Commutativity of control vector fields and "inf-commutativity"

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This paper is dedicated to Alex Ioffe, with gratitude.

Abstract. The notion of flows' commutativity for vector fields is here extended to *control vector fields*, i.e. vector fields depending on a parameter. By saying that a family of control vector fields commute we mean that, for every choice of the control functions, the flows of the resulting (time-dependent) vector fields commute. Let us remark that the control vector fields here considered are locally Lipschitz continuous. Hence, even in the trivial case when controls are kept constant, the usual characterization in terms of Lie brackets is not meaningful. In fact, we utilize a notion of (set-valued) bracket introduced in [**RS1**], which is fit for locally Lipschitz vector fields and extends the usual bracket for C^1 vector fields.

The main achievement (see Theorem 2.11) of the paper consists in a twofold characterization of flows's commutativity of control vector fields. On one hand, this property is characterized by means of a zero-bracket–like condition, namely condition **(ccLBZ)** below. On the other hand, commutativity turns out to be equivalent to an invariance condition formulated in terms of *lifts of multi-time paths*.

In particular this result is exploited in order to establish sufficient conditions for the *commutativity of optimal control problems*, here called *inf-commutativity* —see Definitions 3.2 and 3.3 below.

1. Introduction

1.1. The problem. Roughly speaking, one says that the flows of two vector fields f_1 and f_2 commute if, starting from a point y (of a real vector space, or a manifold) and moving first in the direction of f_1 for a time t_1 and then in the direction of f_2 for a time t_2, one reaches the same point that would be achieved by reversing the implementation's order of the two vector fields. Commutativity of vector fields' flows lies at the basis of many applications, e.g. in mechanics or in differential geometry. For instance, in Hamiltonian mechanics, a function K is a

1991 *Mathematics Subject Classification.* 37N35, 49J15.

The author was supported in part by PRIN Project 2007 "Metodi di viscosit, metrici e di controllo in equazioni alle derivate parziali nonlineari".

constant of the motion if it is in involution with the Hamiltonian H, which means that the Hamiltonian flows originated by H and K, respectively, commute.

Commutativity —say, on an Euclidean space $\mathbb{R}^n$— is characterized (at least locally) by the the well-known point-wise condition

$$[f_1, f_2](x) = 0 \qquad \forall x \in \mathbb{R}^n \tag{1.1}$$

where $[f_1, f_2]$ denotes the Lie bracket of f_1 and f_2. We recall that $[f_1, f_2](x) = Df_2(x) \cdot f_1(x) - Df_1(x) \cdot f_2(x)$ [1] and that condition (1.1) is usually stated for C^1 vector fields. In the case of locally Lipschitz vector fields, (1.1) is not even meaningful. However, in [**RS2**], commutativity of flows has been characterized in terms of vanishing of the set-valued Lie bracket introduced in [**RS1**] —see Definition 1.1 below.

In this paper we aim to extend the notion of commutativity to families of (nonsmooth) *control vector fields* and to families of *optimal control problems*. The commutativity concept for control vector fields will be sketched in the next subsection, where we provide a more thorough outline of the paper. As for the notion of commutativity for optimal control problems —which in the present paper will be called *inf-commutativity*— let us illustrate it by means of an example.

Suppose we are given the minimum problems

$$\begin{cases} \inf\left(\int_0^{T_1} l(x(t), c(t))dt + \int_0^{T_2} m(y(t), d(t))dt\right) \\ \dot{x} = f(x, c) \quad x(0) = \bar{x} \qquad \dot{y} = g(y, d) \quad y(0) = x(T_1) \end{cases} \tag{1.2}$$

and

$$\begin{cases} \inf\left(\int_0^{T_2} m(y(t), d(t))dt + \int_0^{T_1} l(x(t), c(t))dt\right) \\ \dot{y} = g(y, d) \quad y(0) = \bar{x} \qquad \dot{x} = f(x, c) \quad x(0) = y(T_2). \end{cases} \tag{1.3}$$

where $c(\cdot)$ and $d(\cdot)$ are controls which range over a given control set A.

The obvious meaning of problem (1.2) is that the infimum is taken over the four-uples (c, x, d, y) verifying the following conditions: i) c and d are controls defined on $[0, T_1]$ and $[0, T_2]$ respectively ; ii) x is the solution of the Cauchy problem $\dot{x} = f(x, c) \quad x(0) = \bar{x}$; iii) y is the solution of the Cauchy problem $\dot{y} = g(y, d) \quad y(0) = x(T_1)$ (where $x(\cdot)$ is as in ii)). The meaning of (1.3) is the same, up to reversing the order of implementation of (f, l) and (g, m).

A natural question is the comparison between the optimal values of the two problems. In particular, one can wonder if the optimal value of problem (1.2) coincides with that of (1.3). In this case we say that the two problems *inf-commute* (see section 3). Of course, we can generalize this question by partitioning the two intervals into several subintervals[2] and *running* these subintervals (and the corresponding control systems) in an arbitrary order. Also one can consider the interactions of more (than two) optimal control systems, and the latter can include final costs as well.

The question of commutativity of optimal control problems can be interesting, for instance, in the framework of switchings systems, where it would mean a kind of invariance of the output with respect to the order of the optimal switchings. Some direct economic applications —se e.g. [**Ro**]— pose the commutativity issue

[1] As is well known, the Lie bracket is a geometric object, and this is its expression in a given system of coordinates.

[2] A generalization of this discrete implementation of different systems is represented by the *lifts of multi-time paths* —see Subsection 1.2.

quite naturally. Yet, our original motivation —which is not necessarily the most important— was in fact the question of the solution's existence for so-called systems of multi-time Hamilton-Jacobi equations —see e.g. [**BT**], [**MR**]. Indeed, it turns out that the existence of a solution for multi-time systems is, in a sense, equivalent to the commutativity of the underlying control problems. Let us point out that the relationship between inf-commutativity and PDE systems is not surprising at all. For instance, it is well-known that in the case of systems of linear first order PDE's the existence issue is intimately connected with commutativity properties of the involved vector fields. [3]

1.2. Outline of the paper. In Section 2 we address the question of commutativity of control vector fields. In short, one says that the flows of a family of control vector fields $f_i : \mathbb{R}^n \times A \to \mathbb{R}^n$, $i = 1, ..., N$, *commute* if the flows of the (time dependent) vector fields obtained by implementing arbitrary control functions $\mathbf{a}_1(\cdot), ..., \mathbf{a}_N(\cdot)$ in $f_1, ..., f_N$, respectively, commute. Theorem 2.11—which is the main result of the paper — characterizes commutativity both in terms of three (equivalent) *zero bracket conditions* and in terms of *lifts of multi-time paths.*

One of these *zero bracket* characterizations states that the (nonsmooth) control vector fields $f_1, ..., f_N$ commute if and only if

$$[f_i(\cdot, \alpha), f_j(\cdot, \beta)]_{set}(x) = \{0\} \qquad \forall x \in \mathbb{R}^n \,, \tag{1.4}$$

for all $i, j = 1, ..., N$ and *all control values* α, β, where $[,]_{set}$ is the afore-mentioned set-valued Lie bracket. Alternatively, (1.4) can be replaced by an analogous condition involving the ordinary bracket at almost all points.

Instead, the characterization dealing with *lifts* establishes an equivalence between commutativity and the invariance with respect to *lifts of multi-time paths.* Let us briefly explain what this means exactly. If $(t_1, ..., t_N), (\tilde{t}_1, ..., \tilde{t}_N) \in \mathbb{R}^N$, $t_i \leq \tilde{t}_i$, $\forall i = 1, ..., N$, a *multi-time path* connecting $(t_1, ..., t_N)$ and $(\tilde{t}_1, ..., \tilde{t}_N)$ is a curve $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$ such that the components $\mathbf{t}_i$ are absolutely continuous, increasing maps that verify $\mathbf{t}_i(S_\alpha) = t_i$ and $\mathbf{t}_i(S_\omega) = \tilde{t}_i$. If $\mathbf{a}_i : [t_i, \tilde{t}_i] \to A$ are L^1 maps and $\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_N)$, the $\mathbf{a}$-*lift* of $\mathbf{t}$ (from a point $y \in \mathbb{R}^n$) is nothing but the solution $x_{\mathbf{a},\mathbf{t}}[y]$ of the Cauchy problem

$$\frac{dx(s)}{ds} = \sum_{i=1}^{N} f_i(x(s), \mathbf{a}_i \circ \mathbf{t}_i(s)) \frac{d\mathbf{t}_i(s)}{ds} \qquad x(S_\alpha) = y.$$

The characterization of commutativity by means of lifts establishes that, for every two multi-time paths $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$ and $\hat{\mathbf{t}} : [\hat{S}_\alpha, \hat{S}_\omega] \to \mathbb{R}^N$ connecting $(t_1, ..., t_N)$ and $(\tilde{t}_1, ..., \tilde{t}_N)$, one has $x_{\mathbf{a},\mathbf{t}}[y](S_\omega) = x_{\mathbf{a},\hat{\mathbf{t}}}[y](\hat{S}_\omega)$.

In order to prove Theorem 2.11 one needs two sets of results which will be proved in n Sections 4 and 5, respectively. The former family of results concerns the continuity properties of the input-output map $(\mathbf{a}, \mathbf{t}) \mapsto x_{\mathbf{a},\mathbf{t}}[y]$. The latter set of results deal with *multi-time control strings*, which are easy-to-handle representations of the pairs $(\mathbf{a}, \mathbf{t})$ when the controls $\mathbf{a}_i$ are piece-wise constant and the

[3] The afore-mentioned question of the *constant of the motions* in Hamiltonian mechanics is another example of this kind of relationship, for the fact that a number of functions are invariant along the trajectories can be expressed by a system of first order partial differential equations. More generally, many questions of *simmetries* and *integrability* of systems are directly connected with existence issues for systems of PDE (see e.g. [**O**])

multi-time paths $\mathbf{t}$ are piece-wise affine with derivatives ranging in the canonical basis of $\mathbb{R}^N$.

It is quite intuitive that *commutativity of control vector fields* is a sufficient condition for inf-commutativity. As a matter of fact, every characterization of the former notion turns out to be a sufficient condition for the latter. The question of inf-commutativity is treated in Section 3. The latter is concluded by a brief discussion on an open question concerning the characterization of inf-commutativity.

1.3. Notation and definitions. If H is a set and $K \subset H$, $\chi_K : H \to \{0,1\}$ will denote the indicator function of K. This means that $\chi_K(h) = 1$ if and only if $h \in K$.

If n is a positive integer, $y = (y_1, ..., y_n)$ a point of the Euclidean space $\mathbb{R}^n$, we shall set

$$|y| = \left(\sum_{i=0,\dots,n} (y_i)^2\right)^{\frac{1}{2}}, \qquad |y|_1 = \sum_{i=0,\dots,n} |y_i|.$$

If R is a non negative real number, we shall use $B[y,R]$ to denote the closed ball (with respect to the norm $|\cdot|$) of center y and radius R. If $E \subset \mathbb{R}^n$, we shall set

$$B[E,r] = \cup_{y\in E} B[y,r]$$

We shall use $\|\cdot\|_\infty$ and $\|\cdot\|$ to denote the C^0 norm and the the L^1 norm (on suitable spaces).

A *control vector field* on $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \times A \to \mathbb{R}^n$, where A is any set, called *set of control values.* If I is a real interval, and $A \subset \mathbb{R}^m$ for some positive integer m, we shall use $\mathcal{B}(I,A)$ [resp. $L^1(I,A)$] to denote the sets of Borel measurable [resp. Lebesgue integrable] maps from I to $\mathbb{R}^m$ that take values in A.

If $y \in \mathbb{R}^n$, I is a real interval, $a : I \to A$ is a map —called *control*—, $t_0 \in I$, and if the Cauchy problem

$$\dot{x}(t) = f(x(t), a(t)) \qquad x(t_0) = y \tag{1.5}$$

has a unique solution on I, for any $t \in I$ we shall use

$$ye^{\int_{t_0}^t f(a)}$$

to denote the value at $t \in I$ of this solution. This notation is borrowed from [**AG**] and [**KS**]. If $f = f(x)$ is independent of the control —i.e., f is an autonomous vector field— we shall write

$$ye^{(t-t_0)f}$$

instead of

$$ye^{\int_{t_0}^t f}.$$

If $N > 1$ is an integer, and, for every $i = 1, ..., N$, $f_i : \mathbb{R}^n \times A \to \mathbb{R}^n$ is a control vector field, I_i is a real interval, $t_{0i} \in I_i$, and $\mathbf{a}_i : I_i \to A$ is a control (and for every $y \in \mathbb{R}^n$ the Cauchy problem

$$\dot{x}(t) = f_i(x(t), \mathbf{a}_i(t)) \qquad x(t_{0i}) = y \tag{1.6}$$

has a unique solution on I_i) we define inductively the product $ye^{\int_{t_{0\,1}}^{t_1} f_1(\mathbf{a}_1)}...e^{\int_{t_{0\,q}}^{t_q} f_q(\mathbf{a}_q)}$ by setting

$$ye^{\int_{t_{0\,1}}^{t_1} f_1(\mathbf{a}_1)}...e^{\int_{t_{0\,q}}^{t_q} f_N(\mathbf{a}_N)} \doteq \Big(ye^{\int_{t_{0\,1}}^{t_1} f_1(\mathbf{a}_1)}...e^{\int_{t_{0\,q-1}}^{t_{q-1}} f_{N-1}(\mathbf{a}_{N-1})}\Big)e^{\int_{t_{0\,N}}^{t_q} f_q(\mathbf{a}_N)},$$

for every $(t_1,...,t_N) \in I_1 \times ... \times I_N$.

If q, n are positive integers and $m : \mathbb{R}^q \to \mathbb{R}^n$ is a locally Lipschitz continuous map, we let $DIFF(m)$ denote the set of points $x \in \mathbb{R}^q$ such that f is differentiable at x. Then Rademacher's Theorem implies that $\mathbb{R}^q \backslash DIFF(m)$ is a subset of zero Lebesgue measure. If $x \in DIFF(m)$ we shall use $Dm(x)$ to denote the Jacobian matrix of m at x.

The ordinary Lie bracket at $x \in \mathbb{R}^n$ of two C^1 vector fields f, g will be denoted by $[f,g]_{ord}(x)$. That is, we shall set

$$[f,g]_{ord}(x) = Dg(x) \cdot f(x) - Df(x) \cdot g(x)$$

If f, g are just locally Lipschitz continuous vector fields, we shall use both the ordinary Lie bracket —which, by Rademacher's theorem is defined almost everywhere— and the following notion of (set-valued) Lie bracket, which has been introduced in [**RS1**].

Definition 1.1. If f, g are locally Lipschitz continuous vector fields and $x \in \mathbb{R}^n$, we define the *Lie bracket* $[f,g](x)$ *of* f *and* g *at* x to be the convex hull of the set of all vectors

$$v = \lim_{j\to\infty} [f,g]_{ord}(x_j), \tag{1.7}$$

for all sequences $\{x_j\}_{j\in\mathbb{N}}$ such that

1. $x_j \in DIFF(f) \cap DIFF(g)$ for all j,
2. $\lim_{j\to\infty} x_j = x$,
3. the limit v of (1.7) exists.

For every $x \in \mathbb{R}^n$, it is clear that $[f,g](x)$ is a *convex*, *compact*, and *nonempty* subset of $\mathbb{R}^n$. Moreover, the *skew-symmetry identity*

$$[f,g](x) = -[g,f](x) \tag{1.8}$$

clearly holds for all $x \in \mathbb{R}^n$. (This means that $[f,g](x) = \{w : -w \in [g,f](x)\}$. In addition, each locally Lipschitz continuous vector field g satisfies the identity

$$[g(x), g(x)] = \{0\} \ \text{ for every } \ x \in \mathbb{R}^n\,. \tag{1.9}$$

Remark 1.2. If both f and g are continuously differentiable in a neighborhood of a point x then $[f,g](x) = \{[f,g]_{ord}(x)\}$. However this is not true in general at the points of $DIFF(f) \cap DIFF(g)$, where only the inclusion $[f,g](x) \supseteq \{[f,g]_{ord}(x)\}$ holds true everywhere.

2. Commutativity of flows of nonsmooth control vector fields

In this section we shall extend the notion of flows' commutativity to (nonsmooth) control vector fields. We shall characterize this commutativity essentially in two ways: first, like in the case of vector fields, commutativity will be shown to be equivalent to each of the three suitable *Constant Control Zero Lie Bracket* conditions introduced in Definition 2.7 below; secondly, commutativity will be characterized in terms of *lifts* of (absolutely continuous) *multi-time paths*—see Definition 2.8 below.

Let n, m, N be positive integers an integer and let $A \subset \mathbb{R}^m$. Let $f_1, ..., f_N$ be a family of control vector fields defined on $\mathbb{R}^n \times A$.

We shall assume the following **structural hypotheses** (H1)-(H2) on the data:

(H1) For every $i = 1, ..., N$ and every compact subset $K \in \mathbb{R}^n$, there exists $L_K \geq 0$ such that
$$|f_i(x,a) - f_i(y,b)| \leq L_K(|x-y| + |a-b|) \qquad \forall (x,a),(y,b) \in K \times A.$$
(H2) [4] There exists a constant C such that
$$f_i(x,\alpha) \leq C(1 + |x| + |\alpha|) \qquad \forall (x,\alpha) \in \mathbb{R}^n \times A$$

Let us give a notion of *commutativity* for control vector fields, which essentially says that the corresponding flows commute for every choice of the control maps $\mathbf{a}_i(\cdot)$.

Definition 2.1. Let us consider *multi-times* [5] $t = (t_1, ..., t_N)$, $\tilde{t} = (\tilde{t}_1, ..., \tilde{t}_N)$ such that $t \leq \tilde{t}$ (i.e. $t_i \leq \tilde{t}_i$, $\forall i = 1, ..., N$). Let us set
$$\mathcal{A}_{[t,\tilde{t}]} = \big(\mathcal{B}([t_1,\tilde{t}_1], A) \cap L^1([t_1,\tilde{t}_1], A)\big) \times ... \times \big(\mathcal{B}([t_N,\tilde{t}_N], A) \cap L^1([t_N,\tilde{t}_N], A)\big)$$
Each element $\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_N) \in \mathcal{A}_{[t,\tilde{t}]}$ will be called a *N-control defined on* $[t, \tilde{t}]$.[6]

Definition 2.2. We say that the flows of the control vector fields $f_1, ..., f_N$ *commute* if for every N-control
$$\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_N) \in \mathcal{A}_{[t,\tilde{t}]}$$
and any permutation $(i_1, ..., i_N)$ of $(1, ..., N)$ one has
$$ye^{\int_{t_1}^{\tilde{t}_1} f_1(\mathbf{a}_1)} ... \, e^{\int_{t_q}^{\tilde{t}_q} f_q(\mathbf{a}_q)} = ye^{\int_{t_{i_1}}^{\tilde{t}_{i_1}} f_{i_1}(\mathbf{a}_{i_1})} ... \, e^{\int_{t_{i_q}}^{\tilde{t}_{i_q}} f_{i_q}(\mathbf{a}_{i_q})}$$

Remark 2.3. When the f_i are independent of the controls —that is, when they are vector fields— this condition reduces to the usual notion of commutativity of flows of vector fields:
$$ye^{(\tilde{t}_1 - t_1)f_1} ... \, e^{(\tilde{t}_q - t_q)f_q} = ye^{(\tilde{t}_{i_1} - t_{i_1})f_{i_1}} ... \, e^{(\tilde{t}_{i_q} - t_{i_q})f_{i_q}}$$

To begin with, let us consider the case of vector fields without controls. Let us state three Zero Lie Bracket conditions which will be used for the characterization of the commutativity of locally Lipschitz continuous vector fields.

[4] Via a standard application of Gronwall's inequality, this *growth* hypothesis guarantees global existence of solutions for the ordinary differential equation we are going to consider. Of course it can be replaced by conditions of different type, one can assume the f_i's verify some Nagumo's type condition.

[5] Motivated by the context, we call *multi-time* each element on $\mathbb{R}^N$.

[6] The need for Borel measurable controls $\mathbf{a}_i$ comes from the fact we want compositions $\mathbf{a}_i \circ \mathbf{t}_i$ with continuous $\mathbf{t}_i$ to be Lebesgue measurable —actually they turn out to be Borel measurable as well.

DEFINITION 2.4. Let $\{g_1, ..., g_q\}$ be a finite family of locally Lipschitz continuous vector fields on $\mathbb{R}^n$. We shall say that:

- the vector fields $g_1, ..., g_q$ verify the *ordinary Zero Lie Bracket* condition $\mathbf{(ZLB)}_{ord}$ if, for every $i, j \in 1, ..., N$,
$$[g_i, g_j]_{ord}(x) = 0 \qquad \forall x \in DIFF(g_i) \cap DIFF(g_j);$$
- the vector fields $g_1, ..., g_q$ verify the *a.e. Zero Lie Bracket* condition $\mathbf{(ZLB)}_{a.e.}$ [7] if, for every $i, j \in 1, ..., N$,
$$[g_i, g_j]_{ord}(x) = 0 \qquad \text{for a.e. } x \in DIFF(g_i) \cap DIFF(g_j);$$
- the vector fields $g_1, ..., g_q$ verify the *set-valued Zero Lie Bracket* condition $\mathbf{(ZLB)}_{set}$ if, for every $i, j \in 1, ..., N$,
$$[g_i, g_j](x) = \{0\} \qquad \forall x \in \mathbb{R}^n.$$

REMARK 2.5. In view of Theorem 2.6 below these conditions are in fact equivalent. In particular the non trivial relation $\mathbf{(ZLB)}_{a.e.} \Rightarrow \mathbf{(ZLB)}_{ord}$ holds true.

Let us recall from [**RS2**] a result valid for vector fields (without control).

THEOREM 2.6. [**RS2**] *Let $\{g_1, ..., g_q\}$ be a finite family of locally Lipschitz continuous, complete*[8]*, vector fields on $\mathbb{R}^n$. Then the following conditions are equivalent:*

i) *the flows of $g_1, ..., g_q$ commute;*
ii) *the vector fields $g_1, ..., g_q$ verify condition $\mathbf{(ZLB)}_{ord}$*
iii) *the vector fields $g_1, ..., g_q$ verify condition $\mathbf{(ZLB)}_{a.e.}$;*
vi) *the vector fields $g_1, ..., g_q$ verify condition $\mathbf{(ZLB)}_{set}$.*

Let us generalize conditions $\mathbf{(ZLB)}_{ord}$, $\mathbf{(ZLB)}_{a.e.}$, and $\mathbf{(ZLB)}_{set}$ to control vector fields as follows.

DEFINITION 2.7. Let $\{f_1, ..., f_N\}$ be a finite family of control vector fields on $\mathbb{R}^n$ satisfying the structural hypotheses (H1)-(H2). We shall say that:

- the vector fields $f_1, ..., f_N$ verify the *ordinary Constant Control Zero Lie Bracket* condition $\mathbf{(ccZLB)}_{ord}$ if, for every choice of the controls values $a_1, ..., a_N \in A$ and every $i, j \in 1, ..., N$,
$$[f_i(x, a_i), f_j(x, a_j)]_{ord} = 0 \qquad \forall x \in DIFF(f_i(\cdot, a_i)) \cap DIFF(f_j(\cdot, a_j));$$
(i.e. the vector fields $f_1(\cdot, a_1), ..., f_N(\cdot, a_N)$ verify $\mathbf{(ZLB)}_{ord}$)
- the vector fields $f_1, ..., f_N$ verify the *a.e. Constant Control Zero Lie Bracket* condition $\mathbf{(ccZLB)}_{a.e.}$ if, for every choice of the controls values $a_1, ..., a_N \in A$ and every $i, j \in 1, ..., N$,
$$[f_i(x, a_i), f_j(x, a_j)]_{ord} = 0 \qquad \text{for a.e.} x \in DIFF(f_i(\cdot, a_i)) \cap DIFF(f_j(\cdot, a_j));$$
(i.e. the vector fields $f_1(\cdot, a_1), ..., f_N(\cdot, a_N)$ verify $\mathbf{(ZLB)}_{a.e.}$);

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[7] Let us mention that for a different purpose —namely the generalization of Frobenius' theorem to Lipschitz continuous vector fields— an *almost everywhere* condition involving ordinary Lie brackets has been exploited in [**S**].

[8] By saying that a vector field f is *complete* we mean that the solution $t \mapsto ye^{tf}$ is defined for all $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For a more general definition of commutativity, including the case of non complete vector fields, we refer to [**RS2**].

- the vector fields $f_1, ..., f_N$ verify the *set-valued Constant Control Zero Lie Bracket* condition $\mathbf{(ccZLB)}_{set}$ if, for every choice of the controls values $a_1, ..., a_N \in A$ and every $i, j \in 1, ..., N$,
$$[f_i(x, a_i), f_j(x, a_j)] = \{0\} \qquad \forall x \in \mathbb{R}^n;$$
(i.e. the vector fields $f_1(\cdot, a_1), ..., f_N(\cdot, a_N)$ verify $\mathbf{(ZLB)}_{set}$)

In Theorem 2.11 below it will be proved that conditions $\mathbf{(ccZLB)}_{ord}$, $\mathbf{(ccZLB)}_{a.e.}$, and $\mathbf{(ccZLB)}_{set}$ are mutually equivalent and characterize commutativity of control vector fields.

Before stating Theorem 2.11, let us introduce the concepts of *multi-time* path and its **a**-lift.

Definition 2.8. Let $S_\alpha < S_\omega$ be real numbers. An, absolutely continuous map
$$\mathbf{t} = (\mathbf{t}_1, ..., \mathbf{t}_N) : [S_\alpha, S_\omega] \to \mathbb{R}^N$$
is called a *multi-time path* if the components $\mathbf{t}_i$ are increasing maps [9] and $|\frac{d\mathbf{t}(s)}{ds}| > 0$ for almost every $s \in [S_\alpha, S_\omega]$. It $t \doteq \mathbf{t}(S_\alpha)$ and $\tilde{t} \doteq \mathbf{t}(S_\omega)$ we say that $\mathbf{t}$ *connects t with $\tilde{t}$.* For every pair $t, \tilde{t} \in \mathbb{R}^N$ such that $t \le \tilde{t}$, the set of all multi-time paths connecting t with $\tilde{t}$ will be denoted by $MT_{t,\tilde{t}}$.

Definition 2.9. A multi-time path $\mathbf{t}$ is called *simple* if it is piece-wise affine and verifies
$$\frac{d\mathbf{t}}{ds}(s) \in \left\{\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_N}\right\},$$
for all s where it is differentiable, where $\left\{\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_N}\right\}$ is the canonical base of $\mathbb{R}^N$.

Definition 2.10. Let $t, \tilde{t} \in \mathbb{R}^N$ be multi-times such that $t \le \tilde{t}$, and let $\mathbf{a}$ be a N-control defined on $[t, \tilde{t}]$. Let $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$ be a multi-time path connecting t with $\tilde{t}$. Let us define the **a**-*lift of* $\mathbf{t}$ *from a point* $y \in \mathbb{R}^n$ as the solution on $[S_\alpha, S_\omega]$ of the Cauchy problem

$$\frac{dx}{ds} = \sum_{i=1}^{N} f_i(x(s), \mathbf{a}_i \circ \mathbf{t}_i(s)) \frac{d\mathbf{t}_i}{ds} \qquad x(S_\alpha) = y \tag{2.1}$$

(We shall see in Theorem 4.3 that this solution exists and is unique.) We shall use $x_{(\mathbf{a},\mathbf{t})}[y](\cdot)$ to denote the **a**-lift of $\mathbf{t}$ from the point $y \in \mathbb{R}^n$

Here is the main theorem, which is also crucial for proving the results of the next section.

Theorem 2.11. *The following statements are equivalent:*

i) *the flows of the control vector fields $f_1, ..., f_N$ commute;*

[9]We say a real map r defined on an interval I is *increasing* [resp. *strictly increasing*] if $\alpha, \beta \in I$ and $\alpha < \beta$ imply $r(\alpha) \le r(\beta)$ [resp. $r(\alpha) < r(\beta)$] .

ii) *if* $t \le \tilde{t}$ *are multi-times* , $\mathbf{a}$ *is a* N*-control defined on* $[t, \tilde{t}]$*, and* $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$, $\hat{\mathbf{t}} : [\hat{S}_\alpha, \hat{S}_\omega] \to \mathbb{R}^N$ *are simple multi-time paths connecting* t *with* $\tilde{t}$*, then*

$$x_{(\mathbf{a},\mathbf{t})}[y](S_\omega) = x_{(\mathbf{a},\hat{\mathbf{t}})}[y](\hat{S}_\omega);$$

iii) *if* $t \le \tilde{t}$ *are multi-times* , $\mathbf{a}$ *is a* N*-control defined on* $[t, \tilde{t}]$*, and* $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$, $\hat{\mathbf{t}} : [\hat{S}_\alpha, \hat{S}_\omega] \to \mathbb{R}^N$ *are multi-time paths connecting* t *with* $\tilde{t}$*, then*

$$x_{(\mathbf{a},\mathbf{t})}[y](S_\omega) = x_{(\mathbf{a},\hat{\mathbf{t}})}[y](\hat{S}_\omega);$$

iv) *the vector fields* $f_1, ..., f_N$ *verify condition* $\mathbf{(ccZLB)}_{ord}$ *;*
v) *the vector fields* $f_1, ..., f_N$ *verify condition* $\mathbf{(ccZLB)}_{a.e.}$ *;*
vi) *the vector fields* $f_1, ..., f_N$ *verify condition* $\mathbf{(ccZLB)}_{set}$*.*

The proof of this theorem requires some preparatory tools which will be provided in Sections 4 and 5. The proof itself will be given in Section 6.

REMARK 2.12. In the particular case when the vector fields are independent of controls, condition iii) can be regarded as a further condition (in terms of lifts) equivalent to those stated in Theorem 2.6.

3. Inf-commutativity

In this section we propose a notion of *commutativity* for optimal control problems, which we shall call *inf-commutativity*.

Let N be a positive integer, and let $\{f_1, ..., f_N\}$ be a family of N control vector fields, on which we assume the same hypotheses as in section 2.

DEFINITION 3.1. Let us consider two multi-times $t \le \tilde{t}$, and let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a function. Let $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$ be a multi-time path connecting t with $\tilde{t}$.

For any $y \in \mathbb{R}^n$, let us consider the optimal control problem

$$\mathcal{P}_{\mathbf{t}}[\varphi, y] \qquad \text{minimize } \left\{\varphi\left(x_{(\mathbf{a},\mathbf{t})}[y](S_\omega)\right) \quad \mathbf{a} \in \mathcal{A}_{[t,\tilde{t}]}\right\},$$

Moreover, let us consider the value function

$$V_{\mathbf{t}}[\varphi, y] \doteq \inf_{\mathbf{a} \in \mathcal{A}_{[\mathbf{t},\tilde{\mathbf{t}}]}} \varphi\left(x_{(\mathbf{a},\mathbf{t})}[y]S_\omega()\right).$$

A N-control $\breve{\mathbf{a}}$ for problem $\mathcal{P}^{\mathcal{F}}_{\mathbf{t}}[\varphi, y]$ will be called *optimal* if

$$\varphi\left(x_{(\breve{\mathbf{a}},\mathbf{t})}[y](S_\omega)\right) = V_{\mathbf{t}}[\varphi, y]$$

DEFINITION 3.2. We say that the flows of the control vector fields $f_1, ..., f_N$ *inf-commute* if for any map $\varphi : \mathbb{R}^n \to \mathbb{R}$, any $y \in \mathbb{R}^n$, any pair $(t, \tilde{t}) = \big((t_1, ..., t_N), (\tilde{t}_1, ..., \tilde{t}_N)\big)$, $t \le \tilde{t}$, and any two multi-time paths $\mathbf{t}, \hat{\mathbf{t}}$ connecting t and $\tilde{t}$, one has

$$V_{\mathbf{t}}[\varphi, y] = V_{\hat{\mathbf{t}}}[\varphi, y]$$

DEFINITION 3.3. We say that the flows of the control vector fields $f_1, ..., f_N$ *strongly inf-commute* if both the following conditions hold true:

i) the flows of the control vector fields $= f_1, ..., f_N$ inf-commute;

ii) for any map $\varphi : \mathbb{R}^n \to \mathbb{R}$, any $y \in \mathbb{R}^n$, any pair $(t, \tilde{t}) = \big((t_1, ..., t_N), (\tilde{t}_1, ..., \tilde{t}_N)\big)$, $t \leq \tilde{t}$, and any two multi-time paths $\mathbf{t}, \hat{\mathbf{t}}$ connecting t and $\tilde{t}$, an N-control $\mathbf{a}$ *is optimal for problem* $\mathcal{P}^{\mathcal{F}}_{\mathbf{t}}[\varphi, y]$ *if and only if* $\mathbf{a}$ *is optimal for problem* $\mathcal{P}^{\mathcal{F}}_{\hat{\mathbf{t}}}[\varphi, y]$.

Remark 3.4. Definitions 3.2 and 3.3 deal only with the so-called Mayer problems —say, problems with a final cost. Actually, this is not restrictive, in that it is possible to consider problems including a current cost as well, by means of a standard enlargement of the state-space's dimension —see e.g. the example below.

Example 1. Going back to the example proposed in the Introduction, one can consider the following question:

$\mathcal{Q}_1$*Does the infimum value of the problem*

$$\begin{cases} \inf \left(\int_0^{T_1} l(x(t), c(t))dt + \int_0^{T_2} m(y(t), d(t))dt \right) \\ \dot{x} = f(x, c) \quad x(0) = \bar{x} \qquad \dot{y} = g(y, d) \quad y(0) = x(T_1) \end{cases} \tag{3.1}$$

coincide with the infimum value of the problem

$$\begin{cases} \inf \left(\int_0^{T_2} m(y(t), d(t))dt + \int_0^{T_1} l(x(t), c(t))dt \right) \\ \dot{y} = g(y, d) \quad y(0) = \bar{x} \qquad \dot{x} = f(x, c) \quad x(0) = y(T_2) \ \ ? \end{cases} \tag{3.2}$$

A further reasonable question is :

$\mathcal{Q}_2$*Provided* (c, d) *is an optimal pair of controls for (3.1), is* (c, d) *an optimal pair for (3.2) as well?*

Let us consider the enlarged state-space $\mathbb{R}^n \times \mathbb{R}$ and the control vector fields f_1, f_2 obtained by supplementing f and g with new components l and m, respectively. If the flows of the control vector fields f_1, f_2 inf-commute [resp. strongly inf-commute], the answer to the former [resp. the latter] question is positive. In fact, stating that the two infima coincide is equivalent to taking (in Definition 3.2 [resp. 3.3]) $(\bar{x}, 0)$ as initial point, $\varphi(x, z) = z \quad \forall (x, z) \in \mathbb{R}^n \times \mathbb{R}$, and

$$\begin{aligned} \mathbf{t}(s) &= \int_0^s \left[\chi_{(0,T_1)}(\xi) \frac{\partial}{\partial t_1} + \chi_{(T_1, T_1+T_2)}(\xi) \frac{\partial}{\partial t_2} \right] d\xi, \\ \hat{\mathbf{t}}(s) &= \int_0^s \left[\chi_{(T_2, T_2+T_1)}(\xi) \frac{\partial}{\partial t_1} + \chi_{(0,T_2)}(\xi) \frac{\partial}{\partial t_2} \right] d\xi, \end{aligned} \qquad s \in [0, T_1 + T_2].$$

In Remark **??** below we shall provide answers to questions $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Remark 3.5. Because of the arbitrariness of the map φ in Definition 3.2 one could suspect that the mere commutativity —see Definition 2.2— coincides with inf-commutativity. Actually commutativity implies strong inf-commutativity —see Proposition 3.6 below— but the two condition are not equivalent, as shown in the counterexample in Remark 3.8 below.

Proposition 3.6. If the flows of the control vector fields $= \{f_1, ..., f_N\}$ commute, then they strongly inf-commute.

Proof. Let φ, $y \in \mathbb{R}^n$, $(t, \tilde{t})$, $t \leq \tilde{t}$, and let $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$, $\hat{\mathbf{t}} : [\hat{S}_\alpha, \hat{S}_\omega] \to \mathbb{R}^N$ belong to $MT_{t,\tilde{t}}$. Let us consider *the reachable set from* y *of the lift of* $\mathbf{t}$:

$$\mathcal{R}_{\mathbf{t}}[y] = \left\{ x_{(\mathbf{a},\mathbf{t})}[y](S_\omega), \ \ \mathbf{a} \quad \text{is a } N\text{-control for the problem } \mathcal{P}^{\mathcal{F}}_{\mathbf{t}}[\varphi, y] \right\}$$

Let us define $\mathcal{R}_{\hat{\mathbf{t}}}[y]$, *the reachable set from* y *of the* $\mathcal{F}$*-lift of* $\hat{\mathbf{t}}$, in a similar way. By Theorem 2.11 it follows that

$$\mathcal{R}_{\mathbf{t}}[y] = \mathcal{R}_{\hat{\mathbf{t}}}[y],$$

so,in particular,

$$\inf \varphi\left(x_{(\mathbf{a},\mathbf{t})}[y](S_\omega)\right) = \inf_{z\in\mathcal{R}_{\mathbf{t}}[y]} \varphi(z) = \inf_{z\in\mathcal{R}_{\hat{\mathbf{t}}}[y]} \varphi(z) = \inf \varphi\left(x_{(\mathbf{a},\hat{\mathbf{t}})}[y](\hat{S}_\omega)\right).$$

Hence the control vector fields $f_1, ..., f_N$ inf-commute. Let us check that they strongly inf-commute. Let $\check{\mathbf{a}}$ be an optimal N-control for problem $\mathcal{P}^{\mathcal{F}}_{\mathbf{t}}[\varphi, y]$. By Theorem 2.11, for every N-control $\mathbf{a}$, one has

$$x_{(\mathbf{a},\mathbf{t})}[y]S_\omega() = x_{(\mathbf{a},\hat{\mathbf{t}})}[y](\hat{S}_\omega)$$

Hence

$$\varphi\left(x_{(\check{\mathbf{a}},\hat{\mathbf{t}})}[y](\hat{S}_\omega)\right) = \varphi\left(x_{(\check{\mathbf{a}},\mathbf{t})}[y](S_\omega)\right) = \inf \varphi\left(x_{(\mathbf{a},\mathbf{t})}[y](S_\omega)\right) = \inf \varphi\left(x_{(\mathbf{a},\hat{\mathbf{t}})}[y](\hat{S}_\omega)\right),$$

that is, $\check{\mathbf{a}}$ is an optimal N-control for problem $\mathcal{P}^{\mathcal{F}}_{\hat{\mathbf{t}}}[\varphi, y]$ as well. □

By the previous Proposition and Theorem 2.11 we get:

THEOREM 3.7. *Let us assume hypothesis* $(\mathbf{ccZLB})_{a.e.}$ *(or, equivalently,* $(\mathbf{ccZLB})_{set}$*). Then the flows of the control vector fields* $= f_1(x, a), ..., f_N(x, a)$ *strongly inf-commute.*

REMARK 3.8. Let us point out that condition $(\mathbf{ccZLB})_{a.e.}$ (or, equivalently, $(\mathbf{ccZLB})_{set}$) allows one to address problems with nonsmooth data. Moreover it is invariant by changes of coordinates on $\mathbb{R}^n$. However, it must be observed that its validity depends on the *parameterization* of the set-valued maps $F_i(x) \doteq \{y \in \mathbb{R}^n \mid y = f_i(x, a) \quad a \in A\}$. In other words, if $F_i(x) \doteq \{y \in \mathbb{R}^n \mid y = \tilde{f}_i(x, a) \quad a \in A\}$ for an n-uple $(\tilde{f}_1, ..., \tilde{f}_N) \neq (f_1, ..., f_N)$, the validity of $(\mathbf{ccZLB})_{a.e.}$ for $(f_1, ..., f_N)$ does not imply that the same property holds for $(\tilde{f}_1, ..., \tilde{f}_N)$ as well.

REMARK 3.9. The dependence of $(\mathbf{ccZLB})_{a.e.}$ (or, equivalently, $(\mathbf{ccZLB})_{set}$) on the parameterization of the f_i's explains why $(\mathbf{ccZLB})_{a.e.}$ is *not necessary* even for the mere inf-commutativity. Indeed, let us consider the case when $n = 1$, $N = 2$, $A = [-1, 1] \times [0, 1]$,

$$f_1(x, a_1, a_2) = a_1 \qquad f_2(x, a_1, a_2) = a_1(1 - a_2 \sin^2 x)$$

Notice that f_1, f_2 *do not* verify condition $(\mathbf{ccZLB})_{a.e.}$. Indeed

$$[f_1(x, a_1, a_2), f_2(x, a_1, a_2)] = -2a_1a_2 \sin x \cos x$$

Yet, *the flows of control the vector fields* f_1, f_2 *do inf-commute.* In fact, the following stronger fact holds true for f_1 and f_2:

If $\check{\mathbf{a}}(\cdot) : [\eta, \beta] \to A$ *is a Borel measurable map and* $\check{x}(\cdot)$ *is the solution of a Cauchy problem*

$$\dot{x} = f_2(x, \check{\mathbf{a}}), \qquad x(\eta) = y$$

then there exists a Borel measurable map $\overline{\mathbf{a}}(\cdot)$ *such that, if* $\hat{x}(\cdot)$ *is the solution of*

$$\dot{x} = f_1(x, \overline{\mathbf{a}}) \qquad x(\eta) = y,$$

then

$$\hat{x}(t) = x(t) \qquad \forall t \in [\eta, \beta].$$

Indeed, it is sufficient to set

$$\bar{\mathbf{a}} = (\bar{\mathbf{a}}_1(t), \bar{\mathbf{a}}_2(t)) \doteq \big(\check{\mathbf{a}}_1(t)(1 - \check{\mathbf{a}}_2(t)\sin^2\check{x}(t))\ ,\ 0\big)$$

Hence it is easy to check that, for any multi-time path $\mathbf{t}$ and any $y \in \mathbb{R}^n$,

$$\mathcal{R}_{\mathbf{t}}[y] = \overline{\mathcal{R}}_{\mathbf{t}}[y] \tag{3.3}$$

where $\mathcal{R}_{\mathbf{t}}[y]$ and $\overline{\mathcal{R}}_{\mathbf{t}}[y]$ are the *reachable sets of* $\mathbf{t}$ — see the proof of Proposition 3.6— corresponding to the control vector fields $\{f_1, f_2\}$ and $\{\bar{f}_1, \bar{f}_2\}$, respectively, where $\bar{f}_1 = \bar{f}_2 = f_1$. Since f_1 is control-independent, $\bar{f}_1$ and $\bar{f}_2$ commute, so they inf-commute. Hence, by (3.3), f_1 and f_2 commute as well.

Remark 3.10. Since condition $(\mathbf{ccZLB})_{a.e.}$ is equivalent to commutativity, the example of the previous remark shows also that $(\mathbf{ccZLB})_{a.e.}$ is stronger than inf-commutativity. We refer to Subsection 3.1 for short remarks about the question of the identification of a *necessary* (and sufficient) condition for inf-commutativity.

Remark 3.11. (*Working out the previous example*) By applying Theorem 3.7 to Example 1 we obtain, in particular, that the infimum of problem (1.2) coincides with the infimum of problem (1.3) provided $f_1 = (f, l)$ $f_2 = (g, m)$ verify condition $(\mathbf{ccZLB})_{a.e.}$

In the particular case of Calculus of Variations —i.e., when $f(x, a) = g(x, a) = a$— this condition reduces to the fact that for every $v, w \in \mathbb{R}^n$ one has

$$\langle Dl(x, v), w\rangle = \langle Dl(x, w), v\rangle$$

for all x such that both $l(\cdot, s)$ and $m(\cdot, r)$ are differentiable at x. In turn, it is easy to verify that this condition is equivalent to the existence of Lipschitz continuous 1-forms $A(x) = (A_1(x), ..., A_n(x))$, $B(x) = (B_1(x), ..., B_n(x))$, such that

$$l(x, r) = \langle A(x), v\rangle \qquad m(x, s) = \langle B(x), w\rangle \qquad \forall x, v, w \in \mathbb{R}^n$$

and

$$\frac{\partial A_i}{\partial x_j}(x) = \frac{\partial B_j}{\partial x_i}(x)$$

for almost every $x \in \mathbb{R}^n$.

3.1. An open question. As it has been shown in Remark 3.8, condition **(ccLBZ)** is not necessary for inf-commutativity. In a sense, this should have been expected, for condition **(ccLBZ)** affects all trajectories and not only optimal trajectories.

We are not moving any step in this delicate direction. Yet we wish to briefly illustrate the nature of the problem. As a matter of fact, a condition on optimal trajectories should likely involve the adjoint variables arising in necessary conditions fit for these minimum problems. The appropriate condition for commutativity should be likely a condition on the associated *characteristic vector fields*

$$X_{H_i} = \left(\frac{\partial H_i}{\partial p}\ ,\ -\frac{\partial H_i}{\partial x}\ ,\ \frac{\partial H_i}{\partial p}p - H_i\right) \qquad H_i = \sup_{a\in A}\{p \cdot f_i(x, a)\}$$

Roughly speaking, this would mean that the Lie bracket $[X_{H_i}, X_{H_j}]$ should be zero, which causes serious drawbacks from the regularity viewpoint. Indeed, in general the vector fields X_{H_i} are not even continuous. Some hint could come from the fact that, provided all function have the sufficient regularity, one has

$$[X_{H_i}, X_{H_j}] = X_{\{H_i,H_j\}}$$

where $\{H, K\}$ denotes the Poisson bracket of H and K, i.e.,

$$\{H, K\} = \sum_1^N \frac{\partial H}{\partial x_i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial x_i}\frac{\partial H}{\partial p_i}$$

Notice that $[X_{H_i}, X_{H_j}] = 0$ if and only if $\{H_i, H_j\} = 0$, so one could try to use the latter condition, which in principle requires less regularity. [10]. As a matter of fact it is easy to prove that **(ccLBZ)** implies

$$\{H_i, H_j\} = 0 \qquad a.e. \tag{3.4}$$

Incidentally, let us notice that in the example of Remark 3.8 condition (3.4) is verified (while **(ccLBZ)** *is not*).

Hence it is reasonable to conjecture that (some weak form of) (3.4) might characterize the inf-commutativity. At the moment we are unable to go beyond this conjectural level and leave the question as an open problem.

4. The input-output map $(\mathbf{a}, \mathbf{t}) \mapsto x_{(\mathbf{a},\mathbf{t})}[y]$

In this section we will investigate (for a fixed $y \in \mathbb{R}^n$) some continuity properties of the input-out map

$$(\mathbf{a}, \mathbf{t}) \mapsto x_{(\mathbf{a},\mathbf{t})}[y].$$

These properties, besides being interesting e.g. in controllability questions, turn out to be essential essential in the proof of Theorem 2.11.

Let us begin by observing that the equation

$$\frac{dx(s)}{ds} = \sum_{i=1}^N f_i(x(s), \mathbf{a}_i \circ \mathbf{t}_i(s))\frac{d\mathbf{t}_i(s)}{ds} \tag{4.1}$$

is invariant with respect to changes of the parameter s. More precisely, one has:

LEMMA 4.1. *Let $s : [L_\alpha, L_\omega] \to [S_\alpha, S_\omega]$ is a stricly increasing, absolutely continuous map. Then a map $x : [S_\alpha, S_\omega] \to \mathbb{R}^n$ is a solution of (4.1) if and only if the map $z \doteq x \circ s$ is (absolutely continuous and is) a solution of*

$$\frac{dz(\sigma)}{d\sigma} = \sum_{i=1}^N f_i(z(\sigma), \mathbf{a}_i \circ \tilde{\mathbf{t}}_i(\sigma))\frac{d\tilde{\mathbf{t}}_i(\sigma)}{d\sigma} \tag{4.2}$$

on $[L_\alpha, L_\omega]$, where $\tilde{\mathbf{t}}(\sigma) \doteq \mathbf{t}(s(\sigma))$.

.

In view of the uniqueness properties of equations (4.1) and (4.2), the proof of this result is trivial as soon as one exploits the following fact:

LEMMA 4.2. *If a map $s : [a, b] \to [c, d]$ is stricly increasing, absolutely continuous, and such that $s'(\sigma) > 0$ for almost every $\sigma \in [a, b]$, then then the inverse map s^{-1} turns out to be a stricly increasing and absolutely continuous map as well.*

[10]An indication in this direction comes also from some results concerning existence of solutions for Hamilton-Jacobi multi-time systems (see [**BT**], [**LR**],[**MR**]), which, as we have mentioned in the Introduction, is closely related to the question of inf-commutativity.

Proof of Lemma 4.2. Of course it is not restrictive to assume that $s(\cdot)$ is onto. For any measurable subset $E \subset \mathbb{R}$, let $meas(E)$ denote the Lebesgue measure of E. We have to show that if $B \subset [a,b]$ is a measurable set such that $meas(s(B)) = 0$ then $meas(B) = 0$ Indeed

$$0 = meas(s(B)) = \int \chi_B(\sigma) s'(\sigma)\, d\sigma$$

implies that $\chi_B(\sigma)s'(\sigma) = 0$ almost everywhere. Since $s'(\sigma) > 0$ for almost every $\sigma \in [a,b]$ this implies that $meas(B) = 0$.

Let $t, \bar t \in \mathbb{R}^N$, $t < \bar t$. Let $MT^{\#}_{[t,\bar t]} \subset MT_{[t,\bar t]}$ denote the family of multi-time paths (connecting t and $\bar t$ and) parameterized on the interval $[0,1]$. In view of Lemma 4.1, this turns out to be not restrictive for the purpose of proving Theorem 2.11.

Let us define the set $\mathcal{A}P_{[t,\bar t]}$ of *admissible policies between* t *and* $\bar t$ by setting

$$\mathcal{A}P_{[t,\bar t]} \doteq \mathcal{A}_{[t,\bar t]} \times MT^{\#}_{[t,\bar t]} \subset \left(\Pi_{i=1} L^1([t_i, \bar t_i], A)\right) \times C^0([0,1], \mathbb{R}^N),$$

and let us endow it with the topology induced by the product topology.

Theorem 4.3. *Let $y \in \mathbb{R}^n$, $t, \bar t \in \mathbb{R}^N$, $t \le \bar t$. For every $(\mathbf{a}, \mathbf{t}) \in \mathcal{A}P_{[t,\bar t]}$ there exists a unique solution $x_{(\mathbf{a},\mathbf{t})}[y]$ of (2.1).*

Moreover, the input-output functional

$$\mathcal{S}^y_{[t,\bar t]} : \mathcal{A}P_{[t,\bar t]} \to C^0([0,1], \mathbb{R}^n)$$

defined by

$$\mathcal{S}^y_{[t,\bar t]}(\mathbf{a}, \mathbf{t}) = x_{(\mathbf{a},\mathbf{t})}[y]$$

is continuous.

We shall prove this theorem as an application of the following version of Banach's fixed point Theorem.

Theorem 4.4. *Let X be a Banach space with norm $\|\cdot\|$, M a metric space, and let $\Phi : M \times X \to X$ be a continuous function such that*

$$\|\Phi(m,x) - \Phi(m,z)\| \le L\|x - z\| \qquad \forall\, m \in M,\ x, z \in X,$$

for a suitable $L < 1$. Then, for every $m \in M$ there exists a unique $x(m) \in X$ such that

$$x(m) = \Phi(m, x(m)). \tag{4.3}$$

The map $m \to x(m)$ is continuous: more precisely, it satisfies

$$\|x(m) - x(m')\| \le (1-L)^{-1} \|\Phi(m, x(m')) \ - \Phi(m', x(m'))| \tag{4.4}$$

for all $m, m' \in M$.

In order to prove Theorem 4.3 let us set

$$X = \{x(\cdot) \in C^0([0,1], \mathbb{R}^n) \mid x(0) = y\} \qquad M = \mathcal{A}P_{[t,\bar t]}$$

and let us consider the functional

$$\Phi_{[t,\bar t]}[\cdot] : M \times X \to X$$

$$\Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z](s) \doteq y + \sum_{i=1}^{N} \int_0^s f_i(z(\sigma),\mathbf{a}_i \circ \mathbf{t}_i(\sigma)) \frac{d\mathbf{t}_i}{d\sigma}\, d\sigma \qquad \forall s \in [0,1]$$

In order to apply Theorem 4.4 we need to prove the following result on the continuity properties of the functional $\Phi_{[t,\bar{t}]}$.

PROPOSITION 4.5. *Let us assume that [(H1)-(H2) are verified and] there exists a constant L_1 such that*

$$|f(x,a) - f(y,b)| \le L_1(|x-y| + |a-b|) \qquad \forall (x,a),(y,b) \in \mathbb{R}^n \times A. \tag{4.5}$$

Then the functional $\Phi_{[t,\bar{t}]}$ is continuous and verifies

$$\|\Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z] - \Phi_{[t,\bar{t}]}[(\tilde{\mathbf{a}},\mathbf{t}),z]\|_\infty \le L_1 \sum_{i=1}^{N} \|\mathbf{a}_i - \tilde{\mathbf{a}}_i\|_1. \tag{4.6}$$

for all $\mathbf{a},\tilde{\mathbf{a}} \in \mathcal{A}_{[t,\bar{t}]}$ and $\mathbf{t} \in MT^{\#}_{[t,\bar{t}]}$. Moreover, if one endows the space X with the norm $\|x\|_* = \sup_{s\in[0,1]} e^{-2L_1 s}|x(s)|$ [11] then one has

$$\|\Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z_1] - \Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z_2]\|_* \le \frac{1}{2}\|z_1 - z_2\|_* \tag{4.7}$$

for all $(\mathbf{a},\mathbf{t}) \in \mathcal{A}P_{[t,\bar{t}]}$.

PROOF. The proof of (4.7) is straighforward, so we omit it. Let us examine the dependence in the variable $\mathbf{a}$. Let $(\mathbf{a},\mathbf{t}),(\tilde{\mathbf{a}},\mathbf{t}) \in M$ and $z \in X$. Then, for every $s \in [0,1]$ one has

$$\begin{gathered}
|\Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z](s) - \Phi_{[t,\bar{t}]}[(\tilde{\mathbf{a}},\mathbf{t}),z](s)| \le \\
\sum_{i=1}^N \int_0^s |f_i(z(\sigma),\mathbf{a}_i \circ \mathbf{t}_i(\sigma)) - f_i(z(\sigma),\tilde{\mathbf{a}}_i \circ \mathbf{t}_i(\sigma))|\mathbf{t}_i'(\sigma)\, d\sigma \le \\
L_1 \sum_{i=1}^N \int_0^s |\mathbf{a}_i \circ \mathbf{t}_i(\sigma) - \tilde{\mathbf{a}}_i \circ \mathbf{t}_i(\sigma)|\mathbf{t}_i'(\sigma)\, d\sigma = \\
L_1 \sum_{i=1}^N \int_0^{\mathbf{t}_i(s)} |\mathbf{a}_i(\xi) - \tilde{\mathbf{a}}_i(\xi)|\, d\xi \le L_1 \sum_{i=1}^N \|\mathbf{a}_i - \tilde{\mathbf{a}}_i\|_1
\end{gathered}$$

By taking the maximum over $[0,1]$ we obtain (4.6).

Finally, let us examine the continuity in $\mathbf{t}$. For this purpose let us set

$$MT^{\#,+}_{[t,\bar{t}]} \doteq \{\mathbf{t} \in MT^{\#}_{[t,\bar{t}]} \mid \forall i = 1,...,N \ \ \mathbf{t}_i \text{ is either constant or strictly increasing}\}$$

Since $MT^{\#,+}_{[t,\bar{t}]}$ is dense $MT^{\#}_{[t,\bar{t}]}$, Lemma 4.6 below implies that, for every $\mathbf{a}$ and z, the map

$$\mathbf{t} \mapsto \Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z]$$

is continuous. Since $(\mathbf{a},z) \to \Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z]$ is continuous in $(\mathbf{a},z)$, uniformly with respect to $\mathbf{t}$, we can conclude that $\Phi_{[t,\bar{t}]}$ is continuous. □

In order to conclude the proof of Proposition 4.7 we need to prove the following result:

[11] This norm is equivalent to the usual C^0 norm $\|\cdot\|_\infty$.

LEMMA 4.6. *For every* $((\mathbf{a},\mathbf{t}),z) \in \mathcal{A}P_{[t,\bar{t}]}$ *and every sequence* $\mathbf{t}^n$ *in* $MT^{\#,+}_{[t,\bar{t}]}$ *such that*

$$\lim_{n\to\infty} \|\mathbf{t}^n - \mathbf{t}\|_\infty = 0$$

one has

$$\lim_{n\to\infty} \|\Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}^n),z] - \Phi_{[t,\bar{t}]}[(\mathbf{a},\mathbf{t}),z]\|_\infty = 0$$

This result will be proved after Lemma 4.7 below, which concerns the asymptotic behavior of the inverses of a sequence of monotone maps converging to a given function. For any map $\phi : A \to B$ and any subset $C \subseteq B$, let us use ϕ^{-1} to denote the counter image of C.

LEMMA 4.7. *Let* $I, J \subset \mathbb{R}$ *be compact intervals and let* $g : I \to J$ *be a continuous, surjective, increasing map. Let* $g_k : I \to J$, $k \in \mathbb{N}$, *be a sequence of continuous, surjective, strictly increasing maps such that*

$$\lim_{k\to\infty} \|g_k - g\|_\infty = 0.$$

Then, for every $\xi \in J$ *one has*

$$\lim_{k\to\infty} d\left(g_k^{-1}(\xi)\,,\, g^{-1}(\{\xi\})\right) = 0$$

where d *denotes the usual distance between a point and a set.*

PROOF. Let us observe that $g^{-1}(\{\xi\}))$ is a compact interval. Assume by contradiction that the thesis is false. Then, by possibly passing to a subsequence, there exists η such that either

$$g_k^{-1}(\xi) \le \min g^{-1}(\{\xi\})) - \eta \tag{4.8}$$

or

$$g_k^{-1}(\xi) \ge \max g^{-1}(\{\xi\})) + \eta \tag{4.9}$$

for all $k \in \mathbb{N}$ Let us continue the proof by assuming that (4.8) holds —the proof in the alternative case being akin.

Let us set $s_k = g_k^{-1}(\xi)$, and let us observe that by (4.8) there exists $\nu > 0$ such that

$$g(s_k) \le \xi - \nu \tag{4.10}$$

for all $k \in \mathbb{N}$. Indeed, this follows from $g(s_k) \le g(\min g^{-1}(\{\xi\})) - \eta) < \xi$.

On the other hand,

$$|\xi - g(s_k)| = |g_k(s_k) - g(s_k)|$$

converges to zero, for g_k converges to g uniformly. This contradicts (4.10), which concludes the proof. □

Proof of Lemma 4.6. Let Z be the maximal subset of $\{1,...,N\}$ such that $t_i > \tilde{t}_i$ for every $i \in Z$. Let us observe that for every $j \in \{1,...,N\}\backslash Z$, every $k \in \mathbb{N}$, and every $s \in [0,1]$, $\mathbf{t}_j^k(s) = \mathbf{t}_j(s) = t_j = \bar{t}_j$. For every $i \in Z$ and every integer $k \in \mathbb{N}$, let us consider the map

$$s_i^k \doteq (\mathbf{t}_i^k)^{-1} \circ \mathbf{t}_i : [0,1] \to [0,1],$$

and let us set

$$W_i \doteq \{\xi \in [t_i, \bar{t}_i] \mid \mathbf{t}_i^{-1}(\xi)\,\text{is a non trivial interval}\} \qquad J_i \doteq \mathbf{t}_i^{-1}(W_i)$$

Notice that W_i is a countable set, so J_i is a countable union of compact, pair-wise disjoint intervals. Moreover, for every $i \in Z$ and every $s \in [0, S]\backslash J_i$ one has

$$\lim_{k\to\infty} |s - s_i^k(s)| = 0 \tag{4.11}$$

Indeed, if $s \in [0,1]\backslash J_i$ then $\{s\} = (\mathbf{t}_i)^{-1}(\{\mathbf{t}_i(s)\})$. Hence, by Lemma 4.7 one obtains

$$\lim_{k\to\infty} |s - s_i^k(s)| = \lim_{k\to\infty} d\Big((\mathbf{t}_i)^{-1}(\{\mathbf{t}_i(s)\})\,,\,(\mathbf{t}_i^k)^{-1}(\mathbf{t}_i(s))\Big) = 0$$

Now, for every $s \in [0,1]$ and any $k \in \mathbb{N}$, by applying the change of variable $\sigma = s_i^k(\alpha)$ we obtain

$$\begin{gathered}\Big|\Phi_{[t,\bar t]}((\mathbf{a},\mathbf{t}^k),z)(s) - \Phi_{[t,\bar t]}((\mathbf{a},\mathbf{t}),z)(s)\Big| \leq \\ \textstyle\sum_{i\in Z}\Big|\int_0^s f_i\Big(z(\sigma),\mathbf{a}_i\circ\mathbf{t}_i(\sigma)\Big)(\mathbf{t}_i)'(\sigma)\,d\sigma - \int_0^s f_i\Big(z(\sigma),\mathbf{a}_i\circ\mathbf{t}_i^k(\sigma)\Big)(\mathbf{t}_i^k)'(\sigma)\,d\sigma\Big| = \\ \textstyle\sum_{i\in Z}\Big|\int_0^s f_i\Big(z(\sigma),\mathbf{a}_i\circ\mathbf{t}_i(\sigma)\Big)(\mathbf{t}_i)'(\sigma)\,d\sigma - \int_0^{\alpha_i^k(s)} f_i\Big(z(s_i^k(\alpha)),\mathbf{a}_i\circ\mathbf{t}_i(\alpha)\Big)(\mathbf{t}_i)'(\alpha)\,d\alpha\Big| \leq \\ \mathcal{I}^k + \mathcal{W}_s^k\,,\end{gathered} \tag{4.12}$$

where

$$\alpha_i^k(s) = \min \mathbf{t}_i^{-1}(\{\mathbf{t}_i^k(s)\})\ ,$$

$$\mathcal{I}^k = \sum_{i\in Z}\int_0^1 \Big|f_i\Big(z(\sigma),\mathbf{a}_i\circ\mathbf{t}_i(\sigma)\Big) - f_i\Big(z(s_i^k(\sigma)),\mathbf{a}_i\circ\mathbf{t}_i(\sigma)\Big)\Big|(\mathbf{t}_i)'(\sigma)\,d\sigma\ ,$$

and

$$\mathcal{W}^k = \sum_{i\in Z}\int_{\min\{s,\alpha_i^k(s)\}}^{\max\{s,\alpha_i^k(s)\}} \Big|f_i\Big(z(s_i^k(\sigma)),\mathbf{a}_i\circ\mathbf{t}_i(\sigma)\Big)\Big|(\mathbf{t}_i)'(\sigma)\,d\sigma.$$

Setting, for every $i \in Z$,

$$D_i = \{s \in [0,1] \mid \mathbf{t}_i'(s) > 0\}$$

one has

$$\mathcal{I}^k \leq \sum_{i\in Z} L_1 \int_0^1 |z(\sigma) - z(s_i^k(\sigma))|\chi_{D_i}\mathbf{t}_i'(\sigma)\,d\sigma$$

Clearly $D_i \subset [0,1]\backslash J_i$, so by (4.11) (and by the continuity of z) for every $\sigma \in D_i$ one has

$$\lim_{k\to\infty} (z(\sigma) - z(s_i^k(\sigma))\mathbf{t}_i'(\sigma) = 0$$

Hence by Lebesgue's Dominated Convergence Theorem one obtains

$$\lim_{k\to\infty} \mathcal{I}^k = 0 \tag{4.13}$$

Moreover, since $\forall i \in Z$ $\mathbf{t}_i^k$ converges uniformly to $\mathbf{t}_i$, one has

$$\lim_{k\to\infty} \mathcal{W}_s^k = \lim_{k\to\infty} \sum_{i\in Z}\int_{\min\{\mathbf{t}_i(s),\mathbf{t}_i^k(s)\}}^{\max\{t(s),t_k(s)\}} \Big|f_i\Big(z((\mathbf{t}_i^k)^{-1}(\xi)),\mathbf{a}_i(\xi)\Big)\Big|d\xi = 0$$

Notice that, by the absolute continuity of the maps

$$\tau \mapsto \int_{t_i}^{\tau} \Big|f_i\Big(z((\mathbf{t}_i^k)^{-1}(\xi)),\mathbf{a}_i(\xi)\Big)\Big|d\xi\,,$$

this limit is equal to zero, *uniformly with respect to* s. That is, there exists a sequence $\mathcal{W}^k$ such that

$$\mathcal{W}^k_s \leq \mathcal{W}^k \quad \forall (s,k) \in [0,1]\times\mathbb{N} \qquad \text{and} \qquad \lim_{k\to\infty} \mathcal{W}^k = 0\,. \tag{4.14}$$

By (4.12), (4.13), and (4.14), taking the maximum in s over $[0,1]$ we obtain the thesis. □

Proof of Theorem 4.3. Let us begin by proving the theorem under the additional hypothesis (4.5). Since a solution of (2.1) is a fixed point of the functional

$$z \mapsto \Phi_{[t,\bar t]}[(\mathbf{a},\mathbf{t}),z][y]$$

by (4.7) and Theorem 4.4 we obtain that there exists a unique solution $x_{(\mathbf{a},\mathbf{t})}[y]$. Moreover, since in view of Theorem 4.4 one has

$$\|x_{(\mathbf{a},\mathbf{t})}[y] - x_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]\|_* \leq 2\|\Phi_{[t,\bar t]}[(\mathbf{a},\mathbf{t}),x_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]] - \Phi_{[t,\bar t]}[(\tilde{\mathbf{a}},\tilde{\mathbf{t}}),x_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]]\|_* \tag{4.15}$$

by Proposition 4.5 one obtains

$$\begin{gathered}\|x_{(\mathbf{a},\mathbf{t})}[y] - x_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]\|_* \leq \\ 2\textstyle\sum_{i+1}^N \|\mathbf{a}-\tilde{\mathbf{a}}\|_1 + 2\left\|\Phi_{[t,\bar t]}[(\mathbf{a},\tilde{\mathbf{t}}),x_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]] - \Phi_{[t,\bar t]}[(\tilde{\mathbf{a}},\tilde{\mathbf{t}}),x_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]]\right\|_\infty\end{gathered} \tag{4.16}$$

which, by the continuity of $\Phi_{[t,\bar t]}$ implies the thesis of the theorem (when (4.5) is assumed).

Let us prove that the thesis is still valid under the only structural hypotheses (H1)-(H2). This will be done by means of standard cut-off function arguments. Let $\tilde{\mathbf{a}},\mathbf{a} \in \mathcal{A}_{[t,\bar t]}$, $\tilde{\mathbf{t}} \in MT^\#_{[t,\bar t]}$, $y\in\mathbb{R}^n$, and let $\phi_R:\mathbb{R}^n\to\mathbb{R}$ be a C^∞ function such that $\phi=1$ on $B[y\,,\,R]$ and $\phi=0$ on the complement of $B[y\,,\,R+1]$, where R is a positive number to be determined a posteriori. Let us consider the control vector fields $f_1^R,\dots,f_N^R$ defined, for every $i=1,\dots,N$, by

$$f_i^R(x,a) = \phi_R(x) f_i(x,a) \qquad \forall (x,a)\in\mathbb{R}^n\times A$$

In particular, these control vector fields verify hypothesis (4.5), with $L_1 = L_{B[y,R+1]}$. Hence, by the first part of the proof, for every $(\mathbf{a},\mathbf{t})\in\mathcal{A}P_{[t,\bar t]}$ there exists a unique solution of the Cauchy problem

$$\frac{dx(s)}{ds} = \sum_{i=1}^N f_i^R(x(s),\mathbf{a}_i\circ\mathbf{t}_i(s))\frac{d\mathbf{t}_i(s)}{ds} \qquad x(0)=y$$

which will be denoted by $x^R_{(\mathbf{a},\mathbf{t})}[y]$. Let us fix $\tilde{\mathbf{t}}\in MT^\#_{[t,\bar t]}$ and $\tilde{\mathbf{a}}\in\mathcal{A}_{[t,\bar t]}$. Then for all $\mathbf{a}\in\mathcal{A}_{[t,\bar t]}$ and $\mathbf{t}\in MT^\#_{[t,\bar t]}$, by (4.16), one has

$$\|x^R_{(\mathbf{a},\mathbf{t})}[y] - x^R_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]\|_\infty \leq 2L_1\|\sum_{i=1}^N \tilde{\mathbf{a}}_i - \mathbf{a}_i\|_1 + \omega(\|\mathbf{t}-\tilde{\mathbf{t}}\|_\infty), \tag{4.17}$$

where $\omega=\omega(\delta)$ is a positive, strictly increasing function converging to 0 when δ tends to 0.

By the structural hypothesis (H2) and by a standard application of Gronwall's inequality there exists a number M (depending on $\tilde{\mathbf{a}}$ and independent of R) such that

$$x^R_{(\tilde{\mathbf{a}},\tilde{\mathbf{t}})}[y]([0,1]) \subset B[y,M]\,, \tag{4.18}$$

Let $\delta > 0$ be such that

$$2L_1 N\delta + \omega(\delta) < 1 \tag{4.19}$$

Then, by (4.17)-(4.19), for all $\mathbf{a}, \mathbf{t}$ such that

$$\|\mathbf{a}_i - \tilde{\mathbf{a}}_i\|_1 < \delta \quad \forall i = 1, ..., N \qquad \|\mathbf{t} - \tilde{\mathbf{t}}\|_\infty < \delta\,, \tag{4.20}$$

one has

$$x^R_{(\mathbf{a},\mathbf{t})}[y]([0,1]) \subset B[y, M+1]\,.$$

Therefore, if we choose R to be equal to $M+1$ it follows that, for all $\mathbf{a}, \mathbf{t}$ satisfying (4.20), $x^R_{(\mathbf{a},\mathbf{t})}[y]$ are solutions of the original equation as well (corresponding to $(\tilde{\mathbf{a}}, \tilde{\mathbf{t}})$ and $(\mathbf{a}, \mathbf{t})$, respectively). By the (obvious) local uniqueness of the solution it follows that

$$x^R_{(\mathbf{a},\mathbf{t})}[y] = x_{(\mathbf{a},\mathbf{t})}[y].$$

for all $\mathbf{a}, \mathbf{t}$ satisfying (4.20). Hence the thesis follows from the first part of the theorem. □

5. Multi-time control strings

Let us introduce the notion of *multi-time control string*, which will be exploited in the next section in order to prove Theorem 2.11. Multi-time control strings are discrete, easy-to-handle, representations of the pairs $(\mathbf{a}, \mathbf{t})$, where $\mathbf{a}$ is an N-control whose components are piecewise constant and $\mathbf{t}$ is a simple multi-time path.

DEFINITION 5.1. A *multi-time control string* is a four-uple

$$\lambda = \Big(R, (\sigma_1, ..., \sigma_R), h, (\alpha_1, ..., \alpha_R)\Big),$$

where R is a positive integer, $(\sigma_1, ..., \sigma_R)$ is a R-uple of non-negative real numbers, h is a map from $\{1, ..., R\}$ into $\{1, ..., N\}$, and $\alpha_\nu \in A$, for every $\nu = 1, ..., R$. To every multi-time control string λ let us associate the number

$$S_\lambda = \sum_{\nu=1}^{R} \sigma_\nu$$

The set of all multi-time control strings will be denoted by $\mathcal{C}$.

DEFINITION 5.2. Let $\lambda = \Big(R, (\sigma_1, ..., \sigma_R), h, (\alpha_1, ..., \alpha_R)\Big)$ be a multi-time control string, and let us set $s_0 = 0$, $s_\nu = \sum_{\mu=1}^{\nu} \sigma_\mu$ for every $\nu = 1, ..., R$. Let $y \in \mathbb{R}^n$. By *trajectory starting at y and corresponding to the multi-time control string λ.* we mean the solution of the Cauchy problem [12]

$$\frac{dx}{ds}(s) = \sum_{\nu=1}^{R} \chi_{(s_{\nu-1}, s_\nu)}(s) f_{h(\nu)}(x(s), \alpha_\nu) \qquad x(0) = y$$

This map will be denoted by $x[\lambda, y](\cdot)$

[12] This means that at each interval $(s_{\nu-1}, s_\nu)$ the state evolves according to the vector field $f_{h(\nu)}(\cdot, \alpha_\nu)$.

5.0.1. *Multi-time control strings and lifts.* We are going to establish a one-to-one correspondence between the set of multi-time control strings and the class of pairs $(\mathbf{a}, \mathbf{t})$, such that the components $\mathbf{a}_i$ of the N-control $\mathbf{a}$ are piece-wise constant maps and $\mathbf{t}$ is a simple multi-time path.

Let us begin with the notion of i-projection of a multi-time string.

Definition 5.3. For every $i = 1, ..., N$ let us define the map

$$\pi_i : \mathcal{C} \to \mathcal{C}$$

by setting, for every $\lambda = \Big(R, (\sigma_1, ..., \sigma_R), h, (\alpha_1, ..., \alpha_R)\Big)$,

$$\pi_i(\lambda) = \Big(R_i, (\sigma_{\nu_1^i}, ..., \sigma_{\nu_{R_i}^i}), h_i, (\alpha_{\nu_1^i}, ..., \alpha_{\nu_{R_i}^i})\Big),$$

where $\{\nu_1^i, ..., \nu_{R_i}^i\}$ coincides with the subset $h^{-1}(i) \subset \{1, ..., R\}$ endowed with the natural order, R_i is its cardinality, and $h_i = h_{|_{h^{-1}(i)}}$ —that is, h_i is constantly equal to i. The map π_i will be called the i-projection of $\mathcal{C}$.

Given a multi-time control string $\lambda = \Big(R, (\sigma_1, ..., \sigma_R), h, (\alpha_1, ..., \alpha_R)\Big)$, let us define a pair $(\mathbf{a}^\lambda, \mathbf{t}^\lambda)$, where $\mathbf{a}^\lambda$ is an N-control and $\mathbf{t}^\lambda$ is a simple multi-time path defined as follows:

Definition 5.4. For every $i = 1, ..., N$, let us consider its i-projection

$$\pi_i(\lambda) = \Big(R_i, (\sigma_{\nu_1^i}, ..., \sigma_{\nu_{R_i}^i}), h_i, (\alpha_{\nu_1^i}, ..., \alpha_{\nu_{R_i}^i})\Big),$$

and let us set $\tilde{t}^i = \sum_{k=1}^{R_i} \sigma_{\nu_k^i}$. Let us define the piecewise-constant control

$$\mathbf{a}_i^\lambda = \sum_{k=1}^{R_i} \alpha_{\nu_k^i}\, \chi_{(t_{k-1}^i, t_k^i)} \ : [0, \tilde{t}^i] \to A\ ,$$

where we have set

$$t_0^i = 0, \qquad t_k^i = \sum_{l=1}^{k} \sigma_{\nu_l^i} \qquad \forall k = 1, ..., R_i,$$

and let us define the N-control $\mathbf{a}^\lambda$ by setting

$$\mathbf{a}^\lambda = \{a_1^\lambda, ..., a_N^\lambda\}.$$

Moreover, let us consider the simple multi-time path

$$\mathbf{t}^\lambda(s) = \int_0^s \sum_{\nu=1}^{R} \chi_{(S_{\nu-1}, S_\nu)}(\xi) \frac{\partial}{\partial t_{h(\nu)}}\, d\xi,$$

where

$$S_0 = 0, \qquad S_\nu = \sum_{\mu=1}^{\nu} \sigma_\mu \quad \forall \nu = 1, ..., R$$

Conversely, let $\mathbf{a} = (\mathbf{a}_1, ..., \mathbf{a}_N)$ be N-control such that $\mathbf{a}_1, ..., \mathbf{a}_N$ are piece-wise constant maps defined on intervals $[0, \tilde{t}_1]$,..., $[0, \tilde{t}_N]$, respectively, and let

$$\mathbf{t}(s) = \int_{S_0}^s \left(\sum_{\nu=1}^{R-1} \chi_{(S_\nu, S_{\nu+1})}(\xi) \frac{\partial}{\partial t_{h(\nu)}} \right) d\xi$$

be a simple multi-time path connecting $(0,...,0)$ with $(\tilde{t}_1,...,\tilde{t}_N)$. Let us assume that the choice of $S_1,...,S_R$ is such that if $h(\nu)=i$ then the map $\mathbf{a}_i$ is constant on the interval $]S_{\nu-1},S_\nu]$. Let us call **a**-*fit* such a multi-time path.

Let us construct a multi-time control string from the pair $(\mathbf{a},\mathbf{t})$ as follows:

Definition 5.5. Call α_ν the constant value of $\mathbf{a}_i$ on $]S_{\nu-1},S_\nu]$. For any $\nu = 1,...,R$, let us set $\sigma_\nu = S_\nu - S_{\nu-1}$, and let us define the multi-time control string $\lambda^{(\mathbf{a},\mathbf{t})}$ by setting

$$\lambda^{(\mathbf{a},\mathbf{t})} = \big(R,(\sigma_1,...,\sigma_R),h,(\alpha_1,...,\alpha_R)\big)$$

The following trivial result establishes a connection between multi-time control strings and the above pairs of N-controls and simple multi-time paths :

Lemma 5.6. *Let y be a point in $\mathbb{R}^n$, and let λ be a multi-time control string. Then*

$$x_{(\mathbf{a}^\lambda,\mathbf{t}^\lambda)}[y](\cdot) = x[\lambda,y](\cdot)$$

Conversely, let $\mathbf{a}=\{\mathbf{a}_1,...,\mathbf{a}_N\}$ be an N-control made of constant maps defined on intervals $[0,\tilde{t}_1]$,..., $[0,\tilde{t}_N]$, respectively, and let $\mathbf{t}$ be an $\mathbf{a}$-fit simple multi-time path. Then

$$x[\lambda^{(\mathbf{a},\mathbf{t})},y](\cdot) = x_{(\mathbf{a},\mathbf{t})}[y](\cdot)$$

Remark 5.7. The assumption of **a**-fitness made on **t** is not restrictive. Indeed, it is easy to verify that up to a refinement of the partition $S_1,...,S_R$ we can always find a multi-time path $\hat{\mathbf{t}}$ verifying this assumption and such that the $\mathbf{a}^\lambda$-$\mathcal{F}$-lifts at y of the multi-time paths $\mathbf{t}$ and $\hat{\mathbf{t}}$ do coincide.

5.1. Concatenation, equivalence, and ordering.

Definition 5.8. For every pair of multi-time control strings

$$\lambda = \Big(R,(\sigma_1,...,\sigma_R),h,(\alpha_1,...,\alpha_R)\Big)$$
$$\tilde\lambda = \Big(\tilde R,(\tilde\sigma_1,...,\tilde\sigma_{\tilde R}),\tilde h,(\tilde\alpha_1,...,\tilde\alpha_{\tilde R})\Big)$$

let us define the *concatenation of* λ *and* $\tilde\lambda$ as the multi-time control string

$$\lambda\star\tilde\lambda = \Big(R+\tilde R,(\hat\sigma_1,...,\hat\sigma_{R+\tilde R}),h\star\tilde h,(\hat\alpha_1,...,\hat\alpha_{R+\tilde R})\Big)$$

where

$$(\hat\sigma_1,...,\hat\sigma_{R+\tilde R}) = (\sigma_1,...,\sigma_R,\tilde\sigma_1,...,\tilde\sigma_{\tilde R}) \ ,$$

and $h\star\tilde h(\nu) = h(\nu)$ if $\nu \le R$ and $h\star\tilde h(\nu) = \tilde h(\nu - R)$ if $\nu > R$. Moreover, for every finite set of multi-time control strings $\lambda_1,...,\lambda_d$, let us define the *concatenation* $\lambda_1\star...\star\lambda_d$ by the obvious associativity of the operation $\star$.

Definition 5.9. Let $\lambda = \Big(R,(\sigma_1,...,\sigma_R),h,(\alpha_1,...,\alpha_R)\Big)$ be a multi-time control string. We say that $\tilde\lambda = \Big(\tilde R,(\tilde\sigma_1,...,\tilde\sigma_{\tilde R}),\tilde h,(\tilde\alpha_1,...,\alpha_{\tilde R})\Big)$ *is simply equivalent to* λ, and we write if $\lambda\sim\tilde\lambda$, if either they coincide or one of the following two conditions are verified:

1) there exist $\nu \in \{1, ..., R\}$, such that $\tilde{R} = R+1$, $(\sigma_i, \alpha_i) = (\tilde{\sigma}_i, \tilde{\alpha}_i)$ for every $i \leq \nu - 1$, $(\sigma_i, \tilde{\alpha}_i) = (\tilde{\sigma}_{i+1}, \tilde{\alpha}_{i+1})$ for every $i \geq \nu + 1$, $\sigma_\nu = \tilde{\sigma}_\nu + \tilde{\sigma}_{\nu+j}$, and $\tilde{\alpha}_{\nu+j} = \alpha_\nu$ for $j = 0, 1$;
2) λ can be obtained from $\tilde{\lambda}$ in the same way $\tilde{\lambda}$ has been obtained from λ in 1).

We say that λ and $\tilde{\lambda}$ are *equivalent* if there exists a finite number of multi-time control strings $\lambda_1, ..., \lambda_r$ such that

$$\lambda \sim \lambda_1 \sim .... \sim \lambda_r \sim \tilde{\lambda}$$

We write $\tilde{\lambda} \simeq \lambda$ to mean that $\tilde{\lambda}$ is equivalent to λ.

Clearly $\simeq$ *is an equivalence relation on the set* $\mathcal{C}$ *of multi-time control strings.*

The next (trivial) Proposition implies the map $\lambda \mapsto x[\lambda, y](\cdot)$ induces a well-defined map on the quotient $\mathcal{C}/\simeq$.

Proposition 5.10. *If* $\tilde{\lambda} \simeq \lambda$ *then*

$$x[\tilde{\lambda}, y](s) = x[\lambda, y](s)$$

for all $y \in \mathbb{R}^n$ *and* $s \in [0, S_\lambda] = [0, S_{\tilde{\lambda}}]$.

5.1.1. *Time-reordering of a multi-time control string.* Let us introduce the notion of *time-reordering* of a multi-time control string $\lambda \in \mathcal{C}$. Successively we shall investigate the effects of time-reordering on trajectories.

Definition 5.11. Let

$$\lambda = \Big(R, (\sigma_1, ..., s_R), h, (\alpha_1, ..., \alpha_R)\Big),$$

be a multi-time control string, and let $\nu \in \{1, ..., R-1\}$ be such that $h(\nu) \neq h(\nu+1)$. We define the *ν-th elementary time-reordering of* λ as the multi-time control string

$$\omega_\nu(\lambda) = \Big(R, (\sigma_{\rho(1)}, ..., \sigma_{\rho(R)}), h, (\alpha_{\rho(1)}, ..., \alpha_{\rho(R)})\Big)$$

where $\rho(\mu) = \mu$ if $\mu \notin \{\nu, \nu+1\}$, and $\rho(\nu) = \nu + 1$, $\rho(\nu+1) = \nu$.

Remark 5.12. The condition

$$h(\nu) \neq h(\nu + 1)$$

means that we do not allow the interchange of adjacent intervals corresponding to the same time component. Roughly speaking, within the evolution of the same time component we keep the given order.

Definition 5.13. Let λ be a multi-time control string. A control $\tilde{\lambda}$ is called *a time-reordering* of λ if either $\tilde{\lambda} = \lambda$ or there exists a set of elementary time-reordering $\nu_1, ..., \nu_Q$ such that λ

$$\tilde{\lambda} = \omega_{\nu_Q} \circ ... \circ \omega_{\nu_1}(\lambda)$$

We shall write $\tilde{\lambda} \bowtie \lambda$ to mean that $\tilde{\lambda}$ is a time-reordering of λ.

Clearly $\bowtie$ *is an equivalence relation on the set* $\mathcal{C}$ *of multi-time control strings.*

The next theorem implies that under hypothesis **(ccLBZ)** the map $\lambda \mapsto x[\lambda, y](\cdot)$ induces a well-defined map on the quotient $\mathcal{C}/\bowtie$.

THEOREM 5.14. *Let us assume hypothesis* **(ccLBZ)**. *Let* $\lambda, \hat{\lambda} \in \mathcal{C}$ *such that of* $\lambda \bowtie \hat{\lambda}$. *Then*

$$x[y,\lambda](S) = x[y,\hat{\lambda}](S)$$

where $S = S_\lambda (= S_{\hat{\lambda}})$.

PROOF. Since $\hat{\lambda}$ is obtained by λ after applying a finite number of elementary time-reordering, it is sufficient to prove the theorem in the event when $\hat{\lambda}$ is an elementary time-reordering of λ. So, in view of Lemma 5.6 the thesis easily follows from Theorem 2.11. □

5.1.2. $(j_1,...,j_N)$*-ordered multi-time control strings.* If $(j_1,...,j_N)$ is a permutation of $(1,...,N)$, we call $(j_1,...,j_N)$*-ordered* those multi-time control strings such that each time component is *run* only once, according to the order $(j_1,...,j_N)$. More precisely:

DEFINITION 5.15. For each permutation $(j_1,...,j_N)$ of the N-uple $(1,...,N)$, let us define $\mathcal{C}_{j_1,...,j_N} \subset \mathcal{C}$ as the set of those multi-time control strings λ such that

$$\lambda = \pi_{j_1}(\lambda) \star ... \star \pi_{j_N}(\lambda)$$

where, for every $i = 1,...,N$, π_i is the i-projection introduced in Definition 5.3. $\mathcal{C}_{j_1,...,j_N}$ will be called *the set of* $(j_1,...,j_N)$*-ordered multi-time control strings.*

DEFINITION 5.16. The map

$$Pr_{j_1,...,j_N} : \mathcal{C} \to \mathcal{C}_{j_1,...,j_N}$$

defined by

$$Pr_{j_1,...,j_N}(\lambda) = \pi_{j_1}(\lambda) \star ... \star \pi_{j_N}(\lambda)$$

will be called the $(j_1,...,j_N)$-projection of $\mathcal{C}$.

Let us state, without proof, some trivial properties of the map $Pr_{j_1,...,j_N}$.

LEMMA 5.17. *The following properties are verified:*

i) *The map* $Pr_{j_1,...,j_N}$ *induces the identity on the set* $\mathcal{C}_{j_1,...,j_N}$

ii) *If* $\lambda, \tilde{\lambda} \in \mathcal{C}$ *and* $\lambda \bowtie \tilde{\lambda}$, *then*

$$Pr_{j_1,...,j_N}(\lambda) = Pr_{j_1,...,j_N}(\tilde{\lambda}).$$

iii) *For every* $\lambda_1, \lambda_2 \in \mathcal{C}$ *one has*

$$Pr_{j_1,...,j_N}(\lambda_1 \star \lambda_2) = Pr_{j_1,...,j_N}(Pr_{j_1,...,j_N}(\lambda_1) \star \lambda_2) =$$
$$Pr_{j_1,...,j_N}(\lambda_1 \star Pr_{j_1,...,j_N}(\lambda_2)) = Pr_{j_1,...,j_N}\left(Pr_{j_1,...,j_N}(\lambda_1) \star Pr_{j_1,...,j_N}(\lambda_2)\right).$$

Moreover one has:

THEOREM 5.18. *Let* $\lambda = \Big(R, (\sigma_1,...,\sigma_R), h, (\alpha_1,...,\alpha_R)\Big)$ *be a multi-time control string. Then* $Pr_{j_1,...,j_N}(\lambda) \bowtie \lambda$.

PROOF. We shall proceed by induction on the number R. The thesis is trivial for $R = 2$. Let R be greater than 2 and let us consider the multi-time control string

$$\hat{\lambda} = \Big(R-1, (\sigma_1,...,\sigma_{R-1}), h, (\alpha_1,...,\alpha_{R-1})\Big)$$

(where, with a small notational abuse, we have written h to mean the restriction of h to $\{1, ..., R-1\}$). By the inductive hypothesis $Pr_{j_1,...,j_N}(\hat\lambda) \bowtie \hat\lambda$, that is, there exist $\nu_1, ..., \nu_Q \in \{1, ..., R-1\}$ such that

$$Pr_{j_1,...,j_N}(\hat\lambda) = \omega_{\nu_Q} \circ ... \circ \omega_{\nu_1}(\hat\lambda) \tag{5.1}$$

Let us set $\tilde\sigma_1 = \sigma_R$, $\tilde c = \alpha_R$, $\tilde h(1) = h(R)$, and let us define the multi-time control string

$$\xi = \Big(1, \tilde\sigma_1, \tilde h, \tilde c)\Big) \ ,$$

so, in particular, $\lambda = \hat\lambda \star \xi$. Hence, in view of iii) of the previous lemma, one has

$$Pr_{j_1,...,j_N}(\lambda) = Pr_{j_1,...,j_N}(\hat\lambda \star \xi) = Pr_{j_1,...,j_N}(Pr_{j_1,...,j_N}(\hat\lambda) \star \xi). \tag{5.2}$$

Since $Pr_{j_1,...,j_N}(\hat\lambda) \in \mathcal{C}_{j_1,...,j_N}$, it is trivial to verify that there exists $P \in \{1, ..., R\}$ such that

$$Pr_{j_1,...,j_N}\Big(Pr_{j_1,...,j_N}(\hat\lambda) \star \xi\Big) = \omega_{R-P} \circ \omega_{R-P-1} ... \circ \omega_R \Big(Pr_{j_1,...,j_N}(\hat\lambda) \star \xi\Big). \tag{5.3}$$

(When $P = R$ we mean that the right-hand side coincides with $\Big(Pr_{j_1,...,j_N}(\hat\lambda) \star \xi)$.) Since by (5.1) one has

$$Pr_{j_1,...,j_N}(\hat\lambda) \star \xi = \omega_{\nu_Q} \circ ... \circ \omega_{\nu_1}(\hat\lambda \star \xi) = \omega_{\nu_Q} \circ ... \circ \omega_{\nu_1}(\lambda), \tag{5.4}$$

by (5.2)-(5.4) we obtain

$$Pr_{j_1,...,j_N}(\lambda) = \omega_{R-P} \circ ... \circ \omega_R \circ \omega_{\nu_Q} \circ ... \circ \omega_{\nu_1}(\lambda) \ ,$$

so the theorem is proved. □

COROLLARY 5.19. *Let $(j_1, ..., j_N)$ be any permutation of $(1, ..., N)$ and let $\lambda, \tilde\lambda$ be multi-time control strings. Then the following conditions are equivalent:*

i) $\lambda \bowtie \tilde\lambda$;
ii) $Pr_{j_1,...,j_N}(\lambda) = Pr_{j_1,...,j_N}(\tilde\lambda)$

COROLLARY 5.20. *Let us assume Hypothesis* $(\mathbf{ccZLB})_{a.e.}$ *(or, equivalently,* $(\mathbf{ccZLB})_{set}$*). Let $\lambda \in \mathcal{C}$ and set $\tilde\lambda = Pr_{j_1,...,j_N}(\lambda)$, $S = S_\lambda (= S_{\tilde\lambda})$ Then, for every $y \in \mathbb{R}^n$,*

$$x[y, \lambda](S) = x[y, \tilde\lambda](S)$$

PROOF. By Theorem 5.18 $\tilde\lambda$ is a time-reordering of λ. Hence the result follows from Theorem 5.14 □

6. Proof of Theorem 2.11

Thanks to the results of the Sections 4 and 5 we are now in the condition of proving Theorem 2.11, which is recalled below for the reader's convenience:

Theorem 2.11. *The following statements are equivalent:*

i) *the flows of the control vector fields $f_1, ..., f_N$ commute;*

ii) *if* $t \leq \tilde{t}$ *are multi-times* , $\mathbf{a}$ *is a* N*-control defined on* $[t, \tilde{t}]$*, and* $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$*,* $\hat{\mathbf{t}} : [\hat{S}_\alpha, \hat{S}_\omega] \to \mathbb{R}^N$ *are simple multi-time paths connecting* t *with* $\tilde{t}$*, then*

$$x_{(\mathbf{a},\mathbf{t})}[y](S_\omega) = x_{(\mathbf{a},\hat{\mathbf{t}})}[y](\hat{S}_\omega);$$

iii) *if* $t \leq \tilde{t}$ *are multi-times* , $\mathbf{a}$ *is a* N*-control defined on* $[t, \tilde{t}]$*, and* $\mathbf{t} : [S_\alpha, S_\omega] \to \mathbb{R}^N$*,* $\hat{\mathbf{t}} : [\hat{S}_\alpha, \hat{S}_\omega] \to \mathbb{R}^N$ *are multi-time paths connecting* t *with* $\tilde{t}$*, then*

$$x_{(\mathbf{a},\mathbf{t})}[y](S_\omega) = x_{(\mathbf{a},\hat{\mathbf{t}})}[y](\hat{S}_\omega).;$$

iv) *the vector fields* $f_1, ..., f_N$ *verify condition* $\mathbf{(ccZLB)}_{ord}$ *;*
v) *the vector fields* $f_1, ..., f_N$ *verify condition* $\mathbf{(ccZLB)}_{a.e.}$ *;*
vi) *the vector fields* $f_1, ..., f_N$ *verify condition* $\mathbf{(ccZLB)}_{set}$*.*

Proof. Let us begin by observing that the equivalence of *iv)*, *v)*, and *vi)* is a straightforward consequence of the equivalence of *ii)*, *iii)*, and *iv)* in Theorem 2.6. By the same theorem, considering the case when the controls in Definition 2.2 are constant, we obtain that *i)* implies *iv)*, *v)*, and *vi)*. The implication *ii)* $\Rightarrow$ *i)* is trivial, in that the products of exponentials showing up in Definition 2.2 are nothing but particular case of $\mathbf{a}$-lifts of simple multi-time paths. The implication *iii)* $\Rightarrow$ *ii)* is trivial as well, for in fact *ii)* is just a particular case of *iii)*.

Therefore it is sufficient to prove that *iv)* implies *iii)*. In view of Lemma 4.1 we can conveniently choose the parameterization of each multi-time path. In particular we can assume that both $\mathbf{t}$ and $\hat{\mathbf{t}}$ are parameterized on the interval $[0, 1]$.

Let us begin by considering the case when the multi-times paths $\mathbf{t}$ and $\tilde{\mathbf{t}}$ are *simple* and the $\mathbf{a}_i$ are *piecewise constant.*

Since, without loss of generality, we can assume that both $\mathbf{t}$ and $\hat{\mathbf{t}}$ are $\mathbf{a}$-fit (see Remark 5.7) , let us consider the multi-time control strings $\lambda^{(\mathbf{a},\mathbf{t})}, \lambda^{(\mathbf{a},\hat{\mathbf{t}})}$.

It is easy to check that there exist $\tilde{\lambda}^{(\mathbf{a},\mathbf{t})} \simeq \lambda^{(\mathbf{a},\mathbf{t})}$ and $\tilde{\lambda}^{(\mathbf{a},\hat{\mathbf{t}})} \simeq \lambda^{(\mathbf{a},\hat{\mathbf{t}})}$ such that, for every $i = 1, ..., N$,

$$Pr_i(\tilde{\lambda}^{(\mathbf{a},\mathbf{t})}) = Pr_i(\tilde{\lambda}^{(\mathbf{a},\hat{\mathbf{t}})})$$

Hence

$$Pr_{1,...,N}(\lambda^{(\mathbf{a},\mathbf{t})}) = Pr_{1,...,N}(\lambda^{(\mathbf{a},\hat{\mathbf{t}})})$$

Moreover, by Corollary 5.19,

$$\tilde{\lambda}^{(\mathbf{a},\mathbf{t})} \bowtie \tilde{\lambda}^{(\mathbf{a},\hat{\mathbf{t}})}$$

so, by Proposition 5.10, Theorem 5.14, and Lemma 5.6, we obtain

$$x_{(\mathbf{a},\mathbf{t})}[y](1) = x[\lambda^{(\mathbf{a},\mathbf{t})}, y](1) = x[\tilde{\lambda}^{(\mathbf{a},\mathbf{t})}, y](1) = x[\tilde{\lambda}^{(\mathbf{a},\hat{\mathbf{t}})}, y](1) = x[\lambda^{(\mathbf{a},\hat{\mathbf{t}})}, y](1) = x_{(\mathbf{a},\hat{\mathbf{t}})}[y](1)$$

from which we get the thesis (in the considered particular case).

In order to conclude the proof we shall exploit a density argument. For every $i = 1, ..., N$, let $(\mathbf{a}_i^k)_{k\in\mathbb{N}}$ be a sequence of piece-wise constant controls such that

$$\lim_{k\to\infty} \|\mathbf{a}_i^k - \mathbf{a}_i\|_1 = 0$$

Moreover, let $(\mathbf{t}^k)_{k\in\mathbb{N}}$ and $(\hat{\mathbf{t}}^k)_{k\in\mathbb{N}}$ be sequences of simple multi-time paths (parameterized on $[0,1]$) connecting t with $\tilde{t}$ such that

$$\lim_{k\to\infty} \|\mathbf{t}^k - \mathbf{t}\|_\infty = 0 \qquad \lim_{k\to\infty} \|\tilde{\mathbf{t}}^k - \tilde{\mathbf{t}}\|_\infty = 0$$

The existence of such sequences is guaranteed by Lemma 6.1 below. Then, by the first part of the proof and in view of Theorem 4.3, one has

$$x_{(\mathbf{a},\mathbf{t})}[y](1) = \lim_{k\to\infty} x_{(\mathbf{a}^k,\mathbf{t}^k)}[y](1) = \lim_{k\to\infty} x_{(\mathbf{a}^k,\hat{\mathbf{t}}^k)}[y](1) = x_{(\mathbf{a},\hat{\mathbf{t}})}[y](1)$$

which concludes the proof. $\square$

Lemma 6.1. *Let $t,\tilde{t}\in\mathbb{R}^N$ be such that $t\le\tilde{t}$, and let $\mathbf{t}:[0,1]\to\mathbb{R}^N$ be a multi-time connecting t with $\tilde{t}$. Then there exists a sequence $\mathbf{t}^k : [0,1]\to\mathbb{R}^N$), $k\in N$ of simple multi-time paths connecting t with $\tilde{t}$ and such that*

$$\lim_{k\to\infty} \|\mathbf{t}^k - \mathbf{t}\|_\infty = 0.$$

Proof. The approximating simple multi-time path $\mathbf{t}^k$ will be obtained from $\mathbf{t}$ as follows: i) the interval $[0,1]$ is partitioned in k subinterval of equal length; ii) for every $j=1,...,k$ one replaces $\mathbf{t}$ on the j-th subinterval with the path (whose derivative has modulus equal to 1 and) whose image is given by the union of the N segments connecting, respectively, the N pairs of multi-times

$\left(\mathbf{t}(\frac{(j-1)}{k})\ ,\ \mathbf{t}(\frac{(j-1)}{k}) + \left(\mathbf{t}_1(\frac{j}{k}) - \mathbf{t}_1(\frac{(j-1)}{k})\right)\frac{\partial}{\partial x_1}\right)$

$\left(\mathbf{t}(\frac{(j-1)}{k}) + \left(\mathbf{t}_1(\frac{j}{k}) - \mathbf{t}_1(\frac{(j-1)}{k})\right)\frac{\partial}{\partial x_1}\ ,\ \mathbf{t}(\frac{(j-1)}{k}) + \sum_{r=1,2}\left(\mathbf{t}_r(\frac{j}{k}) - \mathbf{t}_r(\frac{(j-1)}{k})\right)\frac{\partial}{\partial x_r}\right)$

....

....

$\left(\mathbf{t}(\frac{(j-1)}{k}) + \sum_{r=1}^{N-1}\left(\mathbf{t}_r(\frac{j}{k}) - \mathbf{t}_r(\frac{(j-1)}{k})\right)\frac{\partial}{\partial x_r}\ ,\ \mathbf{t}(\frac{j}{k})\right)$

More precisely, let $\mathbf{t}_1,...,\mathbf{t}_N$ be the components of $\mathbf{t}$, and, for any positive integer k, every $j=1,...,k$, and every $h=2...,N$, let us set

$$I^k_{j1} = \Big[\frac{(j-1)}{k}, \frac{(j-1)}{k} + \left(\mathbf{t}_1(\frac{j}{k}) - \mathbf{t}_1(\frac{(j-1)}{k})\right)\Big],$$

$$I^k_{jh} = \Big[\frac{(j-1)}{k} + \sum_{r=1}^{h-1}\left(\mathbf{t}_r(\frac{j}{k}) - \mathbf{t}_r(\frac{(j-1)}{k})\right), \frac{(j-1)}{k} + \sum_{r=1}^{h}\left(\mathbf{t}_r(\frac{j}{k}) - \mathbf{t}_r(\frac{(j-1)}{k})\right)\Big].$$

Let us observe that for every k one has[13]

$$I^k_{11} \le I^k_{12} \le ... \le I^k_{1N} \le I^k_{21} \le ... \le I^k_{2N} \le ... \le I^k_{k1} \le .... \le I^k_{kN}$$

and

$$\cup_{j=1,...,k}(\cup_{h=1,...,N} I^k_{jh}) = [0,S]$$

For every integer $k>0$ let us define the simple multi-time path $\mathbf{t}^k$ by setting

$$\mathbf{t}^k(s) = \sum_{j=1}^{k}\sum_{1}^{N}\int_o^s \chi_{I^k_{jh}}(\sigma)\frac{\partial}{\partial t_h}\,d\sigma.$$

It is easy to verify the following two properties:

[13] where $I\le J$ means that $a\le b$ as soon as $a\in I$ and $b\in J$

a) all the curves $\mathbf{t}^k, \hat{\mathbf{t}}^k$ are *simple* multi-time paths defined on $[0,1]$ and verify $(|\frac{d\mathbf{t}^k}{ds}|_1 = |\frac{d\hat{\mathbf{t}}^k}{ds}|_1 = 1$ and) $\mathbf{t}^k(0) = \hat{\mathbf{t}}^k(0) = \mathbf{t}(0) = \hat{\mathbf{t}}(0), \quad \mathbf{t}^k(1) = \hat{\mathbf{t}}^k(1) = \mathbf{t}(L) = \hat{\mathbf{t}}(1)$;

b) the $\mathbf{t}^k$ converge uniformly to $\mathbf{t}$.

The proof is concluded. □

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Acknowledgments. The author wishes to thank Nicola Fusco and Monica Motta for some useful technical advices.

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