Arch. Rational Mech. Anal. 196 (2010) 97–141
Digital Object Identifier (DOI) 10.1007/s00205-009-0237-6

# *Moving Constraints as Stabilizing Controls in Classical Mechanics*

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*Communicated by* The Editors

**Abstract**

The paper analyzes a Lagrangian system which is controlled by directly assigning some of the coordinates as functions of time, by means of frictionless constraints. In a natural system of coordinates, the equations of motion contain terms which are linear or quadratic with respect to time derivatives of the control functions. After reviewing the basic equations, we explain the significance of the quadratic terms related to geodesics orthogonal to a given foliation. We then study the problem of stabilization of the system to a given point by means of oscillating controls. This problem is first reduced to the weak stability for a related convex-valued differential inclusion, then studied by Lyapunov functions methods. In the last sections, we illustrate the results by means of various mechanical examples.

## 1. Introduction

A mechanical system can be controlled in two fundamentally different ways. In a commonly adopted framework [5,17,27], the controller modifies the time evolution of the system by applying additional forces. This leads to a control problem in standard form, where the time derivatives of the state variables depend continuously on the control function.

In other situations, also physically realistic, the controller acts on the system by directly assigning the values of some of the coordinates by means of time dependent constraints. The evolution of the remaining coordinates can then be determined by solving an "impulsive" control system, where the derivatives of the state variables depend (linearly or quadratically) on the time derivative of the control function. This alternative point of view was introduced independently in [13] and in [25].

Motivated by this second approach, in the present paper we study the following problem of Classical Mechanics:

Consider a system where the state space is a product $\mathcal{Q}\times\mathcal{U}$ of finite-dimensional manifolds $\mathcal{Q}$ and $\mathcal{U}$. Assume that one can prescribe the motion $t \mapsto \mathbf{u}(t) \in \mathcal{U}$ of

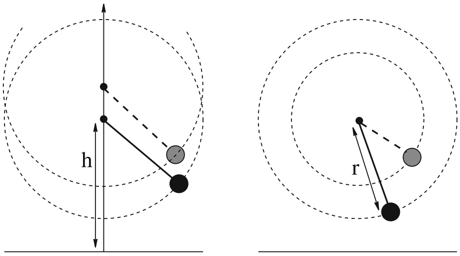


**Fig. 1.** *Left* a pendulum with vertically moving pivot and fixed length. *Right* a pendulum with fixed pivot and variable length

the second component by means of frictionless constraints. Given a point $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$, can one stabilize the system at this point by an oscillatory motion of the control $\mathbf{u}(\cdot)$ around $\bar{\mathbf{u}}$ ?

A well known example where stability is obtained by vibration is provided by a pendulum whose suspension point can oscillate on a vertical guide, as in Fig. 1, left. Calling $\theta$ the angle and $h$ the height of the pivot, in this case we have $(\mathbf{q}, \mathbf{u}) = (\theta, h) \in S^1 \times I$. Here $S^1 = [0, 2\pi]$ with endpoints identified, and $I$ is an open interval. If we take $\bar{\mathbf{q}} = \bar{\theta} = 0$ as the (unstable) upper vertical position of the pendulum, it is well-known (see for example [1,20,21] and references therein) that this configuration can be made stable by rapidly oscillating the pivot around a given value $\bar{\mathbf{u}} = \bar{h}$. More generally, we will show that this system can be asymptotically stabilized at any angle $\bar{\theta}$ with $-\pi/2 < \bar{\theta} < \pi/2$, by a suitable choice of the control function $t \mapsto h(t) = u(t)$.

On the other hand, consider a variable length pendulum, where the pivot is fixed at the origin, but we can assign the radius of oscillation $r$ as a function of time; see Fig. 1, right. The system is again described by two coordinates $(\mathbf{q}, \mathbf{u}) = (\theta, r) \in S^1 \times I$. However, in this case the upright equilibrium position is *not* stabilizable by any oscillatory motion of the radius $r(t)$ around a fixed value.

A major difference between these two systems is that the equation of motion of the first one contains a quadratic term in the time derivative $\dot{\mathbf{u}} \doteq \mathrm{d}\mathbf{u}/\mathrm{d}t$. On the other hand, the equation for the variable-length pendulum is affine with respect to the variable $\dot{\mathbf{u}}$. Actually, the explicit dependence on $\dot{\mathbf{u}}$ here can be entirely removed by a suitable change of coordinates.

To understand the general problem, one has to consider two main issues. The former is *geometric*, and involves the *orthogonal curvature* of the foliation

$$\Lambda \doteq \Big\{ \mathcal{Q} \times \{\mathbf{u}\}, \mathbf{u} \in \mathcal{U} \Big\}. \tag{1.1}$$

Orthogonality is defined here with respect to the Riemannian metric determined by the kinetic energy. The orthogonal curvature is a measure of how a geodesic, which is perpendicular to the leaf $\mathcal{Q} \times \{\mathbf{u}\}$ of the foliation at a given point $(\mathbf{q}, \mathbf{u})$, fails to remain perpendicular to the other leaves it meets. If this curvature is non-zero, then the dynamic equations for $\mathbf{q}$ and for the corresponding momentum $\mathbf{p}$ contain a quadratic term in the time derivative $\dot{\mathbf{u}}$ of the control function. This will be analyzed in detail in Part I, Sections 4, 5.

The latter issue is *analytical*, namely: how to exploit this curvature, that is the quadratic terms in $\dot{\mathbf{u}}$, in order to achieve stabilization. This will be discussed in Part II of this paper. In particular, we study the set of solutions for a system with quadratic, unbounded, controls, making essential use of reparametrization techniques. These, in turn, are combined with arguments involving Lyapunov functions for a convexified system.

Let us point out that there exists a rich literature addressing stabilization of mechanical systems in the classical framework of *force controlled* systems—namely, those mechanical systems where the controls have the physical meaning of forces (see [5–7, 17, 27] and references therein). A link between that framework and the approach adopted in the present work can be established by merely observing that in the latter the actuating forces are nothing but constraint reactions generated by imposing the kinematic control $\mathbf{u}$. Actually, a viewpoint regarding $u$ and its (first and second) time derivatives as controls is already present in the study of the so-called *superarticulated* systems (see in particular [3] and references therein). Such control systems are called *acceleration-controlled* mechanical systems, as opposed to force controlled ones (see also [4, 16] for links between the two kinds of systems). These works are based on the notion of averaging, which is in fact strictly connected with the convexification methods adopted here. In particular, the averaging approach for acceleration-controlled systems lead to stabilization results akin to Theorem 9.2 below, which incidentally is valid for non-potential exogenous forces as well—see also Remark 15.

This paper consists of three parts. In order to keep our exposition as self-contained as possible, in Part I we first describe the mechanical model and recall the basic dynamical equations. In Section 2 we consider a state space $\mathcal{Y} = \mathcal{Q} \times \mathcal{U}$ given by the product of two manifolds. The *controls* will be curves $t \mapsto \mathbf{u}(t)$ taking values in the manifold $\mathcal{U}$. The main physical assumption we are making is that these controls $\mathbf{u}(\cdot)$ are implemented by means of frictionless, time-dependent constraints. One can then derive the equations of motion on the reduced state space $\mathcal{Q}$, where the dynamics depend on $\mathbf{u}$ and on its time derivative $\dot{\mathbf{u}}$, the latter dependence being a polynomial of degree two. In Section 3, we recall the local expression of the control equations in a system of local coordinates adapted to the foliation $\Lambda$ in (1.1). Section 4 contains a survey of some geometrical and functional analytic results concerning the input–output map and the kinetic metric. The main result of Part I appears in Section 5, where we present a new interpretation of the quadratic dependence of the equations of motion on the derivative of the control functions. Our characterization of the quadratic coefficients is given in terms of the concatenation of two geodesics, the second returning to the same leaf of the foliation where the first one started. This generalizes to higher dimensions a result in [22], where the scalar control case is considered.

In Part II we consider a general nonlinear system where the right-hand side is a quadratic polynomial with respect to the time derivatives of the control function.

$$\dot{x} = f(x) + \sum_{\alpha=1}^{m} g_\alpha(x)\, \dot{u}_\alpha + \sum_{\alpha,\beta=1}^{m} h_{\alpha,\beta}(x)\, \dot{u}_\alpha \dot{u}_\beta. \tag{1.2}$$

Using a re-parametrization technique, we show that the stabilization problem for the impulsive control system (1.2) can be reduced to proving a weak stability property for a related differential inclusion with a compact, convex-valued right-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}s}x(s) \in F(x(s)), \tag{1.3}$$

$$F(x) \doteq \overline{co}\left\{ f(x)\, w_0^2 + \sum_{\alpha=1}^{m} g_\alpha(x)\, w_0 w_\alpha + \sum_{\alpha,\beta=1}^{m} h_{\alpha,\beta}(x)\, w_\alpha w_\beta;\right.$$
$$\left. w_0 \in [0,1], \quad \sum_{\alpha=0}^{m} w_\alpha^2 = 1 \right\},$$

where $\overline{co}$ denotes a closed convex hull. Theorems 7.1 and 7.2 relate the weak (asymptotic) stabilizability of the differential inclusion (1.3) with the (asymptotic) stabilizability of the impulsive control system (1.2).

In practical cases, a direct analysis of the multifunction $F$ may be difficult. In Section 7, in addition to (1.3) we thus consider an auxiliary differential inclusion of the form

$$\frac{\mathrm{d}}{\mathrm{d}s}x(s) \in G(x(s)), \tag{1.4}$$

where the multifunction $G$ is derived from (1.2) by neglecting all linear terms, that is, by formally setting $g_\alpha \equiv 0$. As shown by Theorem 6.1, trajectories of (1.3), as well as (1.4), can be approximated by implementing smooth controls, possibly with highly oscillatory behavior. We show that the weak stability of the differential inclusion (1.4) still yields the relevant stabilization properties for the original control system (1.2). Motivated by [35], in Section 8 we also show that the weak stability of the differential inclusion can be established by looking at suitable selections.

In Part III we apply the previous analytic results to the problem of stabilization of mechanical systems, controlled by moving holonomic constraints. Thanks to the particular structure of the quadratic terms that appear in the equations of motion, we show that in many cases one can construct a suitable Lyapunov function, and thus establish the desired stability properties. The paper concludes with some examples, presented in Section 10.

Throughout the paper, our focus is on systems in general form, where the equations of motion depend quadratically on the time derivatives $\dot{u}_\alpha$. In the special case where the dependence is only linear, that is $h_{\alpha,\beta} \equiv 0$ in (1.2), our results still apply; however, controllability and stabilization are best studied by looking at Lie brackets of the vector fields $f$, $g_\alpha$, using standard techniques of geometric control theory [19,37,39].

In addition to [13,25], readers interested in the earlier developments of the theory of control of mechanical systems by moving constraints are referred to [12,14, 15,18,28,29]. A concise survey, also outlining possible applications to swim-like motion in fluids, has recently appeared in [8]. See also the lecture notes [31].

## Part I: Time-dependent holonomic constraints as controls

## 2. Time-dependent constraints as controls

In this section, we recall the general framework of a Lagrangian system subject to additional time-dependent holonomic constraints, which are regarded as *controls*. We refer to [31] for a fully intrinsic derivation of the control equations, which will be presented here in coordinate form.

### *2.1. Structural assumptions*

Let $N$, $M$ be positive integers, and let $\mathcal{Q}$ and $\mathcal{U}$ be manifolds of class $C^2$ and dimension $N$ and $M$, respectively [23,36]. When needed, we shall make the natural identifications of $T(\mathcal{Q}\times\mathcal{U})$, $T^*(\mathcal{Q}\times\mathcal{U})$, and $T(T^*(\mathcal{Q}\times\mathcal{U}))$ with the products $T(\mathcal{Q})\times T(\mathcal{U})$, $T^*(\mathcal{Q})\times T^*(\mathcal{U})$, and $T(T^*(\mathcal{Q}))\times T(T^*(\mathcal{U}))$, respectively.

Let $\mathbf{g}$ be a Riemannian metric on the product manifold $(\mathcal{Q}\times\mathcal{U})$. We shall refer to the Riemannian manifold $(\mathcal{Q}\times\mathcal{U},\mathbf{g})$ as the *original Lagrangian system*, meaning that the *whole state space* is represented by the product manifold $(\mathcal{Q}\times\mathcal{U})$, and the *kinetic energy* $\mathcal{T}$ is the quadratic form defined by

$$\mathcal{T}(\mathbf{q},\mathbf{u})[\mathbf{v},\mathbf{w}] = \frac{1}{2}\mathbf{g}(\mathbf{q},\mathbf{u})\Big((\mathbf{v},\mathbf{w}),(\mathbf{v},\mathbf{w})\Big).$$

$\mathcal{Q}$ and $\mathcal{U}$ are called the *reduced state space* and the *control space*, respectively.[1]

(Regularity of the force). The external force $\mathbf{F}=\mathbf{F}(t,\mathbf{q},\mathbf{u},\mathbf{P},\wp)$ is a function measurable with respect to $t$ and locally Lipschitz with respect to all other variables.

### *2.2. Foliation structure and adapted coordinates*

Let us consider the trivial foliation structure where the set of leaves is

$$\Lambda = \{\mathcal{Q}\times\{\mathbf{u}\} \quad \mathbf{u}\in\mathcal{U}\}\,. \tag{2.1}$$

For every $(\mathbf{q},\mathbf{u})\in\mathcal{Q}\times\mathcal{U}$, we denote by $\Lambda(\mathbf{q},\mathbf{u})\doteq\mathcal{Q}\times\{\mathbf{u}\}$ the *leaf* through $(\mathbf{q},\mathbf{u})$.

We say that a (local) system of coordinates $(\tilde{q},\tilde{u})$ is $\Lambda$-*adapted* if the sets $\{\tilde{u}=\text{constant}\}$ locally coincide with the leaves of the foliation. Of course, the local product coordinates $(q,u)$ are $\Lambda$-adapted. More generally, if $(\tilde{q},\tilde{u})$ are $\Lambda$-adapted, then every system of coordinates $(\hat{q},\hat{u})$ obtained from $(\tilde{q},\tilde{u})$ by means of a local diffeomorphism of the form

$$\hat{q}=\hat{q}(\tilde{q},\tilde{u}) \quad \hat{u}=\hat{u}(\tilde{u}). \tag{2.2}$$

is $\Lambda$-adapted as well.

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[1] Following [25 and 18], one could consider a less trivial foliated structure.

Let us consider the distribution[2] $\Delta$ whose fibers are the tangent spaces to the leaves of $\Lambda$. Namely, one has

$$\Delta_{(\mathbf{q},\mathbf{u})} = T_{\mathbf{q}}\mathcal{Q} \times \{0\}.$$

In our analysis, an important role will be played also by the *orthogonal* distribution

$$\Delta^{\perp}_{(\mathbf{q},\mathbf{u})} = \Big\{ Y \in T_{\mathbf{q}}\mathcal{Q} \times T_{\mathbf{u}}\mathcal{U} \;\; \big| \;\; \mathbf{g}(\mathbf{q},\mathbf{u})(Y,X) = 0 \quad \text{for all} \;\; X \in \Delta_{(\mathbf{q},\mathbf{u})} \Big\}, \tag{2.3}$$

which will also be referred to as the *orthogonal bundle*, for short.

### 2.3. *Admissible input–output pairs*

Consider a control function $t \mapsto \mathbf{u}(t) \in \mathcal{U}$. In this section we define the corresponding output $t \mapsto (\mathbf{q}(t), \mathbf{p}(t))$ as the solution of a certain Cauchy problem. In the following section, we then show that our definition is consistent with the mechanical model, where the control is implemented in terms of frictionless constraints.

For every $(\mathbf{u},\mathbf{w}) \in T\mathcal{U}$, let us define the map $\mathcal{T}^{\mathbf{u},\mathbf{w}} : T\mathcal{Q} \mapsto \mathbb{R}$ by setting

$$\mathcal{T}^{\mathbf{u},\mathbf{w}}(\mathbf{q},\mathbf{v}) \doteq \mathcal{T}(\mathbf{q},\mathbf{u})[\mathbf{v},\mathbf{w}], \tag{2.4}$$

for all $(\mathbf{q},\mathbf{v}) \in T\mathcal{Q}$. This map can be regarded as the kinetic energy of the reduced system when the control takes the value $\mathbf{u}$, with $\dot{\mathbf{u}} = \mathbf{w}$.

Let $I \subset \mathbb{R}$ be an interval, and let $\mathbf{u} : I \mapsto \mathcal{U}$ be an absolutely continuous control function. The (time-dependent) kinetic energy of the reduced system on $\mathcal{Q}$, corresponding to the control $\mathbf{u}(\cdot)$ is described, for all $(\mathbf{q},\mathbf{v}) \in T\mathcal{Q}$ and for almost everywhere $t \in I$, by

$$(t,\mathbf{q},\mathbf{v}) \mapsto \mathcal{T}^{\mathbf{u}(t),\dot{\mathbf{u}}(t)}\big(\mathbf{q},\mathbf{v}\big).$$

The corresponding (time-dependent) Hamiltonian on $T^*\mathcal{Q}$ is

$$(t,\mathbf{q},\mathbf{p}) \mapsto \mathcal{H}^{\mathbf{u}(t),\dot{\mathbf{u}}(t)}(\mathbf{q},\mathcal{P}),$$

where

$$\mathcal{H}^{\mathbf{u},\mathbf{w}}(\mathbf{q},\mathbf{p}) \doteq \sup_{\mathbf{v}\in T_{\mathbf{q}}\mathcal{Q}} \Big\{ \langle \mathbf{p},\mathbf{v}\rangle - \mathcal{T}^{\mathbf{u},\mathbf{w}}(\mathbf{q},\mathbf{v}) \Big\}. \tag{2.5}$$

For every $(\mathbf{q},\mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$, consider the linear map defined by

$$(\mathbf{p},\wp) \mapsto (\mathbf{v},\mathbf{w})(\mathbf{p},\wp) \doteq \operatorname{argmin}\Big\{ \langle (\mathbf{v},\mathbf{w}),(\mathbf{p},\wp)\rangle - \mathcal{T}(\mathbf{q},\mathbf{u})[\mathbf{v},\mathbf{w}] \Big\}.$$

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[2] In our context, the term "distribution" is meant in the sense of differential geometry, namely, a fiber sub-bundle of the tangent bundle $T(\mathcal{Q} \times \mathcal{U})$.

In coordinates, this is nothing but the usual map that transforms momenta into velocities by taking partial derivatives of the Hamiltonian corresponding to the kinetic energy.

Since $\mathbf{g}$ is positive definite, for every $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$ and every $\mathbf{p} \in T^*_{\mathbf{q}}\mathcal{Q}$, the affine function

$$\wp \mapsto (\mathbf{v}, \mathbf{w})(\mathbf{p}, \wp)$$

is invertible. Its inverse will be denoted by

$$\mathbf{w} \mapsto \wp_{(\mathbf{q},\mathbf{u},\mathbf{p})}(\mathbf{w}).$$

Let $(q, u)$ be $\Lambda$-adapted coordinates, and let $(q, u, p, \pi)$ be the corresponding bundle coordinates. Let $(F_i, F_{N+\alpha})$ be the components of the force $\mathbf{F}$, so that

$$\mathbf{F} = F_i \frac{\partial}{\partial p_i} + F_{N+\alpha} \frac{\partial}{\partial \pi_\alpha}. \tag{2.6}$$

Recalling the dimensions of the manifolds $\mathcal{Q}$ and $\mathcal{U}$, we here have $i = 1, \dots, N$ and $\alpha = 1, \dots, M$. The Einstein convention of summing over repeated indices is always used. In addition, we define

$$F_i^{\mathbf{u},\mathbf{w}}(t, \mathbf{q}, \mathbf{p}) \doteq F_i\big(t, \mathbf{q}, \mathbf{u}, \mathbf{p}, \wp_{(\mathbf{q},\mathbf{u},\mathbf{p})}(\mathbf{w})\big) \tag{2.7}$$

and

$$\mathbf{F}^{\mathbf{u},\mathbf{w}}(t, \mathbf{q}, \mathbf{p}) \doteq F_i^{\mathbf{u},\mathbf{w}}(t, \mathbf{q}, \mathbf{p}) \frac{\partial}{\partial p_i}. \tag{2.8}$$

**Remark 1.** Despite (2.4), in general one has

$$\mathcal{H}^{\mathbf{u},\mathbf{w}}(\mathbf{q}, \mathbf{p}) \neq \mathcal{H}\left(\mathbf{q}, \mathbf{u}, \mathbf{p}, \wp_{(\mathbf{q},\mathbf{u},\mathbf{p})}(\mathbf{w})\right).$$

The relation between these two functions is explained in more detail in [31].

**Definition 1.** Let $I \subset \mathbb{R}$ be a time interval. Let

$$\mathbf{u} : I \mapsto \mathcal{U}, \quad (\mathbf{q}, \mathbf{p}) : I \mapsto \mathcal{T}^*\mathcal{Q},$$

be absolutely continuous maps. We say that $\Big(\mathbf{u}(\cdot),\ (\mathbf{q}, \mathbf{p})(\cdot)\Big)$ is an *admissible input–output pair* if $(\mathbf{q}, \mathbf{p})$ is a Carathéodory solution of the control equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big) = X_{\mathcal{H}^{\mathbf{u}(t),\dot{\mathbf{u}}(t)}}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big) + \mathbf{F}^{\mathbf{u}(t),\dot{\mathbf{u}}(t)}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big). \tag{2.9}$$

Here $X_{\mathcal{H}^{\mathbf{u},\dot{\mathbf{u}}}}$ denotes the Hamiltonian vector field corresponding to $\mathcal{H}^{\mathbf{u},\dot{\mathbf{u}}}$ with respect to the symplectic structure on $T^*\mathcal{Q}$.[3]

[3] With reference to $\Gamma$-adapted coordinates $(q^i, u^\alpha)$ and the corresponding bundle coordinates $(q^i, p_i, u^\alpha, \mathbf{p}_\alpha)$, one has $X_{\mathcal{H}^{\mathbf{u},\dot{\mathbf{u}}}} = \dfrac{\partial \mathcal{H}^{u,\dot{u}}}{\partial p_i} dq^i - \dfrac{\partial \mathcal{H}^{u,\dot{u}}}{\partial q^i} dp_i$.

We recall that a Carathéodory solutions of an ODE $\dot{x} = f(t,x)$ is an absolutely continuous function $t \mapsto x(t)$ that satisfies the differential equation at almost everywhere time $t$. Given an initial data

$$\big(\mathbf{q}(\bar{t}), \mathbf{p}(\bar{t})\big) = (\bar{\mathbf{q}}, \bar{\mathbf{p}}), \tag{2.10}$$

and an absolutely continuous control function $t \mapsto \mathbf{u}(t)$, the existence and uniqueness of a corresponding admissible output $(\mathbf{q}(\cdot), \mathbf{p}(\cdot))$ can be obtained from standard ODE theory.

Depending on the geometrical properties of the metric $\mathbf{g}$, the regularity assumptions on the input $\mathbf{u}$ and the output $(\mathbf{q}, \mathbf{p})$ can be considerably weakened.

## 2.4. *Realization of controls as frictionless constraints*

The previous notion of input–output pair is motivated by the fact that we are assuming that the control $\mathbf{u}(\cdot)$ is realized by means of *frictionless* constraints. This is explained by the equivalence of conditions (i) and (iii) in Theorem 2.1 below.

Let us recall the notion of *frictionless constraint reaction* in the Hamiltonian framework.

For every $((\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp)) \in T^*(\mathcal{Q}) \times T^*(\mathcal{U})$, consider the subspace

$$R^{\mathcal{Q}}_{(\mathbf{q},\mathbf{p}),(\mathbf{u},\wp)} = \{0\} \times V_{(\mathbf{u},\wp)}(T^*\mathcal{U}) \subset T_{(\mathbf{q},\mathbf{p})}(T^*(\mathcal{Q})) \times T_{(\mathbf{u},\wp)}(T^*(\mathcal{U}))$$

where $V_{(\mathbf{u},\wp)}(T^*\mathcal{U})$ denotes the *vertical subspace* of $T_{(\mathbf{u},\wp)}(T^*\mathcal{U})$, namely, for given bundle coordinates $(q^i, p_i, u^\alpha, \mathbf{p}_\alpha)$,

$$V_{(\mathbf{u},\wp)}(T^*\mathcal{U}) \doteq \left\{ \frac{\partial}{\partial \wp_\alpha} \quad \alpha = 1, \dots, M \right\}.$$

**Definition 2.** The subspace $R^{\mathcal{Q}}_{(\mathbf{q},\mathbf{p}),(\mathbf{u},\wp)}$ will be called the *subspace of $\mathcal{Q}$-frictionless reactions at* $((\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp))$. The corresponding vector bundle based on $T^*\mathcal{Q} \times T^*\mathcal{U}$ will be called *the vector bundle of $\mathcal{Q}$-frictionless reactions*.

**Remark 2.** Here we are regarding the constraint reactions as a set-valued force, described by the multifunction

$$(\mathbf{q}, \mathbf{p}), (\mathbf{u}, \wp) \mapsto R_{(\mathbf{q},\mathbf{p}),(\mathbf{u},\wp)}.$$

To check that this definition coincides with the usual one it is sufficient to notice that if $\Phi \in R^{\mathcal{Q}}_{\mathbf{q},\mathbf{p},\mathbf{u},\wp}$ and $\Phi = \sum_{r=1}^{N+M} \Phi_r \frac{\partial}{\partial P_r}$ is its local expression, then one has

$$\Big\langle (\Phi_1, \dots, \Phi_N, \Phi_{N+1}, \dots, \Phi_{N+M}) \ , \ (v_1, \dots, v_N, 0, \dots, 0) \Big\rangle = 0, \qquad \text{for all } v_1, \dots, v_N \in \mathbb{R}.$$

Of course, this holds if and only if $\Phi_i = 0$ for all $i = 1, \dots, N$.

Definition 1 is justified by Theorem 2.1 below. Let $Pr_1 : T^*Q \times T^*\mathcal{U} \to T^*\mathcal{Q}$ be the projection on the first factor space, and let $D(Pr_1)$ be its tangent map.

**Theorem 2.1.** [31] *Consider a time interval $I \subset \mathbb{R}$ and let the maps $\mathbf{u} : I \mapsto \mathcal{U}$, $(\mathbf{q}, \mathbf{p}) : I \mapsto T^*\mathcal{Q}$ be twice continuously differentiable. Then the following conditions are equivalent:*

(i) $(\mathbf{u}(\cdot), (\mathbf{q}(\cdot), \mathbf{p}(\cdot)))$ *is an admissible input–output pair, that is,* $(\mathbf{q}, \mathbf{p})$ *verifies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big) = X_{\mathcal{H}^{\mathbf{u}(t),\dot{\mathbf{u}}(t)}}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big) + \mathbf{F}^{\mathbf{u}(t),\dot{\mathbf{u}}(t)}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big). \tag{2.11}$$

(ii) *The path* $\Big(\mathbf{q}(\cdot), \mathbf{p}(\cdot)\Big)$ *is an integral curve of the control system*

$$\begin{aligned}\frac{d}{dt}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big) = D(Pr_1) \cdot \Big[&X_{\mathcal{H}}^{\mathcal{Q}}(t, \mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp) \\ &+ \mathbf{F}^{\mathbf{u}(t),\dot{\mathbf{u}}(t)}\Big(\mathbf{q}(t), \mathbf{p}(t)\Big)\Big]_{|\wp=\wp(\mathbf{q}(t),\mathbf{u}(t),\mathbf{p}(t))}(\dot{\mathbf{u}}(t))\end{aligned}. \tag{2.12}$$

(iii) *There exist selections*

$$t \mapsto \wp(t) \in T^*_{u(t)}(\mathcal{U}) \quad t \mapsto r(t) \in R^{\mathcal{Q}}_{\mathbf{q}(t),\mathbf{p}(t),\mathbf{u}(t),\wp(t)}$$

*such that, for all $t \in I$, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{q}(t), \mathbf{u}(t), \mathbf{p}(t), \wp(t)\Big) = X_{\mathcal{H}} + \mathbf{F} + r(t). \tag{2.13}$$

*The map $r(\cdot)$ in* (2.13) *is called the* constraint reaction corresponding to the motion $(\mathbf{q}, \mathbf{p}, \mathbf{u}, \wp)(\cdot)$.

## 3. The control equation in local coordinates

Consider a $\Lambda$-adapted coordinate chart $(q, u)$ defined on an open set $U$, and let $((q, u), (p, w))$ be the corresponding coordinates on the fiber bundle

$$\bigcup_{(\mathbf{q},\mathbf{u})\in U} \{(\mathbf{q}, \mathbf{u})\} \times (T^*_{\mathbf{q}}\mathcal{Q} \times T_{\mathbf{u}}\mathcal{U}).$$

Let $G = (g_{r,s})_{r,s=1,\dots,N+M}$ be the matrix representing the kinetic metric $\mathbf{g}$, and let $G^{-1} = (g^{r,r})_{r,s=1,\dots,N+M}$ denote its inverse. In the following, we consider the sub-matrices

$$G_1 \doteq (g_{i,j}), \quad G_2 \doteq \left(g_{N+\alpha,N+\beta}\right), \quad (G^{-1})_2 \doteq \left(g^{N+\alpha,N+\beta}\right),$$
$$G_{12} \doteq \left(g_{i,N+\alpha}\right), \quad (G^{-1})_{12} \doteq \left(g^{i,N+\alpha}\right),$$

with the convention that the Latin indices $i, j$ run from 1 to $N$, while the Greek indices $\alpha, \beta$ run from 1 to $M$. For convenience, we also define

$$\begin{aligned}A = (a^{i,j}) \doteq (G_1)^{-1}, \qquad &E = (e_{\alpha,\beta}) \doteq ((G^{-1})_2)^{-1}, \\ &K = (k^i_{N+\alpha}) \doteq (G^{-1})_{12}E.\end{aligned} \tag{3.1}$$

**Proposition 3.1.** [31] *Let* $\mathbf{u}(\cdot) : I \mapsto \mathcal{U}$ *be twice continuously differentiable, and let* $(\mathbf{q}, \mathbf{p}) : I \mapsto T^*\mathcal{Q}$ *be a curve such that* $(\mathbf{u}(\cdot), (\mathbf{q}, \mathbf{p})(\cdot))$ *is an admissible input–output pair for the control equation of motion.*[4]

*Then the the corresponding coordinate maps* $t \mapsto \left(u(t), \begin{pmatrix} q(t) \\ p(t) \end{pmatrix}\right)$ *satisfy the differential equation*

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p \end{pmatrix} + \begin{pmatrix} K\dot u \\ \\ -p^\dagger \frac{\partial K}{\partial q} \dot u \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \frac{1}{2} \dot u^\dagger \frac{\partial E}{\partial q} \dot u \end{pmatrix} + \begin{pmatrix} 0 \\ \\ F^{u(\cdot),\dot u(\cdot)} \end{pmatrix}, \tag{3.2}$$

*where, recalling* (2.8),

$$F^{u(\cdot),\dot u(\cdot)} \doteq \left(F_1^{u(\cdot),\dot u(\cdot)}, \ldots, F_N^{u(\cdot),\dot u(\cdot)}\right). \tag{3.3}$$

For convenience, in (3.2) we write all vectors as column vectors, while the superscript $^\dagger$ denotes transposition. Componentwise, (3.2) reads:

$$\begin{cases} \dot q^i = a^{i,j} p_j + k^i_{N+\alpha} \dot u^\alpha, \\ \dot p_i = -\frac{1}{2} \frac{\partial a^{\ell,j}}{\partial q^i} p_\ell p_j - \frac{\partial k^j_\alpha}{\partial q^i} p_j \dot u^\alpha + \frac{1}{2} \frac{\partial e_{\alpha,\beta}}{\partial q^i} \dot u^\alpha \dot u^\beta + F_i^{u(\cdot),\dot u(\cdot)}. \end{cases} \tag{3.4}$$

(where $i$, $j$, and $\ell$ run from 1 to $N$).

## 4. The Riemannian structure and and the input–output map

The coefficients $\frac{\partial E}{\partial q}$ of the quadratic terms in the dynamic equations (3.2) depend on the interplay between the Riemannian metric $g$ defining the kinetic energy and the foliation $\Lambda$ at (2.1). In this section we review the main results in this direction. To simplify the discussion, throughout this section we shall assume that the additional forces $\mathbf{F}$ vanish identically, so that in (3.3) one has

$$F_{\mathcal{Q}}^{u(\cdot),\dot u(\cdot)} \equiv 0.$$

The following definitions were introduced in [13].

**Definition 3.** A local, $\Lambda$-adapted, system of coordinates $(q, u)$ on $\mathcal{Q} \times \mathcal{U}$ is called *$N$-fit for hyperimpulses* if in (3.4) one has $\partial E/\partial q \equiv 0$. This means that the right-hand side of the corresponding equation of motion is affine with respect to the time derivative $\dot u$

A local, $\Lambda$-adapted, system of coordinates $(q, u)$ on $\mathcal{Q} \times \mathcal{U}$ is called *strongly $N$-fit for hyperimpulses* if in (3.4) one has $\partial E/\partial q \equiv 0$ and $K \equiv 0$. This means that the right-hand side of equation of motion does not explicitly depend on the variable $\dot u$.

Moreover, we shall call *generic* any local, $\Lambda$-adapted, system of coordinates $(q, u)$ which is not $N$-fit for hyperimpulses.

[4] By possibly restricting the size of the interval $I$, we can assume that $(\mathbf{q}(t), \mathbf{u}(t))$ remains inside the domain of the single chart $(q, u)$ for every $t \in I$.

**Remark 3.** The denomination "*N-fit for hyperimpulses*" for a system of coordinates $(q, u)$ refers to the fact that, if the dependence on $\dot{u}$ is only linear, one can then construct solutions $(q(\cdot), p(\cdot))$ also for discontinuous controls $u(\cdot)$. In general, a jump in $u(\cdot)$ will produce a discontinuity in both components $q(\cdot)$ and $p(\cdot)$. For this reason we call it a *hyperimpulse*, distinguished from an *impulse*, which can cause a discontinuity in the component $p(\cdot)$ only.

A first characterization of $N$-fit coordinates was derived in [13]. It is important to observe that the property of being $N$-fit depends only on the metric $g$ and on the foliation $\Lambda$, while it is independent of the particular system of $\Lambda$-adapted coordinates. This allows one to give the following definitions.

**Definition 4.** [29] The foliation $\Lambda$ is called *N-fit for hyperimpulses* if there exists an atlas of $\Lambda$-adapted charts that are also $N$-fit for hyperimpulses. In this case, *all* $\Lambda$-adapted charts are $N$-fit for hyperimpulses.

The foliation $\Lambda$ is called *strongly N-fit for hyperimpulses* if there exists an atlas of $\Lambda$-adapted charts which are strongly $N$-fit for hyperimpulses.

Moreover, the foliation $\Lambda$ will be called *generic* if it is not $N$-fit for hyperimpulses.

The paper [29] established the connection between the $N$-fitness of the foliation $\Lambda$ and the bundle-like property of the metric, introduced in [33,34]. We recall here the main definitions and results.

**Definition 5.** The metric $\mathbf{g}$ is *bundle-like* with respect to the foliation $\Lambda$ if, for one (hence for every) $\Lambda$-adapted chart, it has a local representation of the form

$$\sum_{i,j=1}^{N} g_{i,j}(q,u)\omega^i \otimes \omega^j + \sum_{\alpha,\beta=1}^{M} g_{N+\alpha,N+\beta}(u)dc^\alpha \otimes dc^\beta,$$

where $\omega^1, \ldots, \omega^N$ are linearly independent 1-forms such that, for each $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$ in the domain of the chart, one has

(i) $(\omega^1(\mathbf{q}, \mathbf{u}), \ldots, \omega^N(\mathbf{q}, \mathbf{u}), dc^1(\mathbf{u}), \ldots, dc^M(\mathbf{u}))$ is a basis of the cotangent space $T^*_{\mathbf{q}}\mathcal{Q} \times T^*_{\mathbf{u}}\mathcal{U}$;

(ii) $\langle \omega^i(\mathbf{q}, \mathbf{u}), Y\rangle = 0$, for every $Y \in \Delta^\perp_{(\mathbf{q},\mathbf{u})}$.

We recall that $\Delta^\perp_{(\mathbf{q},\mathbf{u})}$ is the orthogonal bundle, defined at (2.3). If $g$ is bundle-like with respect to the foliation $\Lambda$, the latter is also called a *Riemannian foliation*, because in this case a Riemannian structure can be well defined also on the quotient space. In order to state the next theorem, we recall the notion of *completely integrable* distribution.

**Definition 6.** Let $\mathcal{Y}$ be a manifold of dimension $d$, and let $\Gamma$ be a distribution on $\mathcal{Y}$ of dimension $N \leqq d$. (For every $y \in \mathcal{Y}$, $\Gamma(y)$ is thus an N-dimensional subspace of $T_y\mathcal{Y}$.) We say that the distribution $\Gamma$ is *completely integrable* if, for every $\mathbf{y} \in \mathcal{Y}$, there exists a neighborhood $U$ of $\mathbf{y}$ and a local system of coordinates $(x, z) = (x^1, \ldots, x^N, z^{N+1}, \ldots, z^d)$ such that at each point $y \in U$ one has

$$\Gamma(y) = span\left\{\frac{\partial}{\partial x_i} \quad i = 1, \ldots, N\right\}.$$

**Theorem 4.1.** *On the product space $\mathcal{Y} = \mathcal{Q} \times \mathcal{U}$, consider the natural foliation $\Lambda$ as in (2.1). The following statements are equivalent:*

(i) *The foliation $\Lambda$ is $N$-fit for hyperimpulses.*
(ii) *The metric $\mathbf{g}$ is bundle-like with respect to the foliation $\Lambda$, that is, the foliation $\Lambda$ is Riemannian.*
(iii) *For any $\mathbf{u}, \bar{\mathbf{u}} \in \mathcal{U}$ the map $d_{\mathbf{u},\bar{\mathbf{u}}}(\cdot) : \mathcal{Q} \mapsto \mathbb{R}$ defined by*

$$d_{\mathbf{u}}(\mathbf{q}) \doteq dist\Big((\mathbf{q}, \mathbf{u}), \mathcal{Q} \times \{\bar{\mathbf{u}}\}\Big)$$

*is constant. In other words, given two leaves, the points of one of the two are all at the same distance from the other leaf (this allows one to define a metric on the set of leaves).*
(iv) *If $t \mapsto (\mathbf{q}(t), \mathbf{u}(t))$ is any geodesic curve with respect to the metric $\mathbf{g}$, and if $(\dot{\mathbf{q}}(\tau), \dot{\mathbf{u}}(\tau)) \in \Delta^{\perp}_{(\mathbf{q}(\tau),\mathbf{u}(\tau))}$ at some time $\tau$, then $(\dot{\mathbf{q}}(t), \dot{\mathbf{u}}(t)) \in \Delta^{\perp}_{(\mathbf{q}(t),\mathbf{u}(t))}$ for all $t$. In other words, if a geodesic crosses one of the leaves perpendicularly, then it also crosses perpendicularly every other leaf which it meets.*
(v) *If $(q, u)$ is a $\Lambda$-adapted system of coordinates, then*

$$\frac{\partial g^{N+\alpha,N+\beta}}{\partial q_i} = 0 \quad i = 1, \dots, N, \quad \alpha, \beta = 1, \dots, M, \tag{4.1}$$

*where $G^{-1} = (g^{r,s})$ denotes the inverse of the matrix $G = (g_{r,s})$ representing the metric $\mathbf{g}$ in the coordinates $(q, u)$.*

**Proof.** The equivalence of (i) and (ii) is a trivial consequence of the definitions of bundle-like metric and of $N$-fit system of coordinates. The equivalence of (ii), (iii), and (iv), is a classical result on bundle-like metrics [33]. Moreover, by (3.4), the foliation is fit for jumps if and only if $\partial e_{\alpha,\beta}/\partial q^i \equiv 0$. Recalling that the matrix $(e_{\alpha,\beta})$ is the inverse of $(G^{-1})_2 = (g^{N+\alpha,N+\beta})$, we conclude that (i) is equivalent to (v). □

**Theorem 4.2.** *The following statements are equivalent:*

(i) *The foliation $\Lambda$ is strongly $N$-fit for hyperimpulses.*
(ii) *The foliation $\Lambda$ is $N$-fit for hyperimpulses and the orthogonal bundle $\Delta^{\perp}_{(\mathbf{q},\mathbf{u})}$ in (2.3) is integrable.*
(iii) *There is an atlas such that, for every chart $(q, u)$, one has*

$$\frac{\partial g^{N+\alpha,N+\beta}}{\partial q_i} = 0, \quad g^{i,N+\alpha} = 0 \ \ \textit{for all} \ \ i = 1, \dots, N, \quad \alpha, \beta = 1, \dots, M.$$

Indeed the equivalence of (i) and (ii), formulated in terms of Riemannian foliations, was proved in [33]. The equivalence between (i) and (iii) follows from (3.4). Again, see [33 and 29] for details.

## 5. The quadratic term in the control equation and the orthogonal curvature of the foliation $\Lambda$

As we have seen in the previous section, the $N$-fitness for hyperimpulses of a coordinate system $(q, u)$ can be characterized in terms of geodesics. Indeed, the quadratic terms in the control equation of motion (3.4) are identically zero if and only if any geodesic which perpendicularly crosses one leaf of the foliation $\Lambda$ also has perpendicular intersection with every other leaf it meets.

In the general case, however, the quadratic terms in (3.4) do not vanish. We wish to give here a geometric interpretation of these terms. This will again be achieved by looking at geodesics whose tangent vector initially lies in the orthogonal distribution $\Delta^{\perp}$.

### *5.1. $\mathcal{U}$-orthonormal coordinates*

We shall make an essential use of Proposition 5.1 below, which establishes the existence of a special kind of $\Lambda$-adapted charts. To state it, let us set $(x^1, \ldots, x^{N+M}) \doteq (q^1, \ldots, q^N, u^1, \ldots, u^M)$, and, for every $h, k, r, s = 1, \ldots, N + M$, let us consider the functions

$$\Gamma_{h,r,s} \doteq \frac{1}{2}\left(\frac{\partial g_{h,r}}{\partial x^s} + \frac{\partial g_{h,s}}{\partial x^r} - \frac{\partial g_{r,s}}{\partial x^h}\right), \quad \Gamma^k_{r,s} \doteq g^{kh}\left(\frac{\partial g^{h,r}}{\partial x^s} + \frac{\partial g^{h,s}}{\partial x^r} - \frac{\partial g^{r,s}}{\partial x^h}\right)$$

The $\Gamma^k_{r,s}$ are the well-known Christoffel symbols.

**Proposition 5.1.** *Consider a point $(\bar{\mathbf{q}}, \bar{\mathbf{u}}) \in \mathcal{Q} \times \mathcal{U}$ and an orthonormal basis $\{J_1, \ldots, J_M\}$ of $\Delta^{\perp}(\bar{\mathbf{q}}, \bar{\mathbf{u}})$. Then there exist $\Lambda$-adapted coordinates $(q, u)$, defined on a neighborhood of $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$, such that calling $G = (g_{r,s})_{r,s=1,\ldots,N+M}$ the corresponding kinetic matrix, one has*

(i) *the point $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ has coordinates $(0, 0)$;*

(ii) $g_{r,s}(0, 0) = g^{r,s}(0, 0) = \delta_{r,s}$ *(the Kronecker symbol) for all $r, s = 1, \ldots, N + M$;*

(iii) *for every $w = (w_1, \ldots, w_M) \in \mathbb{R}^M$, the geodesic $(q, u)_w(\cdot)$ issuing from $(\mathbf{q}, \mathbf{u})$ with velocity equal to $w_1 J_1 + \ldots w_M J_M$ has local representation $(q, u)_w(t) = (0, \ldots, 0, tw_1, \ldots, tw_M)$.*

*Moreover, for all indices $i = 1, \ldots, N$ and $\alpha, \beta, \gamma = 1, \ldots, M$, we have*

$$\Gamma_{i,N+\alpha,N+\beta}(0, 0) = \Gamma^i_{N+\alpha,N+\beta}(0, 0) = 0,$$
$$\Gamma_{N+\gamma,N+\alpha,N+\beta}(0, 0) = \Gamma^{N+\gamma}_{N+\alpha,N+\beta}(0, 0) = 0. \tag{5.1}$$

*In turn, this implies*

$$\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0, 0) = \frac{\partial g^{i,N+\beta}}{\partial u^\gamma}(0, 0) + \frac{\partial g^{i,N+\gamma}}{\partial u^\beta}(0, 0), \tag{5.2}$$

$$\frac{\partial g_{N+\alpha,N+\beta}}{\partial u^\gamma}(0, 0) = \frac{\partial g^{N+\alpha,N+\beta}}{\partial u^\gamma}(0, 0) = 0. \tag{5.3}$$

*A chart with the above properties will be called $\mathcal{U}$*-orthonormal at $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$.

**Proof.** We start by considering $\Lambda$-adapted coordinates $(\hat{q}, \hat{u})$, defined on a neighborhood of the point $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$, such that at the point $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ one has $(\hat{q}, \hat{u}) = (0, 0)$ and

$$\frac{\partial}{\partial \hat{u}^\alpha} = J_\alpha, \quad \alpha = 1, \ldots, M,$$

while

$$\left\{ \frac{\partial}{\partial \hat{q}^i}; i = 1, \ldots, N \right\}$$

is an orthonormal basis of the tangent space $T\mathcal{Q}$ at $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$, with respect to the metric $\mathbf{g}$.

To achieve the further property (iii), we need to modify these coordinates using the exponential map. In the following, given a tangent vector $\mathbf{V} \in T_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})}(\mathcal{Q} \times \mathcal{U})$, we denote by $\tau \mapsto \gamma_{\mathbf{V}}(\tau)$ the geodesic curve starting from $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ with velocity $\mathbf{V}$. In other words,

$$\gamma_{\mathbf{V}}(0) = (\bar{\mathbf{q}}, \bar{\mathbf{u}}), \quad \frac{\mathrm{d}\gamma_{\mathbf{V}}}{\mathrm{d}\tau}(0) = \mathbf{V}.$$

The exponential map is then defined by setting

$$\mathrm{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})}(\mathbf{V}) \doteq \gamma_{\mathbf{V}}(1).$$

This is well defined for all vectors $\mathbf{V}$ in a neighborhood of the origin.

Denote by $(\hat{q}, \hat{u})(\mathbf{q}, \mathbf{u})$ the coordinates of a point $(\mathbf{q}, \mathbf{u})$ via the chart $(\hat{q}, \hat{u})$. We now define $(q, u)$ to be the new coordinates of a point $(\mathbf{q}, \mathbf{u})$ provided that

$$\Big(\hat{q}, \hat{u}\Big)(\mathbf{q}, \mathbf{u}) = \Big(\hat{q}, \hat{u}\Big)\left(\mathrm{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})}\Big(\sum_{\alpha=1}^{M} u^\alpha J_\alpha\Big)\right) + \Big(q^1, \ldots, q^N, 0, \ldots, 0\Big). \tag{5.4}$$

Notice that this is well defined, for all $(\mathbf{q}, \mathbf{u})$ in a neighborhood of $(\bar{\mathbf{q}}, \bar{\mathbf{u}})$. Indeed, the map $\rho : \mathbb{R}^{N+M} \mapsto \mathbb{R}^{N+M}$ defined by

$$\begin{aligned}
&\rho(q^1, \ldots, q^N, u^1, \ldots, u^M) \\
&\quad = \Big(\hat{q}, \hat{u}\Big)\left(\mathrm{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})}\Big(\sum_{\alpha=1}^{M} u^\alpha J_\alpha\Big)\right) + \Big(q^1, \ldots, q^N, 0, \ldots, 0\Big).
\end{aligned} \tag{5.5}$$

maps the origin into itself. Moreover, by the properties of the chart $(\hat{q}, \hat{u})$, the Jacobian matrix $\partial\rho/\partial(q^1, \ldots, q^N, u^1, \ldots, u^M)$ at the origin coincides with the identity matrix. This already establishes the properties (i) and (ii).

By construction, $(q, u)(\mathbf{q}, \mathbf{u}) = (0, \ldots, 0, u^1, \ldots, u^M)$ if and only if

$$(\mathbf{q}, \mathbf{u}) = \mathrm{Exp}_{(\bar{\mathbf{q}}, \bar{\mathbf{u}})}\left(\sum_{\alpha=1}^{M} u^\alpha J_\alpha\right).$$

This establishes (iii).

In order to prove (5.2) and (5.3), we observe that the geodesic curves correspond to solutions of the second order equations

$$\begin{cases} \ddot{q}^i = \Gamma^i_{j,\ell}\dot{q}^j\dot{q}^\ell + 2\Gamma^i_{j,N+\alpha}\dot{q}^j\dot{u}^\alpha + \Gamma^i_{N+\alpha,N+\beta}\dot{u}^\alpha\dot{u}^\beta & i = 1,\dots,N \\ \ddot{u}^\gamma = \Gamma^\gamma_{j,\ell}\dot{q}^j\dot{q}^\ell + 2\Gamma^\gamma_{j,N+\alpha}\dot{q}^j\dot{u}^\alpha + \Gamma^\gamma_{N+\alpha,N+\beta}\dot{u}^\alpha\dot{u}^\beta & \gamma = 1,\dots,M \end{cases} \tag{5.6}$$

By the previous construction, for any given $w \in \mathbb{R}^M$ the solution of (5.6) with initial data

$$(q, u, \dot{q}, \dot{u})(0) = (0, 0, 0, w) \tag{5.7}$$

satisfies

$$q(t) \equiv 0, \quad u(t) = tw. \tag{5.8}$$

By (5.6) we obtain

$$0 = \ddot{q}^i(0) = \Gamma^i_{N+\alpha,N+\beta}(0,0)w^\alpha w^\beta$$
$$0 = \ddot{u}^\gamma(0) = \Gamma^\gamma_{N+\alpha,N+\beta}(0,0)w^\alpha w^\beta$$

Since these equalities hold for all initial data $w \in \mathbb{R}^M$ in (5.7), this and property (ii) yield (5.1).

Next, by

$$0 = \Gamma_{i,N+\alpha,N+\beta}(0,0) = \frac{\partial g_{i,N+\beta}}{\partial u^\gamma}(0,0) + \frac{\partial g_{i,N+\gamma}}{\partial u^\beta}(0,0) - \frac{\partial g_{N+\beta,N+\gamma}}{\partial q^i}(0,0)$$

we obtain (5.2), since, by property (ii), one has

$$\frac{\partial g_{i,N+\beta}}{\partial u^\gamma}(0,0) = -\frac{\partial g^{i,N+\beta}}{\partial u^\gamma}(0,0),$$
$$\frac{\partial g_{N+\beta,N+\gamma}}{\partial q^i}(0,0) = -\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0)$$

for all $i = 1,\dots,N$ and $\alpha, \beta = 1,\dots,M$.

Moreover, for every $\alpha, \beta, \gamma = 1,\dots,M$, one has

$$\frac{\partial g_{N+\alpha,N+\beta}}{\partial u^\gamma} = \Gamma_{N+\alpha,N+\gamma,N+\beta} + \Gamma_{N+\beta,N+\gamma,N+\alpha} = 0.$$

Therefore, by the property (ii), (5.3) is proved as well. □

**Remark 4.** As in (3.1), let $(e_{\alpha,\beta})_{\alpha,\beta=1,\dots,M}$ be the inverse of the sub-matrix $(g^{N+\alpha,N+\beta})_{\alpha,\beta=1,\dots,M}$. Then, since at $(q,u) = (0,0)$ we have $g_{r,s} = g^{r,s} = \delta_{r,s}$, it follows

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i}(0,0) = \frac{\partial g_{N+\alpha,N+\beta}}{\partial q^i}(0,0), \qquad \text{for all } i = 1,\dots,N, \quad \alpha,\beta = 1,\dots,M. \tag{5.9}$$

*5.2. The orthogonal curvature of the foliation*

For any $(q, u)$ in the range of a $\Lambda$-adapted chart, consider the quantity

$$\frac{\partial e_{\alpha,\beta}}{\partial q^i} dc^\alpha \otimes dc^\beta \otimes dq^i \tag{5.10}$$

**Definition 7.** We shall refer to the function (5.10) as the *orthogonal curvature tensor of the foliation*. Although the quantity in (5.10) is not a tensor in the strict sense of the word, by (5.11) it still transforms like a tensor with respect to to changes of $\Lambda$-adapted coordinates—see Lemma 5.1 below. Hence, it is intrinsically defined once the foliation $\Lambda$ is given.

**Lemma 5.1.** *The function in* (5.10) *is intrinsically defined with respect to the foliation* $\Lambda$. *This means that if* $(\tilde{q}, \tilde{u})$ *is a* $\Lambda$*-adapted chart then*

$$\frac{\partial \tilde{e}_{\alpha,\beta}}{\partial \tilde{q}^i} = \frac{\partial u^\gamma}{\partial \tilde{u}^\alpha} \frac{\partial u^\delta}{\partial \tilde{u}^\beta} \frac{\partial q^j}{\partial \tilde{q}^i} \frac{\partial e_{\gamma,\delta}}{\partial q^j}. \tag{5.11}$$

**Proof.** Since$(q, u)$ and $(\tilde{q}, \tilde{u})$ are $\Lambda$-adapted, the coordinate transformation $(q, u) \mapsto (\tilde{q}, \tilde{u})$ satisfies $\frac{\partial \tilde{u}}{\partial q} = 0$. Therefore,

$$\tilde{g}^{N+\alpha,N+\beta} = \frac{\partial \tilde{u}^\alpha}{\partial u^\gamma} \frac{\partial \tilde{u}^\beta}{\partial u^\delta} g^{N+\gamma,N+\delta}.$$

By inverting the matrices on both sides of the above identity one obtains

$$\tilde{e}_{\alpha,\beta} = \frac{\partial u^\gamma}{\partial \tilde{u}^\alpha} \frac{\partial u^\delta}{\partial \tilde{u}^\beta} e_{\gamma,\delta},$$

which implies (5.11), because $u = u(\tilde{u})$ is independent of $\tilde{q}$. □

According to Theorem 4.1, the foliation $\Lambda$ is $N$-fit for hyperimpulses if and only if the the corresponding orthogonal curvature is identically equal to zero. We now give a geometric construction which clarifies the meaning of the coefficients $\partial e_{\alpha,\beta}/\partial q^i$ in (5.10), in the general case (see Fig. 2).

Fix any point $(\mathbf{q}, \mathbf{u}) \in \mathcal{Q} \times \mathcal{U}$ and consider any non-zero vector $\mathbf{v} \in \Delta^\perp_{(\mathbf{q},u)}$. Construct the geodesic curve that originates at $(\mathbf{q}, \mathbf{u})$ with speed $\mathbf{v}$, namely

$$s \mapsto \gamma_{\mathbf{v}}(s) \doteq \mathrm{Exp}_{(\mathbf{q},\mathbf{u})}(s\mathbf{v}). \tag{5.12}$$

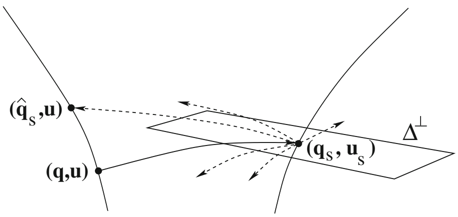


**Fig. 2.** The geodesics involved in the computation of the orthogonal curvature of $\Lambda$

Next, for each $s \neq 0$, consider the orthogonal space $\Delta^{\perp}_{(\mathbf{q}_s,\mathbf{u}_s)}$ at the point $(\mathbf{q}_s, \mathbf{u}_s) = \gamma_{\mathbf{v}}(s)$. Assuming that $s$ is sufficiently small, a transversality argument yields the existence of a unique vector $\mathbf{w} \in \Delta^{\perp}_{(\mathbf{q}_s,\mathbf{u}_s)}$ such that

$$\mathrm{Exp}_{(\mathbf{q}_s,\mathbf{p}_s)}\mathbf{w} = (\hat{\mathbf{q}}_s, \mathbf{u}) \in \mathcal{Q} \times \{\mathbf{u}\}. \tag{5.13}$$

In other words, we are moving back to a point $(\hat{q}_s, \mathbf{u})$ on the original leaf $\mathcal{Q} \times \{\mathbf{u}\}$, following a second geodesic curve. In general, $\hat{\mathbf{q}}_s \neq \mathbf{q}$. We claim that, setting $\sigma \doteq s^2$, the map

$$\sigma \mapsto (\hat{\mathbf{q}}_{\sqrt{\sigma}}, \mathbf{u})$$

defines a unique tangent vector $\mathbf{z}(\mathbf{v}) \in T_{(\mathbf{q},\mathbf{u})}$. Moreover, the map $\mathbf{v} \mapsto \mathbf{z}(\mathbf{v})$ is a homogeneous quadratic map from $\Delta^{\perp}_{(\mathbf{q},\mathbf{u})}$ into the tangent space $T_{(\mathbf{q},\mathbf{u})}\mathcal{Q} \subset T_{(\mathbf{q},\mathbf{u})}(\mathcal{Q} \times \mathcal{U})$. In turn, this determines a unique symmetric bilinear mapping $B : \Delta^{\perp}_{(\mathbf{q},\mathbf{u})} \otimes \Delta^{\perp}_{(\mathbf{q},\mathbf{u})} \mapsto T_{(\mathbf{q},\mathbf{u})}\mathcal{Q}$ such that $B(\mathbf{v},\mathbf{v}) = \mathbf{z}(\mathbf{v})$, namely

$$B(\mathbf{v}_1, \mathbf{v}_2) \doteq \frac{1}{4}\mathbf{z}(\mathbf{v}_1 + \mathbf{v}_2) - \frac{1}{4}\mathbf{z}(\mathbf{v}_1 - \mathbf{v}_2). \tag{5.14}$$

The relation between the bilinear mapping (5.14) and the curvature tensor (5.10) can be best analyzed by using coordinates. Consider an orthonormal basis $(J_1, \ldots, J_M)$ of $\Delta^{\perp}(\mathbf{q}, \mathbf{u})$, together with local $\mathcal{U}$-orthonormal coordinates $(q, u)$, constructed as in Proposition 5.1. If $\mathbf{v} = w_1 J_1 + \cdots + w_M J_M$, then by construction the point $(\mathbf{q}_s, \mathbf{u}_s)$ has coordinates $(0, sw) = (0, \ldots, 0, sw_1, \ldots, sw_M)$. Let $(\hat{q}_w(s), 0)$ be the coordinates of the point $(\hat{\mathbf{q}}_s, \mathbf{u})$, constructed as in (5.13). We now have:

**Theorem 5.1.** *The curve $s \mapsto q_w(s) \in \mathbb{R}^N$ is continuous and satisfies*

$$\lim_{s\to 0} \frac{\hat{q}^i_w(s)}{s^2} = \frac{1}{2}\sum_{\alpha,\beta=1}^{M} \frac{\partial e_{\alpha,\beta}}{\partial q^i} w^\alpha w^\beta \quad i = 1, \ldots, N. \tag{5.15}$$

**Proof.** It is understood that the coefficients $\partial e_{\alpha,\beta}/\partial q^i$ in (5.15) are computed at $(q, u) = (0, 0)$, corresponding to the point $(\mathbf{q}, \mathbf{u})$. In view of (5.9), it suffices to prove that

$$\lim_{s\to 0} \frac{\hat{q}^i_w(s)}{s^2} = \frac{1}{2}\sum_{\alpha,\beta=1}^{M} \frac{\partial g_{N+\alpha,N+\beta}}{\partial q^i} w^\alpha w^\beta. \tag{5.16}$$

In coordinates, the geodesic $\sigma \mapsto \gamma_{\mathbf{w}}(\sigma) = Exp_{(\mathbf{q}_s,\mathbf{u}_s)}(\sigma\mathbf{w})$ is given by a map $\sigma \mapsto (\hat{q}(\sigma), \hat{u}(\sigma))$ which, for suitable adjoint variables $p = (p_1, \ldots, p_N)$, $\pi = (\pi_1, \ldots, \pi_M)$, satisfies the Hamiltonian system

$$\begin{cases} \dot{q}^i = g^{i,j} p_j + g^{i,N+\beta}\pi_\beta \\ \dot{u}^\alpha = g^{N+\alpha,j} p_j + g^{N+\alpha,N+\beta}\pi_\beta \\ \dot{p}_i = -\frac{1}{2}\frac{\partial g^{j,k}}{\partial q^i} p_j p_k - \frac{\partial g^{j,N+\beta}}{\partial q^i} p_j \pi_\beta - \frac{1}{2}\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}\pi_\beta\pi_\gamma \\ \dot{\pi}_\alpha = -\frac{1}{2}\frac{\partial g^{j,k}}{\partial u^\alpha} p_j p_k - \frac{\partial g^{j,N+\beta}}{\partial u^\alpha} p_j \pi_\beta - \frac{1}{2}\frac{\partial g^{N+\beta,N+\gamma}}{\partial u^\alpha}\pi_\beta\pi_\gamma . \end{cases} \tag{5.17}$$

The conditions $\gamma_{\mathbf{w}}(0) = (\mathbf{q}_s, \mathbf{u}_s)$, $\gamma_{\mathbf{w}}(1) \in \mathcal{Q} \times \{\mathbf{u}\}$, and the fact that $\mathbf{w} \in \Delta^{\perp}_{(\mathbf{q}_s, \mathbf{u}_s)}$ imply

$$\begin{cases} q^i(0) = 0, \\ u^\alpha(0) = s w_\alpha, \qquad p_i(0) = 0. \\ u^\alpha(1) = 0. \end{cases} \tag{5.18}$$

For $s$ sufficiently small, the existence and uniqueness of the solution to the two-point boundary value problem (5.17) and (5.18) follows from the implicit function theorem. We now seek an expansion of this solution in powers of $s$.

Call $\bar{\pi} = \pi(0)$, and consider the Cauchy problem for (5.17), with initial data

$$\begin{cases} q(0) = 0, \\ u(0) = sw, \end{cases} \qquad \begin{cases} p(0) = 0, \\ \pi(0) = \bar{\pi}. \end{cases} \tag{5.19}$$

Using the Landau order symbols, our computations can be simplified by observing that

$$\begin{cases} q(\sigma) = \mathcal{O}(s^2), \\ p(\sigma) = \mathcal{O}(s^2), \end{cases} \qquad \begin{cases} u(\sigma) = \mathcal{O}(s), \\ \pi(\sigma) = \mathcal{O}(s), \end{cases} \qquad \text{for all } \sigma \in [0,1]. \tag{5.20}$$

For all $\sigma \in [0,1]$, the solution of the Cauchy problem (5.17), (5.19) thus satisfies

$$\begin{cases} q^i(\sigma) = \int_0^\sigma p_i(t)\,\mathrm{d}t + \int_0^\sigma g^{i,N+\beta}\pi_\beta(t)\,\mathrm{d}t + o(s^2), \\ u^\alpha(\sigma) = sw + \int_0^\sigma \pi_\alpha(t)\,\mathrm{d}t + o(s^2), \\ p_i(\sigma) = -\frac{1}{2}\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) \cdot \int_0^\sigma \pi_\beta(t)\pi_\gamma(t)\,\mathrm{d}t + o(s^2), \\ \pi_\alpha(\sigma) = \bar{\pi} + o(s^2). \end{cases} \tag{5.21}$$

From the second and fourth estimates in (5.21) we deduce

$$u^\alpha(\sigma) = s w_\alpha + \sigma \bar{\pi}_\alpha + o(s^2).$$

Since $u^\alpha(1) = 0$, this implies

$$\bar{\pi}_\alpha = -s w_\alpha + o(s^2).$$

Using this additional information in the third estimate, we obtain

$$p_i(\sigma) = -\frac{1}{2}\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) \cdot \sigma s^2 w_\beta w_\gamma + o(s^2),$$

In turn, the first estimate now yields

$$q^i(1) = -\frac{s^2}{4}\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) w_\beta w_\gamma - \frac{s^2}{2}\frac{\partial g^{i,N+\beta}}{\partial u^\gamma}(0,0) w_\beta w_\gamma + o(s^2).$$

Recalling the identity (5.2), we thus obtain

$$\hat{q}^i(s) \doteq q^i(1) = -\frac{s^2}{2}\frac{\partial g^{N+\beta,N+\gamma}}{\partial q^i}(0,0) w_\beta w_\gamma + o(s^2).$$

In view of property (ii), this establishes (5.16). $\square$

## Part II: Stabilization of control systems with quadratic impulses

### 6. Trajectories of controlled systems with quadratic impulses

We now investigate general control systems of the form:

$$\dot{x} = f(x) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}^\alpha + \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x)\dot{u}^\alpha\dot{u}^\beta. \tag{6.1}$$

Here the state variable $x$ and the control variable $u$ take values in $\mathbb{R}^n$ and in $\mathbb{R}^m$, respectively. We remark that no a priori bounds are imposed on the derivative $\dot{u}$. Our main goal is to understand under which conditions the system can be *stabilized to a given point* $\bar{x}$. In particular, relying on the quadratic dependence on $\dot{u}$ of the right-hand side of (6.1), in Section 8 we shall investigate *vibrational stabilization*, achieved by means of small periodic oscillations of the control function. In Part III, these results will be applied to the stabilization of the mechanical systems discussed in Part I.

Throughout the following we assume that the functions $f$, $g_\alpha$, and $h_{\alpha\beta} = h_{\beta,\alpha}$ are at least twice continuously differentiable. We remark that the more general system

$$\dot{x} = \tilde{f}(t,x,u) + \sum_{\alpha=1}^{m} \tilde{g}_\alpha(t,x,u)\dot{u}^\alpha + \sum_{\alpha,\beta=1}^{m} \tilde{h}_{\alpha\beta}(t,x,u)\dot{u}^\alpha\dot{u}^\beta,$$

where the vector fields depend also on time and on the control $u$, can be easily rewritten in the form (6.1). Indeed, it suffices to work in the extended state space $x \in \mathbb{R}^{1+n+m}$, introducing the additional state variables $x^0 = t$ and $x_{n+\alpha} = u^\alpha$, with equations

$$\dot{x}^0 = 1, \quad \dot{x}_{n+\alpha} = \dot{u}^\alpha \qquad \alpha = 1, \ldots, m.$$

Given the initial condition

$$x(0) = \check{x}, \tag{6.2}$$

for every smooth control function $u : [0,T] \mapsto \mathbb{R}^m$ one obtains a unique solution $t \mapsto x(t; u)$ of the Cauchy problem (6.1) and (6.2). More generally, since the equation (6.1) is quadratic with respect to the derivative $\dot{u}$, it is natural to consider admissible controls in a set of absolutely continuous functions $u(\cdot)$ with derivatives in $\mathbf{L}^2$. For example, for a given $M > 0$, one could allow the controls to belong to

$$\left\{u : [0,T] \mapsto \mathbb{R}^m\,; \quad \int_0^T \left|\dot{u}(t)\right|^2 \mathrm{d}t \leqq M\right\}. \tag{6.3}$$

The main goal of the following analysis is to provide a characterization of the closure of this set of trajectories in terms of an auxiliary differential inclusion. Let us notice that the system (6.1) is naturally connected with the differential inclusion

$$\dot{x} \in \mathcal{F}(x), \tag{6.4}$$

where, for every $x \in \mathbb{R}^n$,

$$\mathcal{F}(x) \doteq \overline{co} \left\{ f(x) + \sum_{\alpha=1}^{m} g_\alpha(x) w^\alpha + \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x) w^\alpha w^\beta; \quad (w^1, \dots, w^m) \in \mathbb{R}^m \right\}. \tag{6.5}$$

Here and in the sequel, for any given subset $A$ of a topological vector space, $\overline{co} A$ denotes the closed convex hull of $A$.

In addition, it will be convenient to work also in an extended state space, using the variable $\hat{x} = \begin{pmatrix} x^0 \\ x \end{pmatrix} \in \mathbb{R}^{1+n}$. For a given $\hat{x}$, consider the set

$$\begin{aligned} F(\hat{x}) \ \doteq \overline{co} \Bigg\{ & \begin{pmatrix} 1 \\ f(x) \end{pmatrix} (a^0)^2 + \sum_{\alpha=1}^{m} \begin{pmatrix} 0 \\ g_\alpha(x) \end{pmatrix} a^0 a^\alpha + \sum_{\alpha,\beta=1}^{m} \begin{pmatrix} 0 \\ h_{\alpha\beta}(x) \end{pmatrix} a^\alpha a^\beta; \\ & a^0 \in [0, 1], \quad \sum_{\alpha=0}^{m} (a^\alpha)^2 = 1 \Bigg\}. \end{aligned} \tag{6.6}$$

Notice that $F$ is a convex, compact valued multifunction on $\mathbb{R}^{1+n}$, Lipschitz continuous with respect to the Hausdorff metric [2].

For a given interval $[0, S]$, the set of trajectories of the *graph differential inclusion*

$$\frac{\mathrm{d}}{\mathrm{d}s} \hat{x}(s) \in F(\hat{x}(s)), \quad \hat{x}(0) = \begin{pmatrix} 0 \\ x^\sharp \end{pmatrix} \tag{6.7}$$

is a non-empty, closed, bounded subset of $\mathcal{C}\big([0, S]; \mathbb{R}^{1+n}\big)$. Consider one particular solution, say $s \mapsto \hat{x}(s) = \begin{pmatrix} x^0(s) \\ x(s) \end{pmatrix}$, defined for $s \in [0, S]$. Assume that $T \doteq x^0(S) > 0$. Since the map $s \mapsto x^0(s)$ is non-decreasing, it admits a generalized inverse

$$s = s(t) \quad \text{iff } x^0(s) = t. \tag{6.8}$$

Indeed, for all but countably many times $t \in [0, T]$ there exists a unique value of the parameter $s$ such that the identity on the right of (6.8) holds. We can thus define a corresponding trajectory

$$t \mapsto x(t) = x\big(s(t)\big) \in \mathbb{R}^n. \tag{6.9}$$

This map is well defined for almost all times $t \in [0, T]$.

To establish a connection between the original control system (6.1) and the differential inclusion (6.7), consider first a smooth control function $u(\cdot)$. As in [32], we define a re-parameterized time variable by setting

$$s(t) \doteq \int_0^t \left( 1 + \sum_{\alpha=1}^{m} (\dot{u}^\alpha)^2(\tau) \right) \mathrm{d}\tau. \tag{6.10}$$

Notice that the map $t \mapsto s(t)$ is strictly increasing. The inverse map $s \mapsto t(s)$ is uniformly Lipschitz continuous and satisfies

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \left(1 + \sum_{\alpha=1}^{m} (\dot{u}^\alpha)^2(t)\right)^{-1}.$$

Now let $x : [0, T] \mapsto \mathbb{R}^n$ be a solution of (6.1) corresponding to the smooth control $u : [0, T] \mapsto \mathbb{R}^m$. We claim that the map $s \mapsto \hat{x}(s) \doteq \begin{pmatrix} t(s) \\ x(t(s)) \end{pmatrix}$ is a solution to the differential inclusion (6.7). Indeed, setting

$$a^0(s) \doteq \frac{1}{\sqrt{1 + \sum_{\beta=1}^{m} (\dot{u}^\beta)^2\big(t(s)\big)}}, \quad a^\alpha(s) \doteq \frac{\dot{u}^\alpha\big(t(s)\big)}{\sqrt{1 + \sum_{\beta=1}^{m} (\dot{u}^\beta)^2\big(t(s)\big)}} \tag{6.11}$$

$(\alpha = 1, \ldots, m)$, one has

$$\begin{cases} \frac{\mathrm{d}t}{\mathrm{d}s} = (a^0)^2(s) \\ \frac{\mathrm{d}x}{\mathrm{d}s} = f\Big(x(s)\Big)(a^0)^2(s) + \sum_{\alpha=1}^{m} g_\alpha\Big(x(s)\Big)\, a^0(s) a^\alpha(s) \\ \qquad\qquad + \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}\Big(x(s)\Big)\, a^\alpha(s) a^\beta(s). \end{cases} \tag{6.12}$$

Hence $\hat{x}(\cdot) = (t(\cdot), x(\cdot))$ verifies (6.7), because, by (6.11),

$$a^0(s) \in [0, 1], \quad \sum_{\alpha=0}^{m} (a^\alpha)^2(s) \equiv 1.$$

Notice that the derivatives $\dot{u}^\alpha$ can now be recovered as

$$\dot{u}^\alpha(t) = \frac{a^\alpha(s(t))}{a^0(s(t))} \quad \alpha = 1, \ldots, m. \tag{6.13}$$

The following theorem shows that every solution of the differential inclusion (6.7) can be approximated by smooth solutions of the original control system (6.1).

**Theorem 6.1.** *Let $\hat{x} = (x^0, x) : [0, S] \mapsto \mathbb{R}^{1+n}$ be a solution to the multivalued Cauchy problem (6.7) such that $x^0(S) = T > 0$. Then there exists a sequence of smooth control functions $u_\nu : [0, T] \mapsto \mathbb{R}^M$ such that the corresponding solutions*

$$s \mapsto \hat{x}_\nu(s) = \begin{pmatrix} t_\nu(s) \\ x_\nu(s) \end{pmatrix}$$

*of the equations (6.11) and (6.12) converge to the map $s \mapsto \hat{x}(s)$ uniformly on $[0, S]$. Moreover, defining the function $x(t) = x(s(t))$ as in (6.9), we have*

$$\lim_{\nu\to\infty} \int_0^T \big|x(t) - x_\nu(t)\big|\, \mathrm{d}t = 0. \tag{6.14}$$

**Proof.** By the assumption, the extended vector fields

$$\hat{f} = \begin{pmatrix} 1 \\ f \end{pmatrix}, \quad \hat{g}_\alpha = \begin{pmatrix} 0 \\ g_\alpha \end{pmatrix}, \quad \hat{h}_{\alpha\beta} = \begin{pmatrix} 0 \\ h_{\alpha\beta} \end{pmatrix}$$

are Lipschitz continuous. Consider the set of trajectories of the control system

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{x} = \hat{f} \cdot (a^0)^2 + \sum_{\alpha=1}^{m} \hat{g}_\alpha a^0 a^\alpha + \sum_{\alpha,\beta=1}^{m} \hat{h}_{ji} a^\alpha a^\beta, \quad \hat{x}(0) = \begin{pmatrix} 0 \\ x^\sharp \end{pmatrix}, \tag{6.15}$$

where the controls $a = (a^0, a^1, \dots, a^m)$ satisfy the pointwise constraints

$$a^0(s) \in [0, 1], \quad \sum_{\alpha=0}^{m} (a^\alpha)^2(s) = 1 \quad s \in [0, S]. \tag{6.16}$$

In the above setting, it is well known [2] that the set of trajectories

$$s \mapsto \hat{x}(s) = \left(x^0, x^1, \dots, x^n\right)(s)$$

of (6.15) and (6.16) is dense on the set of solutions to the differential inclusion (6.7). Hence there exists a sequence of control functions $s \mapsto a_\nu(s) = \left(a_\nu^0, \dots, a_\nu^m\right)(s)$, $\nu \geqq 1$, such that the corresponding solutions $s \mapsto \hat{x}_\nu(s)$ of (6.15) converge to $\hat{x}(\cdot)$ uniformly for $s \in [0, S]$. In particular, this implies the convergence of the first components:

$$x_\nu^0(S) = \int_0^S \left[a_\nu^0(s)\right]^2 \mathrm{d}s \to x^0(S) = T. \tag{6.17}$$

We now observe that the "input–output map" $a(\cdot) \mapsto \hat{x}(\cdot, a)$ from controls to trajectories is uniformly continuous as a map from $\mathbf{L}^1\left([0, S];\ \mathbb{R}^{1+m}\right)$ into $\mathcal{C}\left([0, S];\ \mathbb{R}^{1+n}\right)$. By slightly modifying the controls $a_\nu$ in $\mathbf{L}^1$, we can replace the sequence $a_\nu$ by a new sequence of smooth control functions $\tilde{a}_\nu : [0, S] \mapsto \mathbb{R}^{1+m}$ with the following properties:

$$\tilde{a}_\nu^0(s) > 0 \qquad \text{for all } s \in [0, S],\ \nu \geqq 1. \tag{6.18}$$

$$\int_0^S \left[\tilde{a}_\nu^0(s)\right]^2 \mathrm{d}s = T \qquad \text{for all } \nu \geqq 1, \tag{6.19}$$

$$\lim_{\nu\to\infty} \int_0^S \left|\tilde{a}_\nu(s) - a_\nu(s)\right| \mathrm{d}s = 0. \tag{6.20}$$

This implies the uniform convergence

$$\lim_{\nu\to\infty} \left\|\hat{x}(\cdot, \tilde{a}_\nu) - \hat{x}(\cdot)\right\|_{\mathcal{C}([0,S];\ \mathbb{R}^{1+n})} = 0. \tag{6.21}$$

By (6.18), for each $\nu \geqq 1$ the map

$$s \mapsto x_\nu^0(s) \doteq \int_0^s \left[\tilde{a}_\nu^0(s)\right]^2 \mathrm{d}s$$

is strictly increasing. Therefore it has a smooth inverse $s = s_\nu(t)$. Recalling (6.13), we now define the sequence of smooth control functions $u_\nu : [0, T] \mapsto \mathbb{R}^m$ by setting $u_\nu(t) = \big(u_\nu^1, \ldots, u_\nu^m\big)(t)$, with

$$u_\nu^\alpha(t) = \int_0^t \frac{\tilde{a}_\nu^\alpha(s_\nu(\tau))}{\tilde{a}_\nu^0(s_\nu(\tau))} \mathrm{d}\tau. \tag{6.22}$$

By construction, the solutions $t \mapsto x_\nu(t;\, u_\nu)$ of the original system (6.1) corresponding to the controls $u_\nu$ coincide with the maps $t \mapsto (x_\nu^1, \ldots, x_\nu^n)(s_\nu(t))$, where $\hat{x}_\nu = (x_\nu^0, x_\nu^1, \ldots, x_\nu^n)$ is the solution of (6.15) with control $\tilde{a}_\nu = (\tilde{a}_\nu^0, \ldots, \tilde{a}_\nu^m)$.

To prove the last statement in the theorem, define the increasing functions

$$t(s) = \int_0^s \big[\tilde{a}^0(r)\big]^2 \,\mathrm{d}r, \quad t_\nu(s) = \int_0^s \big[\tilde{a}_\nu^0(r)\big]^2 \,\mathrm{d}r,$$

and let $t \mapsto s(t)$, $t \mapsto s_\nu(t)$ be their inverses, respectively. Notice that each $s_\nu(\cdot)$ is smooth. Moreover,

$$\left|\frac{\mathrm{d}}{\mathrm{d}s} t(s)\right| \leqq 1, \quad \left|\frac{\mathrm{d}}{\mathrm{d}s} t_\nu(s)\right| \leqq 1, \tag{6.23}$$

$$\lim_{\nu\to\infty} \int_0^T \big|s(t) - s_\nu(t)\big| \,\mathrm{d}t = \lim_{\nu\to\infty} \int_0^S \big|t(s) - t_\nu(s)\big| \,\mathrm{d}s = 0. \tag{6.24}$$

Using (6.23), we obtain the estimate

$$\begin{aligned}\int_0^T \big|x(t) - x_\nu(t)\big| \,\mathrm{d}t &= \int_0^T \Big|x(s(t)) - x_\nu(s(t))\Big| \,\mathrm{d}t + \int_0^T \Big|x_\nu(s(t)) - x_\nu(s_\nu(t))\Big| \,\mathrm{d}t \\ &\leqq \int_0^S \big|x(s) - x_\nu(s)\big| \,\mathrm{d}s + C \cdot \int_0^T \big|s(t) - s_\nu(t)\big| \,\mathrm{d}t. \end{aligned} \tag{6.25}$$

Here the constant $C$ denotes an upper bound for the derivative with respect to $s$, for example

$$C \doteq \sup_x \left\{ \big|f(x)\big| + \sum_i \big|g_\alpha(x)\big| + \sum_{\alpha,\beta} \big|h_{\alpha\beta}(x)\big| \right\}, \tag{6.26}$$

where the supremum is taken over a compact set containing the graphs of all functions $x_\nu(\cdot)$. By (6.21) and (6.24), the right-hand side of (6.25) vanishes in the limit $\nu \to \infty$. This completes the proof of the theorem. □

**Remark 5.** For a given time interval $[0, T]$, we are considering controls $u(\cdot)$ in the Sobolev space $W^{1,2}$. The corresponding solutions are absolutely continuous maps; they belong to $W^{1,1}$. Now consider a sequence of control functions $u_\nu$, whose derivatives are uniformly bounded in $\mathbf{L}^2$. Assume that the corresponding re-parameterized trajectories $s \mapsto (t_\nu(s), x_\nu(s))$, constructed as in (6.11) and (6.12), converge to a path $s \mapsto (t(s), x(s))$, providing a solution to (6.7). We wish to point out that, in general, the projection on the state space $t \mapsto x(s(t))$ will have bounded total variation, but it *may well be discontinuous*. Notice that, on the contrary, the uniform limit of the controls $t \mapsto u_\nu(t)$ must be Hölder continuous, because of the uniform $\mathbf{L}^2$ bound on the derivatives.

**Remark 6.** A completely different situation arises when all the vector fields $h_{\alpha\beta}$ vanish identically, so that (6.1) reduces to

$$\dot{x} = f(x) + \sum_{\alpha=1}^{m} g_\alpha(x)\dot{u}^\alpha \tag{6.27}$$

Systems of this form have been extensively studied; see [10,11,24,26,38], or the surveys [8,30] and the references therein. In this case, solutions also can be well defined for general control functions $u(\cdot)$ with bounded variation but possibly discontinuous. We recall that, unless the Lie brackets $[g_\alpha, g_\beta]$ vanish identically, one needs to assign a "graph completion" of the control $u(\cdot)$ in order to uniquely determine the trajectory. Indeed, at each time $\tau$ where $u$ has a jump, one should also specify a continuous path joining the left state $u(\tau-)$ with the right state $u(\tau+)$. See [10] for details.

## 7. Stabilization

In this section we examine various concepts of stability for the impulsive system (6.1) and relate them to the weak stability of the differential inclusion (6.6) and (6.7).

**Definition 8.** We say that the control system (6.1) is *stabilizable* at the point $\bar{x} \in \mathbb{R}^n$ if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. For every initial state $x^\sharp$ with $|x^\sharp - \bar{x}| \leqq \delta$ there exists a smooth control function $t \mapsto u(t) = (u^1, \dots, u^m)(t)$ such that the corresponding trajectory of (6.1) and (6.2) satisfies

$$|x(t,u) - \bar{x}| \leqq \varepsilon \qquad \text{for all } \ t \geqq 0. \tag{7.1}$$

We say that the system (6.1) is *asymptotically stabilizable* at the point $\bar{x}$ if a control $u(\cdot)$ can be found such that, in addition to (7.1), there holds

$$\lim_{t\to\infty} x(t,u) = \bar{x}. \tag{7.2}$$

**Remark 7.** Notice that the point $\bar{x}$ needs not to be an equilibrium point for the vector field $f$.

**Remark 8.** We require here that the stabilizing controls be smooth. As it will become apparent in the sequel, this is hardly a restriction. Indeed, in all cases under consideration, if a stabilizing control $u \in W^{1,2}$ is found, by approximation one one can construct a smooth control $\tilde{u}$ which is still stabilizing.

**Remark 9.** In the above definitions we are not putting any constraints on the control function $u : [0, \infty[ \mapsto \mathbb{R}^m$. In principle, one may well have $|u(t)| \to \infty$ as $t \to \infty$. If one wishes to stabilize the system (6.1) and at the same time keep the control values within a small neighborhood of a given value $\bar{u}$, it suffices to consider the stabilization problem for an augmented system, adding the variables $x^{n+1}, \dots, x^{n+m}$ together with the equations

$$\dot{x}^{n+\alpha} = \dot{u}^\alpha \quad \alpha = 1, \dots, m.$$

Similar stability concepts can be also defined for a differential inclusion

$$\dot{x} \in K(x), \tag{7.3}$$

see for example [35]. We recall that a trajectory of (7.3) is an absolutely continuous function $t \mapsto x(t)$ which satisfies the differential inclusion at almost everywhere time $t$.

**Definition 9.** The point $\bar{x}$ is *weakly stable* for the differential inclusion (7.3) if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. For every initial state $x^\sharp$ with $|x^\sharp - \bar{x}| \leqq \delta$ there exists a trajectory $x(\cdot)$ of (7.3) such that

$$x(0) = x^\sharp, \quad |x(t) - \bar{x}| \leqq \varepsilon \quad \text{for all} \ \ t \geqq 0. \tag{7.4}$$

Moreover, $\bar{x}$ is *weakly asymptotically stable* if, there exists a trajectory which, in addition to (7.4), satisfies

$$\lim_{t\to\infty} x(t) = \bar{x}. \tag{7.5}$$

In connection with the multifunction $F$ defined at (6.6), we consider a second multifunction $F^\diamond$ obtained by projecting the sets $F(\hat{x}) \subset \mathbb{R}^{1+n}$ into the subspace $\mathbb{R}^n$. More precisely, we set

$$\begin{aligned} F^\diamond(x) \ \doteq \overline{co}\Big\{ & f(x)\,(a^0)^2 + \textstyle\sum_{\alpha=1}^m g_\alpha(x)\,a^0 a^\alpha + \sum_{\alpha,\beta=1}^m h_{\alpha\beta}(x)\,a^\alpha a^\beta\,; \\ & \qquad w^0 \in [0,1], \textstyle\sum_{\alpha=0}^m (w^\alpha)^2 = 1 \Big\}. \end{aligned} \tag{7.6}$$

Observe that, if the vector fields $f$, $g_\alpha$ , and $h_{\alpha\beta}$ are Lipschitz continuous, then the multifunction $F^\diamond$ is Lipschitz continuous with compact, convex values. Our first result in this section is:

**Theorem 7.1.** *The impulsive system* (6.1) *is asymptotically stabilizable at the point $\bar{x}$ if and only if $\bar{x}$ is weakly asymptotically stable for the* projected graph differential inclusion

$$\frac{\mathrm{d}}{\mathrm{d}s} x(s) \in F^\diamond(x(s)). \tag{7.7}$$

**Proof.** Let $\bar{x}$ be weakly asymptotically stable for (7.7). Without loss of generality, we can assume $\bar{x} = 0$.

Given $\varepsilon > 0$, choose $\delta > 0$ such that, if $|x^\sharp| \leqq \delta$, then there exists a trajectory $t \mapsto x(s)$ of the differential inclusion (7.7) such that $x(0) = x^\sharp$, $|x(s)| \leqq \varepsilon/2$ for all $t \geqq 0$ and $x(s) \to 0$ as $t \to \infty$. Using the basic approximation property stated in Theorem 6.1, we will construct a smooth control $t \mapsto u(t) = (u^1, \ldots, u^m)(t)$ such that the corresponding trajectory $x(\cdot; u)$ of (6.1) and (6.2) satisfies

$$|x(t)| \leqq \varepsilon \quad \text{for all} \ \ t \geqq 0, \qquad \lim_{t\to\infty} x(t) = 0. \tag{7.8}$$

Define the decreasing sequence of positive numbers $\varepsilon_k \doteq \varepsilon\, 2^{-k}$. For each $k \geqq 0$, choose $\delta_k > 0$ so that, whenever $|x^\sharp| \leqq \delta_k$, there exists a solution to (7.7) with

$$x(0) = x^\sharp, \quad \lim_{s\to\infty} x(s) = 0, \quad |x(s)| < \frac{\varepsilon_k}{2} \qquad \text{for all } \ s \geqq 0. \tag{7.9}$$

Choose a sequence of strictly positive integers $k(1) \leqq k(2) \leqq \cdots$, such that

$$\lim_{j\to\infty} k(j) = \infty, \quad \sum_{j=1}^{\infty} \delta_{k(j)} = \infty. \tag{7.10}$$

Note that the second condition in (7.10) is certainly satisfied if the numbers $k(j)$ grow at a sufficiently slow rate.

Assume $|x^\sharp| \leqq \delta_0$. A smooth control $u$ steering the system (6.1) from $x^\sharp$ asymptotically toward the origin will be constructed by induction on $j$. For $j = 1$, let $x : [0, s_1] \mapsto \mathbb{R}^n$ be a trajectory of the differential inclusion (7.7) such that

$$x(0) = x^\sharp, \qquad |x(s_1)| < \frac{\delta_{k(1)}}{3}, \qquad |x(s)| < \frac{\varepsilon_0}{2} \quad \text{for all } \ s \in [0, s_1].$$

By the definition of $F^\diamond$, there exists a trajectory of the differential inclusion (6.7) having the form $s \mapsto \hat{x}(s) = (x^0(s), x(s))$. Notice that, in order to apply Theorem 6.1 and approximate $x(\cdot)$ with a smooth solution of the control system (6.1) we would need $x^0(s_1) > 0$. This is not yet guaranteed by the above construction. To take care of this problem, we define $s_1' \doteq s_1 + \delta_{k(1)}/3C$, where $C$ provides a local upper bound for the magnitude of the vector field $f$, as in (6.26). We then prolong the trajectory $\hat{x}(\cdot)$ to the larger interval $[0, s_1']$, by setting

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} x^0(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} 1 \\ f(x) \end{pmatrix} \qquad s \in ]s_1, s_1'].$$

This construction achieves the inequalities

$$x^0(s_1') \geqq s_1' - s_1 \geqq \frac{\delta_{k(1)}}{3C}, \quad |x(s_1')| < \frac{2}{3}\delta_{k(1)}.$$

Set $\tau^1 \doteq x^0(s_1')$. By Theorem 6.1, there exists a smooth control $u : [0, \tau^1] \mapsto \mathbb{R}^m$ such that the corresponding solution $s \mapsto (x^0(s, u), x(s, u))$ of (6.11) and (6.12) differs from the above trajectory by less than $\delta_{k(1)}/3$, namely

$$|x^0(s, u) - x^0(s)| < \frac{\delta_{k(1)}}{3}, \quad |x(s; u) - x(s)| < \frac{\delta_{k(1)}}{3} \quad \text{for all } \ s \in [0, s_1'].$$

In particular, setting $x(t, u) \doteq x(s(t), u)$ as in (6.9), this implies

$$|x(\tau_1, u)| < \delta_{k(1)}, \quad |x(t, u)| < \frac{\varepsilon_0}{2} + \frac{\delta_{k(1)}}{3} \leqq \varepsilon_0 \qquad \text{for all } \ t \in [0, \tau_1].$$

The construction now proceeds by induction on $j$. Assume that a smooth control $u(\cdot)$ has been constructed on the time interval $[0, \tau_j]$, in such a way that

$$|x(\tau_j, u)| < \delta_{k(j)}, \qquad |x(t, u)| < \varepsilon_{k(j-1)} \quad \text{for all } \ t \in [\tau_{j-1}, \tau_j]. \tag{7.11}$$

By assumptions, there exists a trajectory $s \mapsto x(s)$ of the differential inclusion (7.7) such that

$$x(0) = x(\tau_j, u), \quad x(s_j)| < \frac{\delta_{k(j+1)}}{3}, \quad |x(s)| < \frac{\varepsilon_{k(j)}}{2} \tag{7.12}$$

for all $s \in [0, s_j]$. This trajectory is extended to the slightly larger interval $[0, s'_j]$, with $s'_j = s_j + \delta_{k(j)}/3C$, by setting

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} x^0(s) \\ x(s) \end{pmatrix} = \begin{pmatrix} 1 \\ f(x) \end{pmatrix} \quad s \in ]s_j, s'_j]. \tag{7.13}$$

Notice that, by (7.12), (7.13), and (6.26), we have

$$x^0(s'_j) \geqq s'_j - s_j \geqq \frac{\delta_{k(j)}}{3C}, \quad |x(s'_j)| < \frac{2}{3}\delta_{k(j+1)}. \tag{7.14}$$

Set $\tau_{j+1} \doteq \tau_j + x^0(s'_j)$. Using again Theorem 6.1, we can extend the control $u : [0, \tau_j] \mapsto \mathbb{R}^M$ to a continuous, piecewise smooth control defined on the larger interval $[0, \tau_{j+1}]$, such that the corresponding solution $s \mapsto x(s, u)$ of (6.1) and (6.2) satisfies

$$|x(\tau_{j+1}, u)| < \delta_{k(j+1)}, \quad |x(t, u)| < \varepsilon_{k(j)} \quad \text{for all } \ t \in [\tau_j, \tau_{j+1}]. \tag{7.15}$$

Notice that, at this stage, the control $u$ is obtained by piecing together two smooth control functions, defined on the intervals $[0, \tau_j]$ and $[\tau_j, \tau_{j+1}]$ respectively. This makes $u$ continuous but possibly not $\mathcal{C}^1$ in a neighborhood of the point $\tau_j$. To fix this problem, we slightly modify the values of $u$ in a small neighborhood of $\tau_j$, so that $u$ becomes smooth also at this point, while the strict inequalities (7.15) still hold.

Having completed the inductive steps for all $j \geqq 1$ we observe that

$$\lim_{j\to\infty} \tau_j = \sum_j \frac{\delta_{k(j)}}{3C} = \infty$$

because of (7.10). As $t \to \infty$, by (7.15) we have $x(t, u) \to 0$. This shows that the impulsive system (6.1) is asymptotically stabilizable at the origin, proving one of the implications stated in the theorem.

The converse implication is obvious, because every solution of the system (6.1) corresponding to a smooth control yields a solution to the differential inclusion (7.7), after a suitable time rescaling.

**Corollary 7.1.** *Let a point $\bar{x}$ be weakly asymptotically stable for the differential inclusion* (6.4), *namely $\dot{x} \in \mathcal{F}(x)$. Then the system* (6.1) *is asymptotically stabilizable at $\bar{x}$.*

**Proof.** Since the point $\bar{x}$ is weakly asymptotically stable for (6.4), then it is asymptotically stable for the differential inclusion (7.7), which, in turn, implies that the impulsive system (6.1) can be stabilized at $\bar{x}$.

### *7.1. Lyapunov functions*

There is extensive literature, in the context of ODEs and of control systems or differential inclusions, relating the stability of an equilibrium state to the existence of a Lyapunov function. We recall below the basic definition, in a form suitable for our applications. For simplicity, we henceforth consider the case $\bar{x} = 0 \in \mathbb{R}^n$, which of course is not restrictive.

**Definition 10.** A scalar function $V$ defined on a neighborhood $\mathcal{N}$ of the origin is a *weak Lyapunov function* for the differential inclusion

$$\dot{x} \in \mathcal{F}(x)$$

if the following holds.

(i) $V$ is continuous on $\mathcal{N}$, and continuously differentiable on $\mathcal{N} \setminus \{0\}$.
(ii) $V(0) = 0$ while $V(x) > 0$ for all $x \neq 0$,
(iii) For each $\delta > 0$ sufficiently small, the sublevel set $\{x\,;\quad V(x) \leqq \delta\}$ is compact.
(iv) At each $x \neq 0$ one has

$$\inf_{y \in \mathcal{F}(x)} \nabla V(x) \cdot y \leqq 0. \tag{7.16}$$

The following theorem relates the stability of the impulsive control system (6.1) to the existence of a Lyapunov function for the differential inclusion (6.4).

**Theorem 7.2.** *Consider the multifunction $\mathcal{F}$ defined at* (6.5)*. Assume that the differential inclusion* (6.4) *admits a Lyapunov function $V = V(x)$ defined on a neighborhood $\mathcal{N}$ of the origin. Then the control system* (6.1) *can be stabilized at the origin.*

**Remark 10.** Notice that the multifunction $\mathcal{F}$ in (6.5) has unbounded values. Yet we can rephrase condition (iv) in the Definition 10 with the following equivalent condition, which is formulated in terms of the bounded multifunction $F$ governing (6.6):
*(iv′) For every $x \in \mathcal{N} \setminus \{0\}$, there exists $\hat{y} = (y_0, y) \in F(x)$ such that*

$$\nabla V(x) \cdot y \leqq 0 \qquad y_0 > 0. \tag{7.17}$$

**Remark 11.** The set of conditions (i)–(iii) and (iv’) represents a slight strengthening of the notion of the weak Lyapunov function when this is applied to the projected graph differential equation (7.7). Yet, let us point out that the weak stability of (7.7) is not enough to guarantee the stabilizability of the control system (6.1), so the condition $y_0 > 0$ in (7.17) plays a crucial role. For example, on $\mathbb{R}^2$, consider the constant vector fields $f = (1, 0)$, $h_{11} = (0, 1)$, $h_{22} = (0, -1)$, $g_1 = g_2 = h_{12} = h_{21} = (0, 0)$. Then, choosing $a^0 = 0$, $a^1 = a^2 = 1/\sqrt{2}$ we see that $(0, 0, 0) \in F(x)$ for every $x \in \mathbb{R}^2$. Hence condition

$$\inf_{y \in F(x)} \nabla V \cdot y \leqq 0$$

is trivially satisfied by any function $V$. However, it is clear that in this case the system (6.1) is not stabilizable at the origin.

**Remark 12.** Theorem 7.2 is somewhat weaker than its counterpart, Theorem 7.1, dealing with asymptotic stability. Indeed, to prove that the impulsive control system (6.1) is stabilizable, we need to assume not only that the differential inclusion (7.7) is weakly stable, but also that there exists a Lyapunov function.

*Proof of Theorem 7.2.* Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$V(x) \leqq 2\delta \quad \text{implies } |x| \leqq \varepsilon.$$

Let an initial state $x^\sharp$ be given, with $V(x^\sharp) \leqq \delta$.

According to Remark 10, for every $x \neq 0$ there exists $(y_0, y) \in F(x)$ such that (7.17) holds. We recall that the multifunction $F$ in (6.6) is Lipschitz continuous, with compact, convex values. Since the set $\Omega \doteq \{x\,;\;\; \delta \leqq V(x) \leqq 3\delta\}$ is compact, by the continuity of $\nabla V$ we can find $\kappa > 0$ such that, for every $x \in \Omega$, there exists $\hat{y} = (y_0, y) \in F(x)$ with

$$\nabla V(x) \cdot y \leqq 0, \quad y_0 \geqq \kappa.$$

The control $u$ will be defined inductively on a sequence of the time intervals $[\tau_{j-1}, \tau_j]$, with $\tau_j \geqq j\kappa$. Set $\tau_0 = 0$. Consider the differential inclusion

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{x}(s) \in \begin{cases} F(x(s)) \cap \{(y_0, y);\; \nabla V(x)\cdot y \leqq 0, y_0 \geqq \kappa\} & \text{if } \delta < V(x) < 2\delta, \\ F(x(s)) & \text{if } V(x) \leqq \delta \text{ or } V(x) \geqq 2\delta, \end{cases} \tag{7.18}$$

with initial data $\hat{x}(0) = (0, x^\sharp)$. The right-hand side of (7.18) is an upper semi-continuous multifunction, with nonempty compact convex values. Therefore (see for example [2]), the Cauchy problem admits at least one solution $s \mapsto \hat{x}(s) = (x^0(s), x(s))$, defined for $s \in [0, 1]$. We observe that this solution satisfies

$$x^0(1) \geqq \kappa, \quad V(x(s)) \leqq \delta \quad \text{for all } \; s \in [0, 1].$$

Hence, by Theorem 6.1 there exists a smooth control $u : [0, \tau_1] \mapsto \mathbb{R}^m$, with $\tau_1 = x^0(1) \geqq \kappa$, such that the corresponding trajectory of (6.1) and (6.2) satisfies

$$V(x(t, u)) < \frac{3}{2}\delta = 2\delta - 2^{-1}\delta \quad \text{for all } \; t \in [0, \tau_1].$$

By induction, assume now that a smooth control $u(\cdot)$ has been constructed on the interval $[0, \tau_j]$ with $\tau_j \geqq \kappa\, j$, and that the corresponding trajectory $t \mapsto x(t, u)$ of the impulsive system (6.1) and (6.2) satisfies

$$V(x(t, u)) \leqq 2\delta - 2^{-j}\delta \quad t \in [0, \tau_j]. \tag{7.19}$$

We then construct a solution $s \mapsto \hat{x}(s) = (x^0(s), x(s))$ of the differential inclusion (7.18) for $s \in [0, 1]$, with initial data $\hat{x}(0) = (0, x(\tau_j, u))$. This function will satisfy

$$x^0(1) \geqq \kappa, \qquad V(x(s)) < 2\delta - 2^{-j}\delta \quad \text{for all } \; s \in [0, 1].$$

Using Theorem 6.1 again, we can prolong the control $u$ to a larger time interval $[0, \tau_{j+1}]$, with $\tau_{j+1} - \tau_j = x^0(1) \geqq \kappa$, in such a way that

$$V(x(t,u)) < 2\delta - 2^{-j-1}\delta \quad t \in [0, \tau_{j+1}]. \tag{7.20}$$

At a first stage, this control $u$ will be piecewise smooth, continuous but not $\mathcal{C}^1$ in a neighborhood of the point $\tau_j$. By a local approximation, we can change its values slightly in a small neighborhood of the point $\tau_j$, making it smooth also at the point $\tau_j$, and preserving the strict inequalities (7.20).

Since $\tau_j \geqq k\,j$ for all $j \geqq 1$, as $j \to \infty$ the induction procedure generates a smooth control function $u(\cdot)$, defined for all $t \geqq 0$, whose corresponding trajectory satisfies $V(x(t,u)) < 2\delta$ for all $t \geqq 0$. This completes the proof of the theorem.

□

Let us consider the 2-homogeneous term of $\mathcal{F}$:

$$\mathcal{F}_2 \doteq f(x) + \overline{co}\left\{ \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x)\, w^\alpha w^\beta\,; \quad (w^1, \ldots, w^m) \in \mathbb{R}^m \right\}$$

In Remark 13 in the next section we will show that $f(x) + \mathcal{F}_2 \subset \mathcal{F}$. Therefore, from Theorem 7.2 we obtain the following result.

**Corollary 7.2.** *Assume that the reduced differential inclusion*

$$\dot{x} \in f(x) + \mathcal{F}_2 \tag{7.21}$$

*admits a Lyapunov function $V = V(x)$ defined on a neighborhood $\mathcal{N}$ of the origin. Then the control system (6.1) can be stabilized at the origin.*

## 8. A selection technique

In the previous section we proved two general results, relating the stability of the control system (6.1) to the weak stability of the differential inclusion (6.4). A complete description of the sets $\mathcal{F}(x)$ in (6.5) may often be very difficult. However, as shown in [35], to establish a stability property it suffices to construct a suitable family of smooth selections. We shall briefly describe this approach.

Let a point $\bar{x} \in \mathbb{R}^n$ be given, and assume that there exists a $\mathcal{C}^1$ selection

$$\begin{aligned}\gamma(x,\xi) \;&\in\; \mathcal{F}_1(x)\\ &\doteq \overline{co}\left\{ \sum_{\alpha=1}^{m} g_\alpha(x)\, w^\alpha + \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x)\, w^\alpha w^\beta\,;\; (w^1, \ldots, w^m) \in \mathbb{R}^m \right\}\end{aligned}$$

depending on an additional parameter $\xi \in \mathbb{R}^d$, such that

$$f(\bar{x}) + \gamma(\bar{x}, \bar{\xi}) = 0. \tag{8.1}$$

for some $\bar{\xi} \in \mathbb{R}^d$. Assuming that $\gamma$ is defined on an entire neighborhood of $(\bar{x}, \bar{\xi})$, consider the Jacobian matrices of partial derivatives computed at $(\bar{x}, \bar{\xi})$:

$$A \doteq \frac{\partial f}{\partial x} + \frac{\partial \gamma}{\partial x}, \quad B \doteq \frac{\partial \gamma}{\partial \xi}.$$

**Theorem 8.1.** *In the above setting, if the linear system with constant coefficients*

$$\dot{x} = Ax + B\xi \tag{8.2}$$

*is completely controllable, then the differential inclusion* (6.4) *and* (6.5) *is weakly asymptotically stable at the point* $\bar{x}$.

We recall that the system (8.2) is completely controllable if and only if the matrices $A, B$ satisfy the algebraic relation $\mathrm{Rank}[B, AB, \ldots, A^{n-1}B] = n$. This guarantees that the system can be steered from any initial state to any final state, within any given time interval [9,37].

To prove the theorem, consider the control system

$$\dot{x} = f(x) + \gamma(x, \xi). \tag{8.3}$$

By a classical result in control theory, the above assumptions imply that, for every point $x^\sharp$ sufficiently close to $\bar{x}$, there exists a trajectory starting from $x^\sharp$ reaching $\bar{x}$ in finite time. In particular, in view of (8.1), the system (8.3) is asymptotically stabilizable at the point $\bar{x}$. Since all trajectories of (8.3) are also trajectories of the differential inclusion (6.4), the result follows. $\square$

**Remark 13.** Toward the construction of smooth selections from the multifunction $\mathcal{F}$ we observe that each closed convex set $\mathcal{F}(x)$ can be equivalently written as

$$\begin{aligned}
\mathcal{F}(x) &\doteq f(x) + \mathcal{F}_1(x) + \mathcal{F}_2(x) \\
&= f(x) + \overline{co}\left\{ \sum_{\alpha=1}^{m} g_\alpha(x) w^\alpha + \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x)\, w^\alpha w^\beta;\, (w^1, \ldots, w^m) \in \mathbb{R}^m \right\} \\
&\quad + \overline{co}\left\{ \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x)\, w^\alpha w^\beta;\, (w^1, \ldots, w^m) \in \mathbb{R}^m \right\}.
\end{aligned} \tag{8.4}$$

Indeed, by definition we have $\mathcal{F}(x) = f(x) + \mathcal{F}_1(x)$. To establish the identity (8.4) it thus suffices to prove that

$$\mathcal{F}_1 + \mathcal{F}_2 \subseteq \mathcal{F}_1. \tag{8.5}$$

Since the set $\mathcal{F}_1(x)$ is convex and contains the origin, for every $(w^1, \ldots, w^m) \in \mathbb{R}^m$ and $\varepsilon \in [0, 1]$ we have

$$y_\varepsilon \doteq \varepsilon \left( \sum_{\alpha=1}^{m} g_\alpha(x) \frac{w^\alpha}{\sqrt{\varepsilon}} + \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x) \frac{w^\alpha w^\beta}{\varepsilon} \right) \in \mathcal{F}_1.$$

Letting $\varepsilon \to 0$ we find

$$\lim_{\varepsilon \to 0+} y_\varepsilon = \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}(x)\, w^\alpha w^\beta. \tag{8.6}$$

Since $\mathcal{F}_1(x)$ is closed, it must contain the right-hand side of (8.6). This proves the inclusion $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Next, observing that $\mathcal{F}_2$ is a cone, for every $y_2 \in \mathcal{F}_2$ and $\varepsilon > 0$ we have $\varepsilon^{-1} y_2 \in \mathcal{F}_2 \subseteq \mathcal{F}_1$. Therefore, if $y_1 \in \mathcal{F}_1$ we can write

$$y_1 + y_2 = \lim_{\varepsilon \to 0+} (1-\varepsilon) y_1 + \varepsilon(\varepsilon^{-1} y_2) \in \mathcal{F}_1$$

because $\mathcal{F}_1$ is closed and convex. This proves (8.5).

**Remark 14.** By Theorem 8.1 and the above remark, one may establish a stability result by constructing suitable selections $\gamma(x,\xi) \in \mathcal{F}_2(x)$ from the cone $\mathcal{F}_2$.

## Part III: Stabilization of mechanical systems

In this part we address the question of how to use some time-dependent holonomic constraints as controls in order to stabilize a mechanical system to a given state.

## 9. Stabilization with vibrating controls

For the reader's convenience, we summarize the results in Section 3. Let $G = (g_{r,s})_{r,s=1,\dots,N+M}$ be the matrix that represents the covariant inertial tensor in a given coordinate chart $(q,u)$. In particular, the kinetic energy of the whole system at a state $(q,u)$ with velocity $(v,w) \in \mathbb{R}^{N+M}$ is given by

$$\mathcal{T} = \frac{1}{2} g_{i,j}(q,u) v^i v^j + g_{i,N+\alpha}(q,u) v^i w^\alpha + \frac{1}{2} g_{N+\alpha,N+\beta}(q,u) w^\alpha w^\beta.$$

Here and in the sequel, $i,j = 1,\dots,N$ while $\alpha,\beta = 1,\dots,M$. By $G^{-1} = (g^{r,s})_{r,s=1,\dots,N+M}$ we denote the inverse of $G$. Moreover, we consider the sub-matrices $G_1 \doteq (g_{i,j})$, $(G^{-1})_2 \doteq (g^{N+\alpha,N+\beta})$, and $(G^{-1})_{12} \doteq (g^{i,N+\alpha})$. Finally, we introduce the matrices

$$A = \left(a^{i,j}\right) \doteq (G_1)^{-1}, \quad E = \left(e_{\alpha,\beta}\right) \doteq ((G^{-1})_2)^{-1}, \quad K = \left(k^i_\alpha\right) \doteq (G^{-1})_{12} E. \tag{9.1}$$

We recall that all the above matrices depend on the variables $q, u$. Concerning the external force, our main assumption will be

**Hypothesis (A)**. *The force $F^{u,w}$ acting on the whole system does not explicitly depend on time, and is affine with respect to the time derivative of the control. Namely*

$$F^{u,w} = F^{u,w}(q,p) = F^u_0(q,p,u) + F^u_1(q,p,u) \cdot w. \tag{9.2}$$

In particular, any positional force (not necessarily conservative) satisfies this hypothesis. Because of (A), the control equations take the form

$$\begin{pmatrix} \dot q \\ \dot p \\ \dot u \end{pmatrix} = \begin{pmatrix} Ap \\ \frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p + F_0^u \\ 0 \end{pmatrix} + \begin{pmatrix} K \\ -p^\dagger \frac{\partial K}{\partial q} + F_1^u \\ 1_M \end{pmatrix} \dot u + \dot u^\dagger \begin{pmatrix} 0 \\ \frac{1}{2} \frac{\partial E}{\partial q} \\ 0 \end{pmatrix} \dot u. \tag{9.3}$$

Our main goal is to find conditions which imply that the system (9.3) is stabilizable at a point $(\bar q, 0, \bar u)$. Two results will be described here. The first one relies on suitable smooth selections from the corresponding set-valued maps, as in Theorem 8.1. The second one is based on the use of Lyapunov functions.

For each $q, u$, consider the cone

$$\Gamma(q,u) \doteq \overline{co} \left\{ w^\dagger \frac{\partial E(q,u)}{\partial q} w; \quad w \in \mathbb{R}^M \right\}. \tag{9.4}$$

Let $\xi \in \mathbb{R}^d$ be an auxiliary control variable, ranging on a neighborhood of a point $\bar\xi \in \mathbb{R}^d$. Aiming to apply Theorem 8.1, let us consider a control system of the form

$$\begin{cases} \dot q = Ap, \\ \dot p = F_0^{\bar u}(q,p) + \gamma(q,p,\bar u,\xi), \end{cases} \tag{9.5}$$

where $\gamma$ is a suitable selection from the cone $\Gamma$. It will be convenient to write (9.5) in the more compact form

$$(\dot q, \dot p) = \Phi(q,p,\bar u,\xi), \tag{9.6}$$

regarding $(q,p) \in \mathbb{R}^{N+N}$ as state variables and $\xi \in \mathbb{R}^d$ as control variable. Assume that

$$F_1^{\bar u}(\bar q, 0) + \gamma(\bar q, 0, \bar u, \bar\xi) = 0. \tag{9.7}$$

By (9.5) this implies $\Phi(\bar q, 0, \bar u, \bar\xi) = 0 \in \mathbb{R}^{2N}$. To test the local controllability of (9.5) at the equilibrium point $(\bar q, 0, \bar u, \bar\xi)$ we look at the linearized system with constant coefficients

$$\begin{pmatrix} \dot p \\ \dot q \end{pmatrix} = \Lambda \begin{pmatrix} p \\ q \end{pmatrix} + \mathcal{B}\xi, \tag{9.8}$$

where

$$\Lambda = \frac{\partial \Phi}{\partial(q,p)} \qquad \mathcal{B} = \frac{\partial \Phi}{\partial \xi}$$

with all partial derivatives being computed at the point $(\bar q, 0, \bar u, \bar\xi)$. We can now state

**Theorem 9.1.** *Assume that a smooth map*

$$(q, p, u, \xi) \mapsto \gamma(q, p, u, \xi) \in \Gamma(q, u) \tag{9.9}$$

*can be chosen in such a way that* (9.7) *holds and so that the linear system* (9.8) *is completely controllable. Then the system* (9.3) *is asymptotically stabilizable at the point* $(\bar q, 0, \bar u)$.

**Proof.** According to Theorem 8.1 and Remark 13, it suffices to show that the control system

$$\begin{pmatrix} \dot q \\ \dot p \\ \dot u \end{pmatrix} = \begin{pmatrix} Ap \\ \frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p + F_0^u \\ 0 \end{pmatrix} + \begin{pmatrix} K \\ -p^\dagger \frac{\partial K}{\partial q} + F_1^u \\ 1_M \end{pmatrix} w$$
$$+ w^\dagger \begin{pmatrix} 0 \\ \frac{1}{2} \frac{\partial E}{\partial q} \\ 0 \end{pmatrix} w + \begin{pmatrix} 0 \\ \gamma(q, p, u, \xi) \\ 0 \end{pmatrix} \tag{9.10}$$

is locally controllable at $(\bar q, 0, \bar u)$. Notice that in (9.10) the state variables are $q, p, u$, while $w, \xi$ are the controls. Computing the Jacobian matrices of partial derivatives at the point $(q, p, u; w, \xi) = (\bar q, 0, \bar u, 0, \bar \xi)$, we obtain a linear system with constant coefficients, of the form

$$\begin{pmatrix} \dot q \\ \dot p \\ \dot u \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & 0 & 0 \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \\ u \end{pmatrix} + \begin{pmatrix} 0 & B_{12} \\ B_{21} & B_{22} \\ 0 & 1_M \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix}$$
$$\doteq \widetilde\Lambda \begin{pmatrix} q \\ p \\ u \end{pmatrix} + \left( \widetilde{\mathcal{B}_1} \widetilde{\mathcal{B}_2} \right) \begin{pmatrix} \xi \\ w \end{pmatrix} \tag{9.11}$$

By assumption, the linear system (9.8) is completely controllable. Therefore

$$\text{Rank} \left[ \mathcal{B}, \ \Lambda\mathcal{B}, \ \ldots, \ \Lambda^{2N-1}\mathcal{B} \right] = 2N. \tag{9.12}$$

We now observe that the matrices $\Lambda, \mathcal{B}$ at (9.8) correspond to the submatrices

$$\Lambda = \begin{pmatrix} \Lambda_{11} & 0 \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ B_{21} \end{pmatrix}. \tag{9.13}$$

Hence from (9.12) it follows

$$\text{span} \left[ \widetilde{\mathcal{B}_1}, \ \widetilde\Lambda \widetilde{\mathcal{B}_1}, \ldots, \widetilde\Lambda^{2N-1} \widetilde{\mathcal{B}_1} \right] = \left\{ \begin{pmatrix} q \\ p \\ 0 \end{pmatrix}; \quad q \in \mathbb{R}^N, \ p \in \mathbb{R}^N \right\}. \tag{9.14}$$

Adding to this subspace the subspace generated by the columns of the matrix $\widetilde{\mathcal{B}_2}$, we obtain the entire space $\mathbb{R}^{2N+M}$. We thus conclude that the linear system (9.11) is completely controllable. In turn, this implies that the nonlinear system (9.10) is asymptotically stabilizable at $(\bar q, 0, \bar u)$, completing the proof. □

By choosing a special kind of selection and relying on the particular structure of (9.5), we can deduce Corollary 9.1 below. To state it, if $k$ is a positive integer such that $kM \geqq N$ and $W = (w_1, \dots, w_k) \in \mathbb{R}^{M\times k}$, let us consider the $N \times kM$ matrix

$$M(u,q,W) \doteq \begin{pmatrix} \frac{\partial e_{1,\beta}}{\partial q^1} w_1^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^1} w_1^\beta, \dots\dots\dots, \frac{\partial e_{1,\beta}}{\partial q^1} w_k^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^1} w_k^\beta \\ \dots\dots \\ \frac{\partial e_{1,\beta}}{\partial q^N} w_1^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^N} w_1^\beta, \dots\dots\dots, \frac{\partial e_{1,\beta}}{\partial q^1} w_k^\beta, \dots, \frac{\partial e_{M,\beta}}{\partial q^N} w_k^\beta \end{pmatrix}. \tag{9.15}$$

**Corollary 9.1.** *Let $k$ be a positive integer and assume that for a given state $(\bar q, \bar u)$ there exists a $k$-tuple $\bar W = (\bar w_1, \dots, \bar w_k) \in (R^M)^k$ such that*

$$\mathrm{Rank}\Big(M(\bar u, \bar q, \bar W)\Big) = N \tag{9.16}$$

*and*

$$\begin{cases} (F_0^u)^1 + \sum_{\alpha,\beta=1}^M \frac{\partial e_{\alpha,\beta}}{\partial q^1} \sum_{r=1}^k \bar w_r^\alpha \bar w_r^\beta = 0 \\ \cdots \\ (F_0^u)^N + \sum_{\alpha,\beta=1}^M \frac{\partial e_{\alpha,\beta}}{\partial q^N} \sum_{r=1}^k \bar w_r^\alpha \bar w_r^\beta = 0, \end{cases} \tag{9.17}$$

*where the involved functions are computed at $(q,p,u) = (\bar q, 0, \bar u)$. Then the system (9.3) is asymptotically stabilizable at the point $(\bar q, 0, \bar u)$.*

**Proof.** Let us observe that the matrices $\Lambda$ and $\mathcal{B}$ in (9.13) have the following form:

$$\mathcal{B} = \begin{pmatrix} 0_{N\times d} \\ \frac{\partial \gamma}{\partial \xi} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 0_{N\times N} & A \\ \frac{\partial(F+\gamma)}{\partial q} & \frac{\partial(F+\gamma)}{\partial p} \end{pmatrix} \tag{9.18}$$

so that, in particular,

$$\Lambda\mathcal{B} = \begin{pmatrix} A \cdot \frac{\partial\gamma}{\partial\xi} \\ \frac{\partial(F+\gamma)}{\partial p} \cdot \frac{\partial\gamma}{\partial\xi} \end{pmatrix} \tag{9.19}$$

Let us set $d = kM$, $\xi = W = (w_1, \dots, w_k)$, and

$$\gamma_i(q,u,W) \doteq \frac{1}{2} \sum_{\ell=1}^k \frac{\partial e_{\alpha,\beta}}{\partial q^i} w_\ell^\alpha w_\ell^\beta \quad i = 1, \dots, N$$

Notice that, by 2-homogeneity $\gamma = (\gamma^1, \dots, \gamma^N)$, is in fact a selection of the set-valued map $\Gamma$ defined in (9.4). In view of Theorem 9.1, to prove asymptotic stability it is sufficient find

$$\bar\xi = \bar W$$

such (9.17) holds and, moreover,

$$\mathrm{Rank}\ [\mathcal{B}, \Lambda\mathcal{B}]\,(\bar q, 0, \bar u, \bar W) = 2N.$$

Since $A$ is a non-singular matrix, by (9.19) the latter condition is equivalent to

$$\operatorname{Rank}\left(\frac{\partial\gamma}{\partial W}\right)(\bar{q}, 0, \bar{u}, \bar{W}) = N. \tag{9.20}$$

In turn, this coincides with (9.16), so the proof is concluded. □

We now describe a second approach, based on Corollary 7.2 and on the construction of a suitable, energy-like, Lyapunov function. Throughout the following we assume that the external force $F$ in (9.2) admits the representation

$$F = F(q, p, u, w) = -\frac{\partial U}{\partial(q, u)} + F^1(q, p, u) \cdot w. \tag{9.21}$$

in terms of a potential function $U = U(q, u)$.

**Definition 11.** Given a $k$-tuple of vectors $W \doteq \{w_1, \ldots, w_k\} \subset \mathbb{R}^M$, the corresponding *asymptotic effective potential* $(q, u) \mapsto U_W(q, u)$ is defined as

$$U_W(q, u) \doteq U(q, u) - \frac{1}{2}\sum_{\ell=1}^{k} w_\ell^\dagger E(q, u) w_\ell$$

$$\left( = U(q, u) - \tfrac{1}{2}\textstyle\sum_{\ell=1}^{k}\sum_{\alpha,\beta=1}^{M} e_{\alpha,\beta}(q, u) w_\ell^\alpha\, w_\ell^\beta \right).$$

**Theorem 9.2.** *Let the external force $F$ have the form* (9.21). *For a given state $(\bar{q}, \bar{u})$, assume that there exist a neighborhood $\mathcal{N}$ of $(\bar{q}, \bar{u})$ and a $k$-tuple $W \doteq \{w_1, \ldots, w_k\} \subset \mathbb{R}^M$, as in Definition 11 which, in addition, satisfy the following property:*

> *There exists a continuously differentiable map $u \mapsto \beta(u)$ defined on a neighborhood of $\bar{u}$ such that the function*
>
> $$(q, u) \mapsto U_W(q, u) + \beta(u)$$
>
> *has a* strict local minimum *at $(q, u) = (\bar{q}, \bar{u})$.*

*Then the system* (9.3) *is stabilizable at $(\bar{q}, 0, \bar{u})$.*

**Remark 15.** This theorem, while being valid for non-conservative forces as well, is similar to stabilization results obtained in the framework of the so-called acceleration-controlled mechanical systems [3]. The main difference between that framework and ours relies on the fact that here we assume that the control variables are actuated by constraint reactions, whereas in [3] the actuating forces are exogenous. Moreover, the averaging methods exploited acceleration-controlled mechanical systems are here subsumed by convexification and selection procedures.

**Proof.** As in Section 8, consider the symmetrized differential inclusion corresponding to (9.3), namely

$$\begin{pmatrix} \dot q \\ \dot p \\ \dot z \end{pmatrix} \in \overline{co} \left\{ \begin{pmatrix} Ap \\ \frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p - \frac{\partial U}{\partial q} \\ 0 \end{pmatrix} + w^\dagger \begin{pmatrix} 0 \\ \frac{1}{2} \frac{\partial E}{\partial q} \\ 0 \end{pmatrix} w, \qquad w \in \mathbb{R}^M \right\}. \tag{9.22}$$

To prove the theorem, it suffices to show that the point $(\bar q, 0, \bar u)$ is a stable equilibrium for the differential equation

$$\begin{pmatrix} \dot q \\ \dot p \\ \dot u \end{pmatrix} = \begin{pmatrix} Ap \\ -\frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p - \frac{\partial U_W}{\partial q} \\ 0 \end{pmatrix}. \tag{9.23}$$

Indeed, by the definition of $U_W$, the right-hand side of (9.23) is a selection of the right-hand side of (9.22). Introducing the Hamiltonian function

$$H_W \doteq \frac{1}{2} pAp^\dagger + U_W,$$

the equation (9.23) can be written in the following Hamiltonian form:

$$\Big(\dot q,\ \dot p,\ \dot u\Big)^\dagger = \left(\frac{\partial H_W}{\partial p}, -\frac{\partial H_W}{\partial q}, 0\right). \tag{9.24}$$

Therefore the map

$$V(q, p, u) \doteq H_W(q, p, u) + \beta(u) \tag{9.25}$$

is a Lyapunov function for (9.23), from which it follows that $(\bar q, 0, \bar z)$ is a stable equilibrium for (9.23). □

## 10. Examples

**Example 1** (Pendulum with oscillating pivot). Let us consider a pendulum with fixed length $r = 1$, whose pivot is moving on the vertical $y$-axis, as shown in Fig. 3, left. Its position is described by two variables: the clockwise angle $\theta$ formed by the pendulum with the $y$-axis, and the height $h$ of the pivot. We now consider $h = u(t)$ to be our control variable, while the evolution of the other variable $\theta = q(t)$ will be determined by the equations of motion. We assume that the control function $t \mapsto u(t)$ can be assigned as a function of time, ranging over a neighborhood of the origin. We assume that both the pendulum and its pivot have unit mass, so that the kinetic matrix $G$ and the matrices in (9.1) take the form

$$G = \begin{pmatrix} 1 & -\sin q \\ -sinq & 2 \end{pmatrix} \qquad A = (1), \qquad E = (1 + \cos^2 q), \qquad K = (\sin q).$$

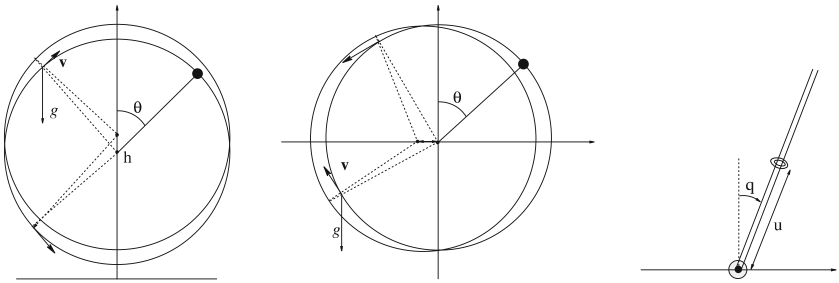


**Fig. 3.** A pendulum whose pivot oscillates vertically (on the *left*) and horizontally (*center*). On the *right*: a bead sliding without friction along a rotating axis

**Remark 16.** To be consistent with the general theory we need to put a mass on the pivot as well. This is needed in order that the matrix $G$ be invertible. On the other hand it is easy to show that the resulting control equations are independent of the mass of the pivot. Actually this should expected, since the motion of the pivot is here considered as a control. Of course, what is not independent of the mass of the pivot is the constraint reaction necessary to produce a given motion of $u$.

Notice that orthogonal curvature of the constraint foliation $\Lambda$, corresponding to the coefficient of $\dot{u}^2$ (see Section 5), is different from zero: $\frac{\mathrm{d}E}{\mathrm{d}q} = -2\sin q \cos q$.

In the presence of gravity acceleration $g$, the control equations for $q$ and the corresponding momentum $p$ are given by

$$\begin{cases} \dot{q} = p + (\sin q)\dot{u} \\ \dot{p} = -\frac{\partial U}{\partial q} - p(\cos q)\dot{u} - (\sin q)(\cos q)\dot{u}^2 , \end{cases} \tag{10.1}$$

where $U(q, u) \doteq g \cos q$ is the gravitational potential.

Using Theorem 9.2, it is easy to check that this system is stabilizable at the upward equilibrium point $(\bar{q}, \bar{p}, \bar{u}) = (0, 0, 0)$. Indeed, choosing $W = \{w\}$ with $w > g$, the corresponding effective potential

$$U_W = g \cos q - \frac{1}{2}(1 + \cos^2 q)w^2.$$

has a strict local minimum at $q = 0$.

To illustrate an application of Theorem 9.1, we now show that the above system is asymptotically stabilizable at every position $(\bar{q}, 0, 0)$ with $0 < |\bar{q}| < \pi/2$. To fix the ideas, assume $\bar{q} > 0$, the other case being entirely similar. For $\xi > 0$, the map $\gamma(q, p, \xi) = -\xi$ provides a smooth selection from the cone

$$\Gamma(q, u) \doteq \overline{co} \left\{ \frac{\partial E(q, u)}{\partial q} w^2; \quad w \in \mathbb{R} \right\} = \{-\xi; \quad \xi \geqq 0\}.$$

The corresponding system (9.5), with $\xi$ as control variable, now takes the form

$$\begin{cases} \dot{q} = p \\ \dot{p} = g \sin q - \xi. \end{cases} \tag{10.2}$$

It is easy to check that $(\bar{q}, \bar{p}, \bar{\xi}) = (\bar{q}, 0, \; g \sin \bar{q})$ is an equilibrium position and the system is locally controllable at this point. Indeed, the linearized control system with constant coefficients is

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g \cos \bar{q} & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \xi.$$

By Theorem 9.1, the system (10.1) is asymptotically stabilizable at $(\bar{q}, 0, 0)$.

By similar arguments one can show that, by means of horizontal oscillations of the pivot, one can stabilize the system at any position of the form $(\bar{q}, 0, 0)$, with $\frac{\pi}{2} \leqq |\bar{q}| \leqq \pi$.

**Example 2** (Sliding bead). Consider the mechanical system represented in Fig. 3 (right), consisting of a bead sliding without friction along a bar, and subject to gravity. The bar can be rotated around the origin, in a vertical plane. Calling $q$ the distance of the bead from the origin, while $u$ is the angle formed by the bar with the vertical line. Regarding $u$ as the controlled variable, in this case the kinetic matrix $G$ and the matrices in (9.1) take the form

$$G = \begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}, \quad A = (1), \quad E = (q^2), \quad K = (0).$$

Again, the orthogonal curvature of the constraint foliation $\Lambda$ does not vanish: $\frac{\mathrm{d}E}{\mathrm{d}q} = 2q$. The control equations for $q$ and the corresponding momentum $p$ are

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -g \, \cos u + q\dot{u}^2. \end{cases} \tag{10.3}$$

This case is easy to understand: by vibrating the angle $u$ one generates a centrifugal force which can contrast the gravitational force. More precisely, the system can be asymptotically stabilized at each $(\bar{q}, \bar{p}, \bar{u}) \in ]0, +\infty[ \times \{0\} \times ] - \pi/2, \pi/2[$ . Indeed, for $q > 0$ we trivially have $\Gamma(q, u) = \{qw^2 ; \;\; w \in \mathbb{R}\} = \{\xi \geqq 0\}$. Hence, if $\cos \bar{u} > 0$, then the control system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -g \, \cos \bar{u} + \xi, \end{cases} \tag{10.4}$$

admits the equilibrium point $(\bar{q}, 0, \bar{\xi})$, with $\bar{\xi} = g \, \cos \bar{u} > 0$. Moreover, this system is completely controllable around this equilibrium point, using $\{\xi \geqq 0\}$ as set of controls. An application of Theorem 9.1 yields the asymptotic stability property.

We remark that here the stabilizing controls cannot be independent of the position $q$ and the velocity $p$. In particular, the approach in Theorem 9.2, based on effective potential, cannot be implemented in this case, because a constant control $w$ cannot stabilize the system

$$\begin{cases} \dot{q} = p, \\ \dot{p} = -g \cos u + q w^2. \end{cases}$$

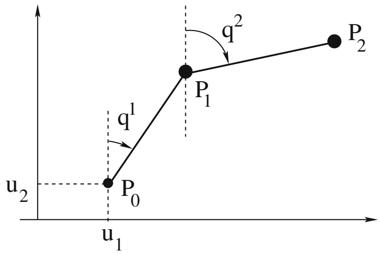


**Fig. 4.** Controlling a double pendulum by moving the pivot at $P_0$

**Example 3** (Double pendulum with moving pivot). We now consider a case where the control $u$ is two-dimensional, hence the cone (9.4) is also two-dimensional. Consider a double pendulum consisting of three point masses $P_0$, $P_1$, $P_2$, such that the distances $|P_0P_1|$, $|P_1P_2|$ are fixed, say both equal to 1. Let these points be subject to the gravitational force and constrained without friction on a vertical plane. Let $(u^1, u^2)$ be the Cartesian coordinates of the pivot $P_0$, and let $q^1, q^2$ the clockwise angles formed by $P_0P_1$ and $P_1P_2$ with the upper vertical half lines centered in $P^0$ and $P^1$, respectively; see Fig. 4. Because of the constraints, the state of the system $\{P_0, P_1, P_2\}$ is thus entirely described by the four coordinates $(q^1, q^2, u^1, u^2)$. The reduced system, obtained by regarding the variables $(u^1, u^2)$ as controls and $(q^1, q^2)$ as state-variables, is two-dimensional. For simplicity we assume that the all three points have unit mass, so that the matrix $G = (g_{rs})$ representing the kinetic energy is given by

$$G = \begin{pmatrix} 2 & \cos(q^1 - q^2) & 2\cos q^1 & -2\sin q^1 \\ \cos(q^1 - q^2) & 1 & \cos q^2 & -\sin q^2 \\ 2\cos q^1 & \cos q^2 & 3 & 0 \\ -2\sin q^1 & -\sin q^2 & 0 & 3 \end{pmatrix}.$$

Moreover, recalling (9.1), we have

$$E = \begin{pmatrix} 1 - \frac{4(\sin q^1)^2}{-3+\cos 3(q^1-q^2)} & -\frac{2\sin 2q^1}{-3+\cos 3(q^1-q^2)} \\ \\ -\frac{2\sin 2q^1}{-3+\cos 3(q^1-q^2)} & 1 - \frac{4(\sin q^1)^2}{-3+\cos 3(q^1-q^2)} \end{pmatrix},$$

$$(F_0^u)^1 = 2g\sin q^1, \quad (F_0^u)^2 = g\sin q^2.$$

Let us observe, as in Remark 16, that the matrix $E$ and the corresponding control equations are independent of the pivot's mass.

**Proposition 10.1.** *For every* $\bar{q}^1 \in ]0, \pi/4[$ *(respectively* $\bar{q}^1 \in ]-\pi/4, 0[$*) there exists* $\delta > 0$ *such that for all* $\bar{q}^2 \in ]-\delta, 0[$ *(respectively* $\bar{q}^2 \in ]-\delta, 0[$*) the system is stabilizable at* $(q^1, q^2, p_1, p_2, u^1, u^2) = (\bar{q}^1, \bar{q}^2, 0, 0, 0, 0)$.
*Moreover, the system is stabilizable at* $(q^1, q^2, p_1, p_2, u^1, u^2) = (0, 0, 0, 0, 0, 0)$.

**Remark 17.** By translational invariance, the result remains true if we replace $(u_1, u_2) = (0, 0)$ with any other value $(\bar{u}^1, \bar{u}^2) \in \mathbb{R}^2$.

*Proof of Proposition 10.1.* Using Corollary 9.1 with $N = M = 2$ and $k = 1$, we deduce that the system can be stabilized at $(\bar{q}^1, \bar{q}^2, \bar{u}^1, \bar{u}^2)$ provided there exist $\bar{w} \in \mathbb{R}^2$ such that

$$\begin{cases} 2g \sin \bar{q}^1 + \sum_{\alpha,\beta=1}^{2} \frac{\partial e_{\alpha,\beta}}{\partial \bar{q}^1} \bar{w}^\alpha \bar{w}^\beta = 0 \\ g \sin \bar{q}^2 + \sum_{\alpha,\beta=1}^{2} \frac{\partial e_{\alpha,\beta}}{\partial \bar{q}^2} \bar{w}^\alpha \bar{w}^\beta = 0 \end{cases} \tag{10.5}$$

and

$$\det \begin{pmatrix} \frac{\partial e_{1,1}}{\partial q^1} \bar{w}^1 + \frac{\partial e_{1,2}}{\partial q^1} \bar{w}^2 & \frac{\partial e_{2,1}}{\partial q^1} \bar{w}^1 + \frac{\partial e_{2,2}}{\partial q^1} \bar{w}^2 \\ \frac{\partial e_{1,1}}{\partial q^2} \bar{w}^1 + \frac{\partial e_{1,2}}{\partial q^2} \bar{w}^2 & \frac{\partial e_{2,1}}{\partial q^2} \bar{w}^1 + \frac{\partial e_{2,2}}{\partial q^2} \bar{w}^2 \end{pmatrix} \neq 0 \tag{10.6}$$

Notice that the latter relation can be written as

$$Q_{\alpha,\beta} \bar{w}^\alpha \bar{w}^\beta \neq 0 \tag{10.7}$$

where the matrix $Q = \left(Q_{\alpha,\beta}\right)$ is defined by

$$Q \doteq \frac{\partial E}{\partial q^1} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\partial E}{\partial q^2}. \tag{10.8}$$

We recall that $E$ denotes the matrix $(e_{\alpha,\beta})$. Moreover, all functions in (10.5)–(10.8) are computed at $(\bar{q}^1, \bar{q}^2)$.

Let us fix $\bar{q}^1 \in ]0, \pi/4[$. In order to establish the existence of a $\delta > 0$ such that for every $\bar{q}^2 \in ] - \delta, 0[$ there is a $\bar{w}$ verifying the relations (10.5), (10.6), we need to study the intersections of the level sets of the quadratic forms $Q$, $\frac{\partial E}{\partial q^1}$, $\frac{\partial E}{\partial q^2}$. An explicit computation yields

$$\frac{\partial E}{\partial q^1} = \begin{pmatrix} \frac{8 \sin q^1 \left(-3 \cos q^1 + \cos(q^1 - 2q^2)\right)}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} & -\frac{4\left(-3 \cos 2q^1 + \cos 2q^2\right)}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} \\ -\frac{4\left(-3 \cos 2q^1 + \cos 2q^2\right)}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} & -\frac{8 \cos q^1 \left(3 \sin q^1 + \sin(q^1 - 2q^2)\right)}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} \end{pmatrix}$$

$$\frac{\partial E}{\partial q^2} = \begin{pmatrix} \frac{8 \sin^2 q^1 \sin(2(q^1 - q^2))}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} & \frac{4 \sin 2q^1 \sin(2(q^1 - q^2))}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} \\ \frac{4 \sin 2q^1 \sin(2(q^1 - q^2))}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} & \frac{8 \cos^2 q^1 \sin(2(q^1 - q^2))}{\left(-3 + \cos(2(q^1 - q^2))\right)^2} \end{pmatrix}$$

In particular, for all $q^1, q^2$ one has

$$\det\left(\frac{\partial E}{\partial q^1}(q^1, q^2)\right) = -\frac{16}{\left(-3 + \cos(2q^1 - 2q^2)\right)^2} < 0,$$

$$\det\left(\frac{\partial E}{\partial q^2}(q^1, q^2)\right) = 0.$$

From the above computations it follows

(i) The quadratic form $w \mapsto w^\dagger \frac{\partial E}{\partial q^1} w$ is indefinite, hence it can be factored as the product of two linear, independent forms. Assume that $\bar{q}^2 \in ]-\bar{q}^1, 0[$, so that $\frac{\partial e_{2,2}}{\partial q^1} < 0$. Hence, for suitable functions $a = a(q^1, q^2)$, $b = b(q^1, q^2)$ such that $a(q^1, q^2) \neq b(q^1, q^2)$ for all $q^1, q^2$, one has

$$\frac{\partial e_{\alpha,\beta}}{\partial q^1} w^\alpha w\beta = \frac{\partial e_{2,2}}{\partial q^1} \left(w^2 - aw^1\right)\left(w^2 - bw^1\right).$$

(ii) If $\bar{q}^2 \in ]-\bar{q}^1, 0[$, the quadratic form $w \mapsto w^\dagger \frac{\partial E}{\partial q^2} w$ is positive semi-definite. It can be expressed as the product of the positive scalar function $\frac{\partial e_{2,2}}{\partial q^1}$ and the square of a linear function. Moreover *this linear function coincides with one of the two linear factors of the quadratic form* $w \mapsto w^\dagger \frac{\partial E}{\partial q^1} w$. This is a trivial consequence of the identity

$$\left(\frac{\partial e_{1,2}}{\partial q^2}\frac{\partial e_{2,2}}{\partial q^1}\right)^2 - 2\frac{\partial e_{2,1}}{\partial q^1}\frac{\partial e_{2,2}}{\partial q^1}\frac{\partial e_{1,2}}{\partial q^2}\frac{\partial e_{2,2}}{\partial q^1}\frac{\partial e_{2,2}}{\partial q^2} + \frac{\partial e_{1,1}}{\partial q^1}\frac{\partial e_{2,2}}{\partial q^1}\left(\frac{\partial e_{2,2}}{\partial q^2}\right)^2 = 0,$$

which can be verified by direct computation. Letting $(w^2 - aw^1)$ be the common factor of the two quadratic forms, we obtain

$$\frac{\partial e_{\alpha,\beta}}{\partial q^2} w^\alpha w^\beta = \frac{\partial e_{2,2}}{\partial q^2}\left(w^2 - aw^1\right)^2.$$

(iii) The quadratic form $w \mapsto w^\dagger Q w$ is semi-definite and, at each $(q^1, q^2)$, *it is proportional to the form* $w^\dagger \frac{\partial E}{\partial q^2} w$. More precisely, one has

$$\begin{aligned} Q_{\alpha,\beta} w^\alpha w^\beta &= \left(\frac{\partial e_{2,2}}{\partial q^1}\cdot\frac{a-b}{2}\right)\frac{\partial e_{\alpha,\beta}}{\partial q^2} w^\alpha w\beta \\ &= \left(\frac{\partial e_{2,2}}{\partial q^1}\cdot\frac{\partial e_{2,2}}{\partial q^2}\cdot\frac{a-b}{2}\right)(w^2 - aw^1)^2. \end{aligned}$$

This is easily deduced by (10.8). Notice that the form $Q_{\alpha,\beta} w^\alpha w^\beta$ is never equal to the null form, because $a(q^1, q^2) \neq b(q^1, q^2)$ for all $q^1, q^2$.

If $S$ is a $2 \times 2$ matrix and $\rho \in \mathbb{R}$, we shall use the notation $\{w^\dagger S w = \rho\} \doteq \{w \in \mathbb{R}^2 ;\ w^\dagger S w = \rho\}$. Since $w^\dagger \frac{\partial E}{\partial q^2} w$ is positive definite and $\sin q^2 < 0$, there exists a real number $\eta > 0$ such that

$$\begin{aligned} &\left\{w^\dagger \frac{\partial E}{\partial q^2} w = -\sin \bar{q}^2 v\right\} \\ &\quad = \left\{w \in \mathbb{R}^2 : (w^2 - aw^1) = \eta\right\} \cup \left\{w \in \mathbb{R}^2 : (w^2 - aw^1) = -\eta\right\}. \end{aligned}$$

Hence, in particular,

$$\left\{w^\dagger \frac{\partial E}{\partial q^2} w = -g \sin \bar{q}^2\right\} \cap \left\{w \in \mathbb{R}^2 : (w^2 - aw^1) = 0\right\} = \emptyset.$$

By (iii) this implies

$$\left\{w^\dagger \frac{\partial E}{\partial q^2} w = -g \sin \bar{q}^2\right\} \cap \{w^\dagger Q w = 0\} = \emptyset. \tag{10.9}$$

Moreover, by (i) the line $\{w \in \mathbb{R}^2 : (w^2 - a w^1) = 0\}$ is an asymptote of the hyperbola $\{w^\dagger \frac{\partial E}{\partial q^1} w = -2g \sin \bar{q}^1\}$. Therefore

$$\left\{w^\dagger \frac{\partial E}{\partial q^1} w = -2g \sin \bar{q}^1\right\} \cap \left\{w^\dagger \frac{\partial E}{\partial q^2} w = -g \sin \bar{q}^2\right\} \neq \emptyset. \tag{10.10}$$

Putting (10.9) and (10.10) together, we establish first statement in Proposition 10.1.

On the other hand, the second statement will be proved by an application of Theorem 9.2. Since $U(q) = g(2\cos q^1 + \cos q^2)$ is a potential, by setting $W = \{(0, \eta)\}$ and $\beta(u) \doteq (u^1)^2 + (u^2)^2$, the effective potential

$$U_W(q, u) \doteq U(q) + \eta^2 e_{2,2}(q) + \beta(u)$$

has a strict minimum at $(q, u) = (0, 0, 0, 0)$, provided that $|\eta|$ is large enough. In view of Theorem 9.2, this implies the that the system is stabilizable at $(q^1, q^2, p_1, p_2, u^1, u^2) = (0, 0, 0, 0, 0, 0)$. $\square$

*Acknowledgments.* The work of the first author was supported by the NSF through grant DMS-0807420, "New problems in nonlinear control". The authors would like to thank F. Cardin and M. Levi for useful conversations.

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(*Received April 10, 2008 / Accepted March 26, 2009*)
*Published online May 26, 2009 – © Springer-Verlag* (*2009*)