
\end{tikzcd}$$

Lemma 3.0.9. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ *of class* C^1. *Let* F *be a finite nonempty subset of* $\mathbb{N} \setminus \{0\}$. *Let*

$$\mathcal{R}[F] := \{\rho \in \mathcal{R} \,:\, \lambda_j[\rho] \neq \lambda_l[\rho] \,,\ \forall j \in F, l \in \mathbb{N} \setminus F\} \,,$$
$$\Theta[F] := \{\rho \in \mathcal{R}[F] \,:\, \lambda_{j_1}[\rho] = \lambda_{j_2}[\rho] \,,\ \forall j_1, j_2 \in F\}.$$

Let $\tilde{\rho} \in \Theta[F]$, $\tilde{u}_1, \tilde{u}_2 \in \mathcal{H}^{1,0}_{\tilde{\rho}}(\Omega)$ *be such that* $p[\tilde{u}_1], p[\tilde{u}_2]$ *are two eigenfunctions corresponding to the eigenvalue* λ_F^{-1} *of the operator* $\tilde{T}_{\rho}$. *Then we have*

$$< d\tilde{T}_{\tilde{\rho}}[\dot{\rho}][p[\tilde{u}_1]], p[\tilde{u}_2] >= \int_{\Omega} \dot{\rho} \tilde{u}_1 \tilde{u}_2 dx, \quad \forall \dot{\rho} \in L^{\infty}(\Omega). \tag{3.0.10}$$

Proof. By standard calculus in Banach spaces it follows

$$\begin{aligned}
&< d_{|\tilde{\rho}} \left((\pi^{\sharp})^{-1} \circ (-\Delta)^{-1} \circ J_{\rho} \circ i \circ \pi^{\sharp}_{\rho} \right) [\dot{\rho}][p[\tilde{u}_1]], p[\tilde{u}_2] >= \\
&< (\pi^{\sharp})^{-1} \circ (-\Delta)^{-1} \circ d_{|\tilde{\rho}} J_{\rho}[\dot{\rho}] \circ i \circ \pi^{\sharp}_{\tilde{\rho}}[p[\tilde{u}_1]], p[\tilde{u}_2] > + \\
&< (\pi^{\sharp})^{-1} \circ (-\Delta)^{-1} \circ J_{\tilde{\rho}} \circ i \circ d_{|\tilde{\rho}} \pi_{\rho}[\dot{\rho}][\tilde{u}_1], p[\tilde{u}_2] >= \\
&-\Delta[(-\Delta)^{-1} \circ J_{\dot{\rho}} \circ i \circ \tilde{u}_1][\tilde{u}_2] + C(-\Delta)[(-\Delta)^{-1} \circ J_{\tilde{\rho}} \circ i \circ [1]][\tilde{u}_2] = \\
&= \int_{\Omega} \dot{\rho} \tilde{u}_1 \tilde{u}_2 dx, \quad \forall \dot{\rho} \in L^{\infty}(\Omega),
\end{aligned}$$

where $C = \frac{\int_{\Omega} \dot{\rho} \tilde{u}_1 dx}{\int_{\Omega} \tilde{\rho} dx}$. The last equality follows by observing that $\int_{\Omega} \tilde{\rho} \tilde{u}_2 dx = 0$. $\square$

We are now able to prove the following

Theorem 3.0.11. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and F a nonempty finite subset of $\mathbb{N}\setminus\{0\}$. Let $\mathcal{R}[F]$, $\Theta[F]$ be defined as in the previous lemma. Then $\mathcal{R}[F]$ is open in $L^\infty(\Omega)$ and the symmetric functions of the eigenvalues*

$$\Lambda_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h\in F\\ j_1<\cdots j_h}} \lambda_{j_1}[\rho]\cdots\lambda_{j_h}[\rho]\,, \qquad h=1,\,\dots,|F| \tag{3.0.12}$$

are real analytic in $\mathcal{R}[F]$. Moreover, if $\rho\in\Theta[F]$ and the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_F[\rho]$ for all $j\in F$, then the differential of $\Lambda_{F,h}$ at ρ is given by the formula

$$d\Lambda_{F,h}[\rho][\dot\rho] = -{\lambda_F}^{h+1}[\rho]\binom{|F|-1}{h-1}\sum_{l\in F}\int_\Omega u_l^2\dot\rho\,dx\,, \tag{3.0.13}$$

for all $\dot\rho\in L^\infty(\Omega)$, where $\{u_l\}_{l\in F}$ is an orthonormal basis for $\lambda_F[\rho]$ in $\mathcal{H}^{1,0}_\rho(\Omega)$.

Proof. The proof is analogous to the proof of Theorem 2.1.27. Here the proof of formula (3.0.13) follows by (3.0.10). □

Remark 3.0.14. *We observe that if $j\in F$, then the restriction of the function which takes $\rho\in\mathcal{R}$ to $\lambda_j[\rho]\in\mathbb{R}$ to $\Theta[F]$ is real analytic. In fact it coincides on $\Theta[F]$ with the real analytic function $\frac{\Lambda_{F,1}[\cdot]}{|F|}$.*

We investigate now the existence of critical mass densities for symmetric functions of the eigenvalues. We have the following theorem.

Theorem 3.0.15. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and $F=\{m,n\}$ with $m,n\in\mathbb{N}$, $m\neq n$. Let $\tilde\rho\in\mathcal{R}[F]$ be continuous and moreover, assume that the solutions of problem (3.0.2) are classic solutions and the nodal domains are stokians. Then for $h=1,2$, $\tilde\rho$ is not a critical mass density for the function which takes $\rho\in\mathcal{R}[F]$ to $\Lambda_{F,h}$ under constraint (2.2.2).*

Proof. Let $\tilde\rho\in\mathcal{R}[F]$ be fixed. Then we have one of the following cases:

i) $\tilde\rho\in\Theta[F]$. Then by 3.0.13 it follows that

$$\begin{aligned} d\Lambda_{F,1}[\tilde\rho][\dot\rho] &= -\lambda_F^2\int_\Omega \dot\rho(u_m^2+u_n^2)dx\,,\\ d\Lambda_{F,2}[\tilde\rho][\dot\rho] &= -\lambda_F^3\int_\Omega \dot\rho(u_m^2+u_n^2)dx. \end{aligned}$$

ii) $\tilde\rho\in\bigcap_{k=1}^2\Theta[F_k]$, where $F_1=\{m\}$, $F_2=\{n\}$. There exists an open neighbourhood in $\mathcal{R}$ of $\tilde\rho$ such that $\mathcal{W}\subseteq\bigcap_{k=1}^2\mathcal{R}[F_k]$. Then

$$\begin{aligned} d\Lambda_{F,1}[\tilde\rho][\dot\rho] &= d(\Lambda_{F_2,1}+\Lambda_{F_1,1})[\tilde\rho][\dot\rho] = -\int_\Omega \dot\rho(\lambda_{F_2}^2u_n^2+\lambda_{F_1}^2u_m^2)dx\,,\\ d\Lambda_{F,2}[\tilde\rho][\dot\rho] &= d(\Lambda_{F_1,1}\Lambda_{F_2,1})[\tilde\rho][\dot\rho] = -\int_\Omega \dot\rho(\lambda_{F_1}\lambda_{F_2}^2u_n^2+\lambda_{F_2}\lambda_{F_1}^2u_m^2)dx, \end{aligned}$$

where $\{u_l\}_{l\in F}$ (respectively, $\{u_l\}_{l\in F_k}$) is an orthonormal basis in $\mathcal{H}^{1,0}_{\tilde\rho}(\Omega)$ of the eigenspace corresponding to $\lambda_F[\tilde\rho]$ (respectively, $\lambda_{F_k}[\tilde\rho]$) and $\lambda_F[\tilde\rho]$ is the common value assumed by all the eigenvalues $\lambda_j[\tilde\rho]$ with $j\in F$ (respectively, $\lambda_{F_k}[\tilde\rho]$ is the value assumed by all the eigenvalue $\lambda_j[\tilde\rho]$ with $j\in F_k$). Suppose now that $\tilde\rho$ is a critical mass density for $\Lambda_{F,h}$, $h=1,2$ under constraint (2.2.2). Then, in both cases, there exist $c_n, c_m>0$, $c>0$ such that

$$\int_\Omega \dot\rho(c_n u_n^2 + c_m u_m^2)dx = c\int_\Omega \dot\rho dx$$

for all $\dot\rho\in L^\infty(\Omega)$. Since $\dot\rho$ is arbitrary, it follows that

$$(c_n u_n^2 + c_m u_m^2) = c\,, \quad a.e. \text{ in } \Omega.$$

Let's study the various cases:

i) $\tilde\rho\in\Theta[F]$, $d\Lambda_{F,1}[\tilde\rho][\dot\rho] = -\lambda_F^2\int_\Omega \dot\rho(u_m^2+u_n^2)dx$ ($d\Lambda_{F,2}[\tilde\rho][\dot\rho] = -\lambda_F^3\int_\Omega \dot\rho(u_m^2 + u_n^2)dx$ is analogous). Then, by differentiating the equality

$$u_m^2 + u_n^2 = C \tag{3.0.16}$$

we obtain

$$u_m\nabla u_m + u_n\nabla u_n = 0 \tag{3.0.17}$$

which implies in particular

$$|\nabla u_m(x)|^2 = \frac{u_n^2(x)}{u_m^2(x)}\,|\nabla u_n(x)|^2\,,$$

for all $x\in\Omega$ such that $u_n(x)\neq 0$. Let's differentiate again in (3.0.17) and use the fact that $-\Delta u_m = \lambda_F\tilde\rho u_m$ and $-\Delta u_n = \lambda_F\tilde\rho u_n$, we obtain

$$|\nabla u_m(x)|^2 + |\nabla u_n(x)|^2 = \lambda_F\tilde\rho\left(u_m^2(x)+u_n^2(x)\right), \tag{3.0.18}$$

hence

$$\left(\frac{u_n^2(x)}{u_m^2(x)}+1\right)|\nabla u_n(x)|^2 = \lambda_F\tilde\rho C, \tag{3.0.19}$$

hence

$$|\nabla u_n(x)|^2 = \lambda_F\tilde\rho u_m(x)^2, \tag{3.0.20}$$

for all $x\in\Omega$ such that $u_m(x)\neq 0$. It is easy to see that (3.0.20) holds also if $x\in\Omega$ is such that $u_m(x)=0$ because in this case u_n^2 has a maximum in x, hence $\nabla u_n(x)=0$, since it is not possible $u_n(x)=0$, see (3.0.16). In this case we have $C=0$ which led to a contradiction. In the same way one can also show

$$|\nabla u_m(x)|^2 = \lambda_F\tilde\rho u_n^2(x).$$

ii) $\tilde{\rho} \in \bigcap_{k=1}^2 \Theta[F_k]$, $d\Lambda_{F,1}[\tilde{\rho}][\dot{\rho}] = -\int_\Omega \dot{\rho}(\lambda_{F_2}^2 u_n^2 + \lambda_{F_1}^2 u_m^2)dx$. By a few computations as in the previous step, by $\lambda_{F_2}^2 u_n^2 + \lambda_{F_1}^2 u_m^2 = C$, using the fact that $-\Delta u_m = \lambda_{F_1}\tilde{\rho} u_m$, $-\Delta u_n = \lambda_{F_2}\tilde{\rho} u_n$, we obtain the following relations:

$$|\nabla u_m(x)|^2 = \frac{\lambda_{F_2}^2}{C\lambda_{F_1}^2}\tilde{\rho}\left(\lambda_{F_1}^3 u_m^2(x) + \lambda_{F_2}^3 u_n^2(x)\right) u_n^2(x); \qquad (3.0.21)$$
$$|\nabla u_n(x)|^2 = \frac{\lambda_{F_1}^2}{C\lambda_{F_2}^2}\tilde{\rho}\left(\lambda_{F_1}^3 u_m^2(x) + \lambda_{F_2}^3 u_n^2(x)\right) u_m^2(x).$$

iii) $\tilde{\rho} \in \bigcap_{k=1}^2 \Theta[F_k]$, $d\Lambda_{F,2}[\tilde{\rho}][\dot{\rho}] = -\int_\Omega \dot{\rho}(\lambda_{F_1}\lambda_{F_2}^2 u_n^2 + \lambda_{F_2}\lambda_{F_1}^2 u_m^2)dx$. By imposing $\lambda_{F_1}\lambda_{F_2}^2 u_n^2 + \lambda_{F_2}\lambda_{F_1}^2 u_m^2 = C$ we obtain

$$|\nabla u_m(x)|^2 = \frac{\lambda_{F_2}^2}{C}\tilde{\rho}\left(\lambda_{F_1}^2 u_m^2(x) + \lambda_{F_2}^2 u_n^2(x)\right) u_n^2(x); \qquad (3.0.22)$$
$$|\nabla u_n(x)|^2 = \frac{\lambda_{F_1}^2}{C}\tilde{\rho}\left(\lambda_{F_1}^2 u_m^2(x) + \lambda_{F_2}^2 u_n^2(x)\right) u_m^2(x).$$

We observe that in all cases, the nodal set of one of the two eigenfunctions coincides with the set where the gradient of the other vanishes. In the first case this follows immediately by (3.0.20) and the properties of $\tilde{\rho}$. But the same statement still holds for the other two cases. In fact in (3.0.21) and (3.0.22) the quantity on the right hand side vanishes only if $u_n = 0$ (respectively $u_m = 0$). This follows by the properties of $\tilde{\rho}$ and by the fact that if the quantities in brackets in the right hand sides vanish in some $x \in \Omega$, since they are non-negative, it would follow that $u_n(x) = 0$ and $u_m(x) = 0$, but this would imply that $u_m, u_n = 0$ on Ω. By the same argument, one can state that there are no points in Ω where both u_m and ∇u_m vanish (respectively u_n and ∇u_n). This implies that nodal sets of u_m are manifolds and coincide with the sets where ∇u_n vanishes. We observe that the nodal sets of the eigenfunctions u of problem (3.0.2) are not empty, since for such functions $\int_\Omega \tilde{\rho} u = 0$, hence u changes its sign on Ω.

Let's consider a nodal domain Ω_m of u_m. The function u_m doesn't change sign on Ω_m. The boundary $\partial\Omega_m$ of Ω_m can be written as $\partial\Omega_m = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \subset \Omega$. First we show that $\Gamma_1 \neq \emptyset$. Assume by contradiction that $\Gamma_1 = \emptyset$. The function $u_n|_{\Omega_m}$ is an eigenfunction of problem (3.0.1) with Ω replaced by Ω_m corresponding to the eigenvalue λ_{F_2}. Indeed the equation $-\Delta u_n = \lambda_{F_2} u_n$ is clearly satisfied on Ω_m and $\frac{\partial u_n}{\partial \nu} = 0$ on $\partial\Omega_m$, since ∇u_n is zero on $\partial\Omega_m$. Since $u_n|_{\Omega_m}$ is non identically zero, it must change sign. Thus, there exist at least two nonempty nodal domains for $u_n|_{\Omega_m}$ in Ω_m. We claim that al least one of them, say Ω_{m_n}, is relatively compact in Ω_m. If this were false, then there would exist at least a point x of $\partial\Omega_m$ such that $u_n(x) = 0$, hence $\nabla u_m(x) = 0$. But we since $\Gamma_1 = \emptyset$ we have $u_m(x) = 0$. Thus $u_n(x) = u_m(x) = 0$, hence $C = 0$, a contradiction. Thus there exists a nodal domain Ω_{m_n} of $u_n|_{\Omega_m}$ such that $\overline{\Omega}_{m_n} \subset \Omega_m$. Now, $u_m|_{\Omega_{m_n}}$ solves problem (3.0.1) with λ_{F_1}, hence it must change sign Ω_{m_n}. But Ω_{m_n} is relatively compact in Ω_m, and on this set u_m has constant sign, a contradiction.

Thus we have proved that $\Gamma_1 \neq \emptyset$. Recall that u_m has constant sign on Ω_m. Moreover, $\frac{\partial u_n}{\partial \nu} = 0$ on Γ_1, while $\nabla u_n = 0$ on Γ_2, since here $u_m = 0$. Then $u_n|_{\Omega_m}$ is solution of problem (3.0.1) with Ω replaced by Ω_m corresponding to the eigenvalue λ_{F_2} and it changes sign on Ω_m. Let Ω_{m_n} be a nodal domain of $u_n|_{\Omega_m}$. By the arguments above we have that $\partial\Omega_{m_n} = \Gamma_{1,n} \cup \Gamma_{2,n}$, where $\emptyset \neq \Gamma_{1,n} \subset \partial\Omega_m$, and $\Gamma_{2,n} \subset \Omega_m$. We claim that there exists at least one nodal domain Ω_{m_n} such that $\Gamma_{1,n} \subset \partial\Omega$. If this were false, the boundary $\partial\Omega_{m_n}$ of each Ω_{m_n} is of the type: $\partial\Omega_{m_n} = (\Gamma_{1,n} \cap \partial\Omega) \cup (\Gamma_{1,n} \cap (\partial\Omega_m \setminus (\partial\Omega \cap \partial\Omega_m))) \cup \Gamma_{2,n}$, and each of these partitions of $\partial\Omega_{m_n}$ is nonempty. Since Ω_{m_n} is connected, $(\Gamma_{1,n} \cap (\partial\Omega_m \setminus (\partial\Omega \cap \partial\Omega_m))) \cap \Gamma_{2,n} \neq \emptyset$. On this set u_m and ∇u_m vanish, a contradiction. Thus there exists Ω_{m_n} such that $\Gamma_{1,n} \subset \partial\Omega$. Then $u_m|_{\Omega_{m_n}}$ is a nontrivial solution of problem (3.0.1) corresponding to the eigenvalue λ_{F_2} and changes its sign on Ω_{m_n}, a contradiction. This concludes the proof. □

We prove now that the function which takes $\rho \in \mathcal{R}$ to $\lambda_j[\rho]$ is continuous with respect to the weak* topology of $L^\infty(\Omega)$. For a fixed $\rho \in \mathcal{R}$ we have the following variational representation of the eigenvalues

$$\lambda_j[\rho] = \inf_{\substack{E \leq (\mathcal{H}^1(\Omega)/\mathbb{R}) \\ \dim E = j}} \sup_{0 \neq u \in E} \frac{\int_\Omega \left|\nabla(\pi_\rho^\sharp u)\right|^2 dx}{\int_\Omega (\pi_\rho^\sharp u)^2 \rho \, dx}, \quad \forall j \in \mathbb{N}. \tag{3.0.23}$$

Remark 3.0.24. *Let $\alpha > 0$ be such that $\rho \geq \alpha$ a.e. in Ω. It is immediate to see that*

$$\lambda_j[\rho] = \inf_{\substack{E \leq \mathcal{H}_\rho^{1,0}(\Omega) \\ \dim E = j}} \sup_{0 \neq u \in E} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 \rho \, dx}, \quad \forall j \in \mathbb{N}, \tag{3.0.25}$$

$$\lambda_j[\alpha] = \inf_{\substack{E \leq \mathcal{H}_\rho^{1,0}(\Omega) \\ \dim E = j}} \sup_{0 \neq u \in E} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega (\pi_\alpha u)^2 \alpha \, dx}, \quad \forall j \in \mathbb{N}. \tag{3.0.26}$$

By observing that $\int_\Omega \alpha(\pi_\alpha u)^2 dx = \int_\Omega \alpha u^2 dx - C\left(\int_\Omega \alpha u dx\right)^2$, where $C = (\alpha\,|\Omega|)^{-1}$, it follows that $\int_\Omega \rho u^2 dx \geq \int_\Omega \alpha(\pi_\alpha u)^2 dx$ for all $u \in \mathcal{H}_\rho^{1,0}(\Omega)$, hence $\lambda_j[\rho] \leq \lambda_j[\alpha]$ for all $j \in \mathbb{N}$. In the same way one can show that if $\beta > 0$ is such that $\rho \leq \beta$ a.e. in Ω, then $\lambda_j[\beta] \leq \lambda_j[\rho]$. It suffices to consider in (3.0.25) and (3.0.26) the space $\mathcal{H}_\beta^{1,0}(\Omega)$ in place of $\mathcal{H}_\rho^{1,0}(\Omega)$.

We need some technical results

Lemma 3.0.27. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $\{\rho_n\}_{n\in\mathbb{N}} \subset L^\infty(\Omega)$, $\rho \in L^\infty(\Omega)$ such that $\rho_n \overset{*}{\rightharpoonup} \rho$. Moreover, let $\alpha := \inf_{n\in\mathbb{N}} \operatorname{ess\,inf}_{x\in\Omega} \rho_n(x) > 0$ and $\beta := \sup_{n\in\mathbb{N}} \|\rho_n\|_{L^\infty(\Omega)} < +\infty$. Then we have*

i) For all $\phi \in (\mathcal{H}^1(\Omega)/\mathbb{R})$, $\pi_{\rho_n}^\sharp \phi \to \pi_\rho^\sharp \phi$ in $L^2(\Omega)$;

ii) if $u_n \rightharpoonup u$ in $(\mathcal{H}^1(\Omega)/\mathbb{R})$ then, (possibly passing to a subsequence) $\pi_{\rho_n}^\sharp u_n \to \pi_\rho^\sharp u$ in $L^2(\Omega)$.

Proof. For the proof of statement $i)$, we observe that if $\tilde\phi \in H^1(\Omega)$ is such that $\int_\Omega \tilde\phi dx = 0$ and $\phi = p[\tilde\phi]$, then it is sufficient to prove

$$\lim_{n\to\infty} \left\| \frac{\int_\Omega \rho_n \tilde\phi dx}{\int_\Omega \rho_n dx} - \frac{\int_\Omega \rho\tilde\phi dx}{\int_\Omega \rho dx} \right\|_{L^2(\Omega)} = 0.$$

By Ḧolder's inequality we have

$$\left\| \frac{\int_\Omega \rho_n \tilde{\phi} dx}{\int_\Omega \rho_n dx} - \frac{\int_\Omega \rho \tilde{\phi} dx}{\int_\Omega \rho dx} \right\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}} \left| \frac{\int_\Omega \rho_n \tilde{\phi} dx}{\int_\Omega \rho_n dx} - \frac{\int_\Omega \rho \tilde{\phi} dx}{\int_\Omega \rho dx} \right|,$$

then the proof of point i) is straightforward.

Now we prove statement ii). Let $\tilde{u}_n, \tilde{u} \in \mathcal{H}^{1,0}(\Omega) := \left\{ u \in \mathcal{H}^1(\Omega) : \int_\Omega u dx = 0 \right\}$ and such that $u_n = p[\tilde{u}_n]$ for all n, $u = p[\tilde{u}]$. We have

$$\left\| \pi^\sharp_{\rho_n} u_n - \pi^\sharp_\rho u \right\|_{L^2(\Omega)} \leq \left\| \pi^\sharp_{\rho_n} u_n - \pi^\sharp_{\rho_n} u \right\|_{L^2(\Omega)} + \left\| \pi^\sharp_{\rho_n} u - \pi^\sharp_\rho u \right\|_{L^2(\Omega)}. \tag{3.0.28}$$

By statement i),the second summand on the right hand side of (3.0.28) goes to zero as n goes to infinity. For the first term, we observe that

$$\begin{aligned}
\left\| \pi^\sharp_{\rho_n} u_n - \pi^\sharp_{\rho_n} u \right\|_{L^2(\Omega)} &= \left\| \tilde{u}_n - \frac{\int_\Omega \rho_n \tilde{u}_n dx}{\int_\Omega \rho_n dx} - \tilde{u} + \frac{\int_\Omega \rho_n \tilde{u} dx}{\int_\Omega \rho_n dx} \right\|_{L^2(\Omega)} \\
&\leq \left\| \tilde{u}_n - \tilde{u} \right\|_{L^2(\Omega)} + \frac{\beta}{\alpha} \left\| \tilde{u}_n - \tilde{u} \right\|_{L^2(\Omega)} \\
&= \left(1 + \frac{\beta}{\alpha} \right) \left\| \tilde{u}_n - \tilde{u} \right\|_{L^2(\Omega)}.
\end{aligned}$$

Since $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}^{1,0}(\Omega)$, it is also bounded in $H^1(\Omega)$ thanks to Poincaré-Wirtinger inequality. Then there exists a subsequence, still denoted by $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ which weakly converges in $H^1(\Omega)$, and strongly in $L^2(\Omega)$ to a certain function $\tilde{w}$. Clearly, $\tilde{w}$ has zero mean. Since $\pi^\sharp$ is a homeomorphism from $(\mathcal{H}^1(\Omega)/\mathbb{R})$ onto $\mathcal{H}^{1,0}(\Omega)$ and the limit is unique, it follows that $\tilde{w} = \tilde{u}$. This concludes the proof. □

We are now ready to prove the following

Proposition 3.0.29. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let $C \subset L^\infty(\Omega)$ be a weakly* compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho\in C} \operatorname{ess\,inf}_{x\in\Omega} \rho(x) > 0$. Then the functions which take $\rho \in C$ to $\lambda_j[\rho]$ are continuous in the weak* topology of $L^\infty(\Omega)$.*

Proof. By Remark 3.0.24 and Lemma 3.0.27, the proof of this proposition follows the line of the proof of Proposition 2.2.8. We show that if ρ_n converges to ρ in the weak* topology of $L^\infty(\Omega)$, then $\lambda_j^n := \lambda_j[\rho_n]$ converges to $\lambda_j := \lambda_j[\rho]$ for all $j \in \mathbb{N}$. As in the first part of the proof of (2.2.8), we have $\lambda_j[\beta] \leq \lambda_j^n \leq \lambda_j[\alpha]$ for suitable $0 < \alpha < \beta < +\infty$ (see Remark 3.0.24). Then we find a sequence $\rho_n \overset{*}{\rightharpoonup} \rho$ such that $\lambda_j^n \to \bar{\lambda}_j$ in $\mathbb{R}$, $u_j^n \rightharpoonup \bar{u}_j$ in $(\mathcal{H}^1(\Omega)/\mathbb{R})$ (here u_j^n is the eigenfunction of T_ρ corresponding to the eigenvalue $(\lambda_j^n)^{-1}$). Clearly we have

$$0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \cdots \leq \bar{\lambda}_j \leq \cdots.$$

By Lemma 3.0.27 we have that

$$\begin{aligned}
\lim_{n\to\infty} & \int_\Omega \nabla(\pi^\sharp_{\rho_n} u_j^n) \cdot \nabla(\pi^\sharp_{\rho_n} \phi) - \lambda_j^n \rho_n \cdot (\pi^\sharp_{\rho_n} u_j^n) \cdot (\pi^\sharp_{\rho_n} \phi) dx \\
= & \int_\Omega \nabla(\pi^\sharp_\rho \bar{u}_j) \cdot \nabla(\pi^\sharp_\rho \phi) - \bar{\lambda}_j \rho \cdot (\pi^\sharp_\rho \bar{u}_j) \cdot (\pi^\sharp_\rho \phi) dx,
\end{aligned}$$

for all $j \in \mathbb{N}$, $\phi \in (\mathcal{H}^1(\Omega)/\mathbb{R})$. Moreover we have

$$\lim_{n\to\infty} \int_\Omega \rho_n \cdot (\pi^\sharp_{\rho_n} u^n_i) \cdot (\pi^\sharp_{\rho_n} u^n_j) dx = \int_\Omega \rho \cdot (\pi^\sharp_\rho \bar{u}_i) \cdot (\pi^\sharp_\rho \bar{u}_j) dx = \delta_{ij}.$$

Then $\{\bar{\lambda}_j\}_{j\in\mathbb{N}} \subseteq \{\lambda_j[\rho]\}_{j\in\mathbb{N}}$. Now the proof of the other inclusion is exactly the same as done in Proposition 2.2.8. □

As a consequence of Theorem 3.0.15, we have

Theorem 3.0.30. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $F = \{m, n\}$ with $m, n \in \mathbb{N}$, $m \neq n$. Let $C \subseteq \mathcal{R}[F]$ be a weakly* compact subset of $L^\infty(\Omega)$ such that $\inf_{\rho\in C} \operatorname{ess\,inf}_\Omega \rho > 0$. Let $M > 0$ and $L_M = \{\rho \in L^\infty(\Omega) : \int_\Omega \rho = M\}$. Then for $h = 1, 2$, the function which takes $\rho \in C \cap L_M$ to $\Lambda_{F,h}[\rho]$ has maxima and minima, and if for such points the solutions of problem (3.0.2) are classic solutions, they must belong to $\partial C \cap L_M$.*

Proof. The proof is identical to that of Theorem 2.2.14. □

4. THE STEKLOV PROBLEM FOR THE LAPLACE OPERATOR

Throughout this chapter Ω is a bounded domain in $\mathbb{R}^N$ of class C^1, $\rho \in \mathcal{R}'$, where $\mathcal{R}' := \{\rho \in L^\infty(\partial\Omega) : \operatorname{ess\,inf}_{x\in\partial\Omega}\rho(x) > 0\}$. We consider the following problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} = \lambda \rho u, & \text{on } \partial\Omega\,, \end{cases} \tag{4.0.1}$$

in the unknowns $u \in C^2(\Omega)\cap C^1(\bar{\Omega})$, $\lambda \in \mathbb{R}$. This problem models a free vibrating membrane whose mass is concentrated at the boundary with surface density ρ. We will consider the weak formulation of the problem

$$\int_\Omega \nabla u \cdot \nabla \phi dx = \lambda \int_{\partial\Omega} \rho u \phi d\sigma\,, \quad \forall \phi \in H^1(\Omega)\,, \tag{4.0.2}$$

in the unknowns $u \in H^1(\Omega)$, $\lambda \in \mathbb{R}$. Actually, we will obtain a problem in $(H^1(\Omega)/\mathbb{R})$ since we need to get rid of the constants, which generate the eigenspace corresponding to the eigenvalue 0. Let Tr the trace operator acting from $H^1(\Omega)$ to $L^2(\partial\Omega)$, which is compact thanks to (1.1.17). We denote by J_ρ the continuous embedding of $L^2(\partial\Omega)$ into $(H^1(\Omega))'$ defined by

$$J_\rho[u][\phi] := \int_{\partial\Omega} \rho u \phi d\sigma, \quad \forall u \in L^2(\partial\Omega), \phi \in H^1(\Omega). \tag{4.0.3}$$

We set

$$H^{1,0}_\rho(\Omega) := \left\{ u \in H^1(\Omega) : \int_{\partial\Omega} \rho u d\sigma = 0 \right\},$$

and we consider on $H^1(\Omega)$ the bilinear form $< u, v >= \int_\Omega \nabla u \cdot \nabla v dx$. We denote by $\mathcal{H}^1(\Omega)$ and $\mathcal{H}^{1,0}_\rho(\Omega)$ the spaces $H^1(\Omega)$ and $H^{1,0}_\rho(\Omega)$ endowed with this form. Moreover, by Poincaré-Wirtinger inequality, it turns out that this bilinear form is indeed a scalar product on $\mathcal{H}^{1,0}_\rho(\Omega)$ whose induced norm is equivalent to the standard one. Next we consider the operator $-\Delta_\rho$ as an operator of $\mathcal{H}^{1,0}_\rho(\Omega)$ to $F(\Omega)$, defined by

$$-\Delta_\rho[u][\phi] = \int_\Omega \nabla u \cdot \nabla \phi dx, \quad \forall u \in \mathcal{H}^{1,0}_\rho(\Omega), \phi \in \mathcal{H}^1(\Omega), \tag{4.0.4}$$

where

$$F(\Omega) := \left\{ G \in (H^1(\Omega))' : G[1] = 0 \right\}.$$

Now the operator $-\Delta_\rho$ considered as an operator acting on the whole space $H^1(\Omega)$, is surjective onto $F(\Omega)$, hence it is injective (and continuous) if restricted to $\mathcal{H}^{1,0}_\rho(\Omega)$ and by Poincaré-Wirtinger its inverse is also continuous. Then $-\Delta_\rho$

turns out to be a homeomorphism of $\mathcal{H}^{1,0}_\rho(\Omega)$ onto $F(\Omega)$. Finally, we define the operator π_ρ from $H^1(\Omega)$ to $H^{1,0}_\rho(\Omega)$ as

$$\pi_\rho[u] := u - \frac{\int_{\partial\Omega} \rho u d\sigma}{\int_{\partial\Omega} \rho d\sigma}.$$

We consider the space $(\mathcal{H}^1(\Omega)/\mathbb{R})$ endowed with the bilinear form

$$< p[u], p[v] >= \int_\Omega \nabla u \cdot \nabla v dx,$$

where p is the canonical projection of $\mathcal{H}^1(\Omega)$ onto $(\mathcal{H}^1(\Omega)/\mathbb{R})$. This bilinear form renders $(\mathcal{H}^1(\Omega)/\mathbb{R})$ a Hilbert space. We denote by $\pi^\sharp_\rho$ the map from $(\mathcal{H}^1(\Omega)/\mathbb{R})$ onto $\mathcal{H}^{1,0}_\rho(\Omega)$ defined by the equality $\pi_\rho = \pi^\sharp_\rho \circ p$, which turns out to be a homeomorphism.

We define the operator T_ρ acting on $(\mathcal{H}^1(\Omega)/\mathbb{R})$ as follows

$$T_\rho := (\pi^\sharp_\rho)^{-1} \circ (-\Delta_\rho)^{-1} \circ J_\rho \circ \mathrm{Tr} \circ \pi^\sharp_\rho. \tag{4.0.5}$$

Then we have the following proposition, whose proof is very similar to the proof of Proposition 3.0.2.

Proposition 4.0.6. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, $\rho \in \mathcal{R}'$. The operator T_ρ is a compact selfadjoint operator in $(\mathcal{H}^1(\Omega)/\mathbb{R})$, whose eigenvalues coincide with the reciprocals of the eigenvalues $\lambda_j[\rho]$ of problem (4.0.2) for all $j \in \mathbb{N}$. Moreover, the set of eigenvalues of problem (4.0.2) is contained in $]0, +\infty[$ and consists of the image of a sequence increasing to $+\infty$. Each eigenvalue has finite multiplicity.*

Remark 4.0.7. *We observe that the pair (λ, u) of the set $\mathbb{R} \times (\mathcal{H}^{1,0}_\rho(\Omega) \setminus \{0\})$ satisfies (4.0.2) if and only if $\lambda > 0$ and the pair $(\lambda^{-1}, p[u])$ of the set $\mathbb{R} \times ((\mathcal{H}^1(\Omega)/\mathbb{R}) \setminus \{0\})$ satisfies the equation*

$$\lambda^{-1} p[u] = T_\rho p[u].$$

As in the previous chapter, we observe that the operator T_ρ can be written in a more suitable way in order to prove real analyticity of symmetric functions of eigenvalues. We consider the operator $-\Delta$ from $\mathcal{H}^{1,0}(\Omega)$ to $F(\Omega)$, where

$$\mathcal{H}^{1,0}(\Omega) := \left\{ u \in \mathcal{H}^1(\Omega) : \int_{\partial\Omega} u d\sigma = 0 \right\}.$$

Then it is easy to prove that this operator is a linear homeomorphism from $\mathcal{H}^{1,0}(\Omega)$ onto $F(\Omega)$. Let π, $\pi^\sharp$ be $\pi_{\rho\equiv 1}$, $\pi^\sharp_{\rho\equiv 1}$ respectively. We define the operator $\tilde{T}_\rho$ from $(\mathcal{H}^1(\Omega)/\mathbb{R})$ to itself by

$$\tilde{T}_\rho u := (-\pi^\sharp)^{-1} \circ (-\Delta)^{-1} \circ J_\rho \circ \mathrm{Tr} \circ \pi^\sharp_\rho u, \quad \forall u \in (\mathcal{H}^1(\Omega)/\mathbb{R}).$$

Then the following diagram commutes

$$
\begin{tikzcd}
(\mathcal{H}^1(\Omega)/\mathbb{R}) \arrow[r, "\pi_\rho^\sharp"] & \mathcal{H}^{1,0}_\rho(\Omega) \arrow[r, "J_\rho \circ \mathrm{Tr}"] & F(\Omega) \arrow[r, "(-\Delta_\rho)^{-1}"] \arrow[dr, "(-\Delta)^{-1}"'] & \mathcal{H}^{1,0}_\rho(\Omega) \arrow[r, "(\pi_\rho^\sharp)^{-1}"] & (\mathcal{H}^1(\Omega)/\mathbb{R}) \\
 & & & \mathcal{H}^{1,0}(\Omega) \arrow[ur, "(\pi^\sharp)^{-1}"'] &
\end{tikzcd}
$$

Lemma 4.0.8. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let F be a finite nonempty subset of $\mathbb{N} \setminus \{0\}$. Let*

$$
\begin{aligned}
\mathcal{R}'[F] &:= \{\rho \in \mathcal{R}' \,:\, \lambda_j[\rho] \neq \lambda_l[\rho]\,,\ \forall j \in F, l \in \mathbb{N} \setminus F\}\,,\\
\Theta'[F] &:= \{\rho \in \mathcal{R}'[F] \,:\, \lambda_{j_1}[\rho] = \lambda_{j_2}[\rho]\,,\ \forall j_1, j_2 \in F\}.
\end{aligned}
$$

Let $\tilde{\rho} \in \Theta'[F]$, $\tilde{u}_1, \tilde{u}_2 \in \mathcal{H}^{1,0}_{\tilde{\rho}}(\Omega)$ be such that $p[\tilde{u}_1], p[\tilde{u}_2]$ are two eigenfunctions corresponding to the eigenvalue λ_F^{-1} of the operator $\tilde{T}_\rho$. Then we have

$$
< d\tilde{T}_{\tilde{\rho}}[\dot{\rho}][p[\tilde{u}_1]], p[\tilde{u}_2] >= \int_{\partial\Omega} \dot{\rho}\tilde{u}_1\tilde{u}_2 d\sigma, \quad \forall \dot{\rho} \in L^\infty(\partial\Omega). \tag{4.0.9}
$$

We are now able to state the following

Theorem 4.0.10. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and F a nonempty finite subset of $\mathbb{N} \setminus \{0\}$. Let $\mathcal{R}'[F]$, $\Theta'[F]$ be defined as in the previous lemma. Then $\mathcal{R}'[F]$ is open in $L^\infty(\partial\Omega)$ and the symmetric functions of the eigenvalues*

$$
\Lambda_{F,h}[\rho] = \sum_{\substack{j_1,\dots,j_h \in F\\ j_1 < \dots j_h}} \lambda_{j_1}[\rho] \cdots \lambda_{j_h}[\rho]\,, \quad h = 1, \dots, |F|\,, \tag{4.0.11}
$$

are real analytic in $\mathcal{R}'[F]$. Moreover, if $\rho \in \Theta'[F]$ and the eigenvalues $\lambda_j[\rho]$ assume the common value $\lambda_F[\rho]$ for all $j \in F$, then the differential of $\Lambda_{F,h}$ at ρ is given by the formula

$$
d\Lambda_{F,h}[\rho][\dot{\rho}] = -{\lambda_F}^{h+1}[\rho] \binom{|F|-1}{h-1} \sum_{l \in F} \int_{\partial\Omega} u_l^2 \dot{\rho}\, d\sigma\,, \tag{4.0.12}
$$

for all $\dot{\rho} \in L^\infty(\partial\Omega)$, where $\{u_l\}_{l \in F}$ is an orthonormal basis for $\lambda_F[\rho]$ in $\mathcal{H}^{1,0}_\rho(\Omega)$.

Remark 4.0.13. *We observe that if $j \in F$, then the restriction to $\Theta'[F]$ of the function which takes $\rho \in \mathcal{R}'$ to $\lambda_j[\rho] \in \mathbb{R}$ is real analytic. In fact it coincides on $\Theta'[F]$ with the real analytic function $\frac{\Lambda_{F,1}[\cdot]}{|F|}$.*

We investigate now the existence of critical mass densities for symmetric functions of the eigenvalues. We have the following Theorem.

Proposition 4.0.14. *Let $B = B^N(0,1)$ be the unit ball in $\mathbb{R}^N$, S_N the $(N-1)$-dimensional measure of ∂B, $F = \{1, ..., N\}$ and $M > 0$. Then the constant mass density ρ_M defined by $\rho_M = \frac{M}{S_N}$ is a critical mass density for $\Lambda_{F,h}$ for $h = 1, ..., N$ under the constraint $\int_{\partial\Omega} \rho d\sigma = M$.*

Proof. It is easy to prove that the set $\{u_i := c_N x_i\}_{i=1}^N$, where $c_N = \left(\int_{\partial\Omega}\rho_M x_i^2 d\sigma\right)^{-1}$ for all $i = 1, ..., N$, is the set of the first N eigenfunction for problem (4.0.2) with constant mass density on the unit ball. Such eigenfunctions correspond to the eigenvalue $\frac{S_N}{M}$, then $\rho_M \in \Theta'[F]$. We have then the following formula

$$d\Lambda_{F,h}[\rho_M][\dot\rho] = -\left(\frac{S_N}{M}\right)^{1+h}\binom{N-1}{h-1}\sum_{i\in F}\int_{\partial\Omega} u_i^2\dot\rho d\sigma,$$

for all $\dot\rho \in L^\infty(\partial\Omega)$. We have to show that for all $h = 1, ...N$ there exists $c_h > 0$ such that

$$\left(\frac{S_N}{M}\right)^{1+h}\binom{N-1}{h-1}\sum_{i\in F}\int_{\partial\Omega} u_i^2\dot\rho d\sigma = c_h\int_{\partial\Omega}\dot\rho d\sigma.$$

But this is immediate, since $\sum_{i\in F} u_i^2 = c_N^2$, hence $c_h = c_N^2\left(\frac{S_N}{M}\right)^{1+h}\binom{N-1}{h-1}$. Then the constant density ρ_M is a critical mass density for $\Lambda_{F,h}[\cdot]$. $\square$

We now prove that the function which takes $\rho \in \mathcal{R}'$ to $\lambda_j[\rho]$ is continuous with respect to the weak* topology of $L^\infty(\partial\Omega)$. For a fixed $\rho \in \mathcal{R}'$ we have the following variational representation of the eigenvalues

$$\lambda_j[\rho] = \inf_{\substack{E\le(\mathcal{H}^1(\Omega)/\mathbb{R})\\ \dim E=j}}\ \sup_{0\ne u\in E}\frac{\int_\Omega\left|\nabla(\pi_\rho^\sharp u)\right|^2 dx}{\int_{\partial\Omega}(\pi_\rho^\sharp u)^2\rho\, d\sigma},\quad \forall j\in\mathbb{N}. \tag{4.0.15}$$

By the same argument used in Remark 3.0.24, we have

Remark 4.0.16. *Let $\rho \in \mathcal{R}'$, $0 < \alpha < \beta < +\infty$ be such that $\alpha \le \rho \le \beta$ a.e. in $\partial\Omega$. Then $\lambda_j[\beta] \le \lambda_j[\rho] \le \lambda_j[\alpha]$.*

Lemma 4.0.17. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1 and $\{\rho_n\}_{n\in\mathbb{N}} \subset L^\infty(\partial\Omega)$, $\rho \in L^\infty(\partial\Omega)$ be such that $\rho_n \overset{*}{\rightharpoonup} \rho$. Moreover, let $\alpha := \inf_{n\in\mathbb{N}}\operatorname{ess\,inf}_{x\in\partial\Omega}\rho_n(x) > 0$ and $\beta := \sup_{n\in\mathbb{N}}\|\rho_n\|_{L^\infty(\partial\Omega)} < +\infty$. Then we have*

i) For all $\phi \in (\mathcal{H}^1(\Omega)/\mathbb{R})$, $\mathrm{Tr}[\pi_{\rho_n}^\sharp\phi] \to \mathrm{Tr}[\pi_\rho^\sharp\phi]$ in $L^2(\partial\Omega)$;

ii) if $u_n \rightharpoonup u$ in $(h^1(\Omega)/\mathbb{R})$ then, (possibly passing to a subsequence) $\mathrm{Tr}[\pi_{\rho_n}^\sharp u_n] \to \mathrm{Tr}[\pi_\rho^\sharp u]$ in $L^2(\partial\Omega)$.

Proof. The proof of statement *i)* is immediate. The proof of statement *ii)* is exactly the same as the one for Lemma 3.0.27: here one uses the compactness of trace operator (in Lemma 3.0.27 we used compactness of embedding of $H^1(\Omega)$ into $L^2(\Omega)$). $\square$

We are now ready to state the following proposition. The proof is as in Proposition 3.0.29.

Proposition 4.0.18. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. Let $C \subset L^\infty(\partial\Omega)$ be a weakly* compact subset of $L^\infty(\partial\Omega)$ such that $\inf_{\rho\in C}\operatorname{ess\,inf}_{x\in\partial\Omega}\rho(x) > 0$. Then the functions which take $\rho \in C$ to $\lambda_j[\rho]$ are continuous in the weak* topology of $L^\infty(\partial\Omega)$.*

We note that the case of Steklov boundary conditions is rather different from the cases analyzed in the previous chapters. In fact, Proposition 4.0.14 shows that there exist critical mass densities for the symmetric functions of the eigenvalues under mass constraint. We are led to investigate the existence of relations between the eigenvalues of the Laplace operator with Steklov boundary conditions and the eigenvalues of the Laplace operator with Neumann boundary conditions. For instance, let's take the unit disc in $\mathbb{R}^2$ and consider the variational representations of the eigenvalues of the two problems (3.0.23) and (4.0.15). We note, given $u \in (\mathcal{H}^1(B)/\mathbb{R})$ and a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of densities in $\mathcal{R}$ such that $\rho_n \equiv \frac{1}{n}$ on $B\left(0, 1-\frac{1}{n}\right)$ and $\rho_n \equiv \frac{M-\frac{\pi}{n}\left(1-\frac{1}{n}\right)^2}{\pi\left(1-\left(1-\frac{1}{n}\right)^2\right)}$ on the remaining part (so that $\int_\Omega \rho_n dx = M \quad \forall n \in \mathbb{N}$), that the denominator in the Rayleigh quotient in (3.0.23) with ρ_n converges to the denominator of Rayleigh quotient in (4.0.15) with density $\rho \equiv \frac{M}{2\pi}$ on $\partial\Omega$. Thus one could expect the spectral convergence of the Neumann problems to the Steklov problem. This will be proved in the sequel. First we need a technical Lemma.

Lemma 4.0.19. *Let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$, $M > 0$, ω_N the volume of B, S_N the $(N-1)$-dimensional measure of ∂B. Let B_n be the ball $B(0, 1-\frac{1}{n})$. Let $\rho_n \in \mathcal{R}$ be defined by*

$$\rho_n(x) := \begin{cases} \frac{1}{n}, & \text{if } x \in B_n, \\ \frac{M-\frac{\omega_N}{n}\left(1-\frac{1}{n}\right)^N}{\omega_N\left(1-\left(1-\frac{1}{n}\right)^N\right)}, & \text{if } x \in B \setminus B_n, \end{cases} \tag{4.0.20}$$

for all $n \in \mathbb{N}$.
Let $\pi^\sharp_{\rho_n}$ the map from $(\mathcal{H}^1(B)/\mathbb{R})$ onto $\mathcal{H}^{1,0}_{\rho_n}(B) := \left\{u \in H^1(B) : \int_B \rho_n u dx = 0\right\}$ defined by the equality $\pi_{\rho_n} = \pi^\sharp_{\rho_n} \circ p$, where

$$\pi_{\rho_n}[u] = u - \frac{\int_B \rho_n u dx}{\int_B \rho_n dx}, \quad \forall u \in H^1(B).$$

Let $\pi^\sharp_0$ the map from $(\mathcal{H}^1(B)/\mathbb{R})$ onto $\mathcal{H}^{1,0}_{\partial B}(B) := \left\{u \in H^1(B) : \int_{\partial B} u d\sigma = 0\right\}$ defined by the equality $\pi_0 = \pi^\sharp_0 \circ p$, where

$$\pi_0[u] = u - \frac{\int_{\partial B} u d\sigma}{S_N}, \quad \forall u \in H^1(B).$$

Then the following statements hold true:

i) For all $\phi \in (\mathcal{H}^1(B)/\mathbb{R})$, $\pi^\sharp_{\rho_n}[\phi] \to \pi^\sharp_0[\phi]$ in $L^2(B)$ (hence also in $\mathcal{H}^1(B)$);

ii) if $u_n \rightharpoonup u$ in $(\mathcal{H}^1(B)/\mathbb{R})$, then (possibly passing to a subsequence) $\pi^\sharp_{\rho_n}[u_n] \to \pi^\sharp_0[u]$ in $L^2(B)$;

iii) assume that $u_n \to u$, $w_n \to w$ in $L^2(B)$, $\mathrm{Tr}[u_n] \to \mathrm{Tr}[u]$, $\mathrm{Tr}[w_n] \to \mathrm{Tr}[w]$ in $L^2(\partial B)$, and such that $\|\nabla u_n\|_{L^2(B)}, \|\nabla u\|_{L^2(B)} \leq C$, $\|\nabla w_n\|_{L^2(B)}, \|\nabla w\|_{L^2(B)} \leq C$ uniformly in $n \in \mathbb{N}$. Then

$$\int_B \rho_n (u_n - u) w_n dx \to 0$$

and

$$\int_B \rho_n (w_n - w) u dx \to 0.$$

Proof. As for statement $i)$ of Lemma 3.0.27, it is sufficient to show that

$$\lim_{n\to+\infty}\left\|\frac{\int_B \rho_n\tilde\phi dx}{M}-\frac{\int_{\partial B}\tilde\phi d\sigma}{S_N}\right\|_{L^2(B)}=0,$$

where $\tilde\phi\in H^1(B)$ is such that $\phi=p[\tilde\phi]$. Since the equality

$$\lim_{n\to+\infty}\int_B \rho_n\tilde\phi dx=\frac{M}{S_N}\int_{\partial\Omega}\tilde\phi d\sigma$$

holds, we have the desired result. We now prove statement $ii)$. Let $\tilde u_n,\tilde u\in\mathcal{H}^{1,0}(B):=\left\{\tilde v\in\mathcal{H}^1(B):\int_B\tilde v dx=0\right\}$ be such that $u_n=p[\tilde u_n]$, $u=p[\tilde u]$. We have

$$\left\|\pi^\sharp_{\rho_n}[u_n]-\pi^\sharp_0[u]\right\|_{L^2(B)}\le\left\|\pi^\sharp_{\rho_n}[u_n]-\pi^\sharp_{\rho_n}[u]\right\|_{L^2(B)}+\left\|\pi^\sharp_{\rho_n}[u]-\pi^\sharp_0[u]\right\|_{L^2(B)}.$$

By statement $i)$ it follows that the second summand in the right hand side goes to zero as $n\to+\infty$. For the first summand, we have

$$\begin{aligned}\left\|\pi^\sharp_{\rho_n}[u_n]-\pi^\sharp_{\rho_n}[u]\right\|_{L^2(B)} &= \left\|\tilde u_n-\frac{\int_B\rho_n\tilde u_n dx}{M}-\tilde u+\frac{\int_B\rho_n\tilde u dx}{M}\right\|_{L^2(B)}\\ &\le \|\tilde u_n-\tilde u\|_{L^2(B)}+\frac{\left\|\int_B\rho_n\left(\tilde u_n-\tilde u\right)dx\right\|_{L^2(B)}}{M}\\ &\le \|\tilde u_n-\tilde u\|_{L^2(B)}+\left(\frac{\omega_N^{\frac12}}{M}\right)\cdot\left|\int_B\rho_n\left(\tilde u_n-\tilde u\right)dx\right|.\end{aligned}$$

Now, if we prove that $\tilde u_n\to\tilde u$ in $L^2(B)$ we are done, since the result follows by statement $iii)$ with $w_n\equiv 1$. Since $\{\tilde u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}^{1,0}(B)$, it is bounded in $H^1(B)$ by Poincaré-Wirtinger inequality. Then, possibly passing to a subsequence, $\{\tilde u_n\}_{n\in\mathbb{N}}$ weakly converges in $H^1(B)$, hence strongly in $L^2(B)$ to a certain function $\tilde w$. Clearly w has zero mean. Since the projection of $(\mathcal{H}^1(B)/\mathbb{R})$ onto $\mathcal{H}^{1,0}(B)$ is a homeomorphism and the limit is unique, it follows that $\tilde w=\tilde u$. Thus $\|\tilde u_n-\tilde u\|_{L^2(B)}\to 0$. Then in order to complete the proof of statement $ii)$ it suffices to prove statement $iii)$.

We make the proof for $N=2$ for the sake of simplicity, but the argument is not restrictive, and can be applied to the N-dimensional ball. Let $\varepsilon=\frac{1}{n}$. Let then $u_\varepsilon(x,y)$, $w_\varepsilon(x,y)\in L^2(B)$, such that $u_\varepsilon\to 0$ in $L^2(B)$ as $\varepsilon\to 0$, and such that the norms of u_ε, w_ε, ∇u_ε, ∇w_ε in $L^2(B)$ are uniformly bounded in ε, and moreover $\mathrm{Tr}[u_\varepsilon]\to 0$ in $L^2(\partial B)$ and $\mathrm{Tr}[u_\varepsilon]$, $\mathrm{Tr}[w_\varepsilon]$ are uniformly bounded in $L^2(\partial B)$. We consider then

$$\lim_{\varepsilon\to 0}\int_B\rho_\varepsilon u_\varepsilon w_\varepsilon dxdy. \tag{4.0.21}$$

We have that

$$\int_B\rho_\varepsilon u_\varepsilon w_\varepsilon dxdy=\varepsilon\int_{B_\varepsilon}u_\varepsilon w_\varepsilon dxdy+C(\varepsilon)\int_{B\setminus B_\varepsilon}u_\varepsilon w_\varepsilon dxdy,$$

where $B_\varepsilon=B(0,1-\varepsilon)$, $C(\varepsilon)=\frac{M-\pi\varepsilon(1-\varepsilon)^2}{\pi\left(1-(1-\varepsilon)^2\right)}$. The first summand clearly goes to zero as $\varepsilon\to 0$. By multiplying and dividing the second summand by ε and observing that $\varepsilon C(\varepsilon) \leq C' < +\infty$ for $\varepsilon \leq \varepsilon_0$, we obtain that

$$\left| \int_B \rho_\varepsilon u_\varepsilon w_\varepsilon dxdy \right| \leq C\varepsilon + C' \cdot \frac{1}{\varepsilon} \int_{B\setminus B_\varepsilon} |u_\varepsilon w_\varepsilon| \, dxdy. \tag{4.0.22}$$

Now consider the second summand in (4.0.22) and pass to polar coordinates (r, θ). We have

$$\int_{B\setminus B_\varepsilon} \frac{1}{\varepsilon} |u_\varepsilon w_\varepsilon| \, dxdy = \int_0^{2\pi} \int_{1-\varepsilon}^1 \frac{r}{\varepsilon} |u_\varepsilon(r,\theta)| \, |w_\varepsilon(r,\theta)| \, drd\theta.$$

We operate a new change of variable, namely $r = 1 - t$ with $0 \leq t \leq \varepsilon$, and denote the functions $u_\varepsilon(1-t,\theta)$, $w_\varepsilon(1-t,\theta)$ by $u_\varepsilon(t,\theta)$ and $w_\varepsilon(t,\theta)$ respectively. We have

$$\int_{B\setminus B_\varepsilon} \frac{1}{\varepsilon} |u_\varepsilon w_\varepsilon| \, dxdy = \int_0^{2\pi} \int_0^\varepsilon \frac{(1-t)}{\varepsilon} |u_\varepsilon(t,\theta)| \, |w_\varepsilon(t,\theta)| \, dtd\theta. \tag{4.0.23}$$

Now, for almost every θ, u_ε, w_ε are absolutely continuous on $[0, \varepsilon]$ and since this set is compact, also their product is absolutely continuous. Let θ be fixed and set $u_\varepsilon(t) = u_\varepsilon(t,\theta)$, $w_\varepsilon(t) = w_\varepsilon(t,\theta)$. We have

$$u_\varepsilon(t)w_\varepsilon(t) = u_\varepsilon(0)w_\varepsilon(0) + \int_0^t \frac{\partial u_\varepsilon}{\partial t'}(t')w_\varepsilon(t') + u_\varepsilon(t')\frac{\partial w_\varepsilon}{\partial t'}(t')dt', \tag{4.0.24}$$

$$u_\varepsilon(t') = u_\varepsilon(0) + \int_0^{t'} \frac{\partial u_\varepsilon}{\partial t''}(t'')dt'',$$

$$w_\varepsilon(t') = w_\varepsilon(0) + \int_0^{t'} \frac{\partial w_\varepsilon}{\partial t''}(t'')dt''.$$

We define $C_1(t,\theta)$ by $C_1(t,\theta) = \left(\int_0^t \left| \frac{\partial u_\varepsilon}{\partial t'}(t',\theta)dt' \right|^2 \right)^{\frac{1}{2}}$. Let ε be fixed, then for a.e. θ, $C_1(t,\theta)$ is increasing in $0 \leq t \leq \varepsilon$ and $C_1(t,\theta) \leq C_1(\varepsilon,\theta)$. From now on we fix θ and denote $C_1(t,\theta)$ by $C_1(t)$. The same considerations hold for $C_2(t,\theta)$ defined by $C_2(t,\theta) = \left(\int_0^t \left| \frac{\partial w_\varepsilon}{\partial t'}(t',\theta)dt' \right|^2 \right)^{\frac{1}{2}}$. Then we have

$$|u_\varepsilon(t')| \leq |u_\varepsilon(0)| + t'^{\frac{1}{2}} C_1(t') \leq |u_\varepsilon(0)| + t^{\frac{1}{2}} C_1(\varepsilon), \tag{4.0.25}$$

$$|w_\varepsilon(t')| \leq |w_\varepsilon(0)| + t'^{\frac{1}{2}} C_2(t') \leq |w_\varepsilon(0)| + t^{\frac{1}{2}} C_2(\varepsilon). \tag{4.0.26}$$

Now, let's consider the right hand side in (4.0.23). By (4.0.25):

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\varepsilon} & \frac{(1-t)}{\varepsilon}\left|u_\varepsilon(t,\theta)\right|\left|w_\varepsilon(t,\theta)\right| dtd\theta \qquad (4.0.27)\\
& \leq \int_0^{2\pi}\int_0^{\varepsilon}\frac{1}{\varepsilon}\left|u_\varepsilon(0,\theta)\right|\left|w_\varepsilon(0,\theta)\right| dtd\theta \\
+ & \int_0^{2\pi}\int_0^{\varepsilon}\frac{(1-t)}{\varepsilon}\int_0^t\left|\frac{\partial u_\varepsilon}{\partial t'}(t',\theta)\right|\left|w_\varepsilon(t',\theta)\right| dt'dtd\theta \\
+ & \int_0^{2\pi}\int_0^{\varepsilon}\frac{(1-t)}{\varepsilon}\int_0^t\left|u_\varepsilon(t',\theta)\right|\left|\frac{\partial w_\varepsilon}{\partial t'}(t',\theta)\right| dt'dtd\theta \leq \\
& \int_0^{2\pi}\left|u_\varepsilon(0,\theta)\right|\left|w_\varepsilon(0,\theta)\right| d\theta \\
+ & \int_0^{2\pi}\int_0^{\varepsilon}\frac{(1-t)}{\varepsilon}\int_0^t\left|\frac{\partial u_\varepsilon}{\partial t'}(t',\theta)\right|(\left|w_\varepsilon(0,\theta)\right| + t^{\frac{1}{2}}C_2(t,\theta))dt'dtd\theta \\
+ & \int_0^{2\pi}\int_0^{\varepsilon}\frac{(1-t)}{\varepsilon}\int_0^t\left|\frac{\partial w_\varepsilon}{\partial t'}(t',\theta)\right|(\left|u_\varepsilon(0,\theta)\right| + t^{\frac{1}{2}}C_1(t,\theta))dt'dtd\theta \leq \\
& \int_0^{2\pi}\left|u_\varepsilon(0,\theta)\right|\left|w_\varepsilon(0,\theta)\right| d\theta \\
+ & \int_0^{2\pi}\int_0^{\varepsilon}\frac{(1-t)}{\varepsilon}C_1(\varepsilon,\theta)t^{\frac{1}{2}}(\left|w_\varepsilon(0,\theta)\right| + t^{\frac{1}{2}}C_2(\varepsilon,\theta))dtd\theta \\
+ & \int_0^{2\pi}\int_0^{\varepsilon}\frac{(1-t)}{\varepsilon}C_2(\varepsilon,\theta)t^{\frac{1}{2}}(\left|u_\varepsilon(0,\theta)\right| + t^{\frac{1}{2}}C_1(\varepsilon,\theta))dtd\theta \leq \\
& \int_0^{2\pi}\left|u_\varepsilon(0,\theta)\right|\left|w_\varepsilon(0,\theta)\right| d\theta \\
+ & \int_0^{2\pi} C_1(\varepsilon,\theta)C_2(\varepsilon,\theta)(\frac{\varepsilon}{2}-\frac{\varepsilon^2}{3}) + C_1(\varepsilon,\theta)\left|w_\varepsilon(0,\theta)\right|(\frac{2\varepsilon^{\frac{1}{2}}}{3}-\frac{2\varepsilon^{\frac{3}{2}}}{5})d\theta \\
+ & \int_0^{2\pi} C_1(\varepsilon,\theta)C_2(\varepsilon,\theta)(\frac{\varepsilon}{2}-\frac{\varepsilon^2}{3}) + C_2(\varepsilon,\theta)\left|u_\varepsilon(0,\theta)\right|(\frac{2\varepsilon^{\frac{1}{2}}}{3}-\frac{2\varepsilon^{\frac{3}{2}}}{5})d\theta.
\end{aligned}$$

Now, since $\int_0^{2\pi} C_1(\varepsilon,\theta)^2 d\theta \leq \|\nabla u_\varepsilon\|^2_{L^2(\Omega)}$, $\int_0^{2\pi} C_2(\varepsilon,\theta)^2 d\theta \leq \|\nabla w_\varepsilon\|^2_{L^2(\Omega)}$ and such quantities are uniformly bounded in ε, and $\mathrm{Tr}[u_\varepsilon]$, $\mathrm{Tr}[w_\varepsilon]$ are uniformly bounded in $L^2(\partial\Omega)$, the second and third summand go to 0 as $\varepsilon \to 0$. Since $\mathrm{Tr}[u_\varepsilon] \to 0$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$, the first summand vanishes as $\varepsilon \to 0$.

Observe now, that for the N-dimensional ball B the same results still hold. Passing to polar coordinates, in (4.0.51) we have to estimate the following quantity:

$$\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\int_0^{\varepsilon}\frac{1}{\varepsilon}\left|u_\varepsilon(\phi_1,...,\phi_{N-1},t)\right|\left|w_\varepsilon(\phi_1,...,\phi_{N-1},t)\right| dV,$$

where

$$dV = (1-t)^{N-1}\sin^{N-2}(\phi_1)\sin^{N-3}(\phi_2)\cdots\sin(\phi_{N-2})d\phi_1\cdots d\phi_{N-1}dt$$

and the calculations are the same as for $N = 2$. This concludes the proof. $\square$

For details of the following results we refer to [2] and [28]. Let's introduce some definition

Definition 4.0.28. *Let H be a real Hilbert space, $\mathcal{K}(H,H)$ the Banach subspace of $\mathcal{L}(H,H)$ of those $T \in \mathcal{L}(H,H)$ which are compact. A set $\mathcal{K} \subset \mathcal{K}(H,H)$ is said to be collectively compact if and only if the set $\{K[x] : K \in \mathcal{K}, x \in B\}$, where B is the unit ball in H, has compact closure. We say that a sequence of compact operators $\{K_n\}_{n\in\mathbb{N}}$ compactly converges to the compact operator K if $\{K_n\}_{n\in\mathbb{N}}$ is collectively compact and $K_n[x_n] \to K[x]$ whenever $x_n \to x$ in H.*

We will need the following

Theorem 4.0.29. *Let H be a real Hilbert space, $\{K_n\}_{n\in\mathbb{N}} \subset \mathcal{K}(H,H)$ compactly convergent to $K \in \mathcal{K}(H,H)$. Then*

$$\lim_{n\to+\infty} \left\|(K_n - K)^2\right\|_{\mathcal{L}(H,H)} = 0$$

Corollary 4.0.30. *In the hypothesis of the previous Theorem, if K_n and K are self-adjoint for all $n \in \mathbb{N}$, then compact convergence of operators implies norm convergence.*

Finally we state the following

Theorem 4.0.31. *Let H be a real Hilbert space, and $\{A_n\}_{n\in\mathbb{N}}$ a sequence of bounded self-adjoint operators converging in norm to the bounded self-adjoint operator A, i.e., $\lim_{n\to+\infty} \|A_n - A\|_{\mathcal{L}(H,H)} = 0$. Then isolated eigenvalues λ of A of finite multiplicity are exactly the limits of eigenvalues of A_n, including multiplicity; moreover the corresponding eigenprojections converge in norm.*

Definition 4.0.32. *Let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$ and $M > 0$. Let $\rho_n \in \mathcal{R}$, $\pi^\sharp_{\rho_n}$, $\pi^\sharp_0$ be defined as in Lemma 4.0.19. We set*

$$\begin{aligned}\tilde{T}_n &:= (\pi^\sharp)^{-1} \circ (-\Delta)^{-1} \circ J_{\rho_n} \circ i \circ \pi^\sharp_{\rho_n},\\ \tilde{T} &:= (\pi^\sharp)^{-1} \circ (-\Delta)^{-1} \circ J_M \circ \mathrm{Tr} \circ \pi^\sharp_0,\end{aligned}$$

where the map $-\Delta$ from $\mathcal{H}^{1,0}(B)$ onto $F(B)$ is defined as in (3.0.8), the map $\pi^\sharp$ from $(\mathcal{H}^1(B)/\mathbb{R})$ onto $\mathcal{H}^{1,0}(B)$ is defined as in the previous chapter, with $\rho \equiv 1$, the maps J_{ρ_n} of $L^2(B)$ into $F(B)$ are defined as in (3.0.3) and the map J_M of $L^2(\partial B)$ into $F(B)$ is defined as in (4.0.3) with $\rho \equiv \frac{M}{S_N}$.

Now we are ready to prove the following

Theorem 4.0.33. *Let $B = B(0,1)$ be the unit ball in $\mathbb{R}^N$, $\tilde{T}$ and $\tilde{T}_n$ be as in Definition 4.0.32. Then the sequence of compact operators $\{\tilde{T}_n\}_{n\in\mathbb{N}}$ compactly converges to the compact operator $\tilde{T}$ in $\mathcal{K}\left((\mathcal{H}^1(B)/\mathbb{R}), (\mathcal{H}^1(B)/\mathbb{R})\right)$.*

Proof. In order to prove the compact convergence of $\tilde{T}_n$ to $\tilde{T}$ we need to verify that

i) $\tilde{T}_n$ and $\tilde{T}$ is compact for all $n \in \mathbb{N}$;

ii) if $\|u_n\|_{(\mathcal{H}^1(B)/\mathbb{R})} \le C$ for all $n \in \mathbb{N}$, then the family $\{\tilde{T}_n u_n\}_{n\in\mathbb{N}}$ is relatively compact;

iii) if $u_n \to u$ in $(\mathcal{H}^1(B)/\mathbb{R})$, then $\tilde{T}_n u_n \to \tilde{T} u$ in $(\mathcal{H}^1(B)/\mathbb{R})$.

The first statement is clearly true by the compactness of i and Tr. For the second statement, first fix $u \in (\mathcal{H}^1(B)/\mathbb{R})$. We have

$$\begin{aligned}\lim_{n\to+\infty}\int_B \rho_n \pi^\sharp_{\rho_n}[u]dx &= \lim_{n\to+\infty}\int_B \rho_n\left(\pi^\sharp_{\rho_n}[u]-\pi^\sharp_0[u]\right)dx\\ &+ \left(\lim_{n\to+\infty}\int_B \rho_n\pi^\sharp_0[u]dx - \frac{M}{S_N}\int_{\partial B}\pi^\sharp_0[u]d\sigma\right)\\ &+ \frac{M}{S_N}\int_{\partial B}\pi^\sharp_0[u]d\sigma.\end{aligned}$$

By Lemma 4.0.19 we have that the first summand goes to zero as $n \to +\infty$, and since the second term converges to zero as $n \to +\infty$, it follows that $\tilde{T}_n u$ is bounded for each $u \in (\mathcal{H}^1(B)/\mathbb{R})$. Thus, by Banach-Steinhaus Theorem, there exists C' such that $\left\|\tilde{T}_n\right\|_{\mathcal{L}((\mathcal{H}^1(\Omega)/\mathbb{R}),(\mathcal{H}^1(B)/\mathbb{R}))} \le C'$ for all $n \in \mathbb{N}$. Moreover, since $\|u_n\|_{(\mathcal{H}^1(B)/\mathbb{R})} \le C$ for all $n \in \mathbb{N}$, possibly passing to a subsequence, we have that $u_n \rightharpoonup u$ in $(\mathcal{H}^1(B)/\mathbb{R})$. This implies that, possibly passing to a subsequence, $\tilde{T}_n u_n \rightharpoonup w$ in $(\mathcal{H}^1(B)/\mathbb{R})$ for $n \to +\infty$. We show that $w = \tilde{T}u$. Let $w_n := \tilde{T}_n u_n$. We have

$$\lim_{n\to+\infty}\int_B \nabla(\pi^\sharp_{\rho_n}[w_n])\cdot\nabla(\pi^\sharp_{\rho_n}[\phi])dx = \int_B \nabla(\pi^\sharp_0[w])\cdot\nabla(\pi^\sharp_0[\phi])dx,$$

for all $\phi \in (\mathcal{H}^1(B)/\mathbb{R})$. On the other hand, we have that

$$\begin{aligned}\int_B \nabla(\pi^\sharp_{\rho_n}[w_n])\cdot\nabla(\pi^\sharp_{\rho_n}[\phi])dx &= -\Delta[(-\Delta)^{-1}\circ J_{\rho_n}\circ i\circ \pi^\sharp_{\rho_n}[u_n]][\pi^\sharp_{\rho_n}[\phi]]\\ &= \int_B \rho_n\pi^\sharp_{\rho_n}[u_n]\pi^\sharp_{\rho_n}[\phi]dx. \qquad (4.0.34)\end{aligned}$$

Then, by Lemma 4.0.19, *iii)* we have

$$\begin{aligned}\lim_{n\to+\infty} < w_n,\phi >_{(\mathcal{H}^1(B)/\mathbb{R})} &= \lim_{n\to+\infty}\int_B \rho_n\pi^\sharp_{\rho_n}[u_n]\pi^\sharp_{\rho_n}[\phi]dx\\ &= \lim_{n\to+\infty}\int_B \rho_n\left(\pi^\sharp_{\rho_n}[u_n]-\pi^\sharp_0[u]\right)\pi^\sharp_{\rho_n}[\phi]dx\\ &+ \lim_{n\to+\infty}\int_B \rho_n\pi^\sharp_0[u]\left(\pi^\sharp_{\rho_n}[\phi]-\pi^\sharp_0[\phi]\right)dx\\ &+ \lim_{n\to+\infty}\int_B \rho_n\pi^\sharp_0[u]\pi^\sharp_0[\phi]dx\\ &= \frac{M}{S_N}\int_{\partial B}\pi^\sharp_0[u]\pi^\sharp_0[\phi]d\sigma\\ &= -\Delta[\pi^\sharp_0\circ(\pi^\sharp_0)^{-1}\circ(-\Delta)^{-1}\circ J_M\circ Tr\circ\pi^\sharp_0[u]][\pi^\sharp_0[\phi]]\\ &= \int_B \nabla(\pi^\sharp_0[u])\cdot\nabla(\pi^\sharp_0[\phi])dx =< \tilde{T}u,\phi >_{(\mathcal{H}^1(B)/\mathbb{R})},\end{aligned}$$

hence $w = \tilde{T}u$. In a similar way one can prove that $\|w_n\|_{(\mathcal{H}^1(B)/\mathbb{R})} \to \|w\|_{(\mathcal{H}^1(B)/\mathbb{R})}$.

In fact

$$\begin{aligned}
\lim_{n\to+\infty}\|w_n\|^2_{(\mathcal{H}^1(B)/\mathbb{R})} &= \lim_{n\to+\infty}\int_B \rho_n\left(\pi^\sharp_{\rho_n}[u_n]-\pi^\sharp_0[u]\right)\pi^\sharp_{\rho_n}[w_n]dx \\
&+ \lim_{n\to+\infty}\int_B \rho_n\pi^\sharp_0[u]\left(\pi^\sharp_{\rho_n}[w_n]-\pi^\sharp_0[w_n]\right)dx \\
&+ \lim_{n\to+\infty}\int_B \rho_n\pi^\sharp_0[u]\left(\pi^\sharp_0[w_n]-\pi^\sharp_0[w]\right)dx \\
&+ \lim_{n\to+\infty}\int_B \rho_n\pi^\sharp_0[u]\pi^\sharp_0[w]dx \\
&= \frac{M}{S_N}\int_{\partial B}\pi^\sharp_0[u]\pi^\sharp_0[w]d\sigma = \|w\|^2_{(\mathcal{H}^1(B)/\mathbb{R})}.
\end{aligned}$$

This proves *ii*). As for point *iii*), let $u_n \to u$ in $(\mathcal{H}^1(B)/\mathbb{R})$. Then there exists C'' such that $\|u_n\|_{(\mathcal{H}^1(B)/\mathbb{R})} \le C''$ for all n. Then, by the same argument used for point *ii*), for each sequence $n_j \to +\infty$, possibly passing to a subsequence, we have $\tilde{T}_{n_j}u_{n_j} \to \tilde{T}u$. Since this is true for each $\{n_j\}_{j\in\mathbb{N}}$, we have the convergence for the whole family, i.e., $\tilde{T}_n u_n \to \tilde{T}u$. This concludes the proof of the Theorem. □

Corollary 4.0.35. *Let B be the unit ball in $\mathbb{R}^N$. Let ρ_n defined as in Lemma 4.0.19. Let $\lambda_j[\rho_n]$ be the eigenvalues of problem (3.0.2) on B for all $j \in \mathbb{N}$. Let $\overline{\lambda}_j$, $j \in \mathbb{N}$ denote the eigenvalues of problem (4.0.2) on B corresponding to the constant surface density $\frac{M}{S_N}$. Then for all $j \in \mathbb{N}$, we have $\lim_{n\to+\infty}\lambda_j[\rho_n] = \overline{\lambda}_j$ for all $j \in \mathbb{N}$.*

Finally we show that this result also holds for bounded domains of $\mathbb{R}^N$ of class C^2. Let M be a parametric hypersurface in $\mathbb{R}^3$ of class C^2, i.e., a $\phi : D \to \mathbb{R}^3$, where D is a bounded open subset of $\mathbb{R}^2$ and $\phi \in C^2(\bar{D})$. Moreover, we assume that $D\phi(u,v)$ is injective at each $(u,v) \in D$. We set

$$M(\varepsilon) := \{\phi(u,v) + t\nu(u,v) : (u,v) \in D, 0 < t < \varepsilon\},$$

where $\nu(u,v)$ is the normal vector to $\phi(u,v)$, given by

$$\nu(u,v) = \frac{\frac{\partial\phi}{\partial u}\wedge\frac{\partial\phi}{\partial v}}{\left|\frac{\partial\phi}{\partial u}\wedge\frac{\partial\phi}{\partial v}\right|}.$$

We consider the map ψ from $D \times]0,\varepsilon[$ onto $M(\varepsilon)$ defined by

$$\psi(u,v,t) = \phi(u,v) + t\nu(u,v)$$

for all $(u,v) \in D$, $t \in]0,\varepsilon[$.

In the sequel we will need the following Lemma. For the sake of completeness we include also statement *ii*).

Lemma 4.0.36. *Let M be a parametric hypersurface and (D,ϕ) a parametrization of M. Assume that $\inf_D\left|\frac{\partial\phi}{\partial u}\wedge\frac{\partial\phi}{\partial v}\right| > 0$. Let $f_\varepsilon, f \in H^1(\Omega)$ for all $\varepsilon > 0$ be such that $f_\varepsilon \to f$ in $H^1(\Omega)$ as $\varepsilon \to 0$. Moreover, assume that ψ is a diffeomorphism for ε sufficiently small. Then we have*

i)

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{M(\varepsilon)} f dV = \int_M f d\sigma; \tag{4.0.37}$$

ii)

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{M(\varepsilon)}(f_\varepsilon - f)\,dV = 0, \tag{4.0.38}$$

where $dV = dxdydz$.

Proof. We consider

$$\frac{1}{\varepsilon}\int_{M(\varepsilon)} f(x,y,z)dxdydz = \frac{1}{\varepsilon}\int_0^\varepsilon\int_D f(u,v,t)\,|\det D\psi|\,dudvdt,$$

and compute the limit as $\varepsilon \to 0$. We observe that

$$\begin{aligned}\det D\psi \;&=\; \det\left[\tfrac{\partial\phi}{\partial u},\;\; \tfrac{\partial\phi}{\partial v},\;\; \nu(u,v)\right] + t\det\left[\tfrac{\partial\nu}{\partial u},\;\; \tfrac{\partial\phi}{\partial v},\;\; \nu(u,v)\right] \\ &-\; t\det\left[\tfrac{\partial\nu}{\partial v},\;\; \tfrac{\partial\phi}{\partial u},\;\; \nu(u,v)\right] + t^2\det\left[\tfrac{\partial\nu}{\partial u,}\;\; \tfrac{\partial\nu}{\partial v},\;\; \nu(u,v)\right].\end{aligned} \tag{4.0.39}$$

Moreover

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^\varepsilon\int_D \quad & f(u,v,t)\,|\det D\psi|\,dudvdt = \\ & \frac{1}{\varepsilon}\int_0^\varepsilon\int_D (f(u,v,t)-f(u,v,0))\,|\det D\psi|\,dudvdt \\ & +\frac{1}{\varepsilon}\int_0^\varepsilon\int_D f(u,v,0)\,|\det D\psi|\,dudvdt.\end{aligned} \tag{4.0.40}$$

For the first summand in the right-hand side of (4.0.40), we observe that for a.e. $(u,v)\in D$, we have
$|f(u,v,t)-f(u,v,0)| \le \int_0^\varepsilon \left|\frac{\partial f}{\partial t'}(u,v,t')\right| dt'$. Then, since $f \in H^1(M(\varepsilon))$, we have

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^\varepsilon\int_D |f(u,v,t)-f(u,v,0)| \quad & |\det D\psi|\,dudvdt \\ & \le \int_D\int_0^\varepsilon \left|\frac{\partial f}{\partial t}(u,v,t)\right| |\det D\psi|\,dtdudv \\ & \le |M(\varepsilon)|^{\frac{1}{2}}\,\|\nabla f\|_{L^2(M(\varepsilon))}\,.\end{aligned}$$

Thus the first summand in the right-hand side of (4.0.40) vanishes as $\varepsilon \to 0$. For the second summand, observe that for $(u,v)\in D$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon |\det D\psi(u,v,t)|\,dt = \left|\det\left[\tfrac{\partial\phi}{\partial u},\;\; \tfrac{\partial\phi}{\partial v},\;\; \nu(u,v)\right]\right| = \left|\frac{\partial\phi}{\partial u}\wedge\frac{\partial\phi}{\partial v}\right|,$$

since the terms in (4.0.39) containing t vanish as $\varepsilon \to 0$. The last quantity is exactly the area element of the surface. Then we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{M(\varepsilon)} f dV = \int_M f d\sigma.$$

Now we prove statement ii). We consider

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{M(\varepsilon)} (f_\varepsilon - f)\,dV &= \frac{1}{\varepsilon}\int_0^\varepsilon \int_D (f_\varepsilon(u,v,t) - f(u,v,t))\,|\mathrm{det}D\psi|\,dudvdt \\
&= \frac{1}{\varepsilon}\int_0^\varepsilon \int_D (f_\varepsilon(u,v,0) - f(u,v,0))\,|\mathrm{det}D\psi|\,dudvdt \\
&+ \frac{1}{\varepsilon}\int_0^\varepsilon \int_D \int_0^t \left(\frac{\partial f_\varepsilon}{\partial t'}(u,v,t') - \frac{\partial f}{\partial t'}(u,v,t')\right) dt'\,|\mathrm{det}D\psi|\,dudvdt \\
&\leq \frac{1}{\varepsilon}\int_0^\varepsilon \int_D |f_\varepsilon(u,v,0) - f(u,v,0)|\,|\mathrm{det}D\psi|\,dudvdt \qquad (4.0.41) \\
&+ \frac{1}{\varepsilon}\int_0^\varepsilon \int_D \int_0^t \left|\frac{\partial f_\varepsilon}{\partial t'}(u,v,t') - \frac{\partial f}{\partial t'}(u,v,t')\right| dt'\,|\mathrm{det}D\psi|\,dudvdt.
\end{aligned}
$$

We set $G_1(u,v) = \left|\det\left[\frac{\partial\nu}{\partial u}, \;\; \frac{\partial\phi}{\partial v}, \;\; \nu(u,v)\right] - \det\left[\frac{\partial\nu}{\partial v}, \;\; \frac{\partial\phi}{\partial u}, \;\; \nu(u,v)\right]\right|$, $G_2(u,v) = \left|\det\left[\frac{\partial\nu}{\partial u,} \;\; \frac{\partial\nu}{\partial v}, \;\; \nu(u,v)\right]\right|$. We have for the first summand of (4.0.41)

$$
\begin{aligned}
\frac{1}{\varepsilon} & \int_0^\varepsilon \int_D |f_\varepsilon(u,v,0) - f(u,v,0)|\,|\mathrm{det}D\psi|\,dudvdt \\
&\leq \frac{1}{\varepsilon}\int_0^\varepsilon \int_D |f_\varepsilon(u,v,0) - f(u,v,0)| \left|\frac{\partial\phi}{\partial u}\wedge\frac{\partial\phi}{\partial v}\right| dudvdt \\
&+ \frac{1}{\varepsilon}\int_0^\varepsilon \int_D |f_\varepsilon(u,v,0) - f(u,v,0)|\, tG_1(u,v)dudvdt \\
&+ \frac{1}{\varepsilon}\int_0^\varepsilon \int_D |f_\varepsilon(u,v,0) - f(u,v,0)|\, t^2G_2(u,v)dudvdt \\
&= \int_M |f_\varepsilon - f|\,d\sigma \\
&+ \frac{\varepsilon}{2}\int_D |f_\varepsilon(u,v,0) - f(u,v,0)|\, G_1(u,v)dudv \\
&+ \frac{\varepsilon^2}{3}\int_D |f_\varepsilon(u,v,0) - f(u,v,0)|\, G_2(u,v)dudv \\
&\leq C(\varepsilon)\int_M |f_\varepsilon - f|\,d\sigma,
\end{aligned}
$$

where

$$
C(\varepsilon) = 1 + \left(\inf_D \left|\frac{\partial\phi}{\partial u}\wedge\frac{\partial\phi}{\partial v}\right|\right)^{-1} \left(\frac{\varepsilon\,\|G_1\|_{L^\infty(D)}}{2} + \frac{\varepsilon^2\,\|G_2\|_{L^\infty(D)}}{3}\right),
$$

and $C(\varepsilon) \to 1$ as $\varepsilon \to 0$. Thus the first summand in (4.0.41) vanishes as $\varepsilon \to 0$ because $f_\varepsilon \to f$ in $L^2(M)$ hence in $L^1(M)$. Now we consider the second summand in (4.0.41). We have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^{\varepsilon}\int_D\int_0^t\left|\frac{\partial f_\varepsilon}{\partial t'}(u,v,t')-\frac{\partial f}{\partial t'}(u,v,t')\right| dt'\,|\det D\psi|\,dudvdt\\
&\le \frac{1}{\varepsilon}\int_0^{\varepsilon}\int_D t^{\frac12}\left(\int_0^t\left|\frac{\partial f_\varepsilon}{\partial t'}(u,v,t')-\frac{\partial f}{\partial t'}(u,v,t')\right|^2 dt'\right)^{\frac12}|\det D\psi|\,dudvdt\\
&\le \frac{C}{\varepsilon}\int_0^{\varepsilon} t^{\frac12}\left(\int_D\int_0^t\left|\frac{\partial f_\varepsilon}{\partial t'}(u,v,t')-\frac{\partial f}{\partial t'}(u,v,t')\right|^2|\det D\psi|\,dt'dudv\right)^{\frac12}dt\\
&\le C\varepsilon^{\frac12}\left\|\nabla\left(f_\varepsilon-f\right)\right\|_{L^2(M(\varepsilon))},
\end{aligned}
$$

where $C=\left(\int_D|\det D\psi|\,dudv\right)^{\frac12}$. This concludes the proof. □

Remark 4.0.42. *Clearly the same result holds for hypersurfaces in $\mathbb{R}^N$. In fact, given a parametrization ϕ from $D\subset\mathbb{R}^{(N-1)}$ to $\mathbb{R}^N$ of the hypersurface M, we define the set $M(\varepsilon)$ in the same way as in the previous case. In this case the normal vector at $\phi(u_1,...,u_{N-1})$ is given by*

$$
\nu(u_1,...,u_{N-1})=\frac{\frac{\partial\phi}{\partial u_1}\wedge\cdots\wedge\frac{\partial\phi}{\partial u_{N-1}}}{\left|\frac{\partial\phi}{\partial u_1}\wedge\cdots\wedge\frac{\partial\phi}{\partial u_{N-1}}\right|}.
$$

The diffeomorphism ψ of $D\times]0,\varepsilon[$ onto $M(\varepsilon)$ is defined as in the previous case. Then, in the computation of $\det D\psi$ *we will obtain*

$$
\begin{aligned}
\det D\psi &= \det\left[\frac{\partial\phi}{\partial u_1}\ \cdots\ \frac{\partial\phi}{\partial u_{N-1}}\ \ \nu(u_1,...,u_{N-1})\right]\\
&+\ t g_1\left(\frac{\partial\phi}{\partial u_1},\frac{\partial\nu}{\partial u_1},...,\frac{\partial\phi}{\partial u_{N-1}},\frac{\partial\nu}{\partial u_{N-1}}\right)+\cdots\\
&+\ t^{N-1}g_{N-1}\left(\frac{\partial\phi}{\partial u_1},\frac{\partial\nu}{\partial u_1},...,\frac{\partial\phi}{\partial u_{N-1}},\frac{\partial\nu}{\partial u_{N-1}}\right),
\end{aligned}
$$

where g_i are suitable compositions of sums and products of the first partial derivatives of ϕ and ν. The first term in the sum is equal to

$$
\left|\frac{\partial\phi}{\partial u_1}\wedge\cdots\wedge\frac{\partial\phi}{\partial u_{N-1}}\right|,
$$

which is the area element of the hypersurface. Now, the extension of Lemma 4.0.36 to hypersurfaces in $\mathbb{R}^N$ is straightforward.

Let Ω be a subset of $\mathbb{R}^N$. We define the set $(\partial\Omega)^{\varepsilon_0}$ by

$$
(\partial\Omega)^{\varepsilon_0}=\left\{x\in\mathbb{R}^N : d(x,\partial\Omega)<\varepsilon_0\right\}.
$$

Theorem 4.0.43. (Tubular neighborhood Theorem). *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Then there exists $\varepsilon_0>0$ such that for each $x\in(\partial\Omega)^{\varepsilon_0}$ there exists a unique couple $(\bar{x},s)\in\partial\Omega\times]-\varepsilon_0,\varepsilon_0[$ such that $x=\bar{x}+s\nu(\bar{x})$; moreover, $\bar{x}$ is the (unique) nearest to x point of the boundary and $s=d(x,\partial\Omega)$. Finally, possibly reducing the value of ε_0, the map $x\to(\bar{x},s)$ is a diffeomorphism of class C^1 of $(\partial\Omega)^{\varepsilon_0}$ onto $\partial\Omega\times]-\varepsilon_0,\varepsilon_0[$.*

Lemma 4.0.44. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2 and $\varepsilon_0 > 0$ as in Theorem 4.0.43. Let $0 < \varepsilon < \varepsilon_0$. We denote by Ω_ε the set $\{x \in \Omega : d(x, \partial\Omega) > \varepsilon\}$. Let $M > 0$ and $\rho_\varepsilon \in \mathcal{R}$ be defined by*

$$\rho_\varepsilon(x) := \begin{cases} \varepsilon, & \text{if } x \in \Omega_\varepsilon, \\ \frac{M-\varepsilon|\Omega_\varepsilon|}{|\Omega\setminus\Omega_\varepsilon|}, & \text{if } x \in \Omega \setminus \Omega_\varepsilon, \end{cases} \tag{4.0.45}$$

for all $0 < \varepsilon < \varepsilon_0$. Let $\pi^\sharp_{\rho_\varepsilon}$ the map from the space $(\mathcal{H}^1(\Omega)/\mathbb{R})$ onto the space $\mathcal{H}^{1,0}_{\rho_\varepsilon}(\Omega) := \left\{u \in H^1(\Omega) : \int_\Omega \rho_\varepsilon u dx = 0\right\}$ defined by the equality $\pi_{\rho_\varepsilon} = \pi^\sharp_{\rho_\varepsilon} \circ p$, where

$$\pi_{\rho_\varepsilon}[u] = u - \frac{\int_\Omega \rho_\varepsilon u dx}{\int_\Omega \rho_\varepsilon dx}, \quad \forall u \in H^1(\Omega).$$

Let $\pi^\sharp_0$ the map from $(\mathcal{H}^1(\Omega)/\mathbb{R})$ onto $\mathcal{H}^{1,0}_{\partial\Omega}(\Omega) := \left\{u \in H^1(\Omega) : \int_{\partial\Omega} u d\sigma = 0\right\}$ defined by the equality $\pi_0 = \pi^\sharp_0 \circ p$, where

$$\pi_0[u] = u - \frac{\int_{\partial\Omega} u d\sigma}{|\partial\Omega|}, \quad \forall u \in H^1(\Omega).$$

Then the following statements hold true:

i) For all $\phi \in (\mathcal{H}^1(\Omega)/\mathbb{R})$, $\pi^\sharp_{\rho_\varepsilon}[\phi] \to \pi^\sharp_0[\phi]$ in $L^2(\Omega)$ (hence also in $\mathcal{H}^1(\Omega)$);

ii) if $u_\varepsilon \rightharpoonup u$ in $(\mathcal{H}^1(\Omega)/\mathbb{R})$, then (possibly passing to a subsequence) $\pi^\sharp_{\rho_\varepsilon}[u_\varepsilon] \to \pi^\sharp_0[u]$ in $L^2(\Omega)$;

iii) assume that $u_\varepsilon \to u$, $w_\varepsilon \to w$ in $L^2(\Omega)$, $\mathrm{Tr}[u_\varepsilon] \to \mathrm{Tr}[u]$, $\mathrm{Tr}[w_\varepsilon] \to \mathrm{Tr}[w]$ in $L^2(\partial\Omega)$, and that $\|\nabla u_\varepsilon\|_{L^2(\Omega)}, \|\nabla u\|_{L^2(\Omega)} \le C$, $\|\nabla w_\varepsilon\|_{L^2(\Omega)}, \|\nabla w\|_{L^2(\Omega)} \le C$ uniformly in $0 < \varepsilon < \varepsilon_0$. Then

$$\int_\Omega \rho_\varepsilon (u_\varepsilon - u) w_\varepsilon dx \to 0$$

and

$$\int_\Omega \rho_\varepsilon (w_\varepsilon - w) u dx \to 0.$$

Proof. The proof of the first two statements follows the same arguments used in the proof of Lemma 4.0.19. Now we prove statement *iii*). It clearly suffices to prove that

$$\lim_{\varepsilon\to 0} \int_\Omega \rho_\varepsilon u_\varepsilon w_\varepsilon dx. \tag{4.0.46}$$

whenever $u_\varepsilon \to 0$ in $L^2(\Omega)$ and $\mathrm{Tr}[u_\varepsilon] \to 0$ in $L^2(\partial\Omega)$. We have that

$$\int_\Omega \rho_\varepsilon u_\varepsilon w_\varepsilon dx = \varepsilon \int_{\Omega_\varepsilon} u_\varepsilon w_\varepsilon dx + C(\varepsilon) \int_{\Omega\setminus\Omega_\varepsilon} u_\varepsilon w_\varepsilon dx,$$

where $C(\varepsilon) = \frac{M-\varepsilon|\Omega_\varepsilon|}{|\Omega\setminus\Omega_\varepsilon|}$. The first summand clearly goes to zero as $\varepsilon \to 0$. By multiplying and dividing the second summand by ε and observing that $\varepsilon C(\varepsilon) \le C' < +\infty$ for $\varepsilon \le \varepsilon_0$, we obtain

$$\left| \int_\Omega \rho_\varepsilon u_\varepsilon w_\varepsilon dx \right| \le C\varepsilon + C' \cdot \frac{1}{\varepsilon} \int_{\Omega\setminus\Omega_\varepsilon} |u_\varepsilon w_\varepsilon| \, dx.$$

We now apply Theorem 4.0.43. Let then $x_0 \in \partial\Omega$ and U_0 be a neighborhood of x_0 in $\mathbb{R}^N$ such that there exists $V_0 \subset \mathbb{R}^{N-1}$ and a parametrization $\phi \in C^2(V_0)$ such that the map ψ from $V_0 \times]0,\varepsilon[$ onto $M(\varepsilon) = \{x \in \Omega : d(x, \partial\Omega \cap U_0) < \varepsilon\}$ defined by

$$\psi(p,t) = \phi(p) + t\nu(p)$$

is a diffeomorphism from $V_0 \times]0,\varepsilon[$ onto $M(\varepsilon)$. Here $\nu(p)$ denotes the unit inner normal at $\phi(p)$. Now we consider

$$\int_{M(\varepsilon)} \frac{1}{\varepsilon} |u_\varepsilon w_\varepsilon| \, dx = \int_{V_0} \int_0^\varepsilon \frac{|\det D\psi|}{\varepsilon} |u_\varepsilon(p,t)| \, |w_\varepsilon(p,t)| \, dt dp. \tag{4.0.47}$$

For almost every $p \in V_0$, $u_\varepsilon(p,t)$, $w_\varepsilon(p,t)$ are absolutely continuous on $[0,\varepsilon]$ and since this set is compact, also their product is absolutely continuous. Let p be fixed. We have

$$u_\varepsilon(t) w_\varepsilon(t) = u_\varepsilon(0) w_\varepsilon(0) + \int_0^t \frac{\partial u_\varepsilon}{\partial t'}(t') w_\varepsilon(t') + u_\varepsilon(t') \frac{\partial w_\varepsilon}{\partial t'}(t') dt', \tag{4.0.48}$$

$$u_\varepsilon(t') = u_\varepsilon(0) + \int_0^{t'} \frac{\partial u_\varepsilon}{\partial t''}(t'') dt'',$$

$$w_\varepsilon(t') = w_\varepsilon(0) + \int_0^{t'} \frac{\partial w_\varepsilon}{\partial t''}(t'') dt''.$$

We observe that, for fixed ε and for almost every $p \in V_0$, the quantity $C_1(t,p) = \left(\int_0^t \left| \frac{\partial u_\varepsilon}{\partial t'}(t',p) \right|^2 dt' \right)^{\frac{1}{2}}$ is increasing in $0 \le t \le \varepsilon$ and $C_1(t,p) \le C_1(\varepsilon,p)$ for all $0 \le t \le \varepsilon$. The same result holds for $C_2(t,p) = \left(\int_0^t \left| \frac{\partial w_\varepsilon}{\partial t'}(t',p) \right|^2 dt' \right)^{\frac{1}{2}}$. Then

$$|u_\varepsilon(t')| \le |u_\varepsilon(0)| + t'^{\frac{1}{2}} C_1(t') \le |u_\varepsilon(0)| + t^{\frac{1}{2}} C_1(\varepsilon), \tag{4.0.49}$$

$$|w_\varepsilon(t')| \le |w_\varepsilon(0)| + t'^{\frac{1}{2}} C_2(t') \le |w_\varepsilon(0)| + t^{\frac{1}{2}} C_2(\varepsilon). \tag{4.0.50}$$

Now, let's consider the right hand side in (4.0.47). By using (4.0.49):

$$\begin{aligned}
\int_{V_0} \int_0^\varepsilon \frac{|\det D\psi|}{\varepsilon} & |u_\varepsilon(p,t)| \, |w_\varepsilon(p,t)| \, dt dp \qquad (4.0.51)\\
& \le \int_{V_0} \int_0^\varepsilon \frac{1}{\varepsilon} |u_\varepsilon(p,0)| \, |w_\varepsilon(p,0)| \, |\det D\psi| \, dt dp \\
& + \|\det D\psi\|_{L^\infty(V_0 \times [0,\varepsilon])} \int_{V_0} \int_0^\varepsilon \frac{1}{\varepsilon} \int_0^t \left| \frac{\partial u_\varepsilon}{\partial t'}(p,t') \right| |w_\varepsilon(p,t')| \, dt' dt dp \\
& + \|\det D\psi\|_{L^\infty(V_0 \times [0,\varepsilon])} \int_{V_0} \int_0^\varepsilon \frac{1}{\varepsilon} \int_0^t |u_\varepsilon(p,t')| \left| \frac{\partial w_\varepsilon}{\partial t'}(p,t') \right| dt' dt dp.
\end{aligned}$$

Now using the same argument in the proof of point *iii*) of Lemma 4.0.19, one can show that the second and third summand vanish as $\varepsilon \to 0$. For the first summand, we observe that by Remark 4.0.42 we have

$$\begin{aligned}
\det D\psi &= \det \left[\frac{\partial \phi}{\partial p_1} \quad \cdots \quad \frac{\partial \phi}{\partial p_{N-1}} \quad \nu(p_1, ..., p_{N-1}) \right] \\
&+ t g_1 \left(\frac{\partial \phi}{\partial p_1}, \frac{\partial \nu}{\partial p_1}, ..., \frac{\partial \phi}{\partial p_{N-1}}, \frac{\partial \nu}{\partial p_{N-1}} \right) + \cdots \\
&+ t^{N-1} g_{N-1} \left(\frac{\partial \phi}{\partial p_1}, \frac{\partial \nu}{\partial p_1}, ..., \frac{\partial \phi}{\partial p_{N-1}}, \frac{\partial \nu}{\partial p_{N-1}} \right),
\end{aligned}$$

where $p = (p_1, ..., p_{N-1})$ and g_i are suitable compositions of sums and products of the first partial derivatives of ϕ and ν. It is not restrictive to assume that $\inf_{V_0} \left| \frac{\partial \phi}{\partial p_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial p_{N-1}} \right| > 0$. Now, using the same argument as in the proof of statement $ii)$ of Lemma 4.0.36, we obtain

$$\begin{aligned}
\frac{1}{\varepsilon} \int_{V_0} \int_0^{\varepsilon} & |u_\varepsilon(p,0)|\, |w_\varepsilon(p,0)|\, |\mathrm{det} D\psi|\, dt dp \\
&\leq \frac{1}{\varepsilon} \int_0^{\varepsilon} \int_{V_0} |u_\varepsilon(p,0)|\, |w_\varepsilon(p,0)| \left| \frac{\partial \phi}{\partial p_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial p_{N-1}} \right| dp dt \\
&+ \sum_{i=1}^{N-1} \frac{1}{\varepsilon} \int_0^{\varepsilon} \int_{V_0} |u_\varepsilon(p,0)|\, |w_\varepsilon(p,0)|\, t^i\, |g_i(p)|\, dp dt \\
&\leq C(\varepsilon) \int_{\partial\Omega \cap U_0} |u_\varepsilon|\, |w_\varepsilon|\, d\sigma,
\end{aligned}$$

where

$$C(\varepsilon) = 1 + \left(\inf_{V_0} \left| \frac{\partial \phi}{\partial p_1} \wedge \cdots \wedge \frac{\partial \phi}{\partial p_{N-1}} \right| \right)^{-1} \cdot \sum_{i=1}^{N-1} \frac{\varepsilon^i}{i+1} \|g_i\|_{L^\infty(D)},$$

and $C(\varepsilon) \to 1$ as $\varepsilon \to 0$. Since $\mathrm{Tr}[u_\varepsilon] \to 0$ in $L^2(\partial\Omega)$ as $\varepsilon \to 0$, it follows that also the first summand vanishes as $\varepsilon \to 0$. Since $\Omega \setminus \Omega_\varepsilon$ can be covered by a finite number of open sets of the type $M(\varepsilon)$, say $\Omega \setminus \Omega_\varepsilon \subset \bigcup_{i=1}^m M_i(\varepsilon)$, we have that

$$\frac{1}{\varepsilon} \int_{\Omega \setminus \Omega_\varepsilon} |u_\varepsilon w_\varepsilon|\, dx \leq \sum_{i=1}^m \frac{1}{\varepsilon} \int_{M_i(\varepsilon)} |u_\varepsilon w_\varepsilon|\, dx.$$

This concludes the proof. □

Corollary 4.0.52. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^2. Let ρ_n defined as in Lemma 4.0.44. Let $\lambda_j[\rho_n]$ be the eigenvalues of problem (3.0.2) for all $j \in \mathbb{N}$. Let $\overline{\lambda}_j$, $j \in \mathbb{N}$ denote the eigenvalues of problem (4.0.2) corresponding to the constant surface density $\frac{M}{|\partial\Omega|}$. Then for all $j \in \mathbb{N}$, we have $\lim_{n \to +\infty} \lambda_j[\rho_n] = \overline{\lambda}_j$ for all $j \in \mathbb{N}$.*

Proof. It is sufficient to repeat the proof of Theorem 4.0.33 by using Lemma 4.0.44. The compact convergence of compact operators $\left\{ \tilde{T}_n \right\}_{n \in \mathbb{N}}$ to the compact operator $\tilde{T}$ in $\mathcal{K}\left((\mathcal{H}^1(\Omega)/\mathbb{R}), (\mathcal{H}^1(\Omega)/\mathbb{R})\right)$ implies norm convergence, and hence spectral convergence by Theorem 4.0.31. □

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