Annali di Matematica pura ed applicata
(IV), Vol. CLXXVII (1999), pp. 117-142

State-Constrained Control Problems with neither Coercivity nor L^1 Bounds on the Controls (*).

M. MOTTA - F. RAMPAZZO

Summary. - *A state-constrained, nonlinear, minimum problem is considered with dynamics depending sublinearly on a control which is not bounded in the L^1 norm. Because of the lack of coercivity, the value map $\mathcal{V}$ fails to be continuous, even in the unconstrained case. However, we prove that under suitable assumptions—which guarantee the continuity of the value maps of the problems with L^1-bounded controls—the value map $\mathcal{V}$ is upper semicontinuous and solves a Bellman equation with a continuous Hamiltonian. Moreover, the map $\tilde{\mathcal{V}}$ obtained by $\mathcal{V}$ by replacing its values at the horizon $t = T$ with the values of the cost function turns out to be the maximal subsolution of the corresponding value problem.*

1. - Introduction.

In dealing with control problems involving unbounded strategies, suitable growth conditions make it possible to define a continuous Hamiltonian. The regularity of the Hamiltonian has obvious consequences, e.g. in the dynamic programming approach, in existence questions, and in the derivation of necessary conditions.

When a coercivity assumption is not in force—a situation which is motivated by several applications, see [10-13, 16, 19, 26, 31, 40, 42, 48]—various kinds of discontinuous behaviours may affect the problem. First of all, it may happen that no optimal trajectories exist among absolutely continuous maps, inasmuch as minimizing sequences often tend to trajectories containing jumps. We recall incidentally that optimal discontinuous trajectories appear naturally in some classical problems of the Calculus of Variations with a slow growth, e.g. in the problem of the minimal surface of revolution. Secondly, neither the value function nor the usual Hamiltonian can be expected to be continuous. This situation is illustrated by the following simple example (see Example 5.1).

(*) Entrata in Redazione il 31 dicembre 1997.

Indirizzo degli AA.: Dipartimento di Matematica Pura e Applicata, Università di Padova, via Belzoni 7, 35131 Padova, Italy; Telefax (39)(49) 8758596.
E-mail: motta@math.unipd.it; rampazzo@math.unipd.it

Let $x[\bar{x}, \xi](\cdot)$ denote the solution of the Cauchy problem

$$\begin{cases} \dot{x} = -|x|\xi & \text{for a.e. } t \in [0, 1], \\ x(0) = \bar{x}, \end{cases}$$

where ξ is a control taking values in $[0, +\infty[$ and consider the problem of minimizing the functional

$$J(\bar{x}, \xi) \doteq |x[\bar{x}, \xi](1) + 1|$$

over controls $\xi \in L^1([0, 1], [0, +\infty[)$. On one hand the «formal» Hamiltonian of this problem, say $\tilde{H}(x, p) = \inf_{\xi \in [0, +\infty[} \{+p|x|\xi\}$, is discontinuous, equal to either 0 or $-\infty$. On the other hand an easy computation shows that the value function $\mathcal{V}(\bar{x}) \doteq \inf_{\xi} J(\bar{x}, \xi)$ is given by

$$\mathcal{V}(\bar{x}) = \begin{cases} 1, & \forall \bar{x} \geq 0, \\ 0, & \forall \bar{x} \in [-1, 0[, \\ |\bar{x} + 1|, & \forall \bar{x} \leq -1. \end{cases}$$

Discontinuities of the value function may also arise from the imposition of state constraints, as is well known in the case of bounded controls. Under the further hypothesis that the L^1 norms of the controls be equibounded, problems with slow growth and state constraints have been already studied from various viewpoints (see [5-7, 18, 27, 28, 32-39, 41, 46, 47]). In particular, it turns out that the corresponding value maps are continuous ([39]), when suitable conditions—see assumption (H1), (H2) in section 2—are assumed on the dynamics at the boundary of the constraint set. In the conventional case—i.e. when the controls take values in a bounded set—such conditions were primarily introduced by H. M. Soner [51]. Successive contributions in this direction are due to P. Loreti [29], I. Capuzzo-Dolcetta and P. L. Lions [14], P. Loreti and E. Tessitore [30], H. Ishii and S. Koike [25].

Obviously, even in the presence of an L^1 bound on the controls, the optimal trajectories have to be searched in a wider class than that of absolutely continuous maps. We remark that, unless commutativity hypotheses on the dynamics are satisfied (see [6, 8, 20, 44, 45, 49, 50]), the class of maps with bounded variation is not sufficiently large to pose the problem well (the above example does not present this difficulty; yet the problem arises as soon as the unbounded control is vector valued). Indeed, because of the slow growth assumption, minimizing sequences of trajectories graphs converge towards *space-time trajectories*. The latter are maps from a pseudo-time interval $[0, 1]$ into space-time which possibly contain arcs of instantaneous evolution (see e.g. [38]). In general a space-time trajectory cannot be identified with the graph of a conventional (possibly discontinuous) trajectory. In order to overcome these difficulties one first embeds the problem into a space-time setting (where t plays the role of a state variable) and then exploits suitable reparametrization techniques. This extended problem turns out to involve only *bounded* controls. Within this space-time setting one is able to formulate a Maximum Principle ([33, 37, 43, 50, 54]) and to establish a Hamilton-Jacobi-Bellman boundary value problem [37, 39] enjoying uniqueness.

In the present paper we remove even the hypothesis of equiboundedness of the L^1 norm of the controls. This implies that a minimizing sequence converges towards objects which are much more involved than space-time trajectories (see e.g. [9]). Moreover, unlike the case with bounded L^1 norm, the value map happens to be discontinuous even in the case where no state constraints are imposed. The above example in fact illustrates such a phenomenon.

Disregarding the problem of representing the limits of minimizing sequences (see [9]), we devote our attention to dynamic programming. The value map $\mathcal{V}$ of the original problem can be approximated by the value maps $\mathcal{V}_K$ of the problems obtained by constraining the L^1 norm of the unbounded control to be less than or equal to K. Indeed, it is straightforward to show that the maps $\mathcal{V}_K$ decrease to $\mathcal{V}$ as K tends to infinity. In particular, this implies that whenever the maps $\mathcal{V}_K$ are continuous the map $\mathcal{V}$ is upper semicontinuous. We recall, incidentally, that the continuity of the maps $\mathcal{V}_K$ is strictly related to the directions of the dynamics on the boundary of the constraint set (see hypotheses (H1) and (H2) below). In particular $\mathcal{V}_K$ is continuous if there are no state constraints. This is the case of the example above, where, for K large enough, $\mathcal{V}_K$ coincides with $\mathcal{V}$ outside a neighborhood of $\bar{x} = 0$.

Via a dynamic programming principle (see Proposition 5.1) stated for the equivalent space-time problem, we prove that $\mathcal{V}$ solves the boundary value problem (BVP) below. The latter must be interpreted in a sense provided by the theory of (discontinuous) viscosity solutions. In the considered example (see Example 5.1) one finds that

i) $\mathcal{V}$ is a viscosity subsolution of

$$- \min_{|(w_0, w)| = 1,\ w_0 \geq 0,\ w \geq 0} \left\{ \frac{\partial \mathcal{V}}{\partial t} w_0 - \frac{\partial \mathcal{V}}{\partial x} |x| w \right\} = 0$$

in $[0, T[\times \mathbb{R}$;

ii) $\mathcal{V}(x) \leq |x + 1|$, $\forall x \in \mathbb{R}$.

Notice that here no state constraints are present (see Definition 5.2 for the general case).

Of course, the question of uniqueness of the solution of the boundary value problem is crucial. In fact, the problem under consideration is quite degenerate: in particular, it does not satisfy a monotonicity hypothesis—the coefficient of $\partial \mathcal{V}/\partial t$ being possibly equal to 0—which is essential in several uniqueness results.

In the general case, where $\mathcal{V}$ is merely upper semicontinuous, simple examples show that one does not have uniqueness, even if $\mathcal{V}$ is continuous (see Example 6.1). However, if we define a map $\tilde{\mathcal{V}}$ by decreeing that $\tilde{\mathcal{V}}$ coincides with the cost function on the (finite) horizon and $\tilde{\mathcal{V}} = \mathcal{V}$ elsewhere, then $\tilde{\mathcal{V}}$ is the maximal upper semicontinuous subsolution of the boundary value problem. This seems to be the best possible characterization of $\mathcal{V}$ by means of the notion of viscosity solution, since examples show that $\mathcal{V}$ does not enjoy minimality properties as a supersolution (see Example 6.2). We remark that other concepts of discontinuous solutions introduced in the recent years are not adeguate to this problem (see Remark 5.1 and the therein quoted references).

The paper is organized as follows. In section 2 we introduce the problem with unbounded controls and its space-time embedding. In section 3 we relate the value function of the problem with the value functions $\mathcal{V}_K$ corresponding to subsets of control functions having L^1 norm less than or equal to K. In section 4 we prove that any space-time trajectory can be approximated by *internal* space-time trajectories, where internal means that the spatial component remains in the interior of the contraint set. This technical result is essential for proving the maximality properties of the value function. Section 5 is devoted to proving that the value function verifies the boundary value problem (BVP) below. In section 6 we give some results which concern the characterization of the value map as the maximal element of the set of subsolutions of the boundary value problem. Finally, motivated by the results of section 6, in the Appendix we discuss some questions related to the continuity of $\mathcal{V}$.

2. - Slow growth control systems and space-time trajectories.

We consider a control system of the form

$$\dot{x} = f(t, x, v, \xi), \qquad x(\bar{t}) = \bar{x}, \tag{2.1}$$

where $0 \le \bar{t} < T$, $t \in [\bar{t}, T]$, and the controls v and ξ take values in a compact set $V \subset \mathbb{R}^q$, and in a closed cone $C \subset \mathbb{R}^m$, respectively. Moreover, the state x is subject to the state-constraint $x \in \overline{\Theta}$, where Θ is an open subset of $\mathbb{R}^n$ and $\overline{\Theta}$ denotes its closure.

We assume f continuous and satisfying the following conditions: for every compact subset $Q \subset \mathbb{R}^n$ there exists a positive $L = L_Q$ such that

$$|f(t, x_1, v, \xi) - f(t, x_2, v, \xi)| \le L(1 + |\xi|)|x_1 - x_2|, \tag{2.2}$$

for all $(t, x_1, v, \xi), (t, x_2, v, \xi) \in [0, T] \times Q \times V \times C$; there exists a constant $A > 0$ such that

$$|f(t, x, v, \xi)| \le A(1 + |\xi|)(1 + |x|), \tag{2.3}$$

for every $(t, x, v, \xi) \in [0, T] \times \mathbb{R}^n \times V \times C$.

Moreover we assume that

there exists a continuous map f^∞, called the recession function *of f, such that*

$$\lim_{r \to +\infty} r^{-1} f(t, x, v, rw) = f^\infty(t, x, v, w) \tag{2.4}$$

uniformly on compact subsets of $[0, T] \times \mathbb{R}^n \times V \times C$. This is actually the main hypothesis on the dynamics. It can be regarded as a *regular and slow growth* assumption.

By the hypotheses on f a unique (Carathèodory) solution to (2.1) exists on $[\bar{t}, T]$ for every control $(v, \xi) \in W(\bar{t}) \doteq \mathcal{B}([\bar{t}, T], V \times C)$, where $\mathcal{B}([a, b], E)$ denotes the set of Borel measurable maps from $[a, b]$ into E which are Lebesgue integrable. Let us denote this solution by $x[\bar{t}, \bar{x}; v, \xi]$, or, whenever the initial data are meant by the context, by $x[v, \xi]$.

Let $W^c(\bar{t}, \bar{x}) \subset W(\bar{t})$ be the subset of controls (v, ξ) such that the corresponding solution agrees with the state constraint, i.e. $x[\bar{t}, \bar{x}; v, \xi](t) \in \overline{\Theta}$ for every $t \in [\bar{t}, T]$. Let $\Psi: \overline{\Theta} \to \mathbb{R}$ be a continuous, bounded map and let us consider the optimal control problem:

$$(\mathcal{P}) \qquad \underset{(v, \xi) \in W^c(\bar{t}, \bar{x})}{\text{minimize}} \; \Psi(x[\bar{t}, \bar{x}; v, \xi](T)) .$$

(Of course all the results of the paper remain valid—up to obvious changes—for a problem involving a Bolza functional). Since neither boundedness assumptions on the controls nor growth assumptions on the vector field are made, in general no absolutely continuous optimal trajectories exist for problem $(\mathcal{P})$.

In [38], in analogy with what had been previously done for the special case where f depended linearly on ξ ([36]), we embed the dynamics (2.1) into a space-time dynamics where time plays the role of a state variable which is *nondecreasing* with respect to the new parameter s.

Let us briefly recall the main points concerning this embedding.

DEFINITION 2.1. – *For every* $(t, x) \in [0, T] \times \mathbb{R}^n$ *and every triple* $(v, w_0, w) \in V \times [0, +\infty[\times C$ *we set*

$$\bar{f}(t, x, v, w_0, w) \doteq \begin{cases} f\left(t, x, v, \dfrac{w}{w_0}\right) \cdot w_0 , & \text{if } w_0 \neq 0 , \\ f^{\infty}(t, x, v, w) , & \text{if } w_0 = 0 . \end{cases}$$

In view of the assumptions on f, $\bar{f}$ is nothing but the continuous extension of $f(t, x, v, w/w_0) \cdot w_0$ to the domain $[0, T] \times \mathbb{R}^n \times V \times [0, +\infty) \times C$. The transformation $f \to \bar{f}$ is the control-theoretical analogue of the transformation which changes a nonparametric integral of the Calculus of Variations into a parametric one—see e.g. [15, sect. 14.2]. Some examples of computation of $\bar{f}$ are provided in [38].

The control system

$$(2.5) \qquad \begin{cases} t'(s) = w_0(s) , \\ x'(s) = \bar{f}(t(s), x(s), v(s), w_0(s), w(s)) , \\ (t(0), x(0)) = (\bar{t}, \bar{x}) , \end{cases}$$

where the parameter s belongs to the standard interval $[0, 1]$ and the superscript denotes differentiation with respect to s, is called the *space-time control system* corresponding to (2.1).

PROPOSITION 2.1. – *If* $(v, \xi) \in W(\bar{t})$ *and* $s \mapsto (t(s), u(s))$ *is a Lipschitz continuous parametrization of the graph of* $t \mapsto u(t) \doteq \int_{\bar{t}}^{t} \xi(\tau)\, d\tau$ *such that* $t'(s) > 0$ *a.e.—e.g.* $(t(s), u(s))$ *is the arc-lenght parametrization of* $(t, u(t))$*—, then, setting* $w_0(s) \doteq t'(s)$, $w(s) \doteq u'(s)$ *and* $(v(s) \doteq v \circ t(s)$, *one has that* $x(t)$ *is the solution to* (2.1) *corresponding to*

(v, ξ) *if and only if* $(t(s), x \circ t(s))$ *is the solution of* (2.5) *corresponding to the control* $(v(s), w_0(s), w(s))$.

Proposition 2.1 gives nothing but the obvious relationship between the trajectories of (2.1) and their (possibly reparametrized) graphs. Yet we allow the control $(v(s), w_0(s), w(s))$ to be a mathematical object more general than a mere reparametrization of the control $(v(t), \xi(t))$. More precisely the *space-time control* (v, w_0, w) is allowed to belong to the set

$$\Gamma(\bar{t}) \doteq \left\{ (v, w_0, w) \in \mathcal{B}([0, 1], V \times [0, +\infty) \times C: \int_0^1 w_0(s)\, ds = T - \bar{t} \right\}.$$

A solution of (2.5) corresponding to a space-time control $(v, w_0, w) \in \Gamma(\bar{t})$ will be called a *space-time trajectory* and will be denoted by $(t, x)[\bar{t}, \bar{x}; v, w_0, w]$ (or, whenever no confusion may arise, by $(t, x)[v, w_0, w]$). We denote by $\Gamma^c(\bar{t}, \bar{x})$ the set of the space-time controls (v, w_0, w) such that $x[\bar{t}, \bar{x}; v, w_0, w](s) \in \overline{\Theta}$ for every $s \in [0, 1]$.

Notice that by allowing $w_0 \equiv 0$ even when $w \not\equiv 0$ we obtain an actual extension of the set $W(\bar{t})$ of the controls of the original system (2.1). Indeed, in view of Proposition 2.1 and Proposition 2.2 below, $W(\bar{t})$ can be identified in a obvious way with the subset $\Gamma^+(\bar{t}) \subset \Gamma(\bar{t})$ defined by

$\Gamma^+(\bar{t}) \doteq \{(v, w_0, w) \in \Gamma(\bar{t})$ satisfying:

$$\text{if } w_0(s) = 0 \text{ for a.e. } s \in [s_1, s_2] \subset [0, 1] \text{ then } w(s) = 0 \text{ for a.e. } s \in [s_1, s_2]\}.$$

PROPOSITION 2.2. – *If* $\sigma: [0, 1] \to [0, 1]$ *is an increasing, surjective map, continuous with its inverse, and* $(v, w_0, w) \in \Gamma(\bar{t})$, *then*

$$x[\bar{t}, \bar{x}; v, w_0, w] \circ \sigma(s) = x[\bar{t}, \bar{x}; v \circ \sigma, (w_0 \circ \sigma) \cdot \sigma', (w \circ \sigma) \cdot \sigma'](s)$$

for every $s \in [0, 1]$.

Moreover let $(v, w_0, w) \in \Gamma^+(\bar{t})$ *and let* $(t, x)(s)$ *denote the corresponding solution to* (2.5). *Then there exists a unique control* $(\tilde{v}, \xi) \in W(\bar{t})$ *such that*

$$(\tilde{v}, \xi) \circ t(s) = \left(v, \frac{w}{w_0}\right)(s)$$

for every $s \in [0, 1]$ *such that* $w_0(s) \neq 0$, *and the solution* $\tilde{x}$ *of* (2.1) *corresponding to* $(\tilde{v}, \xi)$ *verifies*

$$\tilde{x} \circ t(s) = x(s)$$

for every $s \in [0, 1]$.

Since $W(\bar{t})$ is identified with $\Gamma^+(\bar{t})$, the trajectories of (2.5) corresponding to controls in $\Gamma^+(\bar{t})$ will be called *regular*.

In what follows we recall some properties of space-time trajectories and some facts concerning the relationship between the original system (2.1) and its space-time extension (2.5).

PROPOSITION 2.3 [38]. – *Let* $M>0$ *and let* $Q\subset\mathbb{R}^n$ *be a compact set. There are constants* M', $M''>0$, *such that all the space-time trajectories* $(t,x)(\cdot)=$ $=(t,x)[\bar t,\bar x;v,w_0,w](\cdot)$ *with* $(\bar t,\bar x)\in[0,T]\times Q$, $(v,w_0,w)\in\Gamma(\bar t)$, *and* $\|w_0\|_\infty\leq M$, $\|w\|_\infty\leq M$ *satisfy*

$$\begin{cases} |(t,x)(s)|\leq M', & \forall s\in[0,1], \\ |(t,x)(s')-(t,x)(s'')|\leq M''|s'-s''|, & \forall s', s''\in[0,1]. \end{cases} \tag{2.6}$$

As pointed out in [38], in general space-time trajectories corresponding to controls in $\Gamma^c(\bar t,\bar x)$ cannot be approximated by regular trajectories corresponding to controls in $\Gamma^+(\bar t)\cap\Gamma^c(\bar t,\bar x)$. On the contrary, under conditions (H1), (H2) below—which are of the same type as a condition originally introduced by Soner [51] for state-constrained problems—, the set of the regular trajectories corresponding to controls in $\Gamma^+(\bar t)\cap\Gamma^c(\bar t,\bar x)$ is dense in the set of the space-time trajectories associated to $\Gamma^c(\bar t,\bar x)$ in the C^0 topology (see Proposition 2.4).

(H1) There exist a continuous function $\nu_1:[0,T]\times\overline{\Theta}\to V$ and positive constants q_1, r_1 such that for any $(\bar t,\bar x)\in[0,T]\times\overline{\Theta}$ one has

$$B_n[\bar x+hf(\bar t,\bar x,\nu_1(\bar t,\bar x),0);hr_1],\qquad \forall h\in(0,q_1],$$

where $B_l[\bar y;r]$ denotes the closed ball of $\mathbb{R}^l$ with center $\bar y$ and radius r.

(H2) There exist a continuous function $(\nu_2,\omega):[0,T]\times\overline{\Theta}\to V\times(B_m[0,1]\cap C)$ and positive constants q_2, r_2 such that for any $(\bar t,\bar x)\in[0,T]\times\overline{\Theta}$ and any $\overline{w}_0\in[0,1]$ one has

$$B_n[\bar x+h\bar f(\bar t,\bar x,\nu_2(\bar t,\bar x),\overline{w}_0,\omega(\bar t,\bar x));hr_2],\qquad \forall h\in(0,q_2].$$

We remark that for every $(\bar t,\bar x)\in[0,T)\times\overline{\Theta}$ the sets $W^c(\bar t,\bar x)$ and $\Gamma^c(\bar t,\bar x)$ are not empty.

PROPOSITION 2.4 (Density) [38]. – *Assume* (H1), (H2) *and fix* $(\bar t,\bar x)\in[0,T)\times\overline{\Theta}$. *For any* $\varepsilon>0$ *and any control* $(v,w_0,w)\in\Gamma^c(\bar t,\bar x)$ *there is a regular space-time trajectory* $(\check t,\check x)(\cdot)$ *starting from* $(\bar t,\bar x)$ *and such that*

$$\begin{cases} \check x(s)\in\overline{\Theta}, & \forall s\in[0,1], \\ \|(\check t,\check x)-(t,x)\|_\infty\leq\varepsilon, \end{cases} \tag{2.7}$$

where we have set $(t,x)\doteq(t,x)[\bar t,\bar x;v,w_0,w]$ *and* $\|\cdot\|_\infty$ *denotes the sup-norm.*

A sharper density result, involving *internal* trajectories, will be proved in section 4.

The minimum problem $(\mathcal{P})$, once reformulated as

$$\underset{(v,w_0,w)\in\Gamma^+(\bar t)\cap\Gamma^c(\bar t,\bar x)}{\text{minimize}}\quad \Psi(x[v,w_0,w](1)), \tag{$\mathcal{P}$}$$

can be extended to the following space-time problem:

$$(\mathcal{P}_e) \qquad \underset{(v,\, w_0,\, w) \in \Gamma^c(\bar t,\, \bar x)}{\text{minimize}} \Psi(x[v, w_0, w](1)) .$$

Thanks to the density result above we obtain that $(\mathcal{P})$ and $(\mathcal{P}_e)$ have the same infimum value for every $(\bar t, \bar x) \in [0, T) \times \overline{\Theta}$, i.e. the extension is *proper*:

PROPOSITION 2.5. – *Let Ψ be a bounded and continuous real map. Under hypotheses* (H1), (H2), *for every initial condition $(\bar t, \bar x) \in [0, T) \times \overline{\Theta}$ one has*

$$\inf_{(v,\, \xi) \in W^c(\bar t,\, \bar x)} \Psi(x[v, \xi](T)) \left[= \inf_{(v,\, w_0,\, w) \in \Gamma^+(\bar t) \cap \Gamma^c(\bar t,\, \bar x)} \Psi(x[v, w_0, w](1)) \right] = \tag{2.8}$$

$$= \inf_{(v,\, w_0,\, w) \in \Gamma^c(\bar t,\, \bar x)} \Psi(x[v, w_0, w](1)) .$$

3. – The value function.

The value functions of the original problem $(\mathcal{P})$ and of the extended problem $(\mathcal{P}_e)$ are defined on $[0, T) \times \overline{\Theta}$ by

$$\mathcal{F}(\bar t, \bar x) \doteq \inf_{(v,\, \xi) \in W^c(\bar t,\, \bar x)} \Psi(x[\bar t, \bar x; v, \xi](T))$$

and

$$\mathcal{V}(\bar t, \bar x) \doteq \inf_{(v,\, w_0,\, w) \in \Gamma^c(\bar t,\, \bar x)} \Psi(x[\bar t, \bar x; v, w_0, w](1)) ,$$

respectively. Moreover $\mathcal{V}$ is also defined for $\bar t = T$. Since on the basis pf Proposition 2.5 $\mathcal{F}$ and $\mathcal{V}$ coincide on $[0, T) \times \overline{\Theta}$, we shall refer to the *value function* to mean the map $\mathcal{V}$, without further specifications.

The value function $\mathcal{V}$ turns out to be limit of the value functions $\mathcal{V}_K$ of problems $(\mathcal{P}_K)$ obtained by constraining the L^1 norm of ξ to be less than or equal to K. These problems have been investigated in [38, 39], and we refer to these papers for a deeper account on them.

Let us just sketch some facts from [38, 39]. For every $K > 0$ and $(\bar t, \bar x, \bar k) \in [0, T] \times \times \overline{\Theta} \times [0, K]$ let us consider the subset of space-time controls

$$\Gamma^c_{K-\bar k}(\bar t, \bar x) \doteq \left\{ (v, w_0, w) \in \Gamma^c(\bar t, \bar x) \colon \int_0^1 |w(s)|\, ds \leq K - \bar k \right\},$$

and, for every $(\bar t, \bar x, \bar k) \in [0, T] \times \overline{\Theta} \times [0, +\infty[$, let us define the value map

$$\mathcal{V}_K(\bar t, \bar x, \bar k) \doteq \begin{cases} \displaystyle\inf_{(v,\, w_0,\, w) \in \Gamma^c_{K-\bar k}(\bar t,\, \bar x)} \Psi(x[\bar t, \bar x; v, w_0, w](1)) , & \text{if } \bar k \leq K , \\ \mathcal{V}_K(\bar t, \bar x, K), & \text{if } \bar k > K . \end{cases}$$

Observe that the set $\Gamma^c_{K-\bar{k}}(\bar{t},\bar{x})$ is the space-time extension of the subset $W^c_{K-\bar{k}}(\bar{t},\bar{x}) \subset W^c(\bar{t},\bar{x})$ formed by the controls (v,ξ) such that ξ has L^1 norm less than or equal to $K-\bar{k}$. Of course, even in this case one identifies $W^c_{K-\bar{k}}(\bar{t},\bar{x})$ with the subset $\Gamma^c_{K-\bar{k}}(\bar{t},\bar{x}) \cap \Gamma^+(\bar{t})$. $\mathcal{V}_K$ is nothing but the value function of the corresponding minimum problem, here called $(\mathcal{P}_K)$. Actually, $\mathcal{V}_K$ would be defined only for $\bar{k} \leq K$: for technical reasons we extend it to the domain $[0,T]\times\overline{\Theta}\times[0,+\infty[$ by setting $\mathcal{V}_K(\bar{t},\bar{x},\bar{k}) \doteq \mathcal{V}_K(\bar{t},\bar{x},K)$ for $\bar{k}>K$.

PROPOSITION 3.1 [39]. – *Under hypotheses* (H1), (H2) *the map* $\mathcal{V}_K$ *is continuous for every* $K>0$.

The following properties are a straightforward consequence of the definition of $\mathcal{V}_K$.

PROPOSITION 3.2. – *For every* $K>0$ *and every* $(\bar{t},\bar{x},\bar{k}) \in [0,T]\times\overline{\Theta}\times[0,K]$ *one has*

$$\mathcal{V}_K(\bar{t},\bar{x},\bar{k}) = \mathcal{V}_{K-\bar{k}}(\bar{t},\bar{x},0).$$

Moreover for every $(\bar{t},\bar{x},\bar{k}) \in [0,T]\times\overline{\Theta}\times[0,+\infty[$ *the sequence* $\{\mathcal{V}_K(\bar{t},\bar{x},\bar{k})\}_{K\in\mathbb{N}}$ *is nonincreasing.*

In particular the limit $\lim\limits_{K\to\infty} \mathcal{V}_K(\bar{t},\bar{x},\bar{k})$ exists and is bounded below by the infimum of the cost function Ψ. Moreover we have:

PROPOSITION 3.3. – *For every* $(\bar{t},\bar{x},\bar{k}) \in [0,T]\times\overline{\Theta}\times[0,+\infty[$ *one has*

$$\mathcal{V}(\bar{t},\bar{x},) = \mathcal{V}_\infty(\bar{t},\bar{x},\bar{k}) \doteq \lim_{K\to\infty} \mathcal{V}_K(\bar{t},\bar{x},\bar{k}).$$

In particular the limit $\mathcal{V}_\infty$ *is independent of the variable* $\bar{k}$. *Moreover* $\mathcal{V}_\infty (=\mathcal{V})$ *is upper semicontinuous as soon as the maps* $\mathcal{V}_K$ *are continuous.*

PROOF. – Fix $(\bar{t},\bar{x},\bar{k}) \in [0,T]\times\overline{\Theta}\times[0,+\infty[$ and consider a sequence of space-time controls $(v_n, w_{0_n}, w_n) \in \Gamma^c_n(\bar{t},\bar{x})$ such that, setting $x_n \doteq x[\bar{t},\bar{x}; v_n, w_{0_n}, w_n](1)$, one has that the sequence $\Psi(x_n)$ decreases to $\mathcal{V}(\bar{t},\bar{x})$. Since $\Psi(x_n) \geq \mathcal{V}_{n+\bar{k}}(\bar{t},\bar{x},\bar{k}) \geq \mathcal{V}(\bar{t},\bar{x})$ it follows that $\mathcal{V}_{n+\bar{k}}(\bar{t},\bar{x},\bar{k})$ converges to $\mathcal{V}(\bar{t},\bar{x})$ as n goes to ∞. Finally, if the maps $\mathcal{V}_K$ are continuous, $\mathcal{V}_\infty$ is upper semicontinuous because it is the pointwise limit of a nonincreasing sequence of continuous maps.

4. – Approximations with internal trajectories.

For every $(\bar{t},\bar{x}) \in [0,T]\times\overline{\Theta}$ let us denote by $(\Gamma^c_{\mathrm{int}})_K(\bar{t},\bar{x})$ and $\Gamma^c_{\mathrm{int}}(\bar{t},\bar{x})$ the subsets of $\Gamma^c_K(\bar{t},\bar{x})$ and $\Gamma^c(\bar{t},\bar{x})$, respectively, formed by those space-time controls whose corresponding trajectories lie in Θ for all $s\in(0,1]$. Such trajectories will be called *internal*. We now prove a refinement of Proposition 2.4 which will be essential to demonstrate a maximality property of the value function (see section 6). In fact, we show that any space-time trajectory starting from a point $\bar{x}\in\overline{\Theta}$ at a time $\bar{t}<T$ may be approached

by means of internal, regular trajectories. Moreover, if the original trajectory corresponds to a control (v, w_0, w) such that $\|w\|_1 \leq K$, then the controls $(\tilde{v}, \tilde{w}_0, \tilde{w})$ corresponding to the internal approximating trajectories can be chosen to satisfy $\|\tilde{w}\|_1 \leq K$.

THEOREM 4.1. – *Assume* (H1), (H2) *and let* $(\bar{t}, \bar{x}) \in [0, T] \times \overline{\Theta}$, $K > 0$. *For every* $\varepsilon > 0$ *and every control* $(v, w_0, w) \in \Gamma^c_K(\bar{t}, \bar{x})$ *there is a space-time control* $(\bar{v}, \overline{w}_0, \overline{w}) \in (\Gamma^c_{\mathrm{int}})_K(\bar{t}, \bar{x})$ *such that*

$$\|x[v, w_0, w] - x[\tilde{v}, \tilde{w}_0, \tilde{w}]\|_\infty \leq \varepsilon . \tag{4.1}$$

Moreover, if $\bar{t} < T$ *the control* $(\tilde{v}, \tilde{w}_0, \tilde{w})$ *can be chosen in the subset* $(\Gamma^c_{\mathrm{int}})_K(\bar{t}, \bar{x}) \cap \Gamma^+(\bar{t})$.

REMARK 4.1. – It can be easily deduced by the proof of Theorem 4.1 that under the sole hypothesis (H2) an approximation with internal trajectories is still possible. However, if only (H2) is in force, we must allow the approximating control $\tilde{w}$ to have a greater L^1 norm than the L^1 norm of w.

The proof of Theorem 4.1 is a direct consequence of Theorem 4.2 below and is sketched at the end of this section. In turn, the proof of Theorem 4.2 is similar to the proof of Theorem 4.2 of [38]. However the needed changes are quite technical. Hence, for the reader's convenience, we give the proof of Theorem 4.2 in full detail.

Let us fix a compact set $Q \subset \mathbb{R}^n$ and a constant $M > 0$ and let us consider the map

$$\varphi(w) \doteq \max\{|\bar{f}(t, x, v, w_0, w) - \bar{f}(t, x, v, w_0, 0)| : (t, x, v, w_0) \in [0, T] \times Q \times V \times [0, M]\}.$$

We shall make use of the following property of the superposition map $w \mapsto \varphi \circ w$.

LEMMA 4.1 [38]. – *If* $w \in L^1([0, 1], B_m[0, M] \cap C)$ *then* $\varphi \circ w \in L^1([0, 1], \mathbb{R})$, *and for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $w \in L^1([0, 1], B_m[0, M] \cap C)$ *satisfying* $\int_0^1 |w(s)|\, ds \leq \delta$ *one has*

$$\int_0^1 \varphi(w(s))\, ds \leq \varepsilon .$$

THEOREM 4.2. – *Assume* (H1) *and* (H2) *and let* Q *be a compact suset of* $\overline{\Theta}$, $M > 0$. *Then there exists a non negative function* ϱ *continuous and infinitesimal at* 0 *such that, for any* $\varepsilon > 0$, $\bar{y} \doteq (\bar{t}, \bar{x}) \in [0, T] \times Q$ *and any space-time control* $(v, w_0, w) \in \Gamma(\bar{t})$ *with* $\|w_0\|_\infty \leq M$ *and* $\|w\|_\infty \leq M$, *there is an admissible space-time control* $(\check{v}, \check{w}_0, \check{w}) \in \Gamma^c_{\mathrm{int}}(\bar{t}, \bar{x})$ *such that* $\|\check{w}_0\|_\infty \leq M$, $\|\check{w}\|_\infty \leq M$, $\int_0^1 |\check{w}(s)|\, ds \leq a \doteq \int_0^1 |w(s)|\, ds$, *and*

$$|(\check{t}, \check{x})(s) - (t, x)(s)| \leq \varrho(d + \varepsilon), \qquad \forall s \in [0, 1], \tag{4.2}$$

where $(\check{t}, \check{x}) \doteq (t, x)[\bar{t}, \bar{x}; \check{v}, \check{w}_0, \check{w}]$, $(t, x) \doteq (t, x)[\bar{t}, \bar{x}; v, w_0, w]$, *and*

$$d \doteq \sup\{d(x(s), \Theta) : s \in [0, 1]\}. \tag{4.3}$$

Moreover, if $w_0(s) > 0$ *for a.e.* $s \in [0, 1]$ *then* $\check{w}_0(s) > 0$ *as well for a.e.* $s \in [0, 1]$.

PROOF. – In view of the bounds $\|w_0\|_\infty \leq M$, $\|w\|_\infty \leq M$ and by Proposition 2.3, the set of the trajectories of (2.5) corresponding to the space-time controls of $\Gamma(\bar{t})$ is equibounded and equilipschitzian; hence $\bar{f}$ can be replaced by a bounded and uniformly continuous function, which coincides with $\bar{f}$ on a compact set $[0, T] \times (\bar{\Theta} \cap B_n[0, M']) \times \times V \times (B_{1+m}[0, 2M] \cap ([0, +\infty) \times C))$, where M' is defined as in Proposition 2.3. For simplicity we denote this function again by $\bar{f}$. Then, by (H2) it follows that if $|(t, x, v, w_0, w) - (t', x', v', w_0', w')| \leq 6\bar{\delta}$ for $\bar{\delta} > 0$ sufficiently small, one has

$$\begin{cases} |(\nu_2, \omega)(t, x) - (\nu_2, \omega)(t', x')| \leq r_2/2\,, \\ |\bar{f}(t, x, v, w_0, w) - \bar{f}(t', x', v', w_0', w')| \leq r_2/2\,, \end{cases} \tag{4.4}$$

where (ν_2, ω) and r_2 are the same as in hypothesis (H2). Let us consider a space-time control $(v, w_0, w) \in \Gamma(\bar{t})$ such that $\|w_0\|_\infty \leq M$, $\|w\|_\infty \leq M$, and let us set $y \doteq (t, X) \cdot \cdot [\bar{t}, \bar{x}; v, w_0, w]$. The second estimate in (2.6) implies

$$|y(s'') - y(s')| \leq \bar{\delta} \tag{4.5}$$

for every $s', s'' \in [0, 1]$ such that $|s'' - s'| \leq \bar{\delta}/M''$, where M'' is defined as in Proposition 2.3.

For $\varepsilon > 0$ and an arbitrary $s^* \in (0, \bar{\delta}/M'']$ we set

$$\bar{s}_0 \doteq \inf\{s \in [0, s^*] : x(s) \in \Theta^C\} \qquad (\bar{s}_0 = s^* \text{ if } x(s) \in \Theta,\ \forall s \in [0, s^*))\,;$$

$$d \doteq \sup\{\operatorname{dist}(x(s), \Theta) : s \in [0, s^*]\}\,,$$

where the superscript «^{C}» means complementation. In case $\bar{s}_0 < s^*$, or $\bar{s}_0 = s^*$ and $x(\bar{s}_0) \in \partial\Theta$, let us set $s_0 \doteq \max\{\bar{s}_0 - \varepsilon/M'', 0\}$ and let us define the following control

$$\hat{v} \doteq v\chi_{[0, s_0)} + \nu_2(t(s_0), x(s_0))\chi_{[s_0, s_0 + c(d+\varepsilon)]} + v_{c(d+\varepsilon)}\chi_{(s_0 + c(d+\varepsilon), 1]}\,,$$

$$\widehat{w}_0 \doteq w_0\chi_{[0, s_0)} + (T - t(1 - c(d+\varepsilon)))/c(d+\varepsilon)\,\chi_{[s_0, s_0 + c(d+\varepsilon)]} + w_{0\,c(d+\varepsilon)}\chi_{(s_0 + c(d+\varepsilon), 1]}\,,$$

$$\widehat{w} \doteq w\chi_{[0, s_0)} + M\omega(t(s_0), x(s_0))\chi_{[s_0, s_0 + c(d+\varepsilon)]} + w_{c(d+\varepsilon)}\chi_{(s_0 + c(d+\varepsilon), 1]}(s)\,.$$

In the formulas above we have set $c \doteq \min\{4/Mr_2, (1 - s_0)/(d+\varepsilon)\}$ and, for any map g and any constant a, $g_a(s) \doteq g(s - a)$. More precisely, by $0 < s_0 < s^* \leq 1$ and (2.6), if we assume that $\varepsilon \in (0, Mr_2/8]$, we have $(1 - s_0)/(d+\varepsilon) \geq (1 - s^*)/(M''s^* + Mr_2/8)$. Choosing $s^* \leq Mr_2/2(Mr_2 + 4M'')$, we obtain $c = 4/Mr_2$. Furthermore, since $|\omega| \leq 1$ and $(T - t(1 - c(d+\varepsilon)))/c(d+\varepsilon) = c(c(d+\varepsilon))^{-1} \int_{1-c(d+\varepsilon)}^{1} w_0(s)\,ds \leq \|w_0\|_\infty$, one has $\|\widehat{w}_0\|_\infty \leq M$, $\|\widehat{w}\|_\infty \leq M$. Moreover $\widehat{w}_0(s) > 0$ a.e whenever $w_0(s) > 0$ a.e.

Let us set $\tilde{f} \doteq (w_0, \bar{f})$, $\hat{y} = (\hat{t}, \hat{x}) \doteq y[\bar{t}, \bar{x};\ \hat{v}, \widehat{w}_0, \widehat{w}]$, $\bar{v} \doteq \nu_2(t(s_0), x(s_0))$, $\overline{w}_0 \doteq (T - t(1 - c(d+\varepsilon)))/Mc(d+\varepsilon)$ and $\overline{w} \doteq \omega(t(s_0), x(s_0))$. The definition of $(\hat{v}, \widehat{w}_0, \widehat{w})$ and the homogeneity of $\tilde{f}$ with respect to (w_0, w) yield

$$(4.6)\qquad (\hat{t}, \hat{x})(s_0 + c(d+\varepsilon)) = (t(s_0), x(s_0)) + M \int_{s_0}^{s_0 + c(d+\varepsilon)} \tilde{f}(\hat{y}(s), \bar{v}, \overline{w}_0, \overline{w})\, ds\,.$$

Hence, by (4.4)-(4.6) and hypothesis (H2), one obtains that

$$(4.7)\qquad B(\hat{x}(s_0 + c(d+\varepsilon)), 2(d+\varepsilon)) \subset \Theta\,.$$

Moreover, the same argument yields $\hat{x}(s) \in \Theta$ for every $s \in (s_0, s_0 + c(d+\varepsilon)]$. If $s_0 + c(d+\varepsilon) < s^*$, let us consider $s \in (s_0 + c(d+\varepsilon), s^*]$. Then

$$(4.8)\qquad (\hat{t}, \hat{x})(s) = (\hat{t}, \hat{x})(s_0 + c(d+\varepsilon)) + \int_{s_0 + c(d+\varepsilon)}^{s} \tilde{f}(\hat{y}, v_{c(d+\varepsilon)}, w_{0c(d+\varepsilon)}, w_{c(d+\varepsilon)})(s)\, ds =$$

$$= (\hat{t}, \hat{x})(s_0 + c(d+\varepsilon)) + \int_{s_0}^{s - c(d+\varepsilon)} \tilde{f}(\hat{y}(s + c(d+\varepsilon)), v(s), w_0(s), w(s))\, ds =$$

$$= (\hat{t}, \hat{x})(s_0 + c(d+\varepsilon)) + (t, x)_{c(d+\varepsilon)} - (t, x)(s_0) + X_{c(d+\varepsilon)}\,,$$

where

$$X_{c(d+\varepsilon)}(s) \doteq \int_{s_0}^{s - c(d+\varepsilon)} [\tilde{f}(\hat{y}(s + c(d+\varepsilon)), v(s), w_0(s), w(s)) - \tilde{f}(y(s), v(s), w_0(s), w(s))]\, ds\,.$$

By (2.6) and the definition of $X(\cdot)$ one has that

$$|\hat{y}(s_0 + c(d+\varepsilon)) - y(s_0)| \leq M'' c(d+\varepsilon)\,,$$

$$\hat{y}(s + c(d+\varepsilon)) - y(s) = X(s) + [\hat{y}(s_0 + c(d+\varepsilon)) - y(s_0)]\,.$$

Therefore, the hypothesis (2.2) on f together with the regular and slow growth assumption (2.4) imply

$$|X_{c(d+\varepsilon)}(s)| \leq L \int_{s_0}^{s - c(d+\varepsilon)} (w_0(s) + |w(s)|)\, |\hat{y}(s + c(d+\varepsilon)) - y(s)|\, ds \leq$$

$$\leq 2LM \int_{s_0}^{s - c(d+\varepsilon)} [M'' c(d+\varepsilon) + |X(s)|]\, ds\,.$$

Thus by Gronwall's Lemma it follows that

$$(4.9)\qquad |X_{c(d+\varepsilon)}(s)| \leq M'' c(d+\varepsilon)(e^{2LM(s - s_0 - c(d+\varepsilon))} - 1)$$

for all $s \in (s_0 + c(d+\varepsilon), s^*]$. If

$$s^* \leqslant (2LM)^{-1} \ln (1 + Mr_2/4M''),$$

inequality (4.9) yelds

$$|X_{c(d+\varepsilon)}(s)| \leqslant d + \varepsilon, \qquad \forall s \in (s_0 + c(d+\varepsilon), s^*]. \tag{4.10}$$

Let us observe that for any $s \in [0, s^*]$ there exists a point $x_\pi(s) \in \partial\Theta \cap B_n(\bar{x}, 2\bar{\delta})$ such that $d(x(s), \Theta) = |x(s) - x_\pi(s)|$. Since

$$x_{c(d+\varepsilon)}(s) + \widehat{x}(s_0 + c(d+\varepsilon)) - x(s_0) = [x_{c(d+\varepsilon)}(s) - x_{\pi_{c(d+\varepsilon)}}(s)] +$$

$$+ x_{\pi_{c(d+\varepsilon)}}(s) + M \int_{s_0}^{s_0 - c(d+\varepsilon)} \bar{f}(\widehat{y}(s), \bar{v}, \overline{w}_0, \overline{w})\, ds,$$

on the basis of (4.4)-(4.6) it is not diffucult to check that

$$B_n(\widehat{x}(s_0 + c(d+\varepsilon)) + x_{c(d+\varepsilon)}(s) - x(s_0), d+\varepsilon) \subset \Theta. \tag{4.11}$$

Thus by (4.7), (4.8), (4.10) and (4.11) we can conclude that $\widehat{x}(s) \in \Theta$ for all $s \in (0, s^*]$ and

$$|\widehat{y}(s) - y(s)| \leqslant$$

$$\leqslant \begin{cases} M''(s - s_0), & \text{if } s \in [s_0, s_0 + c(d+\varepsilon)], \\ M'' c(d+\varepsilon) + M'' c(d+\varepsilon)(e^{2LM(s - s_0 - c(d+\varepsilon))} - 1), & \text{if } s \in [s_0 + c(d+\varepsilon), 1], \end{cases}$$

i.e.

$$|\widehat{y}(s) - y(s)| \leqslant 2M'' e^{2LM} c(d+\varepsilon), \qquad \forall s \in [0, 1]. \tag{4.12}$$

If $s^* = 1$, the control $(\widehat{v}, \widehat{w}_0, \widehat{w})$ is admissible and satisfies (4.2). Otherwise, let us observe that s^* and all the constants involved in the estimates above are depending only on L, (ν_2, ω) and M. Thus we can repeat exactly the same construction with (v, w_0, w) and the initial point 0 replaced by $(\widehat{v}, \widehat{w}_0, \widehat{w})$ and s^*, respectively. As a result we obtain a new control whose corresponding trajectory belongs to Θ for all $s \in (0, 2s^*]$ and keeps a distance from $(\widehat{t}(s), \widehat{x}(s))$ which is less than or equal to $2M'' e^{2LM} c(d' + \varepsilon)$, where $d' \doteq \sup\{d(\widehat{x}(s), \Theta) : s \in [s^*, 2s^*]\}$. Hence, if now we set $d \doteq \sup\{d(x(s), \Theta) : s \in [0, 2s^*]\}$, by (4.12) it follows that

$$d' \leqslant d + 2M'' e^{2LM} c(d+\varepsilon). \tag{4.13}$$

If $[1/s^*]$ denotes the integer part of $1/s^*$, in at most $N \doteq [1/s^*] + 1$ steps we construct an admissible control, which will be still denoted by $(\widehat{v}, \widehat{w}_0, \widehat{w})$. Moreover, by (4.12) and (4.13), for a suitable constant $C > 0$ the corresponding trajectory $(\widehat{t}, \widehat{x})$ satisfies

$$|(\widehat{t}(s), \widehat{x}(s)) - (t(s), x(s))| \leqslant C(d+\varepsilon), \qquad \forall s \in [0, 1],$$

where d is the same as in (4.3).

However the control $\widehat{w}$ may fail to satisfy the L^1 constraint, i.e. it may happen that

$$\int_0^1 |\widehat{w}|\, ds > a \doteq \int_0^1 |w|\, ds\,.$$

Hence we need a further modification of $\widehat{w}$ to conclude the proof. For this purpose, observe that

$$\int_0^1 |\widehat{w}|\, ds \leq \int_0^1 |w|\, ds + NMc(d+\varepsilon)\,.$$

Set

$$\bar{s} \doteq \sup\left\{ s \in [0,1] \colon \int_0^s |\widehat{w}|\, ds \leq a \right\}$$

and assume that $\bar{s} < 1$. Let us observe that $\int_{\bar{s}}^1 |\widehat{w}|\, ds \leq NMc(d+\varepsilon)$, and in $[\bar{s}, 1]$ let us replace the control $(\widehat{v}, \widehat{w}_0, \widehat{w})$ with $(\widehat{v}, \widehat{w}_0, 0)$. The corresponding trajectory, say $\widehat{y}_{\bar{s}} = (\hat{t}, \widehat{x}_{\bar{s}})$, satisfies

$$|\widehat{x}_{\bar{s}}(s) - \widehat{x}(s)| \leq LM \int_{\bar{s}}^s |\widehat{y}_{\bar{s}}(\sigma) - \widehat{y}(\sigma)|\, d\sigma + \int_{\bar{s}}^s \varphi(w(\sigma))\, d\sigma\,, \tag{4.14}$$

where φ is the same as in Lemma 4.1. The latter implies that, for a suitable increasing function ϱ_1 continuous at 0 and satisfying $\varrho_1(0) = 0$, one as $\int_0^1 \varphi(w(s))\, ds \leq \leq \varrho_1(NMc(d+\varepsilon))$. Thus by applying Gronwall's Lemma we obtain

$$|\widehat{x}_{\bar{s}}(s) - \widehat{x}(s)| \leq \varrho_1(NMc(d+\varepsilon))(e^{LM(s-\bar{s})} - 1)\,.$$

Hence, if $\widehat{x}(s) \in \Theta$, $\forall s \in [\bar{s}, 1]$, we conclude the proof by setting

$$\varrho(d+\varepsilon) \doteq \varrho_1(NMc(d+\varepsilon))\, e^{LM} + C(d+\varepsilon)\,;$$

$$(\check{v}, \check{w}_0, \check{w}) \doteq \begin{cases} (\widehat{v}, \widehat{w}_0, \widehat{w}), & \text{on } [0, \bar{s}], \\ (\widehat{v}, \widehat{w}_0, 0), & \text{on } [\bar{s}, 1], \end{cases}$$

Alternatively, one has $\bar{\bar{s}} \doteq \inf\{s \in [\bar{s}, 1] \colon \widehat{x}(s) \in \Theta^C\} < 1$, $\hat{t}(\bar{\bar{s}}) < T$, and one can regard the system (2.1) on the interval $[\hat{t}(\bar{\bar{s}}), T]$ as a control system driven by the only bounded control v. More precisely, since the function $\widehat{x}(\cdot)$ is constant on any interval $[s', s''] \subset \subset [\bar{\bar{s}}, 1]$ where $\widehat{w}_0 \equiv 0$, by setting e.g.

$$s = \tilde{s}(t) \doteq \min \hat{t}^{\leftarrow}(t)\,, \qquad \tilde{x}(t) \doteq \widehat{x}(\tilde{s}(t))\,, \qquad \tilde{v}(t) \doteq \widehat{v}(\tilde{s}(t))\,,$$

one obtains a trajectory $\tilde{x} = x[\tilde{v}, 0]$ of (2.1) such that $\tilde{x}(t(s)) = \widehat{x}(s)$, $\forall s \in [\bar{\bar{s}}, 1]$.

Hence by using hypothesis (H1) in the t-interval $[\bar{\bar{t}}, T]$, one can apply a procedure similar to the one followed above to modify the control $\tilde{v}$, with ν_1 instead of ν_2. If $\tilde{v}_1$ denotes the resulting control, then the space-time control

$$(\check{v}, \check{w}_0, \check{w}) \doteq \begin{cases} (\hat{v}, \widehat{w}_0, \widehat{w}), & \text{on } [0, \bar{s}], \\ (\hat{v}, \widehat{w}_0, 0), & \text{on } [\bar{s}, \bar{\bar{s}}], \\ (\hat{v}_1, \widehat{w}_0, 0), & \text{on } [\bar{\bar{s}}, 1], \end{cases}$$

where $\hat{v}_1(s) \doteq \tilde{v}_1(\hat{t}(s))$, agrees with the thesis of the theorem.

PROOF OF THEOREM 4.1. – Let $\varepsilon > 0$, $(\bar{t}, \bar{x}) \in [0, T) \times \overline{\Theta}$, $y = y[\bar{t}, \bar{x}; v, w_0, w]$ with $(v, w_0, w) \in \Gamma^c(\bar{t}, \bar{x})$ canonically parametrized and set $M \doteq \|(w_0, w)\|_\infty$, $a \doteq \int_0^1 |w(s)|\, ds$. By Proposition 3.5 in [38] for any $\eta > 0$ there exists a regular control $(v, \tilde{w}_0, w) \in \Gamma^+(\bar{t})$ with $\|\tilde{w}_0\|_\infty \leq M + 1$, such that the corresponding trajectory $\tilde{y} \doteq [\bar{t}, \bar{x}; v, \tilde{w}_0, w]$ satisfies

$$|\tilde{y}(s) - y(s)| \leq \eta .$$

Thus, in view of Theorem 4.2, one can construct a (regular) control $(\check{v}, \check{w}_0, \check{w}) \in \Gamma^c_{\mathrm{int}}(\bar{t}, \bar{x}) \cap \Gamma^+(\bar{t})$, $\check{w}_0 > 0$, such that the corresponding trajectory $\check{y} = y[\bar{t}, \bar{x}; \check{v}, \check{w}_0, \check{w}]$ satisfies

$$|\check{y}(s) - y(s)| \leq |\check{y}(s) - \tilde{y}(s)| + |\tilde{y}(s) - y(s)| \leq \varrho(\eta) + \eta , \qquad \|\check{w}\|_1 \leq a .$$

In order to obtain (4.1) it suffices to choose η so that $\varrho(\eta) + \eta \leq \varepsilon$. Finally, if $\bar{t} = T$, we can apply Theorem 4.2 directly to the control $(v, w_0, w)[\equiv (v, 0, w)]$ to obtain a new space-time control $(\check{v}, \check{w}_0 \check{w})[\equiv (\check{v}, 0, \check{w}) \in \Gamma^c(\bar{t}, \bar{x})$ with $\|\check{w}\|_1 \leq a$ and satisfying (4.1) ■

Let us denote by $(\mathcal{V}_{\mathrm{int}})_K(\bar{t}, \bar{x}, \bar{k})$ and $\mathcal{V}_{\mathrm{int}}(\bar{t}, \bar{x})$ the value functions corresponding to the sets of controls $(\Gamma^c_{\mathrm{int}})_{K-\bar{k}}(\bar{t}, \bar{x})$ and $\Gamma^c_{\mathrm{int}}(\bar{t}, \bar{x})$, respectively:

$$(\mathcal{V}_{\mathrm{int}})_K(\bar{t}, \bar{x}) \doteq \inf_{(v, w_0, w) \in (\Gamma^c_{\mathrm{int}})_{K-\bar{k}}(\bar{t}, \bar{x})} \Psi(x[\bar{t}, \bar{x}; v, w_0, w](1)) ,$$

$$\mathcal{V}_{\mathrm{int}}(\bar{t}, \bar{x}) \doteq \inf_{(v, w_0, w) \in \Gamma^c_{\mathrm{int}}(\bar{t}, \bar{x})} \Psi(x[\bar{t}, \bar{x}; v, w_0, w](1)) .$$

As an obvious consequence of Theorem 4.1 we have that, under hypotheses (H1), (H2),

$$(\mathcal{V}_{\mathrm{int}})_K(\bar{t}, \bar{x}, \bar{k}) = \mathcal{V}_K(\bar{t}, \bar{x}, \bar{k}), \qquad \forall (\bar{t}, \bar{x}, \bar{k}) \in [0, T] \times \overline{\Theta} \times [0, K],$$

$$\mathcal{V}_{\mathrm{int}}(\bar{t}, \bar{x}) = \mathcal{V}(\bar{t}, \bar{x},), \qquad \forall (\bar{t}, \bar{x}) \in [0, T] \times \overline{\Theta} ,$$

the latter inequality holding true even under the sole hypothesis (H2). For the case with bounded controls a similar result has been already proved in [29].

5. – Controls unbounded in the L^1 norm.

Unlike the maps $\mathcal{V}_K$, in general $\mathcal{V}$ fails to be continuous, as it is shown by Example 5.1 below. In fact, we prove that $\mathcal{V}$ is upper semicontinuous and solves a suitable boundary value problem, in which an extended definition of viscosity solution introduced by H. Ishii [23], [24] is involved. In general $\mathcal{V}$ is not the unique solution of this boundary value problem even if it is continuous (see the next section).

Let us begin by recalling the notions of upper and lower semi-continuous envelope of a map. Let φ be a map from a subset E of an Euclidean space $\mathbb{R}^N$ into the extended real line $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. The *upper semicontinuous envelope* φ^* and the *lower semicontinuous envelope* φ_* are defined on the closure $\overline{E}$ of E by

$$\varphi^*(\bar{e}) \doteq \lim_{r\to 0} \sup_{B(\bar{e},\, r)\cap E} \varphi(e), \qquad \varphi_*(\bar{e}) \doteq \lim_{r\to 0} \sup_{B(\bar{e},\, r)\cap E} \varphi(e),$$

respectively. By definition φ^* is upper semicontinuous and φ_* is lower semicontinuous.

In order to state Theorem 5.1, which concerns the dynamic programming equation for problem $(\mathcal{P})$, we need a concept of viscosity subsolution and supersolution valid for discontinuous maps as well—see [24] and [1].

DEFINITION 5.1. – *Let E be a subset of $\mathbb{R}^N$ and let $\mathcal{F}$ be a continuous map from $E \times \mathbb{R}^N$ in $\mathbb{R}$. A function $g: E \to \mathbb{R}$ is called a* viscosity subsolution [resp. supersolution] *of the first order differential equation*

$$\mathcal{F}(z, \nabla g(z)) = 0 \qquad \text{(Eq)}$$

at a point $\bar{z} \in E$ if for any $\lambda \in C^\infty(\mathbb{R}^N)$ such that $\bar{z}$ is a strict local maximum [resp. minimum] on E for $g^* - \lambda$ [resp. $g_* - \lambda$] one has

$$\mathcal{F}(z, \nabla\lambda(\bar{z})) \leq 0 \quad [resp. \geq 0].$$

$g: E \to \mathbb{R}$ is called a viscosity solution *of* (Eq) *at $\bar{z}$ if it is both a viscosity subsolution and a viscosity supersolution.*

REMARK 5.1. – There is a recent and rich literature on the extension of the concept of solution to discontinuous maps. Besides the notion used here [23], let us mentiopn the generalized minimax solutions [53], the envelope solutions [1] and the bilateral solutions [4]. Yet these concepts are not fit for the problems addressed in this paper. For an exaustive account on the question of discontinuous solutions we refer to [1] and to the bibliography therein. Finally, let us mention that another approach to problems with lower semicontinuous value function are provided in [21] and [22], where contingent derivatives are used. However, even that approach does not apply to our case, for our value function is not lower semicontinuous (actually, it is upper semicontinuous).

Let us introduce the Hamiltonian function H for the general case where the L^1 norm of ξ is not bounded. Let $H_{\mathcal{B}}$ be the Hamiltonian for the case with bounded L^1

norm (see [39]), i.e.

$$H_{\mathcal{B}}(t, x, p_t, p_x, p_k) \doteq \min_{(v, w_0, w) \in V \times S_+^m} \{p_t w_0 + p_x \cdot \bar{f}(t, x, v, w_0, w) + p_k |w|\},$$

where $S_+^m \doteq \{(w_0, w) \in \mathbb{R}^{1+m}: |(w_0, \ldots, w_n)| = 1, w_0 \geq 0, w = (w_1, \ldots, w_m) \in C\}$. For every $(t, x, p_t, p_x) \in [0, T] \times \overline{\Theta} \times \mathbb{R}^{1+n}$ we set

$$H(t, x, p_t, p_x) \doteq H_{\mathcal{B}}(t, x, p_t, p_x, 0).$$

We shall consider the following dynamic programming equation

$$\text{(DPE)} \qquad -H(t, x, \nabla_t u, \nabla_x u) = 0.$$

DEFINITION 5.2. – *We shall say that a map $g_1: [0, T] \times \overline{\Theta} \to \mathbb{R}$ is a* subsolution *of the boundary value problem associated to* (DPE)*—briefly, g_1 is a subsolution of* (BVP)*—if*

i) *g_1 is a subsolution of* (DPE) *in $[0, T] \times \Theta$,*

ii) *$g_1^* \leq \Psi$ on $\{T\} \times \overline{\Theta}$.*

We shall say that a map $g_2: [0, T] \times \overline{\Theta} \to \mathbb{R}$ is a supersolution *of the boundary value problem associated to* (DPE)*—briefly, g_2 is a supersolution of* (BVP)*—if*

iii) *g_2 is a supersolution of* (DPE) *in $[0, T] \times \overline{\Theta}$,*

iv) *on $\{T\} \times \overline{\Theta}$, either $(g_2)_* \geq \Psi$ or g_2 is a supersolution of* (DPE).

A map $g: [0, T] \times \overline{\Theta} \to \mathbb{R}$ which is both a subsolution and a supersolution of (BVP), *will be called a* solution *of* (BVP).

THEOREM 5.1. – *The map $\mathcal{V}: [0, T] \times \overline{\Theta} \to \mathbb{R}$ is a solution of* (BVP).

The following example shows that $\mathcal{V}$ is possibly discontinuous—actually, upper semicontinuous—even when no state constraints are in force. Moreover $\mathcal{V}$ is not the unique solution of (BVP).

EXAMPLE 5.1. – Let us consider the problem of minimizing $\Psi(x(1)) \doteq |x(1) + 1|$ over all terminal points of the trajectories $x: [0, 1] \to \mathbb{R}$ satisfying

$$\dot{x} = -|x|\xi, \qquad x(0) = \bar{x}, \qquad \xi(t) \in C \doteq [0, +\infty).$$

Clearly the value function is independent of $\bar{t}$. Let us write x instead of $\bar{x}$, and let us observe that from each $x \geq 0$ one can reach every positive value arbitrarily close the origin. On the other hand, the point $x = -1$ can be reached from every $x \in [-1, 0[$. Note incidentally that the expenditure of $\|\xi\|_1$ which is necessary to steer x to -1 tends to

$+\infty$ as x approaches the origin from the left. A straightforward computation gives:

$$\mathcal{V}(x) = \begin{cases} 1, & \forall x \geq 0, \\ 0, & \forall x \in [-1, 0[, \\ |x+1|, & \forall x \leq -1. \end{cases}$$

In fact, the dynamic programming equation (DPE) reduces to

$$-\min_{(w_0, w) \in S_1 \cap [0, +\infty[^2} \left\{ \frac{\partial \mathcal{V}}{\partial t} w_0 - \frac{\partial \mathcal{V}}{\partial x} |x| w \right\} = 0,$$

and from the condition $\mathcal{V}(x) \leq |x+1|$ one easily checks taht $\mathcal{V}$ is a solution of the boundary value problem (BVP). Yet, it is not unique, in that every map

$$W(x) \doteq \begin{cases} a(x), & \forall x \geq 0, \\ 0, & \forall x \in [-1, 0[, \\ |x+1|, & \forall x \leq -1, , \end{cases}$$

where $a(x)$ is a smooth and nonincreasing map satisfying $0 \leq a(x) \leq 1$, is a solution of (BVP). Let us observe that the map $\tilde{\mathcal{V}}(t, x) \doteq \mathcal{V}(x)$, $\forall (t, x) \in [0, T[\times \overline{\Theta}$, $\tilde{\mathcal{V}}(T, x) \doteq \Psi(x)$, $\forall x \in \overline{\Theta}$ is an upper semicontinuous solution of (BVP). We shall prove that in fact $\tilde{\mathcal{V}}$ is the maximal upper semicontinuous subsolution of (BVP).

The proof of Theorem 5.1 is based on the following Dynamic Programming Principle:

PROPOSITION 5.1 (Dynamic Programming Principle). – *For every* $(\bar{t}, \bar{x}) \in [0, T] \times \overline{\Theta}$ *the value map* $\mathcal{V}$ *verifies*

$$\mathcal{V}(\bar{t}, \bar{x}) = \inf_{(v, w_0, w) \in \Gamma^c(\bar{t}, \bar{x})} \mathcal{V}((t, x)[\bar{t}, \bar{x}; v, w_0, w[(s))$$

for all $s \in [0, 1]$.

PROOF. – Up to a reparametrization argument the proof of this Dynamic Programming Principle is quite standard. However, for the reader convenience, we give a sketch of it.

Fix $\bar{s} \in [0, 1]$ and $\varepsilon > 0$. For any control $(\check{v}, \check{w}_0, \check{w}) \in \Gamma^c(\bar{t}, \bar{x})$ there is a control $(\tilde{v}, \tilde{w}_0, \tilde{w}) \in \Gamma^c(\tau, \xi)$ with $(\tau, \xi) \doteq (t, x)[\bar{t}, \bar{x}; \check{v}, \check{w}_0, \check{w}](\bar{s})$ such that

$$\Psi(x[\tau, \xi; \tilde{v}, \tilde{w}_0, \tilde{w}](1)) \leq \mathcal{V}(\tau, \xi) + \varepsilon.$$

Hence the control (v, w_0, w) defined by $(v, w_0, w)(s) \doteq (\check{v}, \check{w}_0, \check{w})(s)$ for every $s \in [0, \bar{s}]$ and by $(v, w_0, w)(s) \doteq (\tilde{v}, \tilde{w}_0 \cdot (1-\bar{s})^{-1})((s-\bar{s})(1-\bar{s})^{-1})$ for every $s \in]\bar{s}, 1]$ belongs to $\Gamma^c(\bar{t}, \bar{x})$ and moreover $x[\bar{t}, \bar{x}; v, w_0, w](1) = x[\tau, \xi; \tilde{v}, \tilde{w}_0, \tilde{w}](1)$. Since $\mathcal{V}(\bar{t}, \bar{x}) \leq$

$\leq \Psi(x[\bar{t}, \bar{x}, v, w_0, w](1))$, it follows that

$$\mathcal{V}(\bar{t}, \bar{x}) \leq \mathcal{V}(\tau, \xi) + \varepsilon .$$

Now, taking the infimum over the controls $(v, w_0, w) \in \Gamma^c(\bar{t}, \bar{x})$, by the arbitrariness of ε one obtains

$$\mathcal{V}(\bar{t}, \bar{x}) \leq \inf_{(v, w_0, w) \in \Gamma^c(\bar{t}, \bar{x})} \mathcal{V}((t, x)[\bar{t}, \bar{x}; v, w_0, w](s)) \tag{5.1}$$

for all $s \in [0, 1]$.

In order to prove the converse inequality, fix $\bar{s} \in [0, 1]$ and $\varepsilon > 0$ and consider a control $(v, w_0, w) \in \Gamma^c(\bar{t}, \bar{x})$ satisfying

$$\Psi(x[\bar{t}, \bar{x}; v, w_0, w](1)) \leq \mathcal{V}(\bar{t}, \bar{x}) + \varepsilon .$$

Setting $(\tau, \xi) \doteq (t, x)[\bar{t}, \bar{x}; v, w_0, w](\bar{s})$, the control $(\tilde{v}, \tilde{w}_0, \tilde{w})$ defined by $(\tilde{v}, \tilde{w}_0, \tilde{w})(s) \doteq (v, w_0 \cdot (1 - \bar{s}), w \cdot (1 - \bar{s}))(\bar{s} + s(1 - \bar{s}))$ for every $s \in [0, 1]$ is such that $x[\tau, \xi; \tilde{v}, \tilde{w}_0, \tilde{w}](1) = x[\bar{t}, \bar{x}; v, w_0, w](1)$. Since $\mathcal{V}(\tau, \xi) \leq \Psi(x[\tau, \xi; \tilde{v}, \tilde{w}_0, \tilde{w}](1))$, it follows that

$$\mathcal{V}(\tau, \xi) \leq \mathcal{V}(\bar{t}, \bar{x}) + \varepsilon . \tag{5.2}$$

By the arbitrariness of ε, (5.2) together with (5.1) yields the thesis. ∎

In view of the Dynamic Programming Principle above the proof of Theorem 5.1 is similar to the proof Theorem 4.1 in [39], where a similar problem with a bound on the L^1 norm of w was considered. For this reason, we omit it.

6. - Characterizing the value function.

As shown by Example 5.2, in general the value map is not the unique solution of (BVP). Neither uniqueness is achieved when $\mathcal{V}$ is continuous, as the following simplification of Example 5.1 shows.

EXAMPLE 6.1. - Let us consider again the control system

$$\dot{x} = -|x|\,\xi , \qquad x(0) = \bar{x} , \qquad \xi \in C \doteq [0, +\infty[.$$

If we wish to minimize the cost function

$$\Psi(x(1)) \doteq |x(1)| ,$$

we obtain the continuous value function

$$\mathcal{V}(x) = \begin{cases} 0 , & \forall x \geq 0 , \\ -x , & \forall x < 0 . \end{cases}$$

The corresponding Bellman equation is the same as in Example 5.1:

$$-\min_{(w_0,\, w)\in S_1\cap[0,\,+\infty[^2}\left\{\frac{\partial \mathcal{V}}{\partial t}\,w_0-\frac{\partial \mathcal{V}}{\partial x}\,|x|\,w\right\}=0\,.$$

$\mathcal{V}$ is not the unique solution of the corresponding (BVP), for the map

$$\tilde{\mathcal{V}}(t,\,x)\doteq\begin{cases}\mathcal{V}(x)\,, & t<T\,,\\ \Psi(x)\,, & t=T\end{cases}$$

is a solution of (BVP) as well. We remark that in general the value map is not unique even in the class of continuous solutions of (BVP)—see Example 6.2. However observe that $\mathcal{V}$ is maximal among the continuous subsolutions of (BVP), while $\tilde{\mathcal{V}}(t,\,x)$ is maximal among the subsolutions of (BVP). Both this facts agree with the general results below.

Subsolution-properties of $\mathcal{V}$.

Let us set

$$\tilde{\mathcal{V}}(t,\,x)\doteq\begin{cases}\mathcal{V}(t,\,x)\,, & (t,\,x)\in[0,\,T[\,\times\overline{\Theta}\,,\\ \Psi(x)\,, & (T,\,x)\in\{T\}\times\overline{\Theta}\,.\end{cases}$$

Let us observe that $\tilde{\mathcal{V}}$ is upper semicontinuous, for $\mathcal{V}$ is upper semicontinuous and $\mathcal{V}(T,\,x)\leq\Psi(x)$.

THEOREM 6.1. – *The map* $\tilde{\mathcal{V}}$ *solves the* (BVP). *Moreover* $\tilde{\mathcal{V}}$ *is the maximal subsolution of* (BVP) *on* $[0,\,T]\times\overline{\Theta}$.

COROLLARY 6.1. – *The map* $\mathcal{V}$ *is maximal on the set* $[0,\,T[\,\times\overline{\Theta}$ *among the subsolutions of* (BVP).

COROLLARY 6.2. – *If* $\mathcal{V}$ *is continuous then it is maximal on* $[0,\,T]\times\overline{\Theta}$ *among the continuous subsolutions of* (BVP).

COROLLARY 6.3. – *If* $\mathcal{V}(T,\,x)=\Psi(x)$ *for every* $x\in\overline{\Theta}$, *then* $\mathcal{V}$ *is the maximal subsolutions of* (BVP) *on* $[0,\,T]\times\overline{\Theta}$.

The proof of Theorem 6.1 will be given at the end of this section. Corollary 6.1 and 6.3 follow from the same definition of $\tilde{\mathcal{V}}$, while Corollary 6.2 is a straightforward consequence of Corollary 6.1. For analogous results for a problem with unbounded controls without state costraints we refer to [3], where the formal Hamiltonian is used (see also the Appendix).

REMARK 6.1. – The inequality $\mathcal{V}(T,\,x)<\Psi(x)$ means that for a $\bar{t}$ in a neighbourhood of T it is convenient to implement controls $\xi:[\bar{t},\,T]\to C$ having L^1 norm grater than a certain positive constant. On the contrary, the equality $\mathcal{V}(T,\,x)=\Psi(x)$ means that one is not forced to use such controls. Then an obvious condition guaranteeing the equality $\mathcal{V}(T, x) = \Psi(x)$ is the following *non jumping condition at* T, briefly (NJC):

$$\text{(NJC)} \qquad \nabla\Psi(x)\cdot\bar{f}(T, x, v, 0, w) \geq 0\,, \qquad \forall(x, v, w) \in \overline{\Theta}\times V\times C\,.$$

Condition (NJC), which is a sort of compatibility condition (see e.g. [24] and [52]), has to be interpreted in the classical sense whenever Ψ is continuously differentiable. If Ψ is just continuous (NJC) has to be regarded in the sense of the theory of viscosity solutions. This means that on the right-hand side one has to replace the gradient of Ψ with any element of the subdifferential of Ψ, $D^-\Psi$, which is defined as follows:

$$D^-\Psi(x) \doteq \left\{ p \in \mathbb{R}^n : \liminf_{y\to x} \frac{\Psi(y) - \Psi(x) - p\cdot(y-x)}{|y-x|} \geq 0 \right\}.$$

Lack of uniqueness.

Though Corollary 6.1 characterizes the value function on $[0, T[\times\overline{Q}$ in terms of its subsolution properties, a uniqueness result is far to hold true. In fact, since equation (DPE) is highly degenerate—the coefficient of $\mathcal{V}_t$ may happen to be zero—the supersolution properties of $\mathcal{V}$ turn out to be too weak for singling it out among the solutions of (BVP). Even if at $t = T$ $\mathcal{V}$ is continuous and coincides with the cost function, we do not have uniqueness. For example, if $f \equiv 0$, then $\mathcal{V}(t, x)$ coincides with $\Psi(x)$ at any $x \in \overline{\Theta}$. $\mathcal{V}$ verifies the (DPE) which reduces to

$$-\min_{0\leq w_0\leq 1}\left\{ \frac{\partial\mathcal{V}}{\partial t}\, w_0 \right\} = 0$$

and it is straightforward to check that $u(t, x) \doteq \Psi(x) - (T-t)$ is a solution as well. The following example provides a less trivial case (which includes Example 6.1) of non uniqueness.

EXAMPLE 6.2. – Consider any continuous bounded cost function with a dynamics of the form

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\,\xi_i\,.$$

The value function $\mathcal{V}$ is obviously independent of t and it is not difficult to verify that each map $u(t, x) \doteq \mathcal{V}(x) + \varphi(t)$ with φ smooth, bounded, increasing, nonpositive, and such that $\varphi(T) = 0$ is still a solution of the corresponding (BVP).

In order to prove Theorem 6.1 we need the following lemma on the monotonicity of the viscosity subsolutions of (DPE) along internal trajectories.

LEMMA 6.1. – *Let* $Z : [0, T]\times\overline{\Theta}\to\mathbb{R}$ *be a subsolution of* (DPE) on $[0, T[\times\Theta$. *Then* *for every* $(\bar{t}, \bar{x}) \in [0, T[\times \overline{\Theta}$ *and every control* $(v, w_0, w) \in \Gamma^c_{\mathrm{int}}(\bar{t}, \bar{x}) \cap \Gamma^+(\bar{t})$ *the map*

$$s \mapsto Z((t, x)[\bar{t}, \bar{x}; v, w_0, w](s)) \quad \textit{for } s \in [0, 1]$$

is nondecreasing.

This result is well known (see [1], [17]) in the case when Z is continuous. The proof of the general case ([2]) needs some more Calculus within the theory of viscosity solutions.

PROOF OF THEOREM 6.1. – Since $\tilde{\mathcal{V}}^* \equiv \tilde{\mathcal{V}} \equiv \mathcal{V}$ on $[0, T[\times \overline{\Theta}$ and, as it easy to check, $\tilde{\mathcal{V}}_* \equiv \mathcal{V}_*$ on $[0, T] \times \overline{\Theta}$, the fact that $\tilde{\mathcal{V}}$ solves (BVP) is a straightforward consequence of Theorem 5.1.

In order to prove that $\tilde{\mathcal{V}}$ is the maximal subsolution of (BVP), let us begin by observing that on the subset $\{T\} \times \overline{\Theta}$ the result is trivial, in that $Z(T, \bar{x}) \leq \Psi(\bar{x}) \equiv \tilde{\mathcal{V}}(T, \bar{x})$ for all Z and $\bar{x} \in \overline{\Theta}$. Now let us assume by contradiction that there exist a subsolution Z of (BVP) and a constant $\eta > 0$ such that

$$Z(\bar{t}, \bar{x}) - \mathcal{V}(\bar{t}, \bar{x}) = \eta \tag{6.1}$$

for some $(\bar{t}, \bar{x}) \in [0, T[\times \overline{\Theta}$. On the basis of Theorem 4.1 there exists a control $(v, w_0, w) \in \Gamma^c_{\mathrm{int}}(\bar{t}, \bar{x}) \cap \Gamma^+(\bar{t})$ such that

$$\mathcal{V}(\bar{t}, \bar{x}) \geq \Psi(x(1)) - \eta/2\,, \tag{6.2}$$

where $(t, x) \doteq (t, x)[\bar{t}, \bar{x}; v, w_0, w]$. By (6.1), (6.2) it follows that

$$Z(\bar{t}, \bar{x}) \geq \Psi(x(1)) + \eta/2\,. \tag{6.3}$$

Now Lemma 6.1 yields the inequality $Z(T, x(1)) \geq Z((t, x)(0)) \equiv Z(\bar{t}, \bar{x})$ which, together with (6.3), implies

$$\Psi(x(1)) \geq Z(T, x(1)) \geq Z(\bar{t}, \bar{z}) \geq \Psi(x(1)) + \eta/2\,,$$

a contradiction. Hence it is proved that for any subsolution Z of (BVP), $Z \leq \tilde{\mathcal{V}}$ on $[0, T] \times \overline{\Theta}$. ■

Appendix.

When the value function $\mathcal{V}$ is continuous, Corollary 6.2 gives a characterization of $\mathcal{V}$ on the whole domain $[0, T] \times \Theta$. When conditions (H1), (H2) are verified (or, alternatively, the continuity of the maps $\mathcal{V}_K$ is known a priori), discontinuities of $\mathcal{V}$ may only arise because of the occurence of minimizing sequences of controls ξ_n whose L^1 norms tend to infinity (however, even in this case it may happen that $\mathcal{V}$ turns out to be continuous). In fact, if on each compact subset Q of $\overline{\Theta}$ the minimizing controls can be chosen having L^1 norm uniformly bounded by a constant M, then one gets $\mathcal{V}(t, x) = \mathcal{V}_M(t, x)$ for every $(t, x) \in [0, T] \times Q$. Since the maps $\mathcal{V}_M$ are assumed continuous it follows that

$\mathcal{V}$ is continuous as well. Let us remark that in this case we obtain the best possible approximation of $\mathcal{V}$ on compact sets, in that $\mathcal{V}_K \equiv \mathcal{V}$ for a K sufficiently large. Incidentally this makes this approximation more adequate to the problem that the approximation (pursued e.g. in [3] for an infinite horizon problem) by the value functions $\overline{\mathcal{V}}_L$ corresponding to the set of controls $\{\xi\colon \|\xi\|_\infty \leq L\}$, which in general are strictly greater than $\mathcal{V}$. Indeed consider the following simple example on $[0, T] \times \mathbb{R}$.

EXAMPLE A.1.

$$\text{minimize } x^2(T),$$

$$\dot{x} = \xi,$$

$$x(\bar{t}) = \bar{x}, \qquad \xi \in C \doteq \mathbb{R}.$$

Clearly one has $\mathcal{V}(\bar{t}, \bar{x}, 0) = 0$ for each $\bar{x} \in B[0, K]$, while $\mathcal{V}_K(\bar{t}, \bar{x}, 0) = (|\bar{x}| - K)^2$ for every $\bar{x}$ outside $B[0, K]$. Since $\mathcal{V}$ is identically equal to zero then one has $\mathcal{V}_K(\bar{t}, \bar{x}, 0) = \mathcal{V}(\bar{t}, \bar{x})$ in $B[0, K]$. On the other hand it is easy to check that

$$\overline{\mathcal{V}}_L(\bar{t}, \bar{x}) = \max\{0, (|\bar{x}| - L(T - \bar{t}))^2\}.$$

In particular there is no way of invading $[0, T] \times \overline{\Theta}$ with a sequence of compact sets C_L on which the maps $\overline{\mathcal{V}}_L$ coincide with $\mathcal{V}$.

Motivated by the above considerations let us observe that in at least two cases the bound on compact sets of the L^1 norm of the minimizing controls ξ is almost obvious.

The first instance concerns a sort of weak coercitivity assumption. More precisely, we assume that there exist two positive constants C_0, C_1 such that

$$\nabla \Psi(x) \cdot f(t, x, v, \xi) \geq C_0 + C_1 |\xi| \tag{A.1}$$

for all $(t, x, v, \xi) \in [\bar{t}, T] \times \overline{\Theta} \times V \times C$. This condition makes the exploitation of controls ξ with too large L^1 norm unconvenient. Of course condition (A.1) has a classical meaning only if Ψ is regular enough. Otherwise it must be interpreted in the sense provided by the theory of viscosity solutions. It is clear that for a functional of Bolza type

$$\int_{\bar{t}}^{T} l(t, x[v, \xi](t), v(t), \xi(t))\, dt + \psi(x(T))$$

—to which the result of the previous sections can be easly adapted—condition (A.1) is replaced by

$$\nabla \psi(x) \cdot f(t, x, v, \xi) + l(t, x, v, \xi) \geq C_0 + C_1 |\xi|$$

for all $(t, x, v, \xi) \in [\bar{t}, T] \times \overline{\Theta} \times V \times C$. In turn, as soon as $f(t, x, v, \xi) \equiv \xi$, $C = \mathbb{R}^n$, $\Psi \equiv 0$, the above Bolza problem include several classical examples from the Calculus of Variations with slow growth (see e.g. [15]).

A second instance where the L^1 norm of ξ is automatically bounded is represented by the case when too large values of $\|\xi\|_1$ take that state x outside $\overline{\Theta}$. A simple case where such a situation occurs is the following one. Consider m vector fields $g_1, \ldots, g_m$ different from zero at each point $\overline{x} \in \mathbb{R}^n$, and assume that for each $i = 1, \ldots, m$ the solution of the Cauchy problem

$$\dot{x} = g_i(x), \qquad x(0) = \overline{x}$$

is defined for every $t \geq 0$. Moreover assume that the norms of these solutions tend to infinity as t goes to $+\infty$. Let Φ: $\mathbb{R}^n \to \mathbb{R}$ be a coercive map (i.e. $|\Phi(x)| \to +\infty$ as $|x| \to +\infty$) and let r be an increasing bounded map. Let us set $\Psi \doteq r \circ \Phi$ and let us consider the problem

$$\text{minimize}\, \Psi(x(T)), \qquad \dot{x} = g_0(x) + \sum_{i=1}^{n} g_i(x)\, \xi_i, \qquad x(\bar{t}) = \overline{x},$$

with g_0 any regular vector field and the control ξ taking values in the cone $[0, +\infty[^m$. It is easy to prove that for any $\overline{x}$ lying in a compact subset $Q \subset \mathbb{R}^n$, there exists a constant K_Q such that for a suitable minimizing sequence $(\xi_n)_n$ one has $\|\xi\|_1 \leq K_Q$.

Finally let us observe that the value function $\mathcal{V}$ may be continuous even if the minimizing controls ξ_n for some initial conditions have unbounded L^1 norm, as shown by the following example.

EXAMPLE A.2. – Consider the problem

$$(\mathrm{P}_\varepsilon) \qquad \text{minimize}\,\{\arctan x(T)\}, \qquad \dot{x} = -(|x| + \varepsilon)\,\xi, \qquad x(\bar{t}) = \overline{x},$$

where ε is a nonnegative constant, and $x \in \mathbb{R}$, $\xi \in \mathbb{R}$. It is straightforward to verify that for each problem (P_ε) the minimizing sequences $\{\xi_n\}$ of controls satisfy $\|\xi_n\|_1 \to +\infty$. However, whenever $\varepsilon > 0$ one has $\mathcal{V}(x) = -\pi/2$. In particular, $\mathcal{V}$ is continuous. On the contrary, $\mathcal{V}$ has a discontinuity at $x = 0$ as soon as $\varepsilon = 0$.

REFERENCES

[1] M. BARDI - I. CAPUZZO DOLCETTA, *Optional control and viscosity solutions of Hamiltonian-Jacobi Bellman equations*, Birkhäuser, Boston (1997).

[2] M. BARDI - P. SORAVIA, Personal communication.

[3] G. BARLES, *An approach of determinist control problems with unbounded data*, Ann. Ist. Henri Poincaré, 7-4 (1990), pp. 235-258.

[4] E. N. BARRON - R. JENSEN, *Semicontinuous viscosity solutions of Hamiltonian-Jacobi equations with convex Hamiltonian*, Comm. Partial Differential Equations, **15** (1990), pp. 1713-1742.

[5] E. N. BARRON - R. JENSEN - J. L. MENALDI, *Optimal control and differential games with measures*, Nonlinear Analysis, Theory, Methods & Applications, **21**-4 (1993), pp. 214-268.

[6] ALBERTO BRESSAN, *On differential systems with impulsive controls*, Rend. Sem. Mat. Univ. Padova, **78** (1987), pp. 227-236.

[7] ALBERTO BRESSAN - F. RAMPAZZO, *On differential systems with vector-valued impulsive controls*, Boll. Un. Mat. Ital. (7), **2-B** (1988), pp. 641-656.

[8] ALBERTO BRESSAN - F. RAMPAZZO, *Impulsive control systems with commutative vector fields*, Journal of Optimization Theory and Applications, **71** (1991), pp. 67-83.

[9] ALBERTO BRESSAN - F. RAMPAZZO, *Impulsive control systems without commutativity assumption*, Journal of Optimization Theory and Applications, **81** (1991), pp. 435-457.

[10] ALDO BRESSAN, *Hyperimpulsive motions and controllizable coordinates for Lagrangean systems*, Atti Accad. Naz. Lincei, Mem. Cl. Sc. Fis. Mat. Natur., **19** (1991).

[11] ALDO BRESSAN, *On some control problem concerning the ski and the swing*, Atti Acad. Naz. Lincei, Mem. Cl. Sc. Fis. Mat. Natur., Series IX, **1** (1991), pp. 149-196.

[12] ALDO BRESSAN & M. MOTTA, *Some optimization problems with a monotone impulsive character. Approximation by means of structural discontinuities*, Atti Accad. Naz. Lincei, Mem. Cl. Sc. Fis. Mat. Natur., Series IX, **2** (1994), pp. 31-52.

[13] F. CAMILLI - M. FALCONE, *Approximation of control problems involving ordinary and impulsive controls*, Preprint (1995).

[14] I. CAPUZZO-DOLCETTA - P. L. LIONS, *Hamilton-Jacobi equations and state constrained problems*, Trans. Amer. Math. Soc., **318** (1990), pp. 643-668.

[15] L. CESARI, *Optimization-Theory and Applications*, Springer-Verlag, New York, Heidelberg, Berlin (1984).

[16] C. W. CLARK - F. H. CLARKE - G. R. MUNRO, *The optimal exploitation of renewable stocks*, Econometrica, **48** (1979), pp. 25-47.

[17] M. G. CRANDALL - P. L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **277** (1983), pp. 1-42.

[18] G. DAL MASO - F. RAMPAZZO, *On systems of ordinary differential equations with measures as controls*, Differential and Integral Equations, **4** (1991), no. 4, pp. 739-765.

[19] J. R. DORROH - G. FERREYRA, *A multi-state, multi-control problem with unbounded controls*, to appear on SIAM J. Control and Optimization.

[20] V. A. DYKHTA, *Impulse trajectory extension of degenerated optimal control problems, The Liapunov functions methods and applications*, P. BORNE - V. MATROSOV (eds.), J. C. Baltzer AG, Scientific Publishing Co. (1990), pp. 103-109.

[21] H. FRANKOWSKA, *Optimal trajectories associated with a solution of the contingent Hamilton-Jacobi equation*, J. Appl. Math. Optim., **19** (1989), pp. 291-311.

[22] H. FRANKOWSKA, *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations*, SIAM J. Control Optim., **31** (1993), pp. 257-272.

[23] H. ISHII, *Perron's method for Hamilton-Jacobi equations*, Duke Math. J., **55** (1987), pp. 369-384.

[24] H. ISHII, *A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations*, Ann. Sc. Norm. Sup. Pisa (IV), **16** (1989), pp. 105-135.

[25] H. ISHII - S. KOIKE, *A new formulation of state constraints problems for first order PDEs*, SIAM J. Control and Optimization, **365** (1996), pp. 554-576.

[26] D. F. LAWDEN, *Optimal Trajectories for Space Navigations*, Butterworth, London (1963).

[27] LIU - H. J. SUSSMANN, *Limit of high oscillatory controls and the approximation of general paths by admissible trajectories*, Proc. C.D.C. IEEE (1991).

[28] LIU - H. J. SUSSMANN, *A characterization of continuous dependence of trajectories with respect to the input for control-affine systems*, Preprint.

[29] P. LORETI, *Some properties of constrained viscosity solutions of Hamiltonian-Jacobi-Bellman equations*, SIAM J. Control and Optimization, **25** (1987), pp. 1244-1252.

[30] P. LORETI - E. TESSITORE, *Approximation and regularity on constrained viscosity solutions of Hamilton-Jacobi-Bellman equations*, Jour. of Mathematical Systems, Estimation and Control, **4** (1994), pp. 467-483.

[31] J. P. MAREC, *Optimal Space Trajectories*, Elsevier, Amsterdam-Oxford (1979).

[32] B. M. MILLER, *Optimization of dynamic systems with a generalized control,* Automation and Remote Control, **50** (1989), pp. 733-742.
[33] B. M. MILLER, *Condition for the optimality in problems of generalized control. I. Necessary conditions for optimality,* Automation and Remote Control, **53** (1992), pp. 50-58.
[34] B. M. MILLER, *The generalized solutions of ordinary differential equations in the impulse control problems,* Journal of Mathematical Systems, Estimation and Control, 4 (1994), pp. 385-388.
[35] B. M. MILLER, *The generalized solutions of nonlinear optimization problems with impulse control,* SIAM J. Control Optim, **34** (1996), pp. 1420-1440.
[36] M. MOTTA - F. RAMPAZZO, *Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls,* Differential and Integral Equations, 8, 2 (1995), pp. 269-288.
[37] M. MOTTA - F. RAMPAZZO, *Dynamic programming for nonlinear systems driven by ordinary and impulsive controls,* SIAM J. Control and Optimization, **34** (1996), pp. 188-225.
[38] M. MOTTA - F. RAMPAZZO, *Nonlinear system with unbounded controls and state constraints: a problem of proper extension,* NoDEA-Nonlinear Differential Equations and Applications, **3** (1996), pp. 191-216.
[39] M. MOTTA - F. RAMPAZZO, *The value function of slow growth control problem with state constraints,* Journal of Mathematical Systems, Estimation and Control, 8 (1998).
[40] L. W. NEUSTADT, *A general theory of minimum-fuel trajectories,* J. SIAM control, 3 (1965).
[41] F. RAMPAZZO, *Optimal impulsive controls with a constraint on the total variation,* Progress in *Systems and Control Theory, New trends in systems theory,* G. CONTE, A. M. PERDON and B. F. WYMAN (eds.), Boston, Massachusetts (1990), pp. 606-613.
[42] F. RAMPAZZO, *On the Riemannian structure of a Lagrangian system and the problem of adding time-dependent costraints as controls,* Eur. J. Mech., A/Solids, **10** (1991), Gauthier-Villars, pp. 405-431.
[43] R. W. RISHEL, *An extended Pontryagin principle for control systems whose control laws contain measures,* J. SIAM Control, **3** (1965), pp. 191-205.
[44] A. V. SARYCHEV, *Nonlinear systems with impulsive and generalized function controls,* Proc. Conf. on Nonlinear Synthesis, Sopron, Hungary (1989).
[45] W. W. SCHMAEDEKE, *Optimal control theory for nonlinear differential equations containing measures,* J. SIAM Control, **3** (1965), pp. 231-279.
[46] A. N. SESEKIN, *Nonlinear differential equations in the class of functions of bounded variation,* Automation and Remote Control (1990).
[47] A. N. SESEKIN, *Impulse extension in the problem of the optimization of the energy functional,* Automation and Remote Control (1993).
[48] S. P. SETHI, *Dynamic optimal control problems in advertising: a survey,* SIAM Review, **19** (1997), pp. 685-725.
[49] G. N. SILVA - R. B. VINTER, *Measure driven differential inclusions,* J. Math. Anal. Appl., **202** (1996), pp. 727-746.
[50] G. N. SILVA - R. B. VINTER, *Necessary conditions for optimal impulsive control problems,* SIAM J. Control Optim, **35** (1997), pp. 1826-1846.
[51] H. M. SONER, *Optimal control with state-space constraints,* SIAM J. Control and Optimization, **24** (1986), pp. 552-561.
[52] P. SORAVIA, *Optimality principles and representation formulas for viscosity solutions of H-J equations. II. Equations of control problems with state constraints,* Advances in Differential equations (to appear).
[53] A. J. SUBBOTIN, *Discontinuous solutions of a Dirchelet type boundary value problem for first order partial differential equations,* Russian J. Numer. Anal. Math. Modelling, 8 (1993), pp. 145-164.
[54] R. B. VINTER - M. F. L. PEREIRA, *A maximum principle for optimal processes with discontinuous trajectories,* SIAM J. Control Optimization, **26** (1988), no. 1, pp. 205-229.