On the Control of Mechanical systems: *inverse pendulums, Roller Racers, non-euclidean geometries, non linear impulsive equations*

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February 16, 2012

Main references:

- *Moving Constraints as Stabilizing Controls in Classical Mechanics* (with A. Bressan), Arch. Rational Mech. Anal. (2010) (available on my web page)

-Tutorial *Control of Non Holonomic Systems by Active Constraints* SADCO Summer School Imperial College, London September 5-9, 2011 (available on my web page)

Before speaking of analysis, control, and geometry,

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let us begin with

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SOME EXAMPLES of MECHANICAL SYSTEMS:

THREE $\underline{\mathcal{C}\ \ }$ EXAMPLES:

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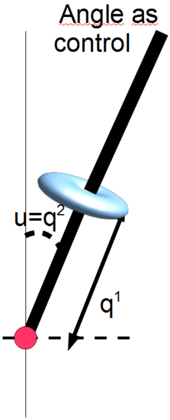
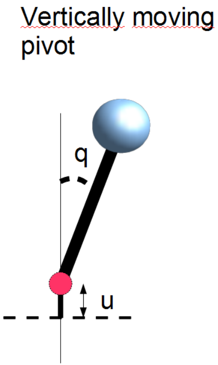
($\mathcal{C}$ stands for *centrifugal*)

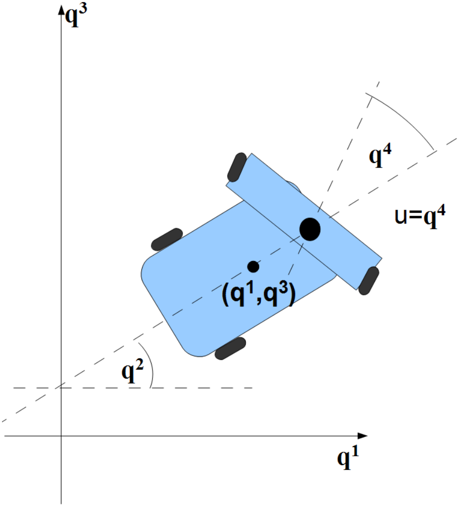
$(1\mathcal{C})$ The angle as control



The control is the angle.

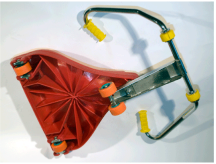
(2$\mathcal{C}$) The pendulum with oscillating pivot





q³
q⁴
u=q⁴
(q¹,q³)
q²
q¹

(The Roller Racer is a well-known toy):



what is shared **by these mechanical systems?**

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NOW

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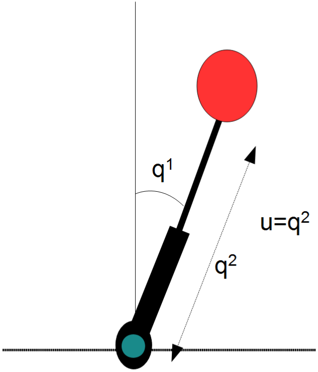
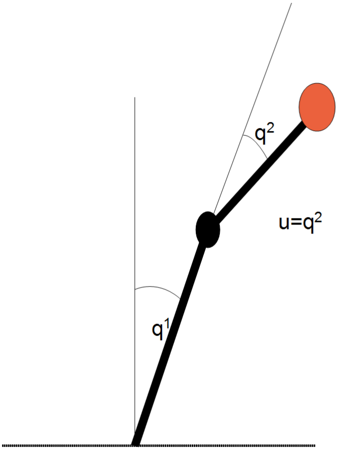


Figure: Length as control

$(2\mathcal{NC})$ The pendulum with a second pendulum as control



OBSERVE: in the former two examples the control can be thought as "shape" of the whole system.

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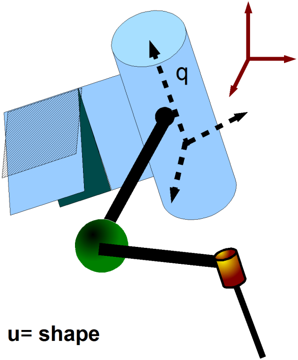
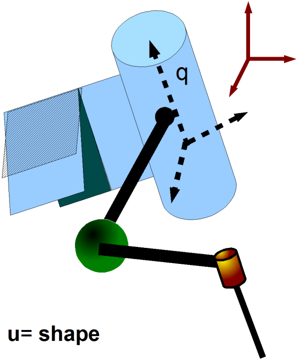
More generally:

$(3\mathcal{NC})$"Shape" as control



u= shape

q= cylinder's position



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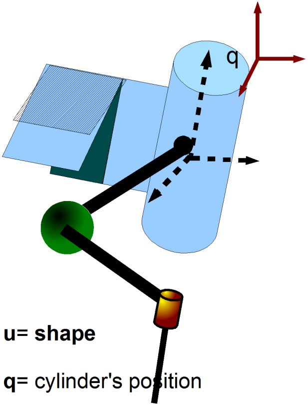
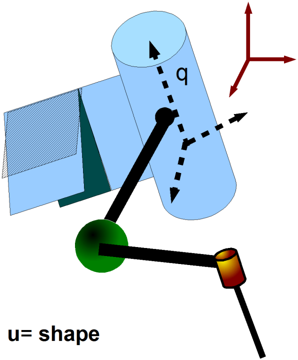
A rigid movement



A rigid movement (the shape u is unchanged)



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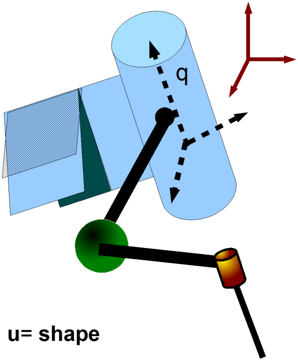

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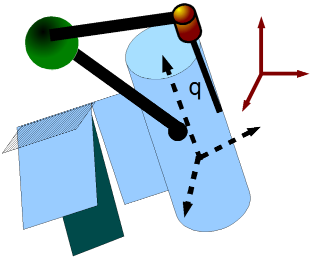
Change of shape



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Change of shape (i.e. change of u)

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by the last three mechanical systems?

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THE GENERAL QUESTION. Investigate **analysis** and **geometry** related to the following program:

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PROBLEM:

WHAT CAN BE SAID ON THE *whole* MOTION $\mathbf{q} = \mathbf{q}(\mathbf{t})$?

Standard goals:

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Optimization

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Optimization
Controllability

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Optimization
Controllability
Stabilizability

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Controllability Somehow related to both Optimization and Stabilizability.

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Vibrational Stabilizability

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(Which is a local way of imposing **moving constraints as controls.**)

The abstract framework:

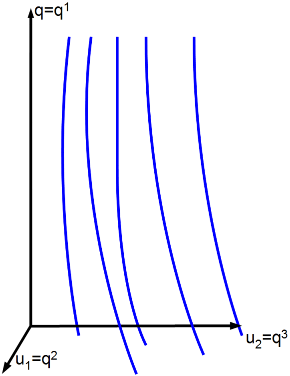

Figure: The foliation $\{u = cost\}$

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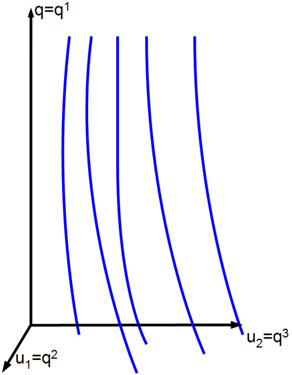
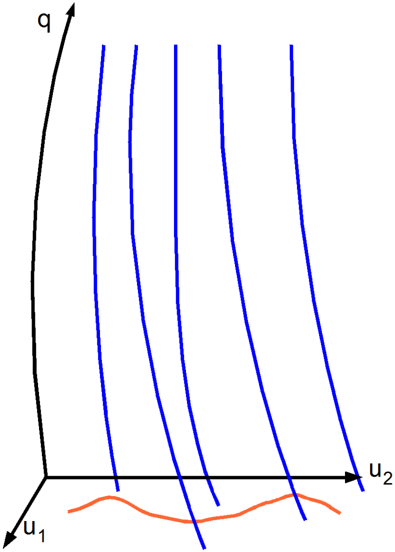


Figure: The foliation $\{u = cost\}$

Given the *projection* **control** $u = u(t)\Big(= (q^2, q^3)(t)\Big)$ **, we aim to analyze the whole motion** $q(t)\Big(= (q^1, q^2, q^3)(t)\Big)$

2-dimensional controls, 1-dimensional leaves



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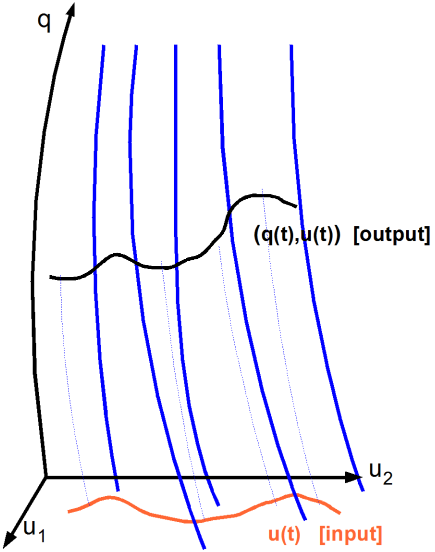

Rapidly oscillating controls. E.g. $u = \epsilon \sin(t/\epsilon)\bar{\mathbf{u}}$

Let us consider

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Which of the previous examples do the job?
The $\mathcal{C}$(=centrifugal) systems
or
the $\mathcal{NC}$ (=non-centrifugal) systems?

(1$\mathcal{C}$) The angle as control

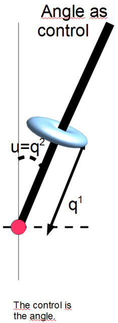

Figure: Oscillations of the angle *do generate* a (centrifugal!) force on the sliding ring

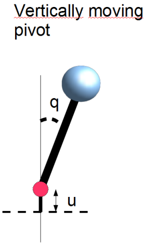

Figure: Oscillations of the pivot *do stabilize* the unstable equilibrium (*Kapiza pendulum*). This means that they *do generate* forces

(3$\mathcal{C}$) The "Roller Racer"

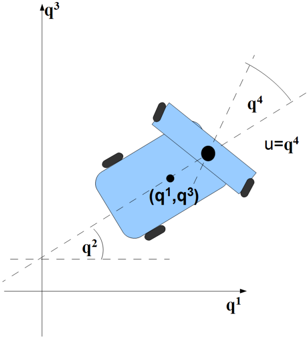

Figure: Oscillations of handlebar generate forward motion

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OSCILLATIONS <u>ARE</u> FORCE-GENERATING.

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- Example 3 (the Roller Racer) is different from the previous ones: the "centrifugal effect" is due to interaction between Riemannian structure and the imposed non-holonomic constraint.

(In both case by "Riemannian structure" we mean the one the configuration manifold inherits from the Kinetic Energy)

So in a *centrifugal* system oscillations generate forces...

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QUESTION: *Is it true that in $\mathcal{NC}$ (=non-centrifugal) systems* "OSCILLATIONS DO NOT GENERATE FORCES"?

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Let us give one more look to what *we have called* $\mathcal{NC}$(=non-centrifugal) systems:

$(1\mathcal{NC})$ The pendulum with length as control

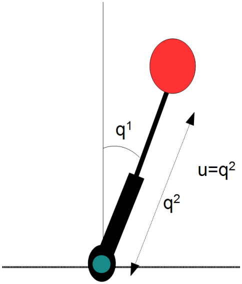


Figure: In fact: it is almost insensitive to small oscillation of length

(2$\mathcal{NC}$) The pendulum with a second pendulum as control

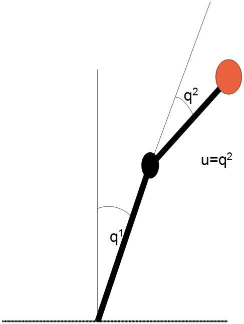
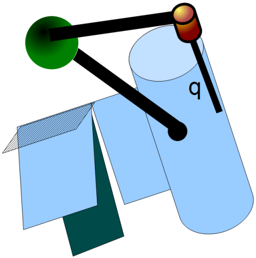


Figure: Again: the first pendulum almost insensitive to small oscillation of the second pendulum

(3$\mathcal{NC}$)"Shape" as control



u= shape

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Change of shape (i.e. change of u)

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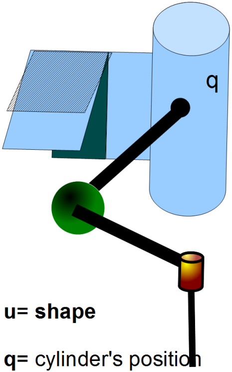

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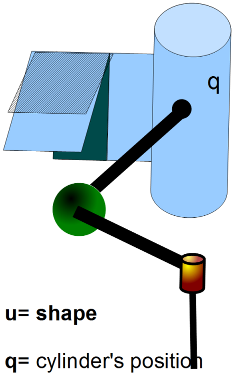

Also in this case,

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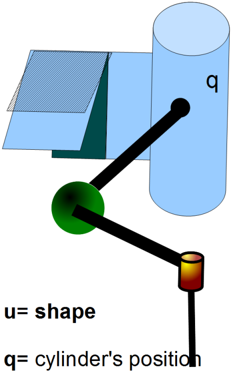

Also in this case, the body is almost insensitive to small rapid oscillation of the control (=the shape)

So,

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QUESTION:

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QUESTION: *Is there some crucial geometric-analytical reason for this behavior's discrepancy between these two classes (centrifugal and non centrifugal) of systems?*

Let us forget mechanics for a while

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and let us consider

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and let us consider
a **particular class of control systems:**

$$\dot{x} = f(x,u) + \sum_{\alpha=1}^{m} g_\alpha(x,u)\dot{u}_\alpha + \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x,u)\dot{u}_\alpha\dot{u}_\beta$$

ODE's depending quadratically on the control derivative $\dot{u}$

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$$x \doteq (x,u),\ \ f \doteq (f,0),\ \ g_\alpha \doteq (g_\alpha, \mathbf{e}_\alpha), h_{\alpha,\beta} \doteq (h_{\alpha,\beta}, 0) \quad)$$

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In view of mechanical applications, we distinguish between:

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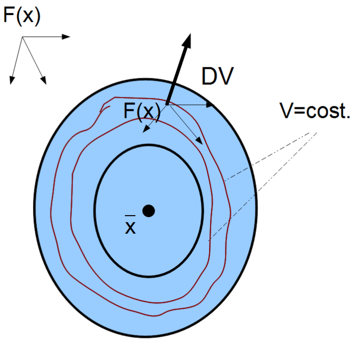

Figure: $\inf_{y \in \mathcal{F}(x)} DV(x) \cdot y \leq 0$

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(**MORAL**: **Rapid oscillations of u cancel out the $\dot{u}$-linear term**.)

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which has *bounded values*

BACK TO MECHANICS

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In particular $\mathcal{H}$ is quadratic in the momenta (p,π)

$$\mathcal{H} = (p,\pi)\mathbf{A}^{-1}(p,\pi)^t \qquad \mathbf{A}^{-1} = (a^{r,s})_{r,s=1,\dots,N+M}$$

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with suitable vector fields $f(q,p,u)$, $g_\alpha(q,p,u)$, $h_{\alpha,\beta}(q,p,u)$ determined by the Kinetic Energy and the applied forces.

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Remark:

The quadratic part $h_{\alpha,\beta}\dot{u}_\alpha\dot{u}_\beta$ is zero

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Stabilizability of mechanical systems

Let us assume that the force F acting on the system is conservative, and let $U = U(q, u)$ be a potential of F .

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(Observe incidentally: $\bar{q}$ might well be an unstable equilibrium.)

- **Idea of the proof:** the choice of the λ_r, w_r selects from the corresponding symmetrized differential inclusion $\dot{x} \in \mathcal{G}$ a conservative mechanical system with potential energy equal to $\widehat{U} = U(\bar{u}, q) - \frac{1}{2}\sum_{r=1}^{k} \lambda_r \sum_{\alpha,\beta=1}^{m} e_{\alpha,\beta}(q, \bar{u}) w_r^{\alpha} w_r^{\beta}$

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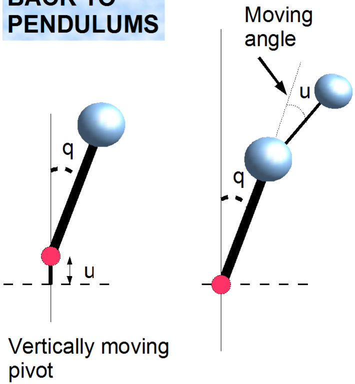
OBSERVE :If $\bar{q}$ is unstable for the frozen control $u = \bar{u}$, a **necessary condition for making $\bar{x}$ stable with rapid oscillations of u** is that
the matrix $e_{\alpha,\beta}$ be q-dependent.
(That is, the quadratic term $h_{\alpha,\beta}$ does not vanish.)

BACK TO PENDULUMS



Is the state q=0 stabilizable by a vibrational u ?

Pendulum with moving pivot

$$\begin{cases} \dot{q} = p + (\sin q)\dot{u} \\ \\ \dot{p} = -\frac{\partial U}{\partial q} - p\cos q\,\dot{u} - (\sin q\cos q)(\dot{u})^2\,, \end{cases}$$

where $U(q,c) \doteq \cos q$.

The kinetic matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -\sin q \\ \\ -\sin q & 2 \end{pmatrix}$$

...so that the 1×1 **matrix** E **is**

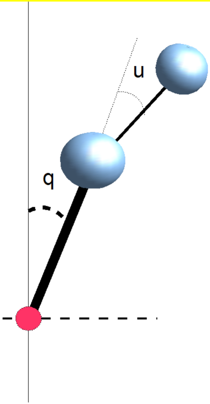
$$E = e_{11} = 1 + \cos^2 q$$

Notice: The necessary condition (**i.e. "E depends on q"**) is verified; Moreover: the **effective potential**

$$U_{\{1\}\{w\}} = \cos q - \frac{1}{2}(1 + (\cos q)^2)w^2.$$

has a minimum at $q = 0$ **as soon as** $w^2 \geq 1$. **Therefore: The system is vibrationally stabilizable at** 0**.**

Why NO stabilizability?



In order to give the answer one needs not even know the equation of motion...

Indeed the kinetic matrix is

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$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{\alpha=\mathbf{1}}^{\mathbf{m}} \mathbf{g}_\alpha(\mathbf{x})\dot{\mathbf{u}}^\alpha$$

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this "explains" the non-stabilizability of the double pendulum

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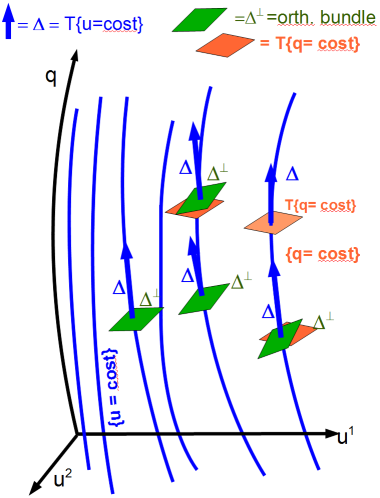
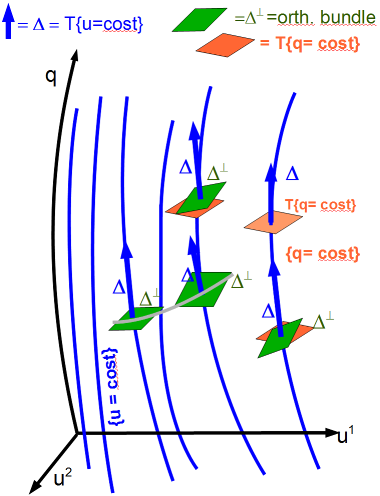
(3) If $\{u = c_1\}$, $\{u = c_2\}$ are leaves, then the "DISTANCE" from the points of $\{u = c_1\}$ to $\{u = c_2\}$ is CONSTANT


= Δ = T{u=cost}
=Δ⊥ =orth. bundle
= T{q= cost}
q
Δ
Δ⊥
T{q= cost}
{q= cost}
{u = cost}
u1
u2



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The "orthogonal curvature"

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}$$

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MEASURES

The "orthogonal curvature"

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MEASURES **how much geodesics which are orthogonal to a leaf $\{u = constant\}$ at a point**

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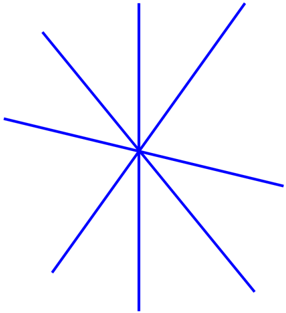
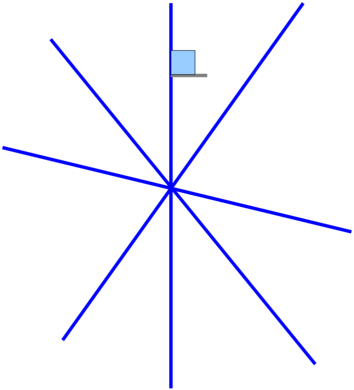
The orthogonal curvature $\frac{\partial e_{\alpha,\beta}}{\partial q^i}$ is a *tensor* with respect to the coordinate transformations that respect the foliation structure.

EXAMPLES



Non-Riemannian Foliation: geodesics lose orthogonality

The angle as control:



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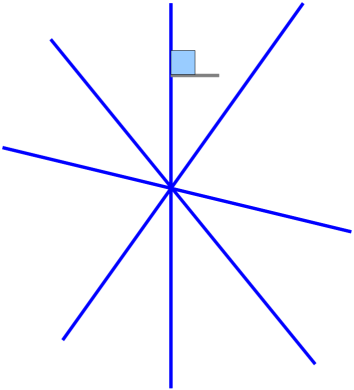

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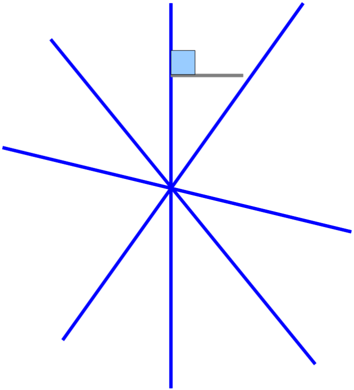

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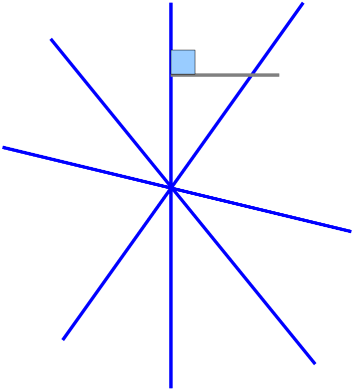

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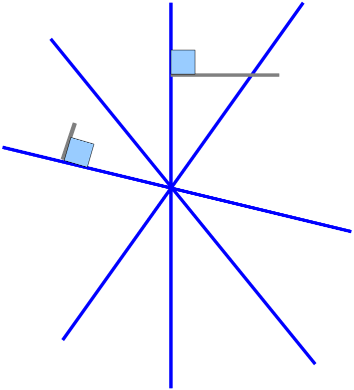

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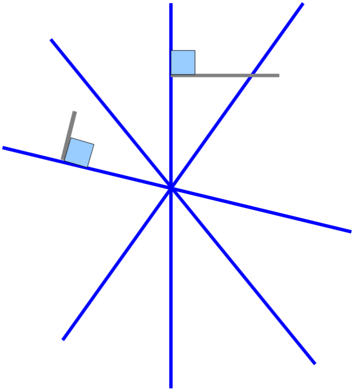

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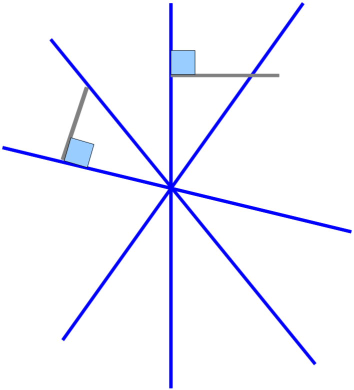

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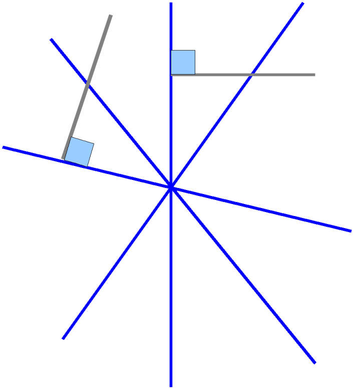

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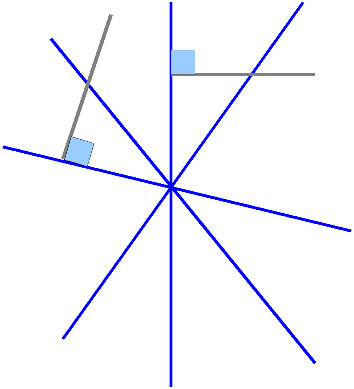

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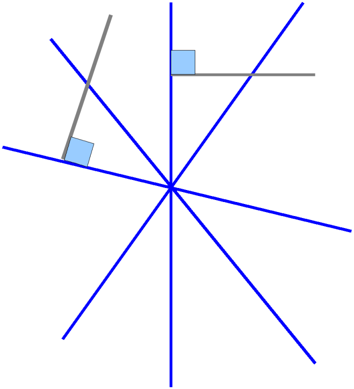

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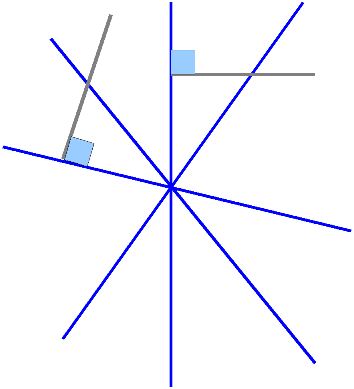

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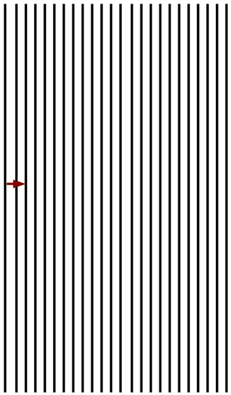
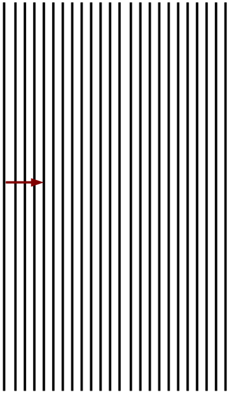
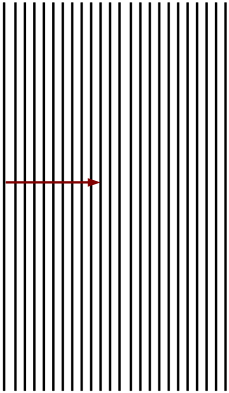
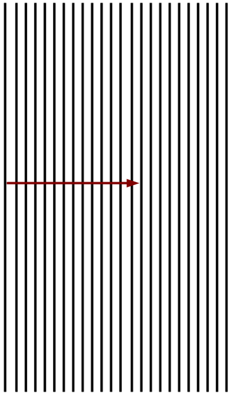
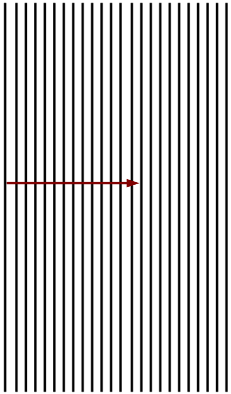
Hence, chance of vibrational stabilization!











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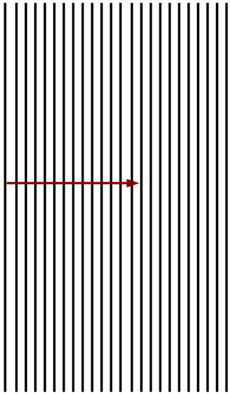

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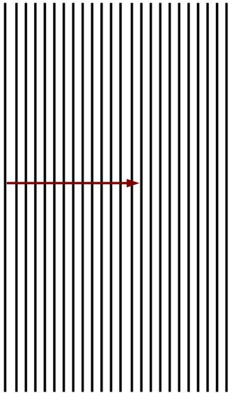

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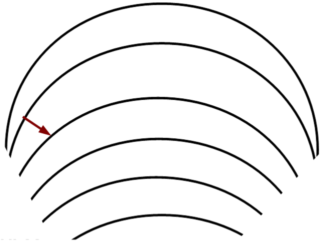
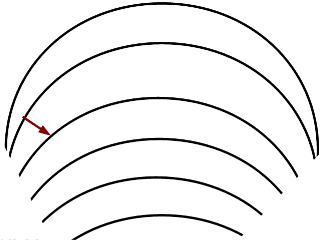


Pendulum with oscillating pivot



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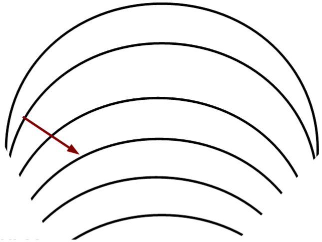

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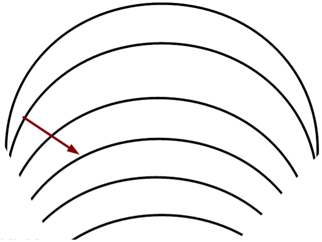

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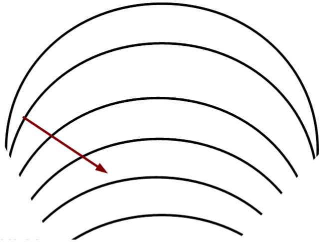

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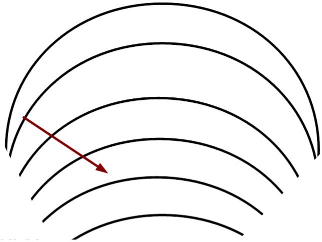

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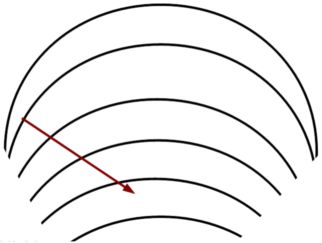

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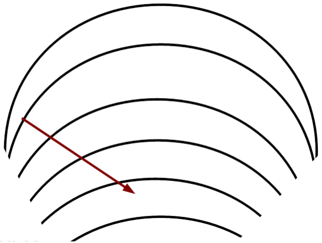

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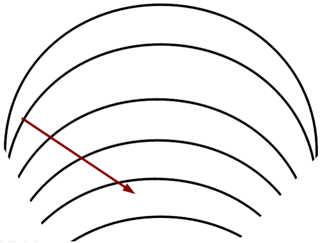

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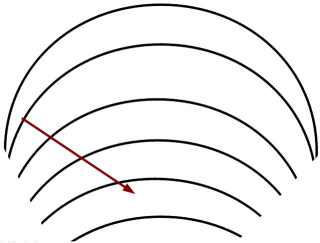

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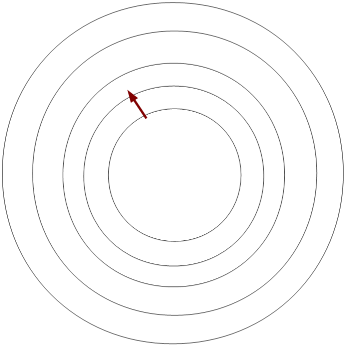
The length of the pendulum as control



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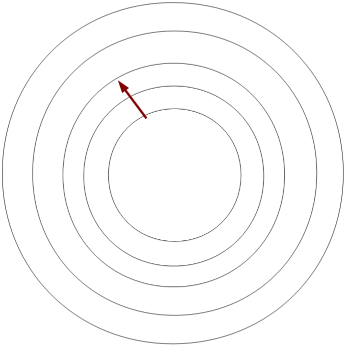

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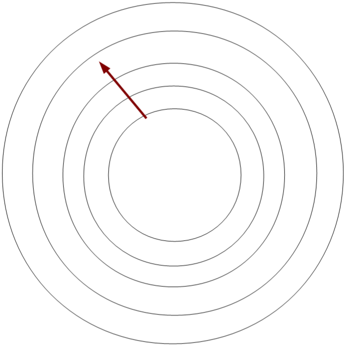

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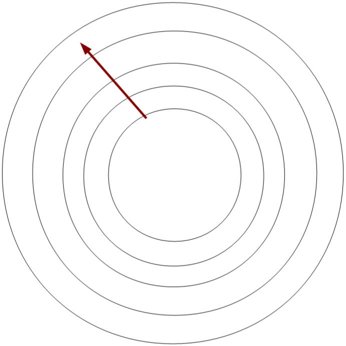

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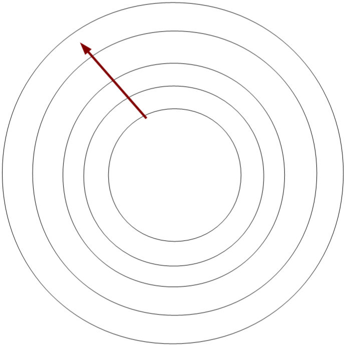

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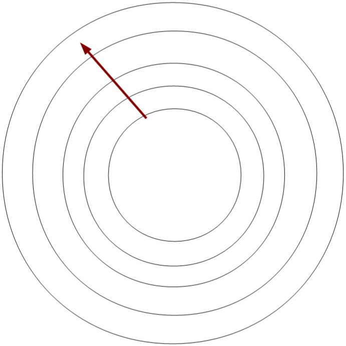

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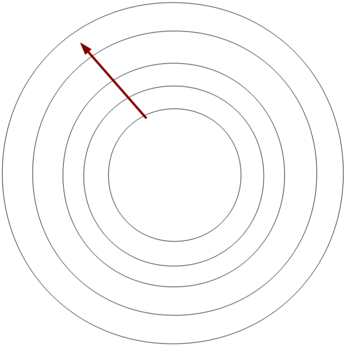

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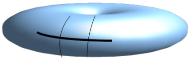

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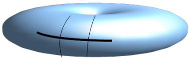

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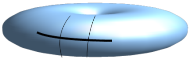

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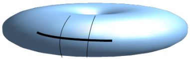

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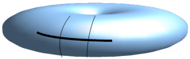

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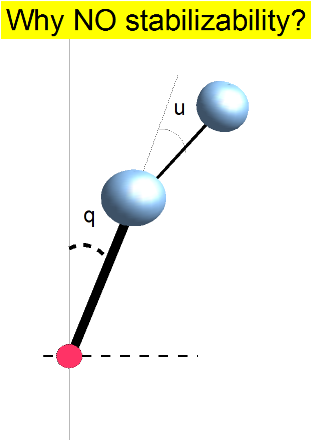
Remember?



What about the **Roller Racer?**

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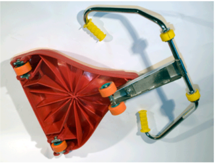
UP TO NOW ALL SYSTEMS WERE HOLONOMIC...



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A *non holonomic constraint* *is a (linear) constraint on the velocity* $\dot{q}$

$$\omega_1(\dot{q}) = 0 \dots \omega_\nu(\dot{q}) = 0$$

which cannot be deduced by differentiation of a constraint $\phi(q) = 0$ **on the configuration** q**.**

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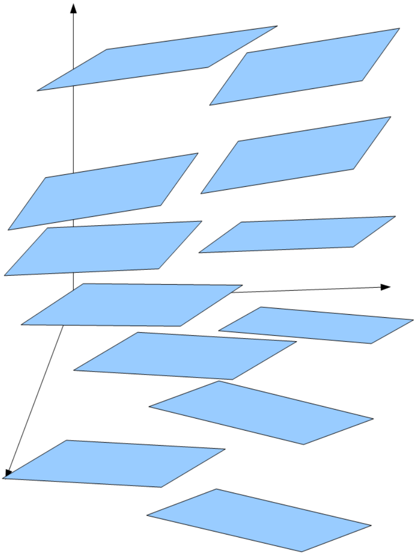
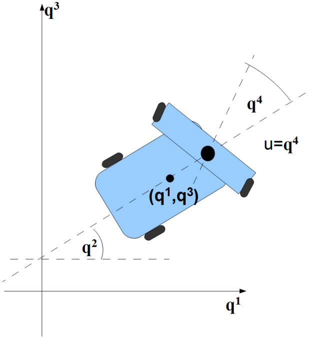
i.e., there is no *foliation* of $\mathcal{Q}$ whose leaves have $\Delta(q)$ as tangent space at any q.

EXAMPLE in $\mathcal{R}^3$



The Roller Racer



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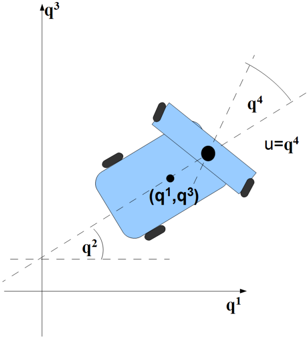
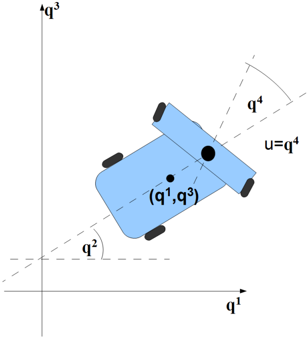
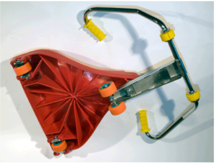


The fact that the velocity of the first body must be directed as the angle q^2 and the analog fact for the velocity of the second body,



The fact that the velocity of the first body must be directed as the angle q^2 and the analog fact for the velocity of the second body, is **non holonomic constraint** on the system.



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One could conjecture that nothing new happens by adding non holonomic constraints (the wheels)

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So one could conjecture that

"the system is affine in $\dot{u}$
if and only if
this was true before the imposition of the non holonomic constraint."

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One could say the non holonomic constraint produces a *"centrifugal" effect*

The control equations for the Roller Racer (on a surface with friction):

$$\begin{cases} \dot{q}^1 &= 2\rho\cos u \sin q^2 \cdot \xi - \frac{J\rho \sin q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{q}^2 &= 2\sin u \cdot \xi - \frac{J\sin^2 u}{\Delta_0} \cdot \dot{u} \\ \dot{q}^3 &= 2\rho\cos q^2 \cos u \cdot \xi - \frac{J\rho \cos q^2 \sin 2u}{2\Delta_0} \cdot \dot{u} \\ \dot{\xi} &= -\sin 2u \left(\frac{(I+J-\rho^2)}{\Delta_1} + \frac{1}{2(\rho^2/\Delta_4 + \sin^2 u)} \right) \cdot \xi \dot{u} - \frac{2J\rho^2 \cos u}{\Delta_1^2} \cdot \dot{u}^2. \end{cases}$$

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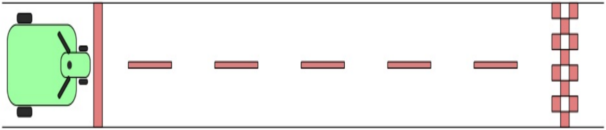
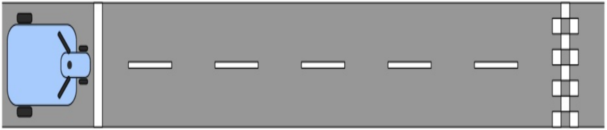
RollerRacers' race . . .





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GRAZIE

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PER L'ATTENZIONE