Nonsmooth Multi-time Hamilton-Jacobi Systems

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ABSTRACT. We establish existence of a solution for systems of Hamilton-Jacobi equations of the form (1.1). A previous result—see [3]—valid for C^1 Hamiltonians is here extended to the case where Hamiltonians are locally Lipschitz continuous. The main tool for dealing with this kind of non-smoothness consists in the interpretation of the existence issue in terms of commutativity of the minimum problems originating the Hamiltonians involved in (1.1). In turn, a sufficient condition for such commutativity is based on a notion of Lie bracket for nonsmooth vector-fields introduced in [20]. Besides existence, we establish uniqueness—actually, a comparison result—, regularity, and four different representations of the solution. Moreover, we prove a front-propagation property in the vector-valued time $(t_1, \dots, t_N)$. The paper also contains results concerning semigroup properties of the solution and the additivity of a suitable defined exponential map.

1. INTRODUCTION

1.1. The problem In this paper we investigate existence and uniqueness of a solution to the so-called *multi-time systems of Hamilton-Jacobi equations*. For a given $T > 0$, these systems have the form

$$\begin{cases} \dfrac{\partial u}{\partial t_1} + H_1(x, D_x u) = 0, \\ \cdots \\ \dfrac{\partial u}{\partial t_N} + H_N(x, D_x u) = 0, \end{cases} \tag{1.1}$$

$(t_1, \dots, t_N, x) \in \,]0, T[^N \times \mathbb{R}^n$, and are associated with an *initial condition*

$$u(0, \dots, 0, x) = \psi(x) \quad \forall\, x \in \mathbb{R}^n. \tag{1.2}$$

By a solution of (1.1) *we mean a map* $u : [0,T]^N \times \mathbb{R}^n \to \mathbb{R}$ *verifying each of the equations in* (1.1) *as a viscosity solution in* $]0,T[^N \times \mathbb{R}^n$—see Definition 2.1. In particular, we are given a vector-valued time variable $t = (t_1,\dots,t_N)$—the so-called *multi-time* variable—, whose dimension coincides with the number of equations of (1.1). If, for a given $i = 1, \dots, N$, we regard the variables t_j such that $j \neq i$ as parameters, the i-th equation is a standard Hamilton-Jacobi equation in the time-variable t_i.

An obvious starting observation is that, as soon as $N > 1$, (1.1) is an *overdetermined system*, namely we have more equations than the dimension (= 1) of the solution's range. It is then natural to expect that in the general case a solution fails to exist. Hence, it is reasonable to look for sufficient conditions on the data in order that a solution does exist (in a sense to be made precise).

System (1.1) can be regarded as a possible nonlinear generalization of overdetermined linear systems, which have been widely investigated both for their importance in physical applications and because of (and thanks to) their differential geometric content—see e.g.[16].

An application in Economics has been proposed by Rochet in [22]. In that paper the solution of (1.1)–(1.2) has the meaning of a *Benefit Function* for a monopolist who wishes to optimize a selling strategy to a population of retailers. In this case the t_i's are parameters affecting the costs of the retailers.

More abstractly, as soon as the Hamiltonians are of the (control-theoretical) form $H_i(x,p) = \sup_{a\in A}\{-\langle p, f_i(x,p)\rangle - \ell_i(x,p)\}$ one can consider a situation where N control systems—the i-th one being characterized by the dynamics-Lagrangian pair (f_i,ℓ_i)—are evolving in *their own times* t_i. A potential application of the results on (1.1)–(1.2) could concern *switching systems*, where the existence of the solution to (1.1)–(1.2) would mean a sort of invariance of the output with respect to the order of implementation of the optimal switchings.

1.2. The case with x-independent Hamiltonians The first contribution explicitly devoted to systems of Hamilton-Jacobi equations is due to P.L. Lions and J.C. Rochet [15], who investigated the case when the Hamiltonians are state-independent, that is $H_i = H_i(p)$, where p denotes the adjoint variable. As in the case of a single equation, this case has the advantage that solutions can be explicitly computed by means of Hopf or Lax-Oleinik formulas. In particular, the existence of a solution is checked by means of a direct verification of the *commutativity* of semigroups generated by the single equations. In fact, the commutativity issue—which later will be interpreted in the sense of *commutativity of optimal control problems*—is intimately related to the question of existence of solutions. In the case of *state-independent* Hamiltonians investigated by Lions and Rochet it turns out that the semigroups always commute, so no quantitative restrictions are to be imposed on the Hamiltonians in order that a solution exists. This can be intuitively regarded as a consequence of the *Euclidean* structure lying behind the fact that the Hamiltonians are state-independent (so-to-speak, the spatial projections of characteristic lines are straight lines).

Still by means of some Hopf-like formulas, recently S. Plaskacz and M. Quincampoix [17] have extended the investigation to the case when the Hamiltonians depend on times and on the solution as well but are independent of x.

1.3. The general case The major contribution to the *non Euclidean case*—i.e., when the Hamiltonians are state-dependent—is due to G. Barles and A. Tourin [3]. They assume two main conditions, which we label $[\mathcal{BT}]_{\mathrm{smooth}}$ and $[\mathcal{BT}]_{\mathrm{ZPB}}$:

Hypothesis $[\mathcal{BT}]_{\mathrm{smooth}}$ (Smoothness of the Hamiltonians)**.** *The Hamiltonians H_i are of class C^1 and convex in the gradient variable.*

Hypothesis $[\mathcal{BT}]_{\mathrm{ZPB}}$ (Zero Poisson Bracket)**.** *for all $i,\ j = 1, \dots, N$ and all $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$*[1]

$$\{H_i, H_j\}(x, p) = 0$$

is verified, where $\{H_i, H_j\}$ denotes the Poisson bracket of H_i and H_j:

$$\{H_i, H_j\} = D_x H_j D_p H_i - D_x H_i D_p H_j.$$

On one hand, the smoothness condition $[\mathcal{BT}]_{\mathrm{smooth}}$ is needed in order to give a classical sense to the Zero Poisson Bracket condition $[\mathcal{BT}]_{\mathrm{ZPB}}$. On the other hand, Barles and Tourin have proven that under hypotheses $[\mathcal{BT}]_{\mathrm{smooth}}$–$[\mathcal{BT}]_{\mathrm{ZPB}}$ there exists a (unique) solution to (1.1)–(1.2).[2] Let us remark that condition $[\mathcal{BT}]_{\mathrm{ZPB}}$ is in fact sufficient for the commutativity of the semigroups corresponding to the equations forming the system (1.1) (the parameter of the i-th semigroup being the i-th time variable t_i). Namely, if $\psi\mathbf{e}^{-t_i H_i}$ denotes the value at time t_i of the solution to the Cauchy problem corresponding to the i-th equation of (1.1) (see Subsection ***2.1***), $[\mathcal{BT}]_{\mathrm{ZPB}}$ implies that

$$\psi\mathbf{e}^{-t_i H_i}\mathbf{e}^{-t_j H_j} = \psi\mathbf{e}^{-t_j H_j}\mathbf{e}^{-t_i H_i}$$

for all $i,\ j = 1, \dots, N$ and all $t_i,\ t_j \in [0, T]$. Although some special situations involving nonsmooth Hamiltonians are treated in [3] (e.g. when one of two Hamiltonians H_i, H_j is smooth and the other one is just Lipschitz continuous, and condition $[\mathcal{BT}]_{\mathrm{ZPB}}$ is verified almost everywhere)[3] by means of regularization techniques, the more general case of (locally) Lipschitz continuous Hamiltonians seems to be outside the range of this approach. On the other hand, the

[1] To be more precise: $(x, p) \in \mathbb{R}^n \times T_x^*\mathbb{R}^n$.

[2] Due to the adopted proof's strategy, some growth and boundedness hypotheses are also assumed in [3] in order to obtain a Lipschitz continuous solution (see Subsection ***5.1***).

[3] A further case treated in [3] is when the regularized Hamiltonians H_i^n are such that $\{H_i^n, H_j^n\} = 0$ for all $i,\ j = 1, \dots, N$ and all $n \in \mathbb{N}$. But the (differential geometric) problem of finding such approximations is, to our knowledge, without an answer up to now, and it may well be that such approximations do not exist in general.

interest for nonsmooth Hamiltonians is obviously justified by the applications to optimal control problems, where Hamiltonians have the form:

$$H_i(x,p) = \sup_{a\in A}\{-\langle f_i(x,a),p\rangle - \ell_i(x,a)\} \quad (i=1,\dots,N). \tag{1.3}$$

Here the f_i are control vector fields, i.e., vector fields depending on a parameter $a \in A$, and the ℓ_i are real functions, sometimes called *Lagrangians*. Notice that, as an effect of the involved maximization, these Hamiltonians happen to be nonsmooth (in p), even in the case where the data f_i, ℓ_i are very regular.

1.4. Our main goal The principal purpose of the present paper is to give existence results[4] for a general case where the smoothness condition $[\mathcal{BT}]_{\text{smooth}}$ is not verified and the commutativity condition $[\mathcal{BT}]_{\text{ZPB}}$ is replaced by a new condition which is meaningful even when the data are nonsmooth.

Actually, hypotheses $[\mathcal{BT}]_{\text{smooth}}$ and $[\mathcal{BT}]_{\text{ZPB}}$ will be replaced by hypotheses $[\mathcal{H}]_{\text{Lip}}$ (or $[\mathcal{H}']_{\text{Lip}}$) and $[\mathcal{H}]_{\text{CCZLB}}$ below, respectively.

Definition 1.1 (Hypothesis $[\mathcal{H}]_{\text{Lip}}$)**.** By saying that the Hamiltonians H_i verify hypothesis $[\mathcal{H}]_{\text{Lip}}$ (*Lipschitz continuous data*) we mean that the Hamiltonians have the form (1.3),[5] and the data f_i, ℓ_i verify the following set of conditions:

For any $i = 1,\dots,N$ the functions $f_i : \mathbb{R}^n \times A \to \mathbb{R}^n$, $\ell_i : \mathbb{R}^n \times A \to \mathbb{R}$ are continuous. Moreover, there is a constant M and, for every $R > 0$, there are some L_R, $M_R > 0$ such that

$$\begin{array}{l} |f_i(x,a) - f_i(y,a)| \le L_R|x-y| \\ |\ell_i(x,a) - \ell_i(y,a)| \le L_R|x-y| \end{array} \quad \forall\,(x,a),\ (y,a) \in B^n(0,R)\times A, \tag{1.4}$$

$$|\ell_i(x,a)| \le M_R \quad \forall\,(x,a) \in B^n(0,R)\times A$$

(where $B^n(0,R)$ is the open ball of radius R of $\mathbb{R}^n$); and

$$|f_i(x,a)| \le M(1+|x|) \quad \forall\,(x,a) \in \mathbb{R}^n \times A.$$

We shall also consider a slightly stronger hypothesis:

Definition 1.2 (Hypothesis $[\mathcal{H}']_{\text{Lip}}$)**.** By saying that the Hamiltonians H_i verify hypothesis $[\mathcal{H}']_{\text{Lip}}$ we mean that all the hypotheses in $[\mathcal{H}]_{\text{Lip}}$ are verified and (1.4) is changed into the stronger condition:

$$\begin{array}{l} |f_i(x,a) - f_i(y,b)| \le L_R|(x,a)-(y,b)| \\ |\ell_i(x,a) - \ell_i(y,b)| \le L_R|(x,a)-(y,b)| \end{array} \quad \forall\,(x,a),\ (y,b) \in B^n(0,R)\times A.$$

[4] The uniqueness issue will be also treated—see Section 3. Let us remark that it is partially based on the recognition of some boundary conditions that are implicit in the definition of solution.

[5] In view of the assumed convexity of the H_i (in the p variable), the fact of considering Hamiltonians of the form (1.3) is not too restrictive (see [9], [13], [18]).

In order to state condition $[\mathcal{H}]_{\mathrm{CCZLB}}$ below, let us consider product coordinates $(x_0, x) \in \mathbb{R}^{1+n}$ and let us define the control vector fields $\hat{f}_i$ on $\mathbb{R}^{1+n}$ by setting, for every $i = 1, \ldots, N$,[6]

$$\hat{f}_i(x,\alpha) = \ell_i(x,\alpha)\frac{\partial}{\partial x_0} + \sum_{j=1}^{n} f_{ij}(x,\alpha)\frac{\partial}{\partial x_j}.$$

Notice that, although the $\hat{f}_i$'s are vector fields on $\mathbb{R}^{1+n}$, they are x_0*-independent*.

Definition 1.3 (Hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$)**.** By saying that the family of pairs (ℓ_i, f_i) verifies hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ (*Constant Control Zero Lie Bracket*) we mean that:

- for any $i = 1, \ldots, N$ the functions $f_i : \mathbb{R}^n \times A \to \mathbb{R}^n$, $\ell_i : \mathbb{R}^n \times A \to \mathbb{R}$ are continuous in (x, a) and locally Lipschitz continuous in x, uniformly with respect to the control;
- for each $\alpha, \beta \in A$ and $i, j = 1, \ldots, N$ one has[7]

$$[\hat{f}_i(x,\alpha), \hat{f}_j(x,\beta)] = 0 \quad \text{for a.e. } x \in \mathrm{DIFF}(\hat{f}_i(\cdot,\alpha)) \cap \mathrm{DIFF}(\hat{f}_j(\cdot,\beta)),$$

where:

(i) for each $k = 1, \ldots, N$ and each $a \in A$ we use $\mathrm{DIFF}(\hat{f}_k(\cdot, a))$ to denote the subset of R^{1+n} on which $\hat{f}_k(\cdot, a)$ is differentiable—which, by Rademacher's Theorem, is a full measure set; and

(ii) for any subset $E \subset \mathbb{R}^{1+n}$, the expression *for a.e.* $x \in E$ means *for every* $x \in E \setminus \mathcal{N}$, where $\mathcal{N}$ is a subset of zero Lebesgue measure.

1.5. An example Let us consider the initial value problem on $[0, T]^2 \times \mathbb{R}^2$

$$\begin{cases} \dfrac{\partial u}{\partial t_1} + H_1(x, D_x u) = 0, \\ \dfrac{\partial u}{\partial t_2} + H_2(x, D_x u) = 0, \end{cases} \tag{1.5}$$

$$u(0,0,x) = \psi(x), \tag{1.6}$$

where

$$H_1(x,p) = -|x_1 - \arctan x_2| p_1 + |p_1|,$$
$$H_2(x,p) = \frac{|x_2 p_1|}{1 + x_2^2} + |x_2 p_2|,$$

[6] We use $(\partial/\partial x_0, \partial/\partial x_1, \ldots, \partial/\partial x_n)$ to denote the canonical basis of $\mathbb{R}^{1+n}$, while f_{ij} stands for the components of f_i with respect to the basis $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ of $\mathbb{R}^n$.

[7] We recall that for each $x \in R^n$ the value of the Lie bracket $[h, k](x)$ of two vector fields h, k which are differentiable at x is defined by

$$[h,k](x) = Dk(x) \cdot h(x) - Dh(x) \cdot k(x).$$

and

$$\psi(x) = |x_1 - \arctan x_2| + |x_2| \quad \forall\, x \in \mathbb{R}^2.$$

Notice that

$$H_i(x,p) = \max_{a\in\{0,-1,1\}} \{-p \cdot f_i(x,a)\}, \quad i = 1,2$$

where the control vector fields f_1, f_2 are defined by

$$f_1(x,a) = \begin{pmatrix} |x_1 - \arctan x_2| + a \\ 0 \end{pmatrix}, \quad f_2(x,a) = \begin{pmatrix} \dfrac{|x_2|a}{1+x_2^2} \\ |x_2|a \end{pmatrix}.$$

Clearly, hypothesis $[\mathcal{H}']_{\mathrm{Lip}}$ is verified. Moreover, for every

$$a, \bar{a} \in \{0,-1,1\} \quad \text{and} \quad x \in \mathrm{DIFF}(f_1(x,a)) \cap \mathrm{DIFF}(f_2(x,\bar{a})),$$

one has

$$[f_1(x,a),\, f_2(x,\bar{a})] = Df_2(x,\bar{a})f_1(x,a) - Df_1(x,a)f_2(x,\bar{a})$$

$$= \begin{pmatrix} 0 & \dfrac{\mathrm{sign}(x_2)(1-x_2^2)\bar{a}}{(1+x_2^2)^2} \\ 0 & \mathrm{sign}(x_2)\bar{a} \end{pmatrix} \cdot \begin{pmatrix} |x_1 - \arctan x_2| + a \\ 0 \end{pmatrix}$$

$$- \begin{pmatrix} \mathrm{sign}(x_1 - \arctan x_2) & \dfrac{-\,\mathrm{sign}(x_1 - \arctan x_2)}{1+x_2^2} \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \dfrac{|x_2|\bar{a}}{1+x_2^2} \\ |x_2|\bar{a} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that the commutativity condition $[\mathcal{H}]_{\mathrm{CCZLB}}$ is verified as well. Hence, by Theorem 5.1 there is a unique viscosity solution U of the initial value problem (1.5)–(1.6). Actually, one can easily check that the function $U : [0,T]^2 \times \mathbb{R}^2$ defined by

$$U(t_1,t_2,x) = -e^{-t_1}(x_1 - \arctan x_2) - (1 - e^{-t_1}) + |x_2|e^{-t_2}, \tag{1.7}$$

if $x_1 - \arctan x_2 \le 1 - e^{t_1}$,

$$U(t_1,t_2,x) = |x_2|e^{-t_2} \tag{1.8}$$

if $1 - e^{t_1} < x_1 - \arctan x_2 < 1 - e^{-t_1}$, and

$$U(t_1, t_2, x) = e^{t_1}(x_1 - \arctan x_2) + (1 - e^{t_1}) + |x_2| e^{-t_2} \tag{1.9}$$

if $x_1 - \arctan x_2 \geq 1 - e^{-t_1}$, is a (viscosity) solution of system (1.5) verifying the initial condition (1.6). In Section ***5.4*** we shall show how the results of the next sections allow one to construct this solution, which, on the basis of Theorem 5.1, is in fact the unique solution.

1.6. Main tools and outline of the paper In order to deal with the different kinds of non-smoothness affecting the gradient variable and the state variable, we shall exploit two separate tools. The first one is the interpretation of the problem in terms of *minimization*, so that the existence problem can be regarded as a question of *commutativity of minimum problems*. In fact, this allows us to skip the problem of the non-smoothness of the Hamiltonians in the variable p.[8]

The second tool allows us to treat the non-regularity in x and consists in a notion of Lie bracket for locally Lipschitz continuous vector fields which has been introduced in [20]. Let us remind that for two smooth vector fields h, g, the condition $[h, g] = 0$ is equivalent to the (local) commutativity of the flows corresponding to h and g, respectively.[9] Let us point out that hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ generalizes this condition in two ways. On one hand, when the controls α and β are fixed, $[\mathcal{H}]_{\mathrm{CCZLB}}$ is a generalization of the above zero bracket condition to nonsmooth vector fields, according to [21]. On the other hand, $[\mathcal{H}]_{\mathrm{CCZLB}}$ extends the zero bracket condition from *vector fields* to *control vector fields*. The characterization of (the suitably defined) flows' commutativity for control vector fields by means of condition $[\mathcal{H}]_{\mathrm{CCZLB}}$ was proved in [19].

The outline of the paper is as follows. In Section 2 we introduce an exponential notation which will prove useful to state various results of the paper. In Section 3 we deal with the uniqueness question. In Section 4 we introduce the notion of commutativity of optimal control problems, which we call *inf-commutativity*. Section 5 contains our main result on the existence of solution to (1.1)–(1.2). Here is a partial statement:

Theorem 5.1 (Partial statement)**.** *Let us assume hypotheses* $[\mathcal{H}]_{\mathrm{Lip}}$ *and* $[\mathcal{H}]_{\mathrm{CCZLB}}$. *Then for any continuous map* ψ, *there exists a unique viscosity solution* $U_\psi = U_\psi(t, x)$ *of the system* (1.1)–(1.2).

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[8]This approach is distinct from the one that focuses on the commutativity of the semigroups associated to the Hamiltonians. Yet the two approaches are obviously connected, as shown e.g. in Theorem 5.1 below.

[9]That is, if xe^{tf} denotes the value at t of the solution to the Cauchy problem $\dot{y} = f(y) y(0) = x$, the validity of $[h, g]$ in a open neighborhood U of x is equivalent to the existence, for every $z \in U$, of some $\varepsilon_z > 0$ such that

$$z e^{th} e^{sg} = z e^{sg} e^{th}$$

for all t, s such that $0 \leq t, s \leq \varepsilon_z$.

Moreover, three different representations of the solution are provided in Theorem 5.1. In particular, the solution can be represented as the composition of the N semigroups corresponding to the single equations (and the fact that this is independent of the order of such composition is due to hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$). For each multi-time $(t_1, \dots, t_N)$ the solution can also be represented as the value function obtained by choosing any *simple* multi-time path connecting $(t_1, \dots, t_N)$ with $(T, \dots, T)$. Actually, if the stronger hypothesis $[\mathcal{H}']_{\mathrm{Lip}}$ is verified, the solution of (1.1)–(1.2) turns out to coincide with the value function obtained by choosing any *absolutely continuous* multi-time path connecting $(t_1, \dots, t_N)$ with $(T, \dots, T)$. Incidentally, as soon as $[\mathcal{H}]_{\mathrm{CCZLB}}$ is assumed, this allows to prove that the above-defined exponential map is commutative and additive:[10]

$$\mathbf{e}^{-t_1 H_1}\mathbf{e}^{-t_2 H_2} = \mathbf{e}^{-t_2 H_2}\mathbf{e}^{-t_1 H_1}, \quad \mathbf{e}^{-t_1 H_1}\mathbf{e}^{-t_2 H_2} = \mathbf{e}^{-t_1 H_1 - t_2 H_2}.$$

In Section 6 we introduce a further value function, which we call the *Best Value*. When hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ is in force, the Best Value coincides (up to reversing time) with the solution to (1.1)–(1.2) (see Theorem 6.3)—and this, in fact, provides a further representation of this solution. Yet the Best Value is perfectly meaningful also when there is no solution of (1.1)–(1.2). So, from the viewpoint of applications, it can be regarded as a possible *replacement* of the (generally non existing) solution of (1.1)–(1.2).

In Section 7 we establish some regularity properties of the solution. Moreover, thanks to the commutativity of the N one-parameter semigroups $\mathbf{e}^{-t_i H_i}$, under hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ the solution of (1.1) is a *N-parameter* semigroup, as shown in Theorem 7.2.

In Section 8 we show that as soon as the Hamiltonians are positively homogeneous in the adjoint variable a *Front Propagation Property* analogous to the one holding true for single equations is valid for multi-time systems as well.

Up to now, with the expression *existence of a solution* we (and the authors of the quoted papers) have meant existence *for every continuous initial data* ψ (subject, in [15] and in [3], only to some regularity conditions). However, even when the commutativity condition $[\mathcal{BT}]_{\mathrm{ZPB}}$ is *not verified*, it may happen that a solution exists only for suitable initial conditions ψ. We devote Section 9 to illustrate an example of such an occurrence, leaving as an open question the search of general sufficient conditions on ψ for the existence of a solution to (1.1)–(1.2) in the case when $[\mathcal{BT}]_{\mathrm{ZPB}}$ is not satisfied.

2. NOTATION AND GENERAL DEFINITIONS

Let M be a positive integer. For any $z \in \mathbb{R}^M$, $|z|$ will denote the usual Euclidean norm of z, while $|z|_1$ will denote the ℓ_1-norm, i.e., $|z|_1 = \sum_{i=1}^M |z_i|$. For any $R > 0$ and for any $z_0 \in \mathbb{R}^M$, $B(z_0, R)$ denotes the open ball $\{z \in \mathbb{R}^M : |z - z_0| < R\}$, and, for any $E \subset \mathbb{R}^M$, $\bar{E}$ stands for the closure of E in $\mathbb{R}^M$. A

[10] See Subsection ***2.1*** for a rigorous definition of the exponential map.

scalar real function $h : [a, b] \to [c, d]$ will be called increasing [resp.: strictly increasing] if for any pair s_0, $s_1 \in [a, b]$ such that $s_0 < s_1$ one has $h(s_0) \le h(s_1)$ [resp.: $h(s_0) < h(s_1)$]. An increasing function $\omega : [0, +\infty[\to [0, +\infty[$, continuous at 0 and such that $\omega(0) = 0$ will be called a *modulus*. A function $\omega : [0, +\infty[^2 \to [0, +\infty[$ will be called a *local modulus* if it is increasing in the second variable and, for every $R \ge 0$, the map $\omega(\cdot, R)$ is a modulus.

If $Q \subset \mathbb{R}^M$ and I is a real interval, $L^1(I, Q)$ will denote the subset of Lebesgue integrable maps from I into $\mathbb{R}^M$ which take values in Q, while $\mathcal{B}(I, Q)$ will denote the subset of the maps belonging to $L^1(I, Q)$ which are Borel measurable.

If $n_1, \dots, n_q$ are positive integers and k is a function from a subset of $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_q}$ to $\mathbb{R}$, for any $i = 1, \dots, q$ we use $D_{z_i}k$ to denote (possibly in a weak sense) the gradient of k with respect to the z_i variable. Moreover, we use Dk to denote the gradient of k with respect to $z = (z_1, \dots, z_q)$.

We will consider solutions to (1.1)–(1.2) in the viscosity sense. Let us recall the definitions of viscosity sub- and supersolution—see e.g. [7].

Definition 2.1. Let q be a positive integer and let $\mathcal{E}$ be a subset of $\mathbb{R}^q$. Let $F : \mathcal{E} \times \mathbb{R}^q \to \mathbb{R}$ and $u : \mathcal{E} \to \mathbb{R}$ be continuous functions, and let $y_0 \in \mathcal{E}$.

The function u is called a *viscosity subsolution* (resp.: a *viscosity supersolution*) of

$$F(y, Du) = 0 \tag{2.1}$$

at y_0 if for every $\varphi \in C^1(\mathbb{R}^q)$ such that y_0 is a local maximum (resp.: minimum) in $\mathcal{E}$ for

$$u(y) - \varphi(y)$$

one has

$$F(y_0, D\varphi(y_0)) \le 0 \quad (\text{resp.: } \ge 0).$$

u is a *viscosity solution* of (2.1) at y_0 if it is both a viscosity subsolution and a viscosity supersolution of (2.1) at y_0.

In particular, for $i = 1, \dots, N$ the i-th equation in (1.1) has the form (2.1) with $q = N + n$, $y = (t, x) \in \mathbb{R}^N \times \mathbb{R}^n$, and $F(t_1, \dots, t_N, x, p_{t_1}, \dots, p_{t_N}, p_x) \doteq p_{t_i} + H_i(x, p_x)$.

2.1. Exponential notation System (1.1) is made of autonomous equations. However, for the sake of completeness, we prefer to define the exponential notation for the more general case of non autonomous equations. In particular, we shall assume that the Hamiltonian is measurable in the time variable, which requires a more general definition of (viscosity) solution. In fact, this is originally due to I. Ishii [11] and we recall it below in the equivalent form stated in [14]. Let us remark that, when H is continuous, the two notions of (sub- and super-) solution do coincide. Moreover, Definition 2.2 turns out to be the natural extension *by density* of Definition 2.1 (see [11], [14], [5]).

Definition 2.2. Let $H = H(t, x, p)$ be Lebesgue-measurable in t and continuous in (x, p). Let $u(t, x)$ be a continuous function on $Q \doteq]0, T[\times \mathcal{O}$. u is a *viscosity subsolution* (resp.: a *viscosity supersolution*) of

$$\frac{\partial u}{\partial t} + H(t, x, D_x u) = 0 \tag{2.2}$$

at $(t_0, x_0) \in Q$ if for every $\varphi \in C^1(\mathcal{O})$ and $b \in L^1([0, T], \mathbb{R})$ such that (t_0, x_0) is a local maximum (resp.:minimum) for

$$u(t, x) + \int_0^t b(s)\, ds - \varphi(x)$$

one has

$$\liminf_{\delta \downarrow 0^+} \operatorname*{ess\,inf}_{|t-t_0|<\delta} \{H(t, x, p) - b(t) : |x - x_0| \le \delta,\ |p - D\varphi(x_0)| \le \delta\} \le 0,$$

respectively

$$\limsup_{\delta \downarrow 0^+} \operatorname*{ess\,sup}_{|t-t_0|<\delta} \{H(t, x, p) - b(t) : |x - x_0| \le \delta,\ |p - D\varphi(x_0)| \le \delta\} \ge 0.$$

$u(t, x)$ is a *viscosity solution* of (2.2) at $(t_0, x_0) \in Q$ if it is both a viscosity subsolution and a viscosity supersolution of (2.2) at (t_0, x_0).

Definition 2.3. We say that $H = H(t, x, p)$ verifies condition $[\mathcal{EU}]$ if for each continuous map $\psi : \mathbb{R}^n \to \mathbb{R}$ and each pair s_0, s_1, $s_0 < s_1$, a viscosity solution $u_\psi(s, x)$ to the Cauchy problem

$$\frac{\partial u}{\partial s} + H(s, x, D_x u) = 0, \quad (s, x) \in]s_0, s_1[\times \mathbb{R}^n, \quad u(s_0, x) = \psi(x) \tag{2.3}$$

exists and is unique.

If H verifies $[\mathcal{EU}]$, for every $s \in [s_0, s_1]$ we shall use the exponential notation

$$\psi \mathbf{e}^{-\int_{s_0}^{s} H(\sigma)\, d\sigma}$$

to denote the continuous map $x \mapsto u_\psi(s, x)$, while

$$[x]\psi \mathbf{e}^{-\int_{s_0}^{s} H(\sigma)\, d\sigma}$$

will stand for its evaluation at x.[11]

Let us prove a parameter-invariance property for the equation in (2.3).

[11] This notation is borrowed from an analogous notation introduced by Agrachev and Gamkrelidze for (finite-dimensional) flows of control vector fields, [1]. Let us notice that it is consistent with a

Lemma 2.4. *Let σ_0, σ_1, s_0, s_1 be real numbers such that $\sigma_0 < \sigma_1$, $s_0 < s_1$, and let $s : [\sigma_0, \sigma_1] \to [s_0, s_1]$ be an absolutely continuous, strictly increasing, surjective map such that $s'(\sigma) > 0$ for a.e. $\sigma \in \,]\sigma_0, \sigma_1[$. Let $H : [s_0, s_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be continuous and let it verify the uniqueness and existence hypothesis $[\mathcal{EU}]$. Then there exists a unique viscosity solution to the Cauchy problem*

$$\tag{2.4} \begin{aligned} &\frac{\partial v}{\partial \sigma}(\sigma, x) + H(s(\sigma), x, Dv(\sigma, x))s'(\sigma) = 0, \\ &(\sigma, x) \in \,]\sigma_0, \sigma_1[\, \times \mathbb{R}^n, \quad v(\sigma_0, x) = \psi(x), \end{aligned}$$

that is, the Hamiltonian $H(s(\cdot), \cdot, \cdot)s'(\cdot)$ satisfies $[\mathcal{EU}]$ in $[\sigma_0, \sigma_1] \times \mathbb{R}^n \times \mathbb{R}^n$. Moreover, the corresponding solutions verify

$$\psi \mathbf{e}^{-\int_{s_0}^{s(\sigma)} H(r)\, dr} = \psi \mathbf{e}^{-\int_{\sigma_0}^{\sigma} (H(s(\eta))s'(\eta))\, d\eta}$$

for all $\sigma \in [\sigma_0, \sigma_1]$.[12]

The proof of this lemma is postponed to the Appendix.

Remark. Let us observe that the property stated in Lemma 2.4 is far beyond what is needed in this paper. In fact, we will only consider the trivial case of C^1 reparametrizations $s(\cdot)$ such that $s' > 0$.

2.1.1. *The autonomous case* In the autonomous case, i.e., when the Hamiltonian H is independent of s, we shall simplify notation by setting, for every real number $r \neq 0$,

$$\mathbf{e}^{rH} \doteq \mathbf{e}^{-\int_0^{|r|} (-rH/|r|)\, d\sigma} \quad \left(= \mathbf{e}^{-\int_{s_0}^{s_0+|r|} (-rH/|r|)\, d\sigma} \quad \forall\, s_0 \in \mathbb{R}\right).$$

There is no ambiguity in using this notation, for, in view of Lemma 2.4, one has

$$\tag{2.5} \mathbf{e}^{rH} = \mathbf{e}^{-\int_0^1 (-rH)\, d\sigma} = \mathbf{e}^{-1(-rH)} = \mathbf{e}^{1(rH)}.$$

---

formal interpretation of the exponential operator. Indeed, if H is given as in (2.2), for every $s \in [s_0, s_1]$ let us define the operator $H(s) : C^1(\mathbb{R}^n) \to C^0(\mathbb{R}^n)$ by setting,

$$H(s) : \varphi \mapsto \varphi H(s) \doteq H(s, \cdot, D_x \varphi(\cdot)).$$

Then, setting $u(s, x) = [x]\psi \mathbf{e}^{-\int_{s_0}^{s} H(\sigma)\, d\sigma}$ and proceeding formally, we obtain:

$$\frac{\partial u}{\partial s}(s, x) = -[x]\psi \mathbf{e}^{-\int_{s_0}^{s} H(\sigma)\, d\sigma} H(s) = -H(s, x, Du(s, x)).$$

[12] In other words, if $u_\psi(s, x)$ is the solution of (2.3) on $[s_0, s_1] \times \mathbb{R}^n$, then

$$v(\sigma, x) \doteq u_\psi(s(\sigma), x)$$

is the (unique) solution of (2.4) on $[\sigma_0, \sigma_1] \times \mathbb{R}^n$.

If H_1, H_2 are given Hamiltonians and $t_1, t_2 \in \mathbb{R}$, then

$$\mathbf{e}^{t_1 H_1} \mathbf{e}^{t_2 H_2} \neq \mathbf{e}^{t_1 H_1 + t_2 H_2}$$

unless suitable hypotheses are made on H_1, H_2. Actually, in order to have an equality, it is sufficient that either $H_1 = H_2$ or $[\mathcal{H}]_{\mathrm{CCZLB}}$ be verified. More precisely, we have the following result.

Proposition 2.5 (Additivity properties of the exponential). *Let H be an Hamiltonian verifying the existence and uniqueness hypothesis $[\mathcal{E}\mathcal{U}]$. Then for every t, $s \in \mathbb{R}$ one has*

$$\mathbf{e}^{tH} \mathbf{e}^{sH} = \mathbf{e}^{tH+sH}. \tag{2.6}$$

Moreover, if H and $\tilde{H}$ are defined by

$$\begin{aligned} H(x,p) &= \sup_{a \in A} \{-\langle f(x,a), p\rangle - \ell(x,a)\}, \\ \tilde{H}(x,p) &= \sup_{a \in A} \{-\langle \tilde{f}(x,a), p\rangle - \tilde{\ell}(x,a)\}, \end{aligned}$$

and the pairs (f, ℓ), $(\tilde{f}, \tilde{\ell})$ verify hypotheses $[\mathcal{H}']_{\mathrm{Lip}}$ and $[\mathcal{H}]_{\mathrm{CCZLB}}$, then for every t, $s \in \mathbb{R}$ one has

$$\mathbf{e}^{tH} \mathbf{e}^{s\tilde{H}} = \mathbf{e}^{tH+s\tilde{H}}. \tag{2.7}$$

While the proof of (2.6) is a straightforward consequence of the definition, (2.7) is non trivial and will be proved in Section 5.

3. COMPARISON AND UNIQUENESS OF SOLUTIONS

While postponing the investigation on existence of solutions to (1.1)–(1.2), let us begin with the (comparison and) uniqueness issue.

Let us fix two positive numbers C and T, and let us set

$$\mathcal{D}_C \doteq \{(t_1, \ldots, t_N, x) : (t_1, \ldots, t_N) \in [0, T[^N \setminus \{(0, \ldots, 0)\},\ |x| < C|T - t|_1\}.$$

The following comparison result states the validity of a so-called *domain of dependence* property. An analogous result is well-known in the case of a single evolution equation (see e.g [4]).

Theorem 3.1 (Domain of dependence). *Let us assume that there exists a modulus ω such that for any $i = 1, \ldots, N$ the Hamiltonian $H_i : \overline{B(0, CT)} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies*

$$\begin{aligned} |H_i(x,p) - H_i(x,q)| &\le C|p-q|, \\ |H_i(x,p) - H_i(y,p)| &\le \omega(|x-y|) + \omega(|x-y|\,|p|) \end{aligned}$$

for all $x, y \in B(0, CT)$, $p, q \in \mathbb{R}^n$.

Let $u_1, u_2 : \overline{\mathcal{D}_C} \to \mathbb{R}$ *be a viscosity sub- and supersolution, respectively, of the multi-time system* (1.1) *at every* (t, x) *in the interior of* $\mathcal{D}_C$. *If they verify* $u_1(0, x) \leq u_2(0, x)$ *for all* $x \in B(0, CT)$, *then* $u_1(t, x) \leq u_2(t, x)$ *for all* $(t, x) \in \mathcal{D}_C$.

Here is the uniqueness result. Notice that, in particular, the hypotheses on the Hamiltonians cover our hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$ on the data of the problem.

Theorem 3.2 (Uniqueness)**.** *Let us assume that there exists* $K > 0$ *such that for any* $i = 1, \ldots, N$, *the Hamiltonian* $H_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is continuous and satisfies*

$$|H_i(x, p) - H_i(x, q)| \leq K(1 + |x|)|p - q|$$

for all $x, p, q \in \mathbb{R}^n$. *Moreover, let us assume that there exists a local modulus* $\omega(\cdot, \cdot)$ *such that for any* $R > 0$ *one has*

$$|H_i(x, p) - H_i(y, p)| \leq \omega(|x - y|, R) + \omega(|x - y|\,|p|, R)$$

for all $p \in \mathbb{R}^n$, $x, y \in B(0, R)$. *Then for any* $T > 0$, *problem* (1.1)–(1.2) *has at most one viscosity solution in* $[0, T]^N \times \mathbb{R}^n$.

We shall prove Theorems 3.1 and 3.2 in the next subsection as corollaries of analogous results concerning the *auxiliary Hamilton-Jacobi equation* (3.1) below.

Remark. Let us remark that neither convexity nor any commutativity hypothesis—like $[\mathcal{BT}]_{\mathrm{ZPB}}$ or its nonsmooth counterpart $[\mathcal{H}]_{\mathrm{CCZLB}}$—are assumed in Theorem 3.2. This is particularly relevant in view of the fact that a solution (for a particular initial datum ψ) may well exist even if no commutativity conditions are verified (see Section 9). Incidentally, Theorem 3.2 improves the uniqueness result contained in [3], in that the latter was proved under the commutativity hypothesis $[\mathcal{BT}]_{\mathrm{ZPB}}$.

3.1. An auxiliary HJ equation Let us consider the auxiliary boundary value problem[13]

$$\max\left\{\frac{\partial u}{\partial t_1} + H_1(x, D_x u), \ldots, \frac{\partial u}{\partial t_N} + H_N(x, D_x u)\right\} = 0 \quad \forall\, (t, x) \in\,]0, T[^N \times \mathbb{R}^n, \tag{3.1}$$

$$\max\left\{\frac{\partial u}{\partial t_1} + H_1(x, D_x u), \ldots, \frac{\partial u}{\partial t_N} + H_N(x, D_x u)\right\} \geq 0 \quad \text{on } \partial_0(\,]0, T[^N \times \mathbb{R}^n), \tag{3.2}$$

$$u(0, \ldots, x) = \psi(x) \quad \forall\, x \in \mathbb{R}^n, \tag{3.3}$$

[13]The boundary condition (3.2)—often called *constrained boundary condition*—means that the solution has to be found among the functions $u : [0, T]^N \times \mathbb{R}^n \to \mathbb{R}$ that are supersolutions of

$$\max\left\{\frac{\partial u}{\partial t_1} + H_1(x, D_x u), \ldots, \frac{\partial u}{\partial t_N} + H_N(x, D_x u)\right\} = 0$$

at each $(t, x) \in \partial_0(\,]0, T[^N \times \mathbb{R}^n)$.

where

$$\partial_0(\,]0,T[^N\times\mathbb{R}^n) = \bigcup_{i=1,\dots,N} \partial_0^i(\,]0,T[^N\times\mathbb{R}^n),$$
$$\partial_0^i(\,]0,T[^N\times\mathbb{R}^n) \doteq \Big\{(t,x)\in(\,[0,T[^N\setminus\{(0,\dots,0)\})\times\mathbb{R}^n : \\ t_i\neq 0, \text{ and } t_j = 0 \text{ for some } j\neq i\Big\}.$$

In Section 6 we shall illustrate a control-theoretical interpretation of the boundary value problem (3.1)–(3.3) in the case where the Hamiltonians are convex in the gradient variable. Roughly speaking, (3.1), is the boundary value problem for the value function—called *Best Value* in Section 6—of the optimal control problem obtained by suitably partitioning the time intervals of the optimal control problems underlying the single equations in (1.1) and by optimizing the implementation's order of such intervals.

Theorem 3.3. *Let us assume that there exists a modulus ω such that, for any $i = 1,\dots,N$, the Hamiltonian $H_i : \overline{B(0,CT)}\times\mathbb{R}^n\to\mathbb{R}$ is continuous and satisfies*

$$|H_i(x,p)-H_i(x,q)| \le C|p-q|,$$
$$|H_i(x,p)-H_i(y,p)| \le \omega(|x-y|)+\omega(|x-y|\,|p|)$$

for all x, $y\in B(0,CT)$, p, $q\in\mathbb{R}^n$.

If u_1 and $u_2 : \overline{\mathcal{D}_C}\to\mathbb{R}$ are a viscosity sub- and supersolution of (3.1) in the interior of $\mathcal{D}_C$ and in $\mathcal{D}_C$, respectively, and $u_1(0,x)\le u_2(0,x)$ for all $x\in B(0,CT)$, then $u_1(t,x)\le u_2(t,x)$ for all $(t,x)\in\mathcal{D}_C$.

Corollary 3.4. *Assume that there exists a $K>0$ such that for any $i = 1,\dots,N$ the Hamiltonian $H_i : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is continuous and satisfies*

$$|H_i(x,p)-H_i(x,q)| \le K(1+|x|)|p-q|$$

for all x, p, $q\in\mathbb{R}^n$. Moreover, assume that there exists a local modulus ω such that for any $i = 1,\dots,N$ and for every $R>0$, one has

$$|H_i(x,p)-H_i(y,p)| \le \omega(|x-y|,R)+\omega(|x-y|\,|p|,R)$$

for all $p\in\mathbb{R}^n$, x, $y\in B(0,R)$. Then for any $T>0$ there is at most one viscosity solution $v : [0,T]^N\times\mathbb{R}^n$ of the auxiliary boundary value problem (3.1), (3.2), (3.3).

Theorem 3.3 and Corollary 3.4 will be proved in the Appendix.

Proof of Theorem 3.1 and Theorem 3.2 These results follow from Theorem 3.3 and Corollary 3.4, respectively, via the following fact:

Proposition 3.5. *Assume that for any $i = 1, \ldots, N$ the Hamiltonian $H_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function. Then, for any $T > 0$, both* (i) *and* (ii) *below hold true:*

(i) *any function $u \in C([0,T]^N \times \mathbb{R}^n)$ which is a viscosity solution of* (1.1) *at any $(t,x) \in]0,T[^N \times \mathbb{R}^n$ verifies* (3.1), (3.2);

(ii) *any viscosity solution of* (3.1) *is a viscosity subsolution of* (1.1) *in $]0,T[^N \times \mathbb{R}^n$.*

Proof. Statement (ii) is trivial. As for (i), it is obvious that u verifies (3.1). Finally, in order to prove that u verifies (3.2), let us remark that for each $i = 1, \ldots, N$ the function

$$K_i(t_1, \ldots, t_N, x, p_{t_1}, \ldots, p_{t_N}, p_x) \doteq p_{t_i} + H_i(x, p_x)$$

is in fact independent of each p_{t_j} such that $j \neq i$. In view of the result in [8], this implies that for any $i = 1, \ldots, N$, u is a supersolution of

$$\frac{\partial u}{\partial t_i} + H_i(x, D_x u) = 0$$

on $\partial_0^i( ]0,T[^N \times \mathbb{R}^n)$, which in turn yields (3.2). ◻

4. Inf-commutativity

By *inf-commutativity* we mean *commutativity of optimal control problems*. This is a generalization—introduced in [19]—of the notion of commutativity for control systems, which, in turn, extends the standard concept of commutativity of vector fields' flows. As a matter of fact, we shall prove existence of a solution to (1.1)–(1.2) by proving that the optimal control problems associated with the pairs (f_i, ℓ_i) do *commute*. Before giving a formal definition of inf-commutativity (see Definition 4.6 below), let us illustrate this notion by means of a simple example.

Suppose we are given the minimum problems

$$\begin{cases} \inf\left(\int_0^{T_1} \ell(x(t_1), c(t_1))\, dt_1 + \int_0^{T_2} m(y(t_2), d(t_2))\, dt_2 \right) \\ \dot{x} = f(x,c), \quad x(0) = \bar{x}, \quad \dot{y} = g(y,d), \quad y(0) = x(T_1) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \inf\left(\int_0^{T_2} m(y(t_2), d(t_2))\, dt_2 + \int_0^{T_1} \ell(x(t_1), c(t_1))\, dt_1 \right) \\ \dot{y} = g(y,d), \quad y(0) = \bar{x}, \quad \dot{x} = f(x,c), \quad x(0) = y(T_2). \end{cases} \tag{4.2}$$

where $c(\cdot)$ and $d(\cdot)$ are controls which range over a given control set A.

The obvious meaning of problem (4.1) is that the infimum is taken over the four-uples (c, x, d, y) verifying the following:

(i) c and d are controls defined on $[0, T_1]$ and $[0, T_2]$, respectively;
(ii) x is the solution over $[0, T_1]$ of the Cauchy problem $\dot{x} = f(x, c)$, $x(0) = \bar{x}$;
(iii) y is the solution over $[0, T_2]$ of the differential equation $\dot{y} = g(y, d)$ taking the final value $x(T_1)$ as its own initial condition: $y(0) = x(T_1)$.

The meaning of (4.2) is analogous, up to an exchange between (f, ℓ) and (g, m).

One can wonder if the infimum value of problem (4.1) coincides with that of (4.2). This would mean that the order according to which one implements the two control systems does not affect the final infimum value. If this holds true for any choice of $\bar{x}$ and of the times T_1, T_2, we shall say that the two control systems (f, ℓ), (g, m) *inf-commute*. Of course, we can generalize this question by partitioning the two intervals into several subintervals and *running* these subintervals (and the corresponding control systems) in whatever order. Also one can consider the interactions of more (than two) optimal control systems, and the latter can include final costs as well.

We shall see in Section 5 that the existence of a solution to the multi-time problem (1.1)–(1.2) is practically equivalent to such a commutation property.

4.1. Multi-time control problems Let us consider a partial order $\leq$ on $\mathbb{R}^N$ by saying that $(t_1, \dots, t_N) = t \leq \bar{t} = (\bar{t}_1, \dots, \bar{t}_N)$ if $t_i \leq \bar{t}_i$ for all $i = 1, \dots, N$. Let us denote the set $[t_1, \bar{t}_1] \times \cdots \times [t_N, \bar{t}_N]$, which we call the *interval between t and $\bar{t}$*, by $[t, \bar{t}\,]$.

Definition 4.1. Let $S > 0$ and let $t, \bar{t} \in \mathbb{R}^N$, $t \leq \bar{t}$. A map

$$\tau = (\tau_1, \dots, \tau_N) : [0, S] \to \mathbb{R}^N$$

is called a *multi-time path connecting the multi-times t and $\bar{t}$* if it is absolutely continuous, for every $i = 1, \dots, N$ the map τ_i is increasing, $|(d\tau/ds)(s)| > 0$ for almost every $s \in [0, S]$, and $\tau(0) = t$, $\tau(S) = \bar{t}$.

Definition 4.2. A *simple multi-time path* is a piece-wise affine multi-time path $\tau : [0, S] \to \mathbb{R}^N$ such that[14]

$$\frac{d\tau}{ds}(s) \in \left\{ \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_N} \right\}$$

for all s where it is differentiable.

Definition 4.3. Let us consider multi-times $t, \bar{t} \in \mathbb{R}^N$ such that $t \leq \bar{t}$. An N-uple

$$\mathbf{a} = (a_1, \dots, a_N) \in \mathcal{B}([t_1, \bar{t}_1], A) \times \cdots \times \mathcal{B}([t_N, \bar{t}_N], A)$$

will be called an *N-control defined on* $[t, \bar{t}\,]$.

[14]We recall that $(\partial/\partial t_1, \dots, \partial/\partial t_N)$ denotes the canonical basis of $\mathbb{R}^N$.

Definition 4.4. Let us assume that the N pairs (ℓ_i, f_i) verify hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$. Let t, $\bar{t} \in \mathbb{R}^N$ be multi-times such that $t \le \bar{t}$, and let $\mathbf{a}$ be an N-control defined on $[t, \bar{t}\,]$. Let $\tau : [0, S] \to \mathbb{R}^N$ be a multi-time path such that

$$\tau(0) = t \quad \tau(S) = \bar{t}.$$

Let us define the $\mathbf{a}$-*lift of* τ *at a point* $(x_0, x) \in \mathbb{R}^{1+n}$ as the solution of the Cauchy problem

$$\text{(4.3)} \qquad \frac{d(y_0, y)}{ds} = \sum_{i=1}^{N} (\ell_i, f_i)(y(s), a_i \circ \tau_i(s)) \frac{d\tau_i}{ds}, \quad (y_0, y)(0) = (x_0, x).$$

The $\mathbf{a}$-lift of τ at a point $(x_0, x) \in \mathbb{R}^{1+n}$ will be denoted by $(y_0, y)_{(\mathbf{a},\tau)}[x_0, x](\cdot)$. When $x_0 = 0$ we will write, in short, $(y_0, y)_{(\mathbf{a},\tau)}[x](\cdot)$.

Remark. Notice that, because of hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$, the $\mathbf{a}$-lift of τ is well-defined, for all $\mathbf{a}$ and τ.

Definition 4.5. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a function and let $\tau : [0, S] \to \mathbb{R}^N$ be a multi-time path connecting two multi-times t, $\bar{t}$ such that $t \le \bar{t}$.

For any $x \in \mathbb{R}^n$ let us consider the minimization problem

$$\text{minimize} \quad \left\{ \psi(y(S)) + \int_0^S \left(\sum_{i=1}^{N} \ell_i(y(s), a_i \circ \tau_i(s)) \frac{d\tau_i}{ds}(s) \right) ds \right\}$$

where the infimum is searched over all N-controls $\mathbf{a} \in \mathcal{B}([\,\bar{t}_1, \bar{t}_1], A) \times \cdots \times \mathcal{B}([\,\bar{t}_N, \bar{t}_N], A)$, and, for every such $\mathbf{a}$, $y(\cdot)$ stands for $y_{(\mathbf{a},\tau)}[x](\cdot)$.

Moreover, for any $0 \le s \le S$ let us set

$$V_\psi^\tau(s, x) \doteq \inf \left\{ \psi(y_{(\mathbf{a},\tau)}[x](S)) + \int_s^S \left(\sum_{i=1}^{N} \ell_i(y_{(\mathbf{a},\tau)}[x](\sigma), a_i \circ \tau_i(\sigma)) \frac{d\tau_i}{d\sigma} \right) d\sigma \right\}.$$

Let us introduce the notion of ψ-*inf-commutativity*:

Definition 4.6. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be an arbitrary continuous map. We say that the flows of the control vector fields $(\ell_1, f_1)(x, a)$, $\ldots$, $(\ell_N, f_N)(x, a)$ *inf-commute at* ψ [resp. *simply inf-commute at* ψ] if for any $x \in \mathbb{R}^n$, any pair $(t, \bar{t}) = ((t_1, \ldots, t_N), (\bar{t}_1, \ldots, \bar{t}_N))$, $t \le \bar{t}$, and any two [resp. simple] multi-time paths τ, $\hat{\tau} : [0, S] \to \mathbb{R}^N$ connecting t and $\bar{t}$, one has

$$V_\psi^\tau(0, x) = V_\psi^{\hat{\tau}}(0, x)$$

We say that the flows of the control vector fields $(\ell_1, f_1)(x,a), \dots, (\ell_N, f_N)(x,a)$ *inf-commute* [resp. *simply inf-commute*] if they inf-commute at ψ [resp. simply inf-commute at ψ] for any continuous map $\psi : \mathbb{R}^n \to \mathbb{R}$.

Theorem 4.7 ([19]). *Assume the regularity hypothesis* $[\mathcal{H}]_{\mathrm{Lip}}$ *and the zero Lie bracket hypothesis* $[\mathcal{H}]_{\mathrm{CCZLB}}$. *Then the flows of the control vector fields* $(\ell_1, f_1)(x,a)$, $\dots$, $(\ell_N, f_N)(x,a)$ simply *inf-commute.*

Moreover, if hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$ *is replaced by the stronger hypothesis* $[\mathcal{H}']_{\mathrm{Lip}}$, *then the flows of the control vector fields* $(\ell_1, f_1)(x,a)$, $\dots$, $(\ell_N, f_N)(x,a)$ *inf-commute.*

Remark. A case where one has ψ-inf-commutativity without having commutativity will be discussed in Section 9.

5. EXISTENCE AND SOME REPRESENTATIONS OF A SOLUTION

For any continuous initial data ψ and any multi-time path τ connecting some $t \in [0,T]^N$ to $(T, \dots, T)$, let us define the map

$$W_\psi^\tau(s,x) = V_\psi^\tau(S-s,x) \quad \forall\, (s,x) \in [0,S] \times \mathbb{R}^n.$$

Theorem 5.1 (Existence and representations). *Let us assume hypotheses* $[\mathcal{H}]_{\mathrm{Lip}}$ *and* $[\mathcal{H}]_{\mathrm{CCZLB}}$. *Then for any continuous map* ψ, *there exists a unique viscosity solution* $U_\psi = U_\psi(t,x)$ *of the problem* (1.1)–(1.2).

Moreover, for any simple *multi-time path* $\tau : [0,S] \to \mathbb{R}^N$ *such that* $\tau(0) = t = (t_1, \dots, t_N)$ *and* $\tau(S) = (T, \dots, T)$ *and any permutation* $(i_1, \dots, i_N)$ *of* $(1, \dots, N)$, *one has the following two representations of this solution:*

(1) $\quad U_\psi(t,x) = W_\psi^\tau(0,x)$

(2) $\quad U_\psi(t,x) = [x]\psi \mathbf{e}^{-t_N H_N} \cdots \mathbf{e}^{-t_1 H_1} \quad (= [x]\psi \mathbf{e}^{-t_{i_N} H_{i_N}} \dots \mathbf{e}^{-t_{i_1} H_{i_1}})$

for all $(t,x) \in [0,T]^N \times \mathbb{R}^n$.

If hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$ *is replaced by the stronger regularity assumption* $[\mathcal{H}']_{\mathrm{Lip}}$, *then representation* (1) *above holds for* all*—i.e., not necessarily simple—multi-time paths* $\tau : [0,S] \to \mathbb{R}^N$ *such that* $\tau(0) = t = (t_1, \dots, t_N)$ *and* $\tau(S) = (T, \dots, T)$ *and one also has the following third representation of the solution of* (1.1)–(1.2)*:*

(3) $\quad U_\psi(t,x) = [x]\psi \mathbf{e}^{-t_N H_N - \cdots - t_1 H_1} \quad (= [x]\psi \mathbf{e}^{-t_{i_N} H_{i_N} - \cdots - t_{i_1} H_{i_1}})$.

This theorem will be proved in Subsections ***5.2***, ***5.3***, while in the next subsection we make some comments on the commutativity hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$.

5.1. Some remarks on the hypotheses

Neither bounded fields nor controllability Although our main goal was the removal of the smoothness assumption on the Hamiltonians, notice that neither we are assuming that the vector fields or the Lagrangian are bounded, nor we are making some controllability hypothesis. As a matter of fact, these kinds of hypotheses were made in [3][15] in order to exploit a priori bounds on the solution's gradient which is essential for the proof's strategy adopted there.

The commutativity hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ The main achievement of Theorem 5.1 is the fact that the Hamiltonians are not assumed to be smooth. Hence, it is reasonable to investigate the relation occurring between hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ and hypothesis $[\mathcal{BT}]_{\mathrm{ZPB}}$, which was made in [3] to prove existence.

For any $i = 1, \ldots, N$, let us define the i-th unminimized Hamiltonian

$$h_i(x, p, a) \doteq \langle -p, f_i(x, a)\rangle - \ell_i(x, a) \quad \forall\, (x, p, a) \in \mathbb{R}^n \times \mathbb{R}^n \times A.$$

By definition, the Hamiltonian H_i is given by

$$H_i(x, p) = \sup_{a \in A} h_i(x, p, a).$$

Under assumption $[\mathcal{H}]_{\mathrm{Lip}}$ the function h_i is locally Lipschitz in x and differentiable in p uniformly with respect to a. Hence by Rademacher's Theorem for any $a \in A$ there is a set $\mathcal{N}_{i,a} \subset \mathbb{R}^n$ of zero Lebesgue measure such that $h_i(\cdot, \cdot, a)$ is differentiable at every $(x, p) \in (\mathbb{R}^n \setminus \mathcal{N}_{i,a}) \times \mathbb{R}^n$.

Actually, in the result below we add a uniformity condition on the set of differentiability points.

Proposition 5.2. *Let us assume hypotheses* $[\mathcal{H}]_{\mathrm{Lip}}$ *and* $[\mathcal{H}]_{\mathrm{CCZLB}}$. *Moreover, let the following additional conditions be verified:*

(i) *for any* $i = 1, \ldots, N$ *there is a set* $\mathcal{N}_i \subset \mathbb{R}^n$ *of zero Lebesgue measure such that for all* $p \in \mathbb{R}^n$ *the map* $h_i(\cdot, p, a)$ *is differentiable in* $\mathbb{R}^n \setminus \mathcal{N}_i$, *uniformly with respect to* a (*that is, the sets* $\mathcal{N}_{i,a} \subset \mathbb{R}^n$ *defined above do not depend on* a);

(ii) *for any* $i = 1, \ldots, N$, *the map* $a \to D_x h_i(x, p, a)$ *is continuous at every* $(x, p) \in (\mathbb{R}^n \setminus \mathcal{N}_i) \times \mathbb{R}^n$;

(iii) *the control set A is compact.*

Then the zero Lie bracket hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ *implies that the zero Poisson bracket hypothesis* $[\mathcal{BT}]_{\mathrm{ZPB}}$ *is verified almost everywhere.*

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[15] The "coercivity" assumption

$$\lim_{|p| \to +\infty} H_i(x, p) = +\infty \quad \text{uniformly for } x \in \mathbb{R}^n,$$

which was made in [3], is in fact a first order controllability hypothesis on the underlying optimal control problem.

Proof. Let us fix an arbitrary $i \in \{1, \dots, N\}$, and let us consider the set

$$M_i(x,p) \doteq \operatorname*{argmax}_{a \in A} h_i(x,p,a) \doteq \{b \in A : h_i(x,p,b) = \max_{a \in A} h_i(x,p,a)\}$$

which, by (iii) turns out to be non-empty for all $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$. Furthermore, for all $(x,p) \in (\mathbb{R}^n \setminus \mathcal{N}_i) \times \mathbb{R}^n$ let us define the subset $Y_i(x,p)$ by setting

$$\begin{aligned} Y_i(x,p) \doteq \{(D_x h_i(x,p,a), D_p h_i(x,p,a)) : a \in M_i(x,p)\} \\ (= \{(-D_x f_i(x,a)p - D_x \ell_i(x,a), -f_i(x,a)) : a \in M_i(x,p)\}). \end{aligned}$$

Under hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$ the Hamiltonian $H_i(\cdot,\cdot)$ is locally Lipschitz continuous. Hence by Rademacher's Theorem there exists a subset $\tilde{\mathcal{N}}_i \subset \mathbb{R}^n \times \mathbb{R}^n$ of zero Lebesgue measure such that $H_i(\cdot,\cdot)$ is differentiable at every $(x,p) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \tilde{\mathcal{N}}_i$. By hypothesis (i), it follows that at every $(x,p) \in ((\mathbb{R}^n \setminus \mathcal{N}_i) \times \mathbb{R}^n) \setminus \tilde{\mathcal{N}}_i$ both $H_i(\cdot,\cdot)$ and $h_i(\cdot,\cdot,a)$ are differentiable (uniformly with respect to a). Hence by well known properties of marginal functions (see e.g. Proposition 2.13, II, in [4]), for every $(x,p) \in ((\mathbb{R}^n \setminus \mathcal{N}_i) \times \mathbb{R}^n) \setminus \tilde{\mathcal{N}}_i$ the set $Y_i(x,p)$ is a singleton and

$$\begin{aligned} D_x H_i(x,p) &= D_x h_i(x,p,a) = -D_x f_i(x,a)p - D_x \ell_i(x,a), \\ D_p H_i(x,p) &= D_p h_i(x,p,a) = -f_i(x,a) \end{aligned}$$

for any $a \in M_i(x,p)$. Hence for any $i,\ j \in \{1,\dots,N\}$, $i \neq j$ and for any $(x,p) \in ((\mathbb{R}^n \setminus (\mathcal{N}_i \cup \mathcal{N}_j)) \times \mathbb{R}^n) \setminus (\tilde{\mathcal{N}}_i \cup \tilde{\mathcal{N}}_j)$ one has

$$\{H_j, H_i\}(x,p) = (p,1)[\hat{f}_j(x,a_j), \hat{f}_i(x,a_i)] \tag{5.1}$$

for arbitrary $a_i \in M_i(x,p)$ and $a_j \in M_j(x,p)$. By $[\mathcal{H}]_{\mathrm{CCZLB}}$ and the regularity hypothesis (i), at any $x \in \mathbb{R}^n \setminus (\mathcal{N}_i \cup \mathcal{N}_j)$ one has

$$[\hat{f}_j(x,a_j), \hat{f}_i(x,a_i)] = 0 \quad \forall\, a_j,\ a_i \in A.$$

Therefore (5.1) implies that

$$\{H_i, H_j\}(x,p) = 0 \quad \text{for all } (x,p) \in ((\mathbb{R}^n \setminus (\mathcal{N}_i \cup \mathcal{N}_j)) \times \mathbb{R}^n) \setminus (\tilde{\mathcal{N}}_i \cup \tilde{\mathcal{N}}_j),$$

which concludes the proof. □

5.2. Preliminary results for the proof of Theorem 5.1 Theorem 5.1 will be proved in the next subsection as a consequence of Theorems 5.4 and 5.6 below.

To begin with, to every multi-time $t = (t_1, \dots, t_N) \in [0,T]^N$ we can attach canonically special simple multi-time paths, namely those which run each time component only once.

Definition 5.3 (Time-ordered value functions)**.** For any permutation $\iota = \{i_1, \dots, i_N\}$ of $\{1, \dots, N\}$ and any $t = (t_1, \dots, t_N)$ such that $0 \le t \le (T, \dots, T)$, let us set $S_0 \doteq 0$, $S_j \doteq \sum_{r=1}^{j}(T - t_{ir})$ for any $j = 1, \dots, N$, $S \doteq S_N$, and let us consider the simple multi-time path

$$\tau_\iota^t(s) \doteq \tau_{i_1,\dots,i_N}^{t_1,\dots,t_N}(s) = (t_1, \dots, t_N) + \int_0^s \sum_{j=1}^{N} \chi_{]S_{j-1},S_j]}(\xi) \frac{\partial}{\partial t_{i_j}} \, d\xi$$

for all $s \in [0, S]$. Correspondingly, for any arbitrary continuous map ψ and any $x \in \mathbb{R}^n$, let us define the value function

$$V_\psi^{i_1,\dots,i_N}(t, x) \doteq V_\psi^{\tau_\iota^t}(0, x),$$

where $V_\psi^{\tau_\iota^t}$ is defined according to Definition 4.5. We will refer to the maps $V_\psi^{i_1,\dots,i_N}$ as *time-ordered value functions*.

Remark. For any ψ, there are exactly $N!$ *time-ordered* value functions. Notice that they are defined on $[0, T]^N \times \mathbb{R}^n$, while the functions $V_\psi^{\tau_\iota^t}$ are defined on sets of the form $[0, S] \times \mathbb{R}^n$.

For each permutation $\iota = (i_1, \dots, i_N)$, let us define the map $W_\psi^{i_1,\dots,i_N}$ on $[0, T]^N \times \mathbb{R}^n$ by setting

$$W_\psi^{i_1,\dots,i_N}(t_1, \dots, t_N, x) = V_\psi^{i_1,\dots,i_N}(T - t_1, \dots, T - t_N, x).$$

Theorem 5.4. *Let us assume hypothesis* $[\mathcal{H}]_{\mathrm{Lip}}$. *For any permutation* $\{i_1, \dots, i_N\}$ *of* $\{1, \dots, N\}$ *the map* $W_\psi^{i_1,\dots,i_N}$ *is a viscosity solution of the equation*

$$\frac{\partial u}{\partial t_{i_1}}(t, x) + H_{i_1}(x, D_x u(t, x)) = 0$$

on $]0, T]^N \times \mathbb{R}^n$. *Moreover, it verifies the boundary condition*

$$W_\psi^{i_1,\dots,i_N}(0, \dots, 0, x) = \psi(x) \quad \forall\, x \in \mathbb{R}^n.$$

Furthermore, if the function ψ *is locally Lipschitz continuous, the map* $W_\psi^{i_1,\dots,i_N}$ *is locally Lipschitz continuous as well. Finally, if the function* ψ *is globally Lipschitz continuous and the dynamics* f_i *and the Lagrangians* ℓ_i *are bounded and globally Lipschitz continuous in* x, *uniformly with respect to the controls, then the map* $W_\psi^{i_1,\dots,i_N}$ *turns out to be globally Lipschitz continuous.*

Proof. This theorem is an easy consequence of well known results on evolution equations (see e.g. Sect. III in [4]) and of the representation formula for the map $W_\psi^{i_1,\dots,i_N}$ proved in Theorem 5.6 below. □

In order to prove Theorem 5.6 below, we shall make use of the following dynamic programming principle, whose proof is omitted, because it can be easily obtained by arguing as in the proof of the dynamic programming principle for finite horizon problems (see e.g. Proposition 3.2, III in [4]).[16]

Proposition 5.5 (Dynamic Programming Principle). *Let us assume hypothesis* $[\mathcal{H}]_{\mathrm{Lip}}$ *and let* ψ *be an arbitrary continuous function. Let us fix* $(t,x) \in [0,T]^N \times \mathbb{R}^n$, $t \neq (T,\dots,T)$, *and a permutation* $\iota = \{i_1,\dots,i_N\}$ *of* $\{1,\dots,N\}$. *Let us set* $S_0 \doteq 0$, $S_j \doteq \sum_{r=1}^{j}(T - t_{i_r})$ *for any* $j = 1,\dots,N$ *and* $S = S_N$, *and* $\tau(\cdot) = \tau_\iota^t(\cdot)$. *Then for any* $0 < \sigma < S$ *one has*

$$V_\psi^{i_1,\dots,i_N}(t,x) = \inf\Big\{ \int_0^\sigma \Big(\sum_{j=1}^{N} \ell_{i_j}(y(s), a_{i_j} \circ \tau_{i_j}(s))\chi_{[S_{i_{j-1}},S_{i_j}]}(s)\Big)\, ds + V_\psi^{i_1,\dots,i_N}(\tau(\sigma), y(\sigma))\Big\},$$

where the infimum is taken over the set of controls

$$\mathbf{a} = (a_1,\dots,a_N) \in \mathcal{B}([t_1,T],A) \times \cdots \times \mathcal{B}([t_N,T],A),$$

and $y(\cdot) = y_{(\mathbf{a},\tau)}[x](\cdot)$.

Theorem 5.6. *Let us assume hypothesis* $[\mathcal{H}]_{\mathrm{Lip}}$. *Let* ψ *be an arbitrary continuous function and let* $(i_1,\dots,i_N)$ *be any permutation of* $(1,\dots,N)$. *Then for every* $(t_1,\dots,t_N) \in [0,T]^N$ *and any* $x \in \mathbb{R}^n$ *one has*

$$W_\psi^{i_1,\dots,i_N}(t_1,\dots,t_N,x) = [x]\psi \mathbf{e}^{-t_{i_N} H_{i_N}} \mathbf{e}^{-t_{i_{N-1}} H_{i_{N-1}}} \cdots \mathbf{e}^{-t_{i_1} H_{i_1}}.$$

Proof. For every $(\tau,x) \in [0,T] \times \mathbb{R}^n$ let us consider the value function

$$V_\psi^j(\tau,x) \doteq \inf\Big\{\psi(y(T)) + \int_\tau^T \ell_j(y(\xi),a(\xi))\, d\xi\Big\} \tag{5.2}$$

where the infimum is taken over the set of controls $a(\cdot) \in \mathcal{B}([\tau,T],A)$ and $y(\cdot)$ stands for the solution of[17]

$$\frac{dy}{d\xi}(\xi) = f_j(y(\xi),a(\xi)), \quad y(\tau) = x. \tag{5.3}$$

[16]Let us notice, however, that this dynamic programming principle is not completely standard because of the dependence of $V^{i_1,\dots,i_N}(t,x)$ on t through the t-parameter family of multi-time paths $\{\tau_\iota^t\}_t$.

[17]It is convenient to notice that, since the problem is autonomous, V_ψ^j can be equivalently defined as $V_\psi^j(\tau,x) = \inf\{\psi(y(T-\tau)) + \int_0^{T-\tau} \ell_j(y(\xi),a(\xi))\, d\xi\}$, where the control $a(\cdot)$ ranges over the set $\mathcal{B}([0,T-\tau],A)$ and y solves the equation in (5.3) with initial condition $y(0) = x$.

It is well-known that for every $\tau \in [0,T]$ one has

$$[x]\psi\mathbf{e}^{-\tau H_j} = V_\psi^j(T-\tau,x) \quad \forall\, x \in \mathbb{R}^n. \tag{5.4}$$

Up to renaming indexes, we can limit ourselves to prove the theorem for $(i_1,\dots,i_N) = (1,\dots,N)$. Moreover, since e^{0H_j} coincides with the identity map, without loss of generality, we can consider only points $(t,x) \in]0,T]^N \times \mathbb{R}^n$. We argue by induction on N. For $N = 1$, the thesis follows from (5.4) with $j = 1$. Let $N > 1$. For any continuous map φ, any $(t_1,\dots,t_{N-1}) \in [0,T]^{N-1}$, any $x \in \mathbb{R}^n$, and any $j = 1,\dots,N-1$ let us set $S_0 \doteq 0$, $S_j \doteq \sum_{r=1}^{j}(T-t_r)$ and let us introduce the value function

$$\begin{aligned}
&V_\varphi^{1,\dots,N-1}(t_1,\dots,t_{N-1},x)\\
&\qquad = \inf\left\{ \int_0^{S_{N-1}} \sum_{j=1}^{N-1} \ell_j(\tilde{y}(s),\tilde{a}_j\circ\tilde{\tau}_j(s))\chi_{]S_{j-1},S_j]}(s)\,ds + \varphi(\tilde{y}(S_{N-1}))\right\},
\end{aligned}$$

where the infimum is taken over the set of controls

$$\tilde{\mathbf{a}} = (\tilde{a}_1,\dots,\tilde{a}_{N-1}) \in \mathcal{B}([t_1,T],A)\times\cdots\times\mathcal{B}([t_{N-1},T],A),$$

$\tilde{\tau}$ is defined by

$$\tilde{\tau}(s) = (t_1,\dots,t_{N-1}) + \int_0^s \sum_{j=1}^{N-1} \chi_{]S_{j-1},S_j]}(\xi)\frac{\partial}{\partial t_j}\,d\xi$$

for all $s \in [0,S_{N-1}]$, and, for any control $\tilde{\mathbf{a}}$, $\tilde{y}(\cdot)$ denotes the solution of

$$\frac{dy}{ds} = \sum_{j=1}^{N-1} f_j(y(s),\tilde{a}_j\circ\tilde{\tau}_j(s))\chi_{]S_{j-1},S_j]} \quad s \in]0,S_{N-1}[\,,\ y(0) = x.$$

Furthermore, let us set

$$W_\varphi^{1,\dots,N-1}(t_1,\dots,t_{N-1},x) = V_\varphi^{1,\dots,N-1}(T-t_1,\dots,T-t_{N-1},x).$$

By the inductive hypothesis it follows that

$$W_\varphi^{1,\dots,N-1}(t_1,\dots,t_{N-1},x) = [x]\varphi\mathbf{e}^{-t_{N-1}H_{N-1}}\dots\mathbf{e}^{-t_1H_1} \tag{5.5}$$

for all $(t_1,\dots,t_{N-1},x) \in [0,T]^{N-1}\times\mathbb{R}^n$. Since

$$V_\psi^{1,\dots,N}(T,\dots,T,t_N,x) = V_\psi^N(t_N,x) \quad \forall\, t_N \in [0,T],\ x \in \mathbb{R}^n,$$

choosing $\sigma = S_{N-1}$ in Proposition 5.5, we get

$$(5.6)\quad \begin{aligned} &V_\psi^{1,\dots,N}(t_1,\dots,t_N,x) \\ &\quad = \inf\Big\{ \int_0^{S_{N-1}} \sum_{j=1}^{N-1} \ell_j(y(s), a_j\circ(\tau^t)_j(s))\chi_{[S_{j-1},S_j]}(s)\,ds \\ &\qquad\qquad + V_\psi^N(t_N, y(S_{N-1}))\Big\} \end{aligned}$$

where:

(i) the infimum is taken over the set of controls

$$\mathbf{a} = (a_1,\dots,a_N) \in \mathcal{B}([t_1,T],A)\times\cdots\times\mathcal{B}([t_N,T],A);$$

(ii) $\tau^t = \tau_{1,\dots,N}^{t_1,\dots,t_N}$ and $(\tau^t)_j$ denotes its j-th component;

(iii) for any control $\mathbf{a}$, we have written $y(\cdot)$ instead of $y_{(\mathbf{a},\tau^t)}[x](\cdot)$.

Since the restrictions to the interval $[0,S_{N-1}]$ of the functions $\tau^t(\cdot)$, $\mathbf{a}(\cdot)$, and $y(\cdot)$ can be identified, respectively, with the functions $\tilde\tau(\cdot)$, $\tilde{\mathbf{a}}(\cdot)$, and $\tilde y(\cdot)$ defined above, by (5.6) it follows that (see footnote [17])

$$(5.7)\qquad V_\psi^{1,\dots,N}(t_1,\dots,t_N,x) = V_{V_\psi^N(t_N,\cdot)}^{1,\dots,N-1}(t_1,\dots,t_{N-1},x).$$

Thus (5.7), (5.4) and (5.5)—where we choose $\varphi(\cdot) \doteq V_\psi^N(t_N,\cdot)$—yield that

$$\begin{aligned} W_\psi^{1,\dots,N}(t_1,\dots,t_N,x) &= [x]V_\psi^N(t_N,\cdot)\mathbf{e}^{-t_{N-1}H_{N-1}}\cdots\mathbf{e}^{-t_1H_1} \\ &= [x]\psi\mathbf{e}^{-t_NH_N}\mathbf{e}^{-t_{N-1}H_{N-1}}\cdots\mathbf{e}^{-t_1H_1}, \end{aligned}$$

which concludes the proof. □

Remark. The relevant property, stated in Theorem 5.6, that for any permutation $(i_1,\dots,i_N)$ of $(1,\dots,N)$ the map $W_\psi^{i_1,\dots,i_N}$ may be defined as composition of elements of N one-parameter semigroups of solutions generated by the single Hamilton-Jacobi equations belonging to the multi-time system (1.1) holds even if a solution to (1.1)–(1.2) does not exist. In particular, it is independent of the zero Lie bracket hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$. Incidentally, this implies that we do not need hypothesis $[\mathcal{H}]_{\mathrm{CCZLB}}$ also in Theorem 5.4.

5.3. Proof of Theorem 5.1 Let us fix a point $(t,x) \in [0,T]^N\times\mathbb{R}^n$. By Theorem 4.7 we easily obtain that for any simple multi-time path $\tau : [0,S] \to \mathbb{R}^N$ such that $\tau(0) = (t_1,\dots,t_N)$ and $\tau(S) = (T,\dots,T)$ and any permutation $(i_1,\dots,i_N)$ of $(1,\dots,N)$, the maps $W_\psi^\tau(0,x)$, $W_\psi^{i_1,\dots,i_N}(t,x)$, and $W_\psi^{1,\dots,N}(t,x)$ do coincide. Actually, in view of Theorem 5.4, the map

$$(t,x) \mapsto U_\psi(t,x) = W_\psi^{1,\dots,N}(t,x)$$

(or, equivalently, each of the maps $W_\psi^{i_1,\dots,i_N}$) is a viscosity solution of the problem (1.1)–(1.2). This proves the existence issue and the representation (1)—while uniqueness comes from Theorem 3.2.

Representation (2) has been proved in Theorem 5.6.

If hypothesis $[\mathcal{H}']_{\mathrm{Lip}}$ is assumed, Theorem 4.7 ensures that the representation formula (1) for the solution of (1.1)–(1.2) holds also for *any* multi-time path.

In order to prove (3), for any C^1 multi-time path $\tau : [0,S] \to \mathbb{R}^N$ joining some $t \in [0,T]^N$ to $(T,\dots,T)$ let us introduce the Hamiltonian

$$\mathcal{H}_\tau(s,x,p) \doteq \sum_{i=1}^{N} H_i(x,p)\frac{d\tau_i}{ds}(s),$$

defined for all $(s,x,p) \in [0,S] \times \mathbb{R}^n \times \mathbb{R}^n$. From well known uniqueness and representation results (see e.g. Theorem 3.17, III in [4]) it follows that, under hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$, the Cauchy problem

$$\begin{aligned} &\frac{\partial v}{\partial s}(s,x) + \mathcal{H}_\tau(s,x,D_x v(s,x)) = 0, \\ &(s,x) \in\,]0,S[\, \times \mathbb{R}^n, \quad v(0,x) = \psi(x) \end{aligned} \tag{5.8}$$

has a unique viscosity solution, which, according to the notation introduced in Section 2, is denoted by $[x]\psi\mathbf{e}^{-\int_0^s \mathcal{H}_\tau(r)\,dr}$. Moreover,

$$[x]\psi\mathbf{e}^{-\int_0^s \mathcal{H}_\tau(r)\,dr} = W_\psi^\tau(s,x) \quad \forall\,(s,x) \in [0,S] \times \mathbb{R}^n.$$

This allows us to prove (3): indeed, choosing

$$\tau(s) \doteq (t_1,\dots,t_N) + s(T-t_1,\dots,T-t_N) \quad s \in [0,1],$$

one has

$$\mathcal{H}_\tau = -t_1H_1 - \cdots - t_NH_N,$$

so, by representation (1),

$$U_\psi(t,x) = W_\psi^\tau(0,x) = [x]\psi\mathbf{e}^{-t_NH_N\cdots -t_1H_1}.$$

Proof of the second part of Proposition 2.5. The additivity property (2.7) is a straightforward consequence of the representation formulas (2) and (3) in Theorem 5.1. □

***5.4. Solving the initial value problem* (1.5)–(1.6)** In the Introduction we have considered the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t_1} + H_1(x, D_x u) = 0, \\ \dfrac{\partial u}{\partial t_2} + H_2(x, D_x u) = 0, \end{cases}$$
$$u(0,0,x) = \psi(x),$$

where

$$\psi(x) = |x_1 - \arctan x_2| + |x_2| \qquad \forall x \in \mathbb{R}^2$$

and

$$H_i(x,p) = \max_{a \in \{0,-1,1\}} \{-p \cdot f_i(x,a)\}, \qquad i = 1,2,$$
$$f_1(x,a) = \begin{pmatrix} |x_1 - \arctan x_2| + a \\ 0 \end{pmatrix}, \qquad f_2(x,a) = \begin{pmatrix} \dfrac{|x_2|a}{1+x_2^2} \\ \\ |x_2|a \end{pmatrix}.$$

After having observed that this problem verifies the hypotheses of Theorem 5.1, we have given an explicit solution U (see (1.7), (1.8), (1.9)). We now show how this solution can be constructed on the basis of the results of the previous subsections. For every $(t_1, t_2, x) = (t, x) \in [0,T]^2 \times \mathbb{R}^2$ let us set $S_1 \doteq (T - t_1)$ and $S \doteq (T - t_1) + (T - t_2)$ and let us consider the simple multi-time path

$$\tau(s) = (t_1, t_2) + \int_0^s \left(\chi_{]0,S_1]}(\xi) \frac{\partial}{\partial t_1} + \chi_{]S_1,S]}(\xi) \frac{\partial}{\partial t_2} \right) d\xi$$

for all $s \in [0, S]$. Correspondingly, for any $x \in \mathbb{R}^2$ let us introduce the time ordered value function

$$V^{1,2}(t_1, t_2, x) \doteq \inf\{\psi(y(S))\},$$

where:

(i) the infimum is searched over all 2-controls

$$\mathbf{a} = (a_1, a_2) \in \mathcal{B}([t_1, T], \{0,-1,1\}) \times \mathcal{B}([t_2, T], \{0,-1,1\});$$

and

(ii) for every such $\mathbf{a}$, $y(\cdot)$ stands for the solution of the Cauchy problem

$$\frac{dy}{ds} = f_1(y(s), a_1 \circ \tau_1(s)) \chi_{]0,S_1]}(\xi) + f_2(y(s), a_1 \circ \tau_2(s)) \chi_{]S_1,S]}(\xi), \qquad y(0) = x.$$

Notice that for every 2-control $\mathbf{a}$ one has

$$y_2(s) = x_2, \quad \frac{dy_1}{ds} = |y_1(s) - \arctan x_2| + a_1 \circ \tau_1(s), \qquad \forall\, s \in]0, S_1]$$

and that the map $E(x) \doteq x_1 - \arctan x_2$ evaluated along $y(s)$ remains constant for all $s \in [S_1, S]$. Hence, one easily deduces that the optimal policy consists in:

(i) minimizing $|y_1(s) - \arctan x_2|$ for $s \in [0, S_1]$; and

(ii) minimizing $|y_2(s)|$ for $s \in]S_1, S]$.

A direct computation yields:

$$V^{1,2}(t_1, t_2, x) = -e^{-(T-t_1)}(x_1 - \arctan x_2) - (1 - e^{-(T-t_1)}) + |x_2| e^{-(T-t_2)}$$

when $x_1 - \arctan x_2 \le 1 - e^{T-t_1}$;

$$V^{1,2}(t_1, t_2, x) = |x_2| e^{-(T-t_2)}$$

when $1 - e^{T-t_1} < x_1 - \arctan x_2 < 1 - e^{-(T-t_1)}$;

$$V^{1,2}(t_1, t_2, x) = e^{T-t_1}(x_1 - \arctan x_2) + (1 - e^{T-t_1}) + |x_2| e^{-(T-t_2)}$$

when $x_1 - \arctan x_2 \ge 1 - e^{-(T-t_1)}$. In view of Theorem 5.1, we obtain the solution U by simply setting, for every $(t_1, t_2, x) \in [0, T]^2 \times \mathbb{R}^2$,

$$U(t_1, t_2, x) = V^{1,2}(T - t_1, T - t_2, x).$$

6. The Best Value

In this section we wish to introduce the (standard) control problem obtained by the whole family of multi-time control problems by considering the multi-time paths as new controls, so that the optimizing strategy consists not only in the choice of the *N-control* $\mathbf{a}$, but also in the choice of the path joining the given initial multi-time $(t_1, \dots, t_N)$ with the end multi-time $(T, \dots, T)$.

The interest for this auxiliary problem is obvious from the point of view of applications. Indeed, unless commutativity hypotheses are assumed (as in Theorem 4.7), the cost depends on the multi-time path followed from $(t_1, \dots, t_N)$ to $(T, \dots, T)$. So, one could be interested to single out the *best choice* of the multi-time path. This provides the multi-time control problem with a reasonable concept of value, the *Best Value*, even when there is no existence for the original multi-time system (1.1)–(1.2).

But the auxiliary optimal control problem we are introducing is interesting also from an other viewpoint. Indeed the corresponding Hamilton-Jacobi equation is exactly equation (3.1), used in Section 3 to prove the uniqueness of the solutions to (1.1)–(1.2). Moreover, as soon as the multi-time problem (1.1)–(1.2) admits a solution, this coincides (up to reversing time) with the Best Value (see Theorem 6.3). In particular this can be used to study the regularity of the solution to (1.1)–(1.2),[18] which (thanks to the uniqueness property of equation (3.1)) can be reduced to the regularity of the value function of a standard optimal control problem.

Let us consider the optimal control problem

$$\mathcal{P}[t,x] \qquad \text{minimize} \quad \Big\{\psi(y(S)) + \int_0^S \Big(\sum_{i=1}^N \ell_i(y(s), a_i \circ \tau_i(s)) w_i(s)\Big)\, ds\Big\},$$

where

$$\begin{cases} \tau'(s) = w(s), \\ y'(s) = \displaystyle\sum_{i=1}^N f_i(y(s), a_i \circ \tau_i(s)) w_i(s), \end{cases} \tag{6.1}$$

$$(\tau, y)(0) = (t, x) \qquad \tau(S) = (T, \dots, T), \tag{6.2}$$

and the minimization is performed over the set of all control pairs $(w, \mathbf{a})$ such that the control $w : [0, S] \to \mathbb{R}^N$ is piecewise constant and ranges over the canonical basis of $\mathbb{R}^N$, $\mathbf{a} = (a_1, \dots, a_N) \in \mathcal{B}([t_1, T], A) \times \cdots \times \mathcal{B}([t_N, T], A)$, and $\tau(S) = (T, \dots, T)$.[19] In the sequel, we will refer to this set as *admissible* control set and we will denote it by $\mathcal{A}(t)$. Furthermore, for any $(w, \mathbf{a})$ the corresponding solution to (6.1)–(6.2) will be denoted by the pair $(\tau_w[t](\cdot), y_{(w,\mathbf{a})}[t,x](\cdot))$. By the definition of w it follows that $\mathcal{A}(t) \neq \emptyset$ for all $t \in [0, T]^N \setminus \{(T, \dots, T)\}$.

Notice also that S *is determined by* t; indeed, one has

$$S = \sum_{i=1}^N (T - t_i).$$

Problem $\mathcal{P}[t,x]$ can be regarded either as a control problem with target $\mathcal{T} = \{(T, \dots, T)\} \times \mathbb{R}^n$ and $S = \inf\{s > 0 : \tau(s) = (T, \dots, T)\}$, or, equivalently, as an exit-time control problem, where S turns out to be the first exit-time of the pair (τ, y) from the region $([0, +\infty[^N \setminus \{(T, \dots, T)\}) \times \mathbb{R}^n$.

Let us consider the value function associated with problem $\mathcal{P}[t,x]$:

$$\mathbf{V}_\psi(t,x) = \inf_{(w,\mathbf{a}) \in \mathcal{A}(t)} \Big\{\psi(y(S)) + \int_0^S \Big(\sum_{i=1}^N \ell_i(y(s), a_i \circ \tau_i(s)) w_i(s)\Big)\, ds\Big\}.$$

The map $\mathbf{V}_\psi(\cdot,\cdot)$ will be called the *Best Value* of the multi-time control problem (1.1)–(1.2).

[18] See the two proofs of Theorem 7.1.

[19] This is equivalent to say that τ is chosen among simple multi-time paths connecting t with $(T, \dots, T)$.

Theorem 6.1 (Regularity of the Best Value)**.** *Let us assume hypothesis* $[\mathcal{H}]_{\mathrm{Lip}}$.

(i) *For any continuous map* ψ, *the map* $\mathbf{V}_\psi(\cdot,\cdot)$ *is continuous on* $[0,T]^N \times \mathbb{R}^n$.

(ii) *If the function* ψ *is locally Lipschitz continuous, then* $\mathbf{V}_\psi(\cdot,\cdot)$ *is locally Lipschitz continuous as well.*

(iii) *If the function* ψ *is globally Lipschitz continuous and the dynamics* f_i *and the Lagrangians* ℓ_i *are bounded and globally Lipschitz continuous uniformly with respect to the controls, then* $\mathbf{V}_\psi(\cdot,\cdot)$ *turns out to be globally Lipschitz continuous.*

Proof. For any t, x, $(w,\mathbf{a}) \in \mathcal{A}(t)$, and $S \in [0,NT]$ we set

$$I(t,x,w,\mathbf{a},S) = \psi(y(S)) + \int_0^S \Big(\sum_{i=1}^N \ell_i(y(s), a_i \circ \tau_i(s)) w_i(s) \Big)\, ds$$

where $\tau(\cdot) = \tau_w[t](\cdot)$ and $y(\cdot) = y_{(w,\mathbf{a})}[t,x](\cdot)$. By applying Gronwall's inequality, one has that, for any $R > 0$ and for all $(s,x) \in [0,S] \times \overline{B(0,R)}$,

$$|y(s)| \le |x| + M(1+|x|)se^{Ms} \le R + M(1+R)NTe^{MNT} \doteq C_1(R), \tag{6.3}$$

where M is the same constant introduced in assumption $[\mathcal{H}]_{\mathrm{Lip}}$.

For a fixed $t \in [0,T]^N$, the regularity properties in the variable x can be deduced by standard arguments. Moreover, such properties turn out to be uniform with respect to the variable t as the latter ranges in $[0,T]^N$.

Let us fix $x \in \mathbb{R}^n$. The proof of the continuity of $\mathbf{V}_\psi(\cdot,x)$ requires some care because of the dependence of the admissible control set $\mathcal{A}(t)$ on t. Let $(\bar{t},x)$, $(\tilde{t},x) \in [0,T]^N \times \mathbb{R}^n$ and suppose that $\mathbf{V}_\psi(\bar{t},x) \ge \mathbf{V}_\psi(\tilde{t},x)$. Since

$$\begin{aligned}\mathbf{V}_\psi(\bar{t},x) - \mathbf{V}_\psi(\tilde{t},x) = {} & [\mathbf{V}_\psi(\bar{t},x) - \mathbf{V}_\psi(\tilde{t}_1,\bar{t}_2,\dots,\bar{t}_N,x)] \\ & + [\mathbf{V}_\psi(\tilde{t}_1,\bar{t}_2,\dots,\bar{t}_N,x) - \mathbf{V}_\psi(\tilde{t}_1,\tilde{t}_2,\bar{t}_3,\dots,\bar{t}_N,x)] \\ & + \cdots + [\mathbf{V}_\psi(\tilde{t}_1,\dots,\tilde{t}_{N-1},\bar{t}_N,x) - \mathbf{V}_\psi(\tilde{t},x)],\end{aligned}$$

we can consider only the case where $\bar{t}_i \neq \tilde{t}_i$ for a given i, and $\bar{t}_j = \tilde{t}_j$ for $j \neq i$. For any $\varepsilon > 0$, let $(w,\mathbf{a}) \in \mathcal{A}(\tilde{t})$ be a control such that

$$\mathbf{V}_\psi(\tilde{t},x) \ge I(\tilde{t},x,w,\mathbf{a},S) - \varepsilon,$$

where $S = \sum_{r=1}^N (T - \tilde{t}_r)$. In general, the control $(w,\mathbf{a})$ does not belong to $\mathcal{A}(\bar{t})$, so let us construct a second control pair $(\bar{w},\bar{\mathbf{a}}) \in \mathcal{A}(\bar{t})$ as follows.

If $\tilde{t}_i < \bar{t}_i$, we consider the control $\bar{\mathbf{a}} = (\bar{a}_1,\dots,\bar{a}_N)$ such that $\bar{a}_j = a_j$ for all $j \neq i$ and $\bar{a}_i(r)$ coincides with the restriction of $a_i(r - (\bar{t}_i - \tilde{t}_i))$ to $[\bar{t}_i, T]$. Furthermore, let us set

$$\bar{s} \doteq \inf\{s > 0 : (\tau_w[\tilde{t}\,])_i(s) > T - (\bar{t}_i - \tilde{t}_i)\} \quad (\le S)$$

and define the control

$$\bar{w}(s) = w(s)\chi_{[0,\bar{s}]}(s) + \left[w(s) - \left\langle w(s), \frac{\partial}{\partial t_i} \right\rangle \frac{\partial}{\partial t_i} \right] \chi_{]\bar{s},S]}(s)$$

for all s. Notice that $\bar{w}$ is not yet an admissible control, in that it takes the value $0 \notin \{\partial/\partial t_1, \dots, \partial/\partial t_N\}$ on a finite number of intervals. However, it is clear that on the intervals where $\bar{w} = 0$ the corresponding solution to (6.1) and the cost function I remain constant. Let us consider the control one obtains from $\bar{w}$ when, loosely speaking, all the intervals where $\bar{w} = 0$ are removed and the remaining intervals are translated backwards, so that, in particular, the corresponding simple multi-time path $\bar{\tau}$ verifies $\bar{\tau}(\bar{S}) = (T, \dots, T)$ at $\bar{S} = S - (\bar{t}_i - \tilde{t}_i)$. There is no danger of confusion in using again the notation $(\bar{w}, \bar{\mathbf{a}})$ for the so obtained control. In particular, $(\bar{w}, \bar{\mathbf{a}})$ turns out to belong to $\mathcal{A}(\bar{t})$.

If $\tilde{t}_i > \bar{t}_i$, let us fix an arbitrary $\bar{a} \in A$ and consider the control $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_N)$ such that $\bar{a}_j = a_j$ for all $j \neq i$, and

$$\bar{a}_i(r) = a_i(r - (\bar{t}_i - \tilde{t}_i))\chi_{[\bar{t}_i, T+(\bar{t}_i - \tilde{t}_i)]}(r) + \bar{a}\chi_{]T+(\bar{t}_i - \tilde{t}_i), T]}(r) \quad \text{for all } r.$$

Moreover, we define the control

$$\bar{w}(s) = w(s)\chi_{[0,S]}(s) + \frac{\partial}{\partial t_i}\chi_{]S,S+(\tilde{t}_i - \bar{t}_i)]}(s)$$

for all s. Clearly, the control pair $(\bar{w}, \bar{\mathbf{a}})$ belongs to $\mathcal{A}(\bar{t})$.

Setting $\tilde{y}(s) = y_{(w,\mathbf{a})}[\tilde{t}, x](s)$ and $\bar{y}(s) = y_{(\bar{w},\bar{\mathbf{a}})}[\bar{t}, x](s)$ for all s, standard estimates yield that

$$|\tilde{y}(s) - \bar{y}(s)| \leq M(1 + |x|)e^{(N-1)TL_{C_1}}|\bar{t}_i - \tilde{t}_i| \doteq C_4|\bar{t}_i - \tilde{t}_i|,$$

and

$$|I(\bar{t}, x, \bar{w}, \bar{\mathbf{a}}, \bar{S}) - I(\tilde{t}, x, w, \mathbf{a}, S)| \leq C_5|\bar{t}_i - \tilde{t}_i| + \omega(C_4|\bar{t}_i - \tilde{t}_i|, C_1),$$

where $C_1 = C_1(|x|)$ is the same as in (6.3), the parameters M, L_{C_1} are the same as in assumption $[\mathcal{H}]_{\mathrm{Lip}}$, $C_5 = NTL_{C_1}C_4 + NM_{C_1}$, and, for any $r > 0$, $\omega(\cdot, r)$ is the modulus of continuity of ψ in $\overline{B(0,r)}$. Hence

$$0 \leq \mathbf{V}_\psi(\bar{t}, x) - \mathbf{V}_\psi(\tilde{t}, x) \leq C_5|\bar{t} - \tilde{t}| + \omega(C_4|\bar{t} - \tilde{t}|, C_1) + \varepsilon,$$

which, by the arbitrariness of ε, proves the continuity of $\mathbf{V}_\psi(\cdot, x)$ for any x. If ψ is locally Lipschitz continuous, the same estimate implies that $\mathbf{V}_\psi(\cdot, x)$ is locally Lipschitz continuous in t. Furthermore, $\mathbf{V}_\psi(\cdot, x)$ turns out to be globally Lipschitz continuous in t whenever ψ is Lipschitz continuous and the functions f_i, ℓ_i are globally Lipschitz continuous in x, uniformly with respect to the controls, i.e., L_R in $[\mathcal{H}]_{\mathrm{Lip}}$ does not depend on R, and moreover, all the f_i, ℓ_i are bounded. $\square$

Theorem 6.2 (A HJ equation for the Best Value)**.** *For any continuous function ψ, the map*

$$\mathbf{W}_\psi(t,x) \doteq \mathbf{V}_\psi(T-t_1,\dots,T-t_N,x) \quad (t,x)\in[0,T]^N\times\mathbb{R}^N$$

is the unique viscosity solution of (3.1)*,* (3.2)*,* (3.3)*.*

Proof. Because of hypothesis $[\mathcal{H}]_{\mathrm{Lip}}$, the uniqueness of the solution is a consequence of Theorem 3.4. By standard results on exit-time value functions (see [12], [2]), $\mathbf{W}_\psi$ turns out to be a viscosity solution of the equation

$$\max_{w\in\{\partial/\partial t_1,\dots,\partial/\partial t_N\}} \sum_{i=1}^N \left[\frac{\partial u}{\partial t_i} + H_i(x,D_x u)\right] w_i = 0 \tag{6.4}$$

in $]0,T]^N\times\mathbb{R}^n$. Moreover, it is a viscosity supersolution of (6.4) on $\partial_0(\,]0,T]^N\times\mathbb{R}^n)$ (see the definition of $\partial_0(\,]0,T]^N\times\mathbb{R}^n)$ in Section 3). Finally, in view of the continuity result established in Theorem 6.1, $\mathbf{W}_\psi$ satisfies the boundary condition

$$\mathbf{W}_\psi(0,x) = \psi(x)$$

for all $x\in\mathbb{R}^n$. Since

$$\begin{aligned}\max_{w\in\{\partial/\partial t_1,\dots,\partial/\partial t_N\}} &\sum_{i=1}^N [p_{t_i} + H_i(x,p)]w_i \\ &= \max\{p_{t_1}+H_1(x,p),\dots,p_{t_N}+H_N(x,p)\},\end{aligned}$$

for all $p_{t_1},\dots,p_{t_N}\in\mathbb{R}$, $p\in\mathbb{R}^n$, and $x\in\mathbb{R}^n$, this concludes the proof. □

As a Corollary of Theorem 5.1 (and the definition of the Best Value) we obtain a further representation formula for the solution of (1.1)–(1.2).

Theorem 6.3. *Let us assume hypotheses $[\mathcal{H}]_{\mathrm{Lip}}$, $[\mathcal{H}]_{\mathrm{CCZLB}}$, and for any continuous map ψ let $U_\psi = U_\psi(t,x)$ be the unique solution of* (1.1)–(1.2)*. Then*

$$U_\psi(t,x) = \mathbf{W}_\psi(t,x) \quad \forall\,(t,x)\in[0,T]^N\times\mathbb{R}^n.$$

7. Regularity of the Solution and Semigroup Properties

Theorem 7.1 (Regularity)**.** *Let us assume hypotheses $[\mathcal{H}]_{\mathrm{Lip}}$, $[\mathcal{H}]_{\mathrm{CCZLB}}$. If the function ψ is locally Lipschitz continuous, the solution U_ψ of* (1.1)–(1.2) *is locally Lipschitz continuous. Moreover, if the function ψ is globally Lipschitz continuous and the vector fields $\hat{f}_i(\cdot,a)$ are globally Lipschitz continuous, uniformly with respect to the controls a, and bounded, then the solution U_ψ turns out to be globally Lipschitz continuous.*

Theorem 7.2 (Semigroup Property). *Let us assume hypotheses* $[\mathcal{H}]_{\mathrm{Lip}}$, $[\mathcal{H}]_{\mathrm{CCZLB}}$. *Then for every continuous map* ψ *and every pair* $(t, \bar{t}) \in [0,T]^N \times [0,T]^N$ *such that* $t + \bar{t} \in [0,T]^N$ *one has*

$$U_{U_\psi(t,\cdot)}(\bar{t}, x) = U_\psi(t + \bar{t}, x) \tag{7.1}$$

for all $x \in \mathbb{R}^N$.

First proof of Theorem 7.1. Since the solution coincides with $W_\psi^{1,\dots,N}(t,x) = V_\psi^{1,\dots,N}(T - t_1, \dots, T - t_N, x)$, the result is a corollary of Theorem 5.4. ◻

Second proof of Theorem 7.1. On the basis of Theorem 3.2 and Proposition 3.5 the solution is unique and coincides with the solution $\mathbf{W}_\psi$ of the auxiliary Hamilton-Jacobi equation (3.1). Hence the regularity of U_ψ is simply a direct consequence of Theorem 6.1. ◻

Proof of Theorem 7.2. By the additivity property (2.7), for every permutation $(i_1, \dots, i_N)$ of $(1, \dots, N)$ one has

$$[x]\psi \mathbf{e}^{-\tau H_i} \mathbf{e}^{-\sigma H_j} = [x]\psi \mathbf{e}^{-\sigma H_j} \mathbf{e}^{-\tau H_i}$$

for all $i, j = 1, \dots, N$, $i \neq j$ $\tau, \sigma \in [0,T]$. Therefore, by (2.6), one has

$$\begin{aligned} U_\psi(t + \bar{t}, x) &= W_\psi^{1,\dots,N}(t + \bar{t}, x) \\ &= [x]\psi \mathbf{e}^{-(t_N + \bar{t}_N)H_N} \cdots \mathbf{e}^{-(t_1 + \bar{t}_1)H_1} \\ &= [x]\psi \mathbf{e}^{-t_N H_N} \mathbf{e}^{-\bar{t}_N H_N} \cdots \mathbf{e}^{-t_1 H_1} \mathbf{e}^{-\bar{t}_1 H_1} \\ &= [x]\psi \mathbf{e}^{-t_N H_N} \cdots \mathbf{e}^{-t_1 H_1} \mathbf{e}^{-\bar{t}_N H_N} \cdots \mathbf{e}^{-\bar{t}_1 H_1} \\ &= W^{1,\dots,N}_{\psi \mathbf{e}^{-t_N H_N} \cdots \mathbf{e}^{-t_1 H_1}}(\bar{t}, x) \\ &= W^{1,\dots,N}_{W_\psi^{1,\dots,N}(t,\cdot)}(\bar{t}, x) \\ &= U_{U_\psi(t,\cdot)}(\bar{t}, x), \end{aligned}$$

which proves (7.1). ◻

8. MULTI-TIME FRONT PROPAGATION

Let us assume Hypotheses $[\mathcal{H}]_{\mathrm{Lip}}$ and $[\mathcal{H}]_{\mathrm{CCZLB}}$. In Section 5 we have proved that under these hypotheses a unique solution exists for the multi-time problem (1.1)–(1.2). In order to keep track of the dependence on the Hamiltonians let us use $U_\psi^{(H_1,\dots,H_N)}$—instead of U_ψ—to denote this solution. In the case of a single Hamiltonian it is well-known that, as soon as the Hamiltonian is positively 1-homogeneous in p, the propagations of the zero level and sub-level depend only on the zero and sub-zero levels of the initial datum (i.e., these propagations coincide for two initial data whose zero level and sub-level are equal). It is then natural to wonder if a similar property—which we call the *Front Propagation Property* holds true for multi-time systems as well. To begin with, let us state rigorously this property. For a given map $\phi : \mathbb{R}^n \to \mathbb{R}$ we shall use $\{\phi(\cdot) < 0\}$ and $\{\phi(\cdot) = 0\}$, respectively, to denote the sets $\{x \in \mathbb{R}^n : \phi(x) < 0\}$ and $\{x \in \mathbb{R}^n : \phi(x) = 0\}$.

Definition 8.1. We say that the Hamiltonians $H_1, \ldots, H_N$ verify the *Front Propagation Property* (FPP) if for every pair of continuous functions ψ, $\tilde{\psi}$ such that

$$\{\psi(\cdot) < 0\} = \{\tilde{\psi}(\cdot) < 0\},$$
$$\{\psi(\cdot) = 0\} = \{\tilde{\psi}(\cdot) = 0\},$$

one has

$$\{U_\psi^{(H_1,\ldots,H_N)}(t_1, \ldots, t_N, \cdot) < 0\} = \{U_{\tilde{\psi}}^{(H_1,\ldots,H_N)}(t_1, \ldots, t_N, \cdot) < 0\},$$
$$\{U_\psi^{(H_1,\ldots,H_N)}(t_1, \ldots, t_N, \cdot) = 0\} = \{U_{\tilde{\psi}}^{(H_1,\ldots,H_N)}(t_1, \ldots, t_N, \cdot) = 0\},$$

for all multi-times $(t_1, \ldots, t_N) \in [0, T]^N$.

Theorem 8.2 (Front propagation for multi-time systems)**.** *Let us assume that each Hamiltonian H_i is positively* 1*-homogeneous.*[20] *Then* (FPP) *is verified for all* $(t_1, \ldots, t_N) \in [0, T]^N$.

Remark. Notice that under our structural hypothesis on the Hamiltonians H_i, the 1-homogeneity hypothesis (in p) is equivalent to the fact that all ℓ_i are equal to zero. Moreover, the 1-homogeneity hypothesis implies that (FPP) holds if and only if for every real number r the property $(\mathrm{FPP})^r$ holds as well, where $(\mathrm{FPP})^r$ is obtained by (FPP) by replacing zero with r (so that, in particular, $(\mathrm{FPP})^0$ coincides with (FPP)).

Proof of Theorem 8.2. Let $t = (t_1, \ldots, t_N) \in [0, T]^N$ and let ψ, $\tilde{\psi}$ be continuous maps such that

$$\{\psi(\cdot) < 0\} = \{\tilde{\psi}(\cdot) < 0\},$$
$$\{\psi(\cdot) = 0\} = \{\tilde{\psi}(\cdot) = 0\}.$$

If $(t_1, \ldots, t_N) = (0, \ldots, 0)$, then there is nothing to prove. If $(t_1, \ldots, t_N) \neq (0, \ldots, 0)$, let us set

$$j = \inf\{i = 1, \ldots, N : t_h = 0 \quad \forall\, h > i\}$$

[20] I.e., $H_i(x, \lambda p) = \lambda H_i(x, p)$ for all $(x, p, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty[$. Sometimes this property is referred to as the *geometric property*.

and let us proceed inductively on j. Let us recall that, by Theorem 5.1,

$$U_{\psi}^{(H_1,\ldots,H_N)}(t_1,\ldots,t_N,x) = [x]\psi\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_NH_N}.$$

If $j = 1$, then $U_{\psi}^{(H_1,\ldots,H_N)}(t_1,\ldots,t_N,x) = [x]\psi\mathbf{e}^{-t_1H1}$, so we are in the case of a single equation for which the property is known to be true (see e.g.[6], [10]). Now assume that the thesis holds true for a $j > 1$ and let us prove that it is valid for $j+1$ as well. Indeed, the inductive hypothesis implies that

$$\begin{aligned}\{x \in \mathbb{R}^n : [x](\psi\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_jH_j}) < 0\}\\ = \{x \in \mathbb{R}^n : [x](\tilde{\psi}\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_jH_j}) < 0\}\end{aligned}$$

and

$$\begin{aligned}\{x \in \mathbb{R}^n : [x](\psi\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_jH_j}) = 0\}\\ = \{x \in \mathbb{R}^n : [x](\tilde{\psi}\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_jH_j}) = 0\}.\end{aligned}$$

Since (FPP) holds for a single equation, we deduce

$$\begin{aligned}&\{U_{\psi}^{(H_1,\ldots,H_N)}(t_1,\ldots,t_N,\cdot) < 0\}\\ &\quad= \{x \in \mathbb{R}^n : [x](\psi\mathbf{e}^{-t_1H1}\cdots\mathbf{e}^{-t_jH_j})\mathbf{e}^{-t_{j+1}H_{j+1}} < 0\}\\ &\quad= \{x \in \mathbb{R}^n : [x](\tilde{\psi}\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_jH_j})\mathbf{e}^{-t_{j+1}H_{j+1}} < 0\}\\ &\quad= \{U_{\tilde{\psi}}^{(H_1,\ldots,H_N)}(t_1,\ldots,t_N,\cdot) < 0\}\end{aligned}$$

and

$$\begin{aligned}&\{U_{\psi}^{(H_1,\ldots,H_N)}(t_1,\ldots,t_N,\cdot) = 0\}\\ &\quad= \{x \in \mathbb{R}^n : [x](\psi\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_jH_j})\mathbf{e}^{-t_{j+1}H_{j+1}} = 0\}\\ &\quad= \{x \in \mathbb{R}^n : [x](\tilde{\psi}\mathbf{e}^{-t_1H_1}\cdots\mathbf{e}^{-t_jH_j})\mathbf{e}^{-t_{j+1}H_{j+1}} = 0\}\\ &\quad= \{U_{\tilde{\psi}}^{(H_1,\ldots,H_N)}(t_1,\ldots,t_N,\cdot) = 0\}\end{aligned}$$

so the thesis is verified for $j+1$ as well, which concludes the proof. □

9. AN OPEN QUESTION

The main achievement of this paper is the replacement of the Zero Poisson Bracket condition $[\mathcal{BT}]_{\mathrm{ZPB}}$ with the Constant Control Zero Lie Bracket condition $[\mathcal{H}]_{\mathrm{CCZLB}}$, which allows to deal with nonsmooth Hamiltonians.

However, it must be noticed that both these conditions imply (under different regularity conditions) existence of a solution to the multi-time problem (1.1)–(1.2) for *every* initial datum ψ. Then a natural question is: if neither $[\mathcal{BT}]_{\mathrm{ZPB}}$ nor $[\mathcal{H}]_{\mathrm{CCZLB}}$ are verified, can a solution exist for some particular choice of the initial datum ψ?

We conclude this paper with an example showing that such a case can actually occur. Moreover, the example suggests that, at least for linear systems, some intrinsic condition must be satisfied by the function ψ in order that a solution u such that $u(0,x) = \psi(x)$ may exist.

Example. Let us consider the multi-time system

$$\begin{cases} \dfrac{\partial u}{\partial t_1} + H_1(x, D_x u) = 0, \\ \dfrac{\partial u}{\partial t_2} + H_2(x, D_x u) = 0, \\ u(0,0,x) = \psi(x), \end{cases} \tag{9.1}$$

where

$$H_1(x,p) = p_2(1 + (x_1)^3), \quad H_2(x,p) = p_1.$$

Notice that one can write

$$H_1(x,p) = p \cdot f_1(x), \quad H_2(x,p) = p \cdot f_2(x)$$

where

$$f_1(x) = (1 + (x_1)^3)\frac{\partial}{\partial x_2}, \quad f_2(x) = \frac{\partial}{\partial x_1}.$$

In particular, one has

$$[f_1, f_2]x = -3(x_1)^2 \frac{\partial}{\partial x_2},$$

which implies

$$\{H_1, H_2\}(x,p) = p \cdot [f_1, f_2]x = -3p_2(x_1)^2.$$

Therefore the commutativity condition $[\mathcal{H}]_{\mathrm{CCZLB}}$ (which in this case is equivalent to the Zero Poisson Bracket condition (ZPB)) is *not* verified for this system. Actually it is easy to verify that there is no solution for $\psi := \pi_2$ where π_2 denotes the projection on the second coordinate, that is, $\pi_2(x) = x_2$.[21] Yet, for every differentiable map $\psi = \psi(x_1, x_3)$ depending only on x_1 and on x_3 it is straightforward to check that the map

$$u(t_1, t_2, x) = \psi(x_1 - t_2, x_3)$$

is a classical solution of (9.1).

[21]Indeed by applying the theory of characteristics, if u were a solution, one would obtain $u(t_1,t_2,x) = \pi_2(xe^{-t_2 f_2}e^{-t_1 f_1})$, where we have used the notation xe^{sg} to denote the solution of the Cauchy problem $\dot{x} = g(x)$ at time s. In particular, one would have

$$0 = \pi_2(0,0,0) = \pi_2 \circ ((0,0,0)e^{-t_1 f_1}e^{-t_2 f_2}e^{t_1 f_1}e^{t_2 f_2}) = -t_1(t_2)^3,$$

which is false for every pair $t_1, t_2 \in \mathbb{R} \setminus \{0\}$.

Remark. In the previous example we have seen that a solution exists provided

$$\frac{\partial \psi}{\partial x_2} = 0.$$

It is not difficult to check that the maps ψ satisfying this relation are in fact the only ones for which a solution exists.

Notice the intrinsic character of the above condition, for it can be written as the Hamilton-Jacobi equation

$$K(x, D\psi(x)) = 0,$$

where $K(x,p) = p \cdot g(x)$ and $g(x) = \partial/\partial x_2$.

Hence it appears plausible that in the general case some sufficient condition can be found on ψ in order that a solution to the multi-time problem (1.1)–(1.2) does exist.

APPENDIX A. PROOF OF LEMMA 2.4

Let us use $u_\psi(s,x)$ to denote the solution to (2.3).

Step 1 Let us begin by assuming that $s(\cdot)$ is C^1 with $s'(\sigma) > 0 \ \forall \sigma \in]\sigma_0, \sigma_1[$. Hence the inverse function of $s(\cdot)$ is C^1 as well and its derivative is strictly positive. Let us denote the inverse function of $s(\cdot)$ by $\sigma = \sigma(\cdot)$. Let us show that $v(\sigma,x) \doteq u_\psi(s(\sigma),x)$ is a viscosity subsolution of (2.4) in $]\sigma_0,\sigma_1[\times \mathbb{R}^n$. Let $\varphi \in C^1([\sigma_0,\sigma_1]\times\mathbb{R}^n)$ and let $(\bar\sigma,\bar x) \in]\sigma_0,\sigma_1[\times\mathbb{R}^n$ be a local maximum point of $v(\sigma,x) - \varphi(\sigma,x)$. Hence setting

$$\bar s \doteq s(\bar\sigma), \quad \tilde\varphi(s,x) \doteq \varphi(\sigma(s),x),$$

one has that $(\bar s,\bar x)$ is a local maximum point of $u(s,x) - \tilde\varphi(s,x)$. Then

$$\frac{\partial \tilde\varphi}{\partial s}(\bar s,\bar x) + H(\bar s,\bar x, D_x\tilde\varphi(\bar s,\bar x)) \le 0.$$

Since $s'(\bar\sigma) > 0$ and $(\partial\tilde\varphi/\partial s)(\bar s,\bar x)s'(\bar\sigma) = (\partial\varphi/\partial\sigma)(\bar\sigma,\bar x)$, multiplying the above expression by $s'(\bar\sigma)$ one obtains that

$$\frac{\partial \varphi}{\partial \sigma}(\bar\sigma,\bar x) + H(s(\bar\sigma),\bar x, D_x\varphi(\bar\sigma,\bar x))s'(\bar\sigma) \le 0,$$

which proves that v is a viscosity subsolution of (2.4) at $(\bar\sigma,\bar x)$. The proof that v is a viscosity supersolution of (2.4) in $]\sigma_0,\sigma_1[\times\mathbb{R}^n$ is akin, so we omit it. Hence v is a viscosity solution of (2.4).

Since $s(\cdot)$ is a C^1 diffeomorphism, by applying the previous arguments in the opposite direction we obtain that v is a viscosity solution of (2.4) if and only if u is a viscosity solution of (2.3). In particular, (2.4) admits a unique solution as soon as this holds true for (2.3).

Step 2 Let $s : [\sigma_0, \sigma_1] \to [s_0, s_1]$ be an absolutely continuous, strictly increasing, surjective map such that $s'(\sigma) > 0$ for a.e. $\sigma \in]\sigma_0, \sigma_1[$. Let us denote by ω the function defined as

$$\omega(\sigma) = \begin{cases} s'(\sigma) & \sigma \in [\sigma_0, \sigma_1], \\ 0 & \sigma \in \mathbb{R} \setminus [\sigma_0, \sigma_1]. \end{cases}$$

Hence $\omega \in L^1(\mathbb{R})$ and $\mathrm{supp}(\omega) = \overline{\{\sigma \in \mathbb{R} : \omega(\sigma) \neq 0\}} \subset [\sigma_0, \sigma_1]$. Moreover, let us set

$$\rho(\sigma) = \begin{cases} e^{1/(\sigma^2-1)} & |\sigma| < 1, \\ 0 & |\sigma| \geq 1, \end{cases}$$

and let us consider the sequence of mollifiers $\{\rho_n\}_n \subset C_c^\infty(\mathbb{R})$ defined by

$$\rho_n(\sigma) \doteq Cn\rho(n\sigma), \quad \text{where } C \doteq \frac{1}{\int_{\mathbb{R}} \rho(r)\, dr}.$$

Then the maps

$$\omega_n(\sigma) \doteq \omega * \rho_n(\sigma) = \int_{\mathbb{R}} \omega(\sigma - r)\rho_n(r)\, dr$$

belong to $C_c^\infty(\mathbb{R})$ for all n and verify

$$\lim_n \int_{\mathbb{R}} |\omega_n(\sigma) - \omega(\sigma)|\, d\sigma = 0.$$

Observe that for all n one has $\omega_n(\sigma) > 0$ for every $\sigma \in]\sigma_0, \sigma_1[$. Hence for any n the map

$$s_n(\sigma) \doteq s_0 + \int_{\sigma_0}^{\sigma} \omega_n(r)\, dr$$

is C^1 from $[\sigma_0, \sigma_1]$ to $s_n([\sigma_0, \sigma_1])$, $s_n'(\sigma) > 0\ \forall\, \sigma \in]\sigma_0, \sigma_1[$. Moreover, the $s_n(\cdot)$'s converge uniformly to $s(\cdot)$ on $[\sigma_0, \sigma_1]$. For any n let us introduce the function $v_n : [\sigma_0, \sigma_1] \times \mathbb{R}^n \to \mathbb{R}$ given by

$$v_n(\sigma, x) \doteq u_\psi(s_n(\sigma), x).$$

Owing to Step 1, for any n the function v_n is the (unique) viscosity solution of

$$\frac{\partial v}{\partial \sigma}(\sigma, x) + H(s_n(\sigma), x, D_x v(\sigma, x))\omega_n(\sigma) = 0, \quad v(\sigma_0, x) = \psi(x).$$

By the continuity of u_ψ and the properties of the ω_n's and s_n's it is now easy to deduce that the v_n's converge uniformly to v on any compact subset of $[\sigma_0, \sigma_1] \times$

$\mathbb{R}^n$ and the $H(s_n(\cdot),x,p)\omega_n(\cdot)$'s converge to $H(s(\cdot),x,p)\omega(\cdot)$ in $L^1([s_0,s_1],C(K))$ for any compact subset $K\subset\mathbb{R}^n\times\mathbb{R}^n$.[22] Thus the stability result in [11], implies that v is a viscosity solution of (2.4), according to Definition 2.2.

In order to prove the uniqueness of the solution to (2.4), let us assume that $w\in C([\sigma_0,\sigma_1]\times\mathbb{R}^n)$ solves (2.4). Let us denote by $\sigma=\sigma(s)$ the inverse function of $s=s(\sigma)$ and let us set $\nu(s,x)=w(\sigma(s),x)$. Let us recall that $\sigma(\cdot)$ turns out to be absolutely continuous and strictly increasing with $\sigma'(s)>0$ for a.e. $s\in[s_0,s_1]$. Therefore by applying the previous arguments in the opposite direction we obtain that ν is a viscosity solution to (2.3). Since problem (2.3) has a unique solution, $\nu(s,x)=u_\psi(s,x)\ \forall\,(s,x)\in[s_0,s_1]\times\mathbb{R}^n$. Hence $w(\sigma,x)=\nu(s(\sigma),x)=u_\psi(s(\sigma),x)=v(\sigma,x)\ \forall\,(\sigma,x)\in[\sigma_0,\sigma_1]\times\mathbb{R}^n$.

APPENDIX B. PROOFS OF THEOREMS 3.3 AND COROLLARY 3.4

These proofs, which we give in detail for the sake of self-consistency, are slight modifications of standard arguments usually exploited in the proof of comparison theorems for single equations (see e.g. Theorem 3.12, III in [4]). We point out that such arguments exploit the natural order on $\mathbb{R}$ and the special role played by the time derivative in the equation. In our case, the time variable is vector valued, but still the corresponding derivatives appear in the equation in a way which allows us to exploit the partial order of $\mathbb{R}^N$.

Proof of Theorem 3.3. Assume by contradiction there are $0<\delta<NT$ and $(\bar t,\bar x)\in\mathcal{D}_C$ such that

$$u_1(\bar t,\bar x)-u_2(\bar t,\bar x)=\delta,\quad |\bar x|\le C|T-\bar t|_1-2\delta. \tag{B.1}$$

Let us choose $M>\sup\{|u_1(t,x)-u_2(s,y)|:(t,x,s,y)\in\mathcal{D}_C^2\}$ $(\ge\delta)$ and $h\in C^1(\mathbb{R})$ such that $h'\le 0$, $h(r)=0$ for $r\le-\delta$, $h(r)=-3M$ for $r\ge 0$. Choose positive parameters ε, η, β and define

$$\begin{aligned}\Phi(t,s,x,y)=u_1(t,x)-u_2(s,y)-\frac{|x-y|^2+\sum_{i=1}^N|t_i-s_i|^2}{2\varepsilon}-\eta|t+s|_1\\ +h((|x|+\beta^2)^{1/2}-C|T-t|_1)+h((|y|+\beta^2)^{1/2}-C|T-s|_1).\end{aligned}$$

Let $(\bar t,\bar s,\bar x,\bar y)\in\overline{\mathcal{D}_C}^2$ be such that

$$\max_{\overline{\mathcal{D}_C}^2}\Phi=\Phi(\bar t,\bar s,\bar x,\bar y).$$

[22] That is,

$$\lim_n\int_{s_0}^{s_1}|H(s_n(r),x(r),p(r))\omega_n(r)-H(s(r),x(r),p(r))\omega(r)|\,dr=0$$

for all pairs of continuous functions $(x,p):[s_0,s_1]\to K$.

We claim that $\min\{|\bar t|_1, |\bar s|_1\} = 0$ or $(\bar t, \bar s, \bar x, \bar y)$ lies in $\mathcal{D}_C^2$, for β and η small enough. Indeed, if, by contradiction, it happens that $|\bar x| = C|T - \bar t|_1$ or $|\bar y| = C|T - \bar s|_1$, by the definition of h we get

$$\Phi(\bar t, \bar s, \bar x, \bar y) \le M - 3M = -2M.$$

On the other hand, by (B.1) it follows that, for any $\beta < \delta$ and $\eta < \delta/4|\bar t|_1$,

$$\max_{\overline{\mathcal{D}_C}^2} \Phi \ge \Phi(\bar t, \bar t, \bar x, \bar x) = \delta - 2\eta|\bar t|_1 + 2h((|\bar x| + \beta^2)^{1/2} - C|T - \bar t|_1) \ge \frac{\delta}{2}, \tag{B.2}$$

which proves the claim.

From the inequality $\Phi(\bar t, \bar t, \bar x, \bar x) + \Phi(\bar s, \bar s, \bar y, \bar y) \le 2\Phi(\bar t, \bar s, \bar x, \bar y)$, we have

$$\frac{|\bar x - \bar y|^2 + \sum_{i=1}^N |\bar t_i - \bar s_i|^2}{\varepsilon} \le u_1(\bar t, \bar x) - u_1(\bar s, \bar y) + u_2(\bar t, \bar x) - u_2(\bar s, \bar y).$$

Hence $|\bar x - \bar y|^2 + \sum_{i=1}^N |\bar t_i - \bar s_i|^2 \le 2M\varepsilon$ and by the continuity of u_1, u_2 it follows that

$$\frac{|\bar x - \bar y|^2 + \sum_{i=1}^N |\bar t_i - \bar s_i|^2}{\varepsilon} \le \omega_1(\varepsilon)$$

for some modulus ω_1.

Furthermore, neither $|\bar t|_1 = 0$ nor $|\bar s|_1 = 0$ for a suitable ε. Indeed, if $|\bar t|_1 = 0$, since $u_1(0, \bar x) \le u_2(0, \bar x)$,

$$\Phi(0, \bar s, \bar x, \bar y) \le u_1(0, \bar x) - u_1(\bar s, \bar y) + u_2(0, \bar x) - u_2(\bar s, \bar y) \le \omega_2(\sqrt{2M\varepsilon}),$$

where ω_2 is the modulus of continuity of u_2 in $\overline{\mathcal{D}_C}$, which contradicts (B.2) as soon as ε is small enough. The proof that $|\bar s|_1 > 0$ is analogous. Hence if we define the test functions

$$\begin{aligned} \phi(t, x) = {} & \frac{|x - \bar y|^2 + \sum_{i=1}^N |t_i - \bar s_i|^2}{2\varepsilon} + \eta|t + \bar s|_1 \\ & - h((|x| + \beta^2)^{1/2} - C|T - t|_1), \\ \gamma(s, y) = {} & -\frac{|\bar x - y|^2 + \sum_{i=1}^N |\bar t_i - s_i|^2}{2\varepsilon} - \eta|\bar t + s|_1 \\ & + h((|y| + \beta^2)^{1/2} - C|T - s|_1), \end{aligned}$$

so that $u_1 - \phi$ has a maximum at $(\bar{t}, \bar{x})$ and $u_2 - \gamma$ has a minimum at $(\bar{s}, \bar{y})$ and use the definition of viscosity sub- and supersolution for (3.1), (3.2), we get that there is some $i = 1, \dots, N$ such that

$$2\eta \le C(h'(Y) + h'(X)) + H_i\left(\bar{y}, \frac{\bar{x} - \bar{y}}{\varepsilon} + h'(Y)\frac{\bar{y}}{(|\bar{y}| + \beta^2)^{1/2}}\right) - H_i\left(\bar{x}, \frac{\bar{x} - \bar{y}}{\varepsilon} - h'(X)\frac{\bar{x}}{(|\bar{x}| + \beta^2)^{1/2}}\right),$$

where $X = (|\bar{x}| + \beta^2)^{1/2} - C|T - \bar{t}\,|_1$ and $Y = (|\bar{y}| + \beta^2)^{1/2} - C|T - \bar{s}|_1$. Now standard calculations together with the hypotheses on the H_i's yield that

$$2\eta \le C(h'(Y) + h'(X)) + \omega(|\bar{x} - \bar{y}|) + \omega\left(\frac{|\bar{x} - \bar{y}|^2}{\varepsilon} + |\bar{x} - \bar{y}|\,|h'(Y)|\right) + C\left|h'(Y)\frac{\bar{y}}{(|\bar{y}| + \beta^2)^{1/2}} + h'(X)\frac{\bar{x}}{(|\bar{x}| + \beta^2)^{1/2}}\right|.$$

Since $h' = -|h'|$, letting ε tend to 0 leads to a contradiction, which completes the proof. □

Proof of Corollary 3.4. As a first step, one easily shows that the comparison result stated in Theorem 3.3 holds also in the set

$$\mathcal{D}_C(x_0) \doteq \left\{(t_1, \dots, t_N, x) : (t_1, \dots, t_N) \in [0, T[^N \setminus \{(0, \dots, 0)\}, \; |x - x_0| < C|T - t|_1\right\}$$

for any x_0, provided the hypotheses on the Hamiltonians H_i introduced there hold for all $x, y \in B(x_0, CT)$ and $u_1(0, x) \le u_2(0, x)$ for all $x \in B(x_0, CT)$.

As a second step, assume that $T < K$. Let both u_1, u_2 be viscosity solutions of (3.1), (3.2), (1.2) in $[0, T]^N \times \mathbb{R}^n$. Fixed $x_0 \in \mathbb{R}^n$, one defines

$$r = \frac{KT(1 + |x_0|)}{1 - KT} > 0 \quad \text{and} \quad C = K(1 + |x_0| + r) > 0.$$

It is now easy to check that for such a choice of C the Hamiltonians H_i satisfy all the hypotheses of Theorem 3.3 in $B(x_0, CT)$. Hence by Step 1 and considering u_1, u_2, as viscosity sub- and supersolutions of (3.1), and of (3.1), (3.2) respectively, it follows that $u_1(t, x) \le u_2(t, x)$ for all $(t, x) \in \mathcal{D}_C(x_0)$. Since $]0, T[^N \times \mathbb{R}^n = \bigcup_{x_0} \mathcal{D}_C(x_0)$, the thesis is proved by interchanging the role of u_1 and u_2.

At this point the proof of the general case $T \ge K$ is obtained by iterating the previous argument on a suitable finite number of N dimensional time intervals of fixed length smaller than $1/K$. □

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