Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 243 (2007) 270–300
Journal of Differential Equations
www.elsevier.com/locate/jde

Frobenius-type theorems for Lipschitz distributions ☆

Franco Rampazzo

Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Trieste 63, 35121, Padova, Italy

Received 18 May 2007; revised 18 May 2007

Available online 28 June 2007

Abstract

We generalize the classical Frobenius Theorem to distributions that are spanned by locally Lipschitz vector fields. The various versions of the involutivity conditions are extended by means of set-valued Lie derivatives—in particular, set-valued Lie brackets—and set-valued exterior derivatives. A PDEs counterpart of these Frobenius-type results is investigated as well.

MSC: 53C12; 58A30; 17B66; 26E25

Keywords: Distribution; Involutivity; Integral manifold; Lipschitz tensor field; Set-valued tensor field

1. Introduction

1.1. The problem and an outline of the paper

1.1.1. The main problem

A *distribution* Δ on a differentiable manifold M is a set-valued function $q \rightsquigarrow \Delta_q$ which maps a point $q \in M$ into a subspace Δ_q of the tangent space $T_q M$. If n is the dimension of M, a distribution Δ with constant dimension $k \leqslant n$ is called (completely) *integrable* if in a neighborhood of any point $q \in M$ one can find local coordinates $(x, y) = (x^1, \dots, x^k, y^1, \dots, y^{n-k})$ such that (i) each level set $L_{\bar{q}} \doteq \{q \mid y(q) = y(\bar{q})\}$ is a k-dimensional submanifold of M, and (ii) the tan-

☆ Research supported in part by the Research Project *Alcuni aspetti analitici e geometrico-differenziali nei problemi di minimo a crescita lenta*, University of Padova, 2003.

E-mail address: rampazzo@math.unipd.it.
URL: http://www.math.unipd.it/~rampazzo.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.05.040

gent space to $L_{\bar{q}}$ at a point $q \in L_{\bar{q}}$ coincides with Δ_q. The sets L_q are called (local) *integral submanifolds* of the distribution Δ. Clearly, the question whether integral manifolds do exist is trivial when $k = 1$, for the problem reduces to a question of solutions' existence for an ODE. On the contrary, if $k > 1$, local integral submanifolds do not exist unless a geometrical condition, namely *involutivity*, is verified. As is well known, the Frobenius Theorem characterizes local integrability by means of involutivity. We recall that a distribution Δ is called *involutive* if for every pair of fields (f, g) *belonging to* Δ,[1] the Lie bracket

$$[f, g] = Dg \cdot f - Df \cdot g$$

belongs to Δ as well.

The minimal assumptions under which the Frobenius Theorem is usually stated include the fact that Δ is of class C^1.[2] In this paper we are going to investigate the (local) existence of integral manifolds in the case when the distribution Δ is only Lipschitz. Let us justify our interest in this topic by means of a few basic considerations:

(A) Local Lipschitz continuity is enough in the case when the distribution Δ is one-dimensional: indeed, there is no bracket condition when $k = 1$, so one is allowed to ignore the problem of the differentiability of the involved vector fields. So it seems natural to investigate the Lipschitz case also for $k > 1$.

(B) Among the areas that could benefit from an extension of the Frobenius Theorem to non-smooth distributions, let us mention Foliation Theory, Geometric Control Theory, and Classical Mechanics. In particular, in Classical Mechanics the Frobenius Theorem can be rephrased as a characterization of those (linear) non-holonomic constraints which can be represented, in fact, as holonomic constraints. Clearly the question of the regularity of the integral submanifolds—i.e., of the state-constraints—is crucial in the determination of the dynamical equations.

(C) From a purely theoretical viewpoint, the extension of the Frobenius Theorem poses some intriguing questions. For instance: *what should the usual Lie bracket be replaced by when the vector fields are not differentiable*? *What about involutivity in this case*?

(D) The usual results of the "smooth" theory do not indicate an obvious way for well-posing the problem in the case when Δ is merely Lipschitz. Indeed, in the case when the f_i's are of class C^1, the Frobenius Theorem states that, provided involutivity is verified, the local integral manifolds—namely the level sets $y = \bar{y}$—are submanifolds *of class* C^1. In fact, this is nothing but a consequence of the Implicit Function Theorem, since the maps y are of class C^1.

Based on that, one could expect that in the case of a Lipschitz distribution, the *integral submanifolds* should be at most locally Lipschitz. Then a problem arises: what does it mean that a submanifold is an *integral submanifold* of a distribution Δ?

The questions in (C) will be dealt with by making use of the set-valued bracket introduced in [12]. In particular, this bracket allows us to extend the involutivity condition into an

[1] We say that a vector field f belongs to Δ if $f(q) \in \Delta_q$ for every $q \in M$.

[2] We recall that a distribution is said to be Lipschitz (respectively of class C^1) if in a neighborhood U of any point $\bar{q} \in M$ there are vector fields $f_1, \dots, f_k$ which are locally Lipschitz (respectively of class C^1) and such that Δ_q is the linear span of the vectors $f_1(q), \dots, f_k(q)$.

everywhere-defined inclusion relation, the *set-involutivity*—see Definition 4.6 below. In a similar way, we extend the classical conditions that, in the smooth case, are equivalent to involutivity and involve exterior derivatives and Lie derivatives. Actually, the need of generalizing Lie derivatives and exterior derivatives motivates the contents of Sections 2 and 3 (see 1.1.2 below).

As for the questions raised in (D)—namely the problem of giving a notion of integrability for a Lipschitz distribution—let us observe that, in principle, (i) something like the tangent bundle does not exist for a Lipschitz submanifold, even though there exist several nonsmooth analogues of the notion of tangent space at a point; and (ii) the problem is made subtler by the fact that one is looking for the existence of a foliation rather than a single integral manifold. Yet, the answer our main result—namely, Theorem 4.11—gives to these problems is, at a first glance, surprisingly simple: if set-involutivity is verified, then local foliations exist and are made of submanifolds that are *of class* $C^{1,1}$!3

Let us point out that, on one hand, the character of our results is *local*. On the other hand, even in the smooth case the passage from the local version of the Frobenius Theorem to the global one is a mainly topological issue.

In Section 5, the classical PDEs parallel of the Frobenius Theorem is generalized as well to the Lipschitz case. In particular, we prove the local existence of vector-valued, locally Lipschitz, solutions of the corresponding Cauchy problems. Obviously, the regularity result included in Theorem 4.11 is interpreted as a regularity property of the level sets of the Cauchy's problem solution.

Partial versions of the (so-called) Frobenius Theorem in the smooth case were originally proved in [6,7], and [8]. In [4] the result was established in the form we know it today. Obviously, one can also refer to several textbooks and lecture notes—see e.g. [11,14]. As for Lipschitz distributions, an integrability theorem is contained in the first part of [15]. Next, this theorem is utilized to prove an interesting result on the existence of global cross sections to Anosov flows. In our view, some points in the integrability result need some elaboration. Yet, we wish to thank the author of [15] for stimulating (e-mail) conversations occurred while the present article was being prepared. An interesting extension of the Frobenius Theorem on metric spaces can be found in [3], where vector fields are replaced with *arc fields*. An involutivity condition in [3] is also expressed in terms of *brackets* of arc fields. Yet, the result in [3] is hardly comparable with the one presented here. Indeed, the mentioned involutivity condition can be reasonably rephrased in terms of vector fields only if the latter are twice differentiable.

1.1.2. Derivatives of Lipschitz tensors

In the standard case of a smooth distribution Δ, there are essentially three kinds of conditions that characterize complete integrability. The first type of condition deals with vector fields belonging to Δ and their Lie brackets. The second variant involves the 1-forms that span the codistribution annihilating Δ and their exterior derivatives. The third kind of characterization deals, in fact, with both vector fields and 1-forms and includes the notion of Lie derivative of a 1-form. Aiming to extend all of these conditions to the case of a Lipschitz distribution, we needfully have to face the question on how Lie brackets, exterior derivatives, and Lie derivatives can be generalized to the case when the tensors to be differentiated are just locally Lipschitz. Actually, Sections 2 and 3 are devoted to such issues. In particular, in Section 2 we introduce the notion of *set-valued, convex, envelope* of a section of a fiber bundle, which turns out to be a

3 We recall that this means that, in a neighborhood of each point, these manifolds are level sets of differentiable maps whose derivatives are locally Lipschitz continuous.

chart-invariant concept. In Section 3 we apply this notion to Lie derivatives and exterior derivatives, with the aim of extending these derivations to the Lipschitz case. Let us point out that the enveloping of Lie and exterior derivatives *works well* because these tensors are *robust*—see Definition 2.11—that is, their envelopes can be reconstructed starting from any full subset of their domain. Incidentally, it turns out that both Clarke's generalized gradient (2.9) and the set-valued Lie bracket introduced in [12][4] are nothing but instances of envelopes of the corresponding classical objects.

1.2. Preliminaries and notation

1.2.1. Lipschitz maps and manifold structures

If r, s are positive integers, $U \subseteq \mathbb{R}^r$, $V \subseteq \mathbb{R}^s$ are open subsets, ℓ is a non-negative integer, we say that a map $m : U \to V$ is of class $C^{\ell,1}$ if m is of class C^ℓ and its ℓth order partial derivatives are locally Lipschitz.

If $r = s$ we say that a map $m : U \to V$ is an *isomorphism of class* C^ℓ if m is a bijection such that both m and m^{-1} are of class C^ℓ. Moreover, an isomorphism $m : U \to V$ of class C^ℓ is called an *isomorphism of class* $C^{\ell,1}$ if both m and m^{-1} are of class $C^{\ell,1}$. Isomorphisms of class $C^{0,1}$ are sometimes called *lipeomorphisms*.

If ℓ is a non-negative integer, we shall say that M is a *manifold of class* C^ℓ [respectively $C^{\ell,1}$] if M is a finite-dimensional, second-countable, Hausdorff, manifold whose transition maps are of class C^ℓ [respectively $C^{\ell,1}$]. A manifold of class $C^{0,1}$ is also called a *a Lipschitz manifold*.

If M is a n-dimensional manifold and $q \in M$, by saying that a pair (U, x) is a *coordinate chart near* q we shall mean that U is an open subset of M containing q and $x : U \to \mathbb{R}^n$ is an element of the maximal atlas of M.

Convention. By *manifold* (with no further specification) we shall mean a manifold of class C^2.

Let M, N be manifolds. If $\ell = 0, 1, 2$ [respectively $\ell = 0, 1$], a map $m : M \to N$ is *of class* C^ℓ [respectively $C^{\ell,1}$] if for every $q \in M$ there exist a coordinate chart (U, x) of M near q and a coordinate chart (V, ψ) of N near $m(q)$ such that $m(U) \subseteq V$ and $\psi \circ m \circ x^{-1} : x(U) \to \psi(V)$ is a map of class C^ℓ [respectively $C^{\ell,1}$]. Maps of class $C^{0,1}$ will be also called *locally Lipschitz maps*. Let n be a positive integer, let M be an n-dimensional manifold, and let $U \subseteq M$ and $A \subseteq \mathbb{R}^n$ be open subsets. If $\ell = 0, 1, 2$ [respectively $\ell = 0, 1$], an homeomorphism of class C^ℓ [respectively $C^{\ell,1}$] $x : U \to A$ such that x^{-1} is of class C^ℓ [respectively $C^{\ell,1}$] as well—where A is endowed with the standard manifold structure induced by $\mathbb{R}^n$—is called a *coordinate chart of class* C^ℓ [respectively $C^{\ell,1}$]. (In particular, the original charts of M coincide with the coordinate charts of class C^2.)

It is clear that the family of coordinate charts of class C^ℓ [*respectively* $C^{\ell,1}$] *on* M *is an atlas which gives* M *a structure of manifold of class* C^ℓ [*respectively* $C^{\ell,1}$]. We call this structure *the natural structure of class* C^ℓ [respectively $C^{\ell,1}$] on the manifold M.

If M is a manifold, we say that a subset $\mathcal{F} \subset M$ *is a full subset*—equivalently: $\mathcal{F}$ *has full measure*— if its complement has (Lebesgue) measure equal to zero.[5] If $M' \subseteq M$ is any subset, we say that *a property* $P(q)$ *holds for almost every* $q \in M'$—or, equivalently, that $P(q)$ *holds*

[4] We remind that the Lie bracket is a special case of Lie derivative.

[5] A subset $\mathcal{N} \subset M$ has zero Lebesgue measure if $x(U \cap S)$ is a subset of $\mathbb{R}^n$ of zero Lebesgue measure whenever (U, x) is a chart of M.

almost everywhere in M'—if there exists a full subset $\mathcal{F} \subseteq M$ such that $P(q)$ holds for every $q \in M' \cap \mathcal{F}$.

Let N, M be manifolds of class C^1 and let $f : M \mapsto N$ be any map. We shall use $\mathit{DIFF}(f)$ to denote the subset of differentiability points of f. The well-known Rademacher theorem states that if f is locally Lipschitz, then it is almost everywhere differentiable on M, that is, $\mathit{DIFF}(f)$ is a full subset of M.

1.2.2. Submanifolds

Definition 1.1. Let n be a non-negative integer, and let M be an n-dimensional manifold. Let $e = 0, 1, 2$ [respectively $e = 0, 1$] and let k be an integer such that $0 \leqslant k \leqslant n$. A subset $N \subset M$ is called *a k-dimensional, embedded submanifold of class* C^e [respectively *of class* $C^{e,1}$] if, for every $q \in N$, there is a coordinate chart (U, x) of class C^e [respectively $C^{e,1}$] near q such that

$$x(U \cap N) = x(U) \cap \left(\mathbb{R}^k \times \{0\}\right), \tag{1}$$

where $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Remark 1.2. It is straightforward to check that a k-dimensional, embedded submanifold of class C^e [respectively $C^{e,1}$] is a manifold of class C^e [respectively $C^{e,1}$] as soon as we endow it with the charts $(U \cap N, x|_{U \cap N})$, where the (U, x) are the charts giving the relation (1.1), as q runs on N. The charts (U, x) as above are called the *submanifold charts*.

It will be useful to exploit the notion of embedded submanifold in terms of *image of a set* (instead of *level set of a function*, as in Definition 1.1):

Proposition 1.3. *Let n, k be integers such that $0 \leqslant k \leqslant n$, and let M be an n-dimensional manifold. Let $A \subseteq \mathbb{R}^k$ be an open subset, and let $f : A \to M$ be a map of class C^2 [respectively $C^{1,1}$] having constant rank k at all points of A. Then, for every $a \in A$ there is a real number $\epsilon > 0$ such that the image $f(]a - \epsilon, a + \epsilon[^k \cap A)$ is a k-dimensional, embedded submanifold of M of class C^2 [respectively $C^{1,1}$].*

Notice that the statement of Proposition 1.3 refers also to a regularity property which will be crucial for our purposes in the case of Lipschitz distributions. We omit the proof of this elementary result, since it relies on the same standard argument as in the C^2 case, see e.g. [10].

2. Set-valued envelopes

2.1. Vector bundles' sections and their envelopes

Let us begin by recalling some basic notions from the theory of set-valued functions.

Definition 2.1. Let M, N be topological spaces, and let $F : M \rightsquigarrow N$ be a set-valued map.[6] Let q be a point of M. F is said to be *upper semi-continuous—shortly: u.s. continuous—at* q if for any $e \in F(q)$ and any neighborhood V of e there is a neighborhood U of q such that

$$\bigcup_{q' \in U} F(q') \subseteq V.$$

[6] A *set-valued map* $F : M \rightsquigarrow N$ is a map from M into the set $\mathcal{P}(N)$ of parts of N.

F is said to be *lower semi-continuous at* q if for every $e \in F(q)$ and every neighborhood V of e there is a neighborhood U of q such that

$$F(q') \cap V \neq \emptyset \quad \forall q' \in U.$$

F is said to be *continuous at* q if it is both upper and lower semi-continuous at q.

F is said to be *upper semi-continuous* [respectively *lower semi-continuous*, *continuous*] if it is upper semi-continuous [respectively lower semi-continuous, continuous] at each $q \in M$.

Let M a manifold, and let E be a M-based vector bundle of class C^1. Let $\pi : E \to M$ be the bundle projection of E into M.

Definition 2.2. Let M' be any subset of the manifold M. A (single-valued) *section of* E *on* M' is a map $s : M' \to E$ such that $\pi \circ s$ coincides with the identity on M'. A *set-valued section of* E *on* M' is a set-valued function $S : M' \rightsquigarrow E$ such that

$$\pi\big(S(q)\big) = \{q\} \quad \forall q \in M'.$$

(We shall use the expression *set-valued section of* E to mean a set-valued section of E on M.)

A set-valued section of E on M' is called *convex* if, for every $q \in M'$,

$$S(q) = \mathrm{co}\big[S(q)\big].^{7}$$

Definition 2.3. Let M' be a dense subset of M, and let $s : M' \to E$ be a section of E on M'. Let us define the *upper semi-continuous envelope of* s as the set-valued section of E

$$s^{+}_{\mathrm{set}} : M \rightsquigarrow E$$

whose graph coincides with the closure of the graph of s.[8] In other words, for every $q \in M$ one has

$$v \in s^{+}_{\mathrm{set}}(q)$$

if and only if there exists a sequence $(q_m)_{m \in \mathbb{N}}$ in M' such that

$$\lim\big(s(q_m)\big) = (q, v).$$

Definition 2.4. The set-valued section $s_{\mathrm{set}} : M \rightsquigarrow E$ that maps any $q \in M$ into the subset

$$s_{\mathrm{set}}(q) \doteq \mathrm{co}\left[s^{+}_{\mathrm{set}}(q)\right]$$

is called the *convex upper semi-continuous envelope of* s.

[7] If X is a real vector space and $A \subset X$, we use co$[A]$ to denote the *the convex hull* of A, that is, the intersection of all convex sets containing A.

[8] In the language of [2], s^{+}_{set} is the upper limit of s.

As for the regularity of set-valued envelopes, we have

Proposition 2.5. *Let M' be a dense subset of M, and let $s : M' \to E$ be a locally bounded section of E on M'. Then both s^+_{set} and s_{set} are upper semi-continuous set-valued maps* (*from M into E*) *with non-empty, compact values.*

Proof. Let us set $K \doteq \bigcup_{q\in M'} s(q)$, and let $\overline{K}$ be the closure of K. Then $s^+_{\mathrm{set}}(q) = \overline{K} \cap E_q$, for every $q \in M$. Since M' is dense in M and s is locally bounded, this implies that $s^+_{\mathrm{set}}(q)$ is non-empty and compact. Moreover, the graph of s^+_{set} is clearly a closed subset of E, so the map s^+_{set} is upper semi-continuous—see e.g. [1]. Since for every $q \in M$ $s_{\mathrm{set}}(q)$ is the convex hull of $s^+_{\mathrm{set}}(q)$, the map s_{set} turns out to be upper semi-continuous as well, with non-empty, compact, values. □

The next, elementary, result concerns an inclusion property which, in its version for *robust* sections—see Proposition 2.13 below—will play an important role in the proofs of Frobenius-type theorems of Section 4.

Proposition 2.6. *Let M' be a dense subset of a M, and let $s : M' \to E$ be a locally bounded section of E on M'. Moreover, let $G : M \rightsquigarrow E$ be a convex, upper semi-continuous, set-valued section with closed values. Then the following conditions are equivalent*:

(i) $s(q) \in G(q)$, *for all* $q \in M'$;
(ii) $s_{\mathrm{set}}(q) \subseteq G(q)$, *for all* $q \in M$.

We omit the trivial proof of this result.

Remark 2.7. If $q \in M'$, both $s^+_{\mathrm{set}}(q)$, and $s_{\mathrm{set}}(q)$ contain the element $s(q)$. However, unless s is continuous at q, they do not coincide, in general, with the singleton $\{s(q)\}$.

Corollary 2.8. *Let M' be a dense subset of M, and let $s : M' \to E$ be a section of E* (*on M'*). *The following conditions are equivalent*:

(i) *the upper semi-continuous envelope $s^+_{\mathrm{set}} : M \rightsquigarrow E$ is single-valued* (*and continuous*);
(ii) *the convex upper semi-continuous envelope of $s_{\mathrm{set}} : M \rightsquigarrow E$ is single-valued* (*and continuous*);
(iii) *the function s is continuous* (on M').

In particular,

$$s(q') = 0 \quad \forall q' \in M'$$

is equivalent to

$$s_{\mathrm{set}}(q) = \{0\} \quad \forall q \in M.$$

Proof. Observe that an upper semi-continuous set-valued map whose images are singletons is continuous. The equivalence of (i) and (ii) is trivial. Moreover, if $\tilde{s}$ denotes the unique continuous extension of s to the whole M, then (i) follows from (iii) by Proposition 2.6, as soon as we set $G(q) \doteq \{\tilde{g}(q)\}$ for all $q \in M$. Finally, let us assume condition (i), and let us notice that, for every

$q \in M'$, one has $\{s(q)\} = s^+_{\mathrm{set}}(q)$. Since s_{set} is continuous on M, its restriction to M' is continuous as well, so s is continuous. $\square$

Example 2.9 *(The Clarke generalized gradient).* Let m, n be positive integers, let $M \subseteq \mathbb{R}^m$ be an open subset, and let $f : M \to M \times \mathbb{R}^n$ be a locally Lipschitz section (of the trivial vector bundle $M \times \mathbb{R}^n$). By Rademacher's theorem the set of differentiability $DIFF(f)$ is a full subset of M. So the derivative Df is a bounded measurable section of the trivial fiber bundle $M \times \mathbb{R}^{n+m}$ defined on the dense set $DIFF(f)$. The set-valued section $(Df)_{\mathrm{set}}$ is nothing but the *Clarke generalized gradient of f*—see [5].

Example 2.10 *(A Lie bracket for Lipschitz vector fields).* Let M be a manifold of class C^2 and let f, g be locally Lipschitz vector fields. Then, in view of Rademacher's theorem, the Lie bracket $[f, g]$ is a locally bounded, measurable section of the tangent bundle TM defined on the full subset $DIFF(f) \cap DIFF(g)$. Hence we can consider the convex u.s. envelope

$$[f, g]_{\mathrm{set}}.$$

This set-valued section *coincides* with the set-valued Lie bracket that was introduced in [12] with the purpose of giving an extension of the Chow's Theorem in the case of locally Lipschitz vector fields. This bracket has been also used in the commutativity results in [13], and is here used in the Frobenius-type result below—see Theorem 4.11.

2.2. Robust envelopes

Definition 2.11. Let M' be a full subset of M, and let $s : M' \to E$ be a section of E on M'. We say that s is *robust* if, for every full subset $\mathcal{F} \subset M'$, one has

$$s_{\mathrm{set}} = (s|_{\mathcal{F}})_{\mathrm{set}}$$

Remark 2.12. Let us notice that, in general, a section needs not to be robust. Consider, for instance, the trivial, $\mathbb{R}$-based, fiber bundle $E = \mathbb{R} \times \mathbb{R}$. Let $s : M \to E$ be defined by

$$s(r) = \begin{cases} (r, 0) & \text{if } r \in \mathbb{Q}, \\ (r, 1) & \text{if } r \notin \mathbb{Q}. \end{cases}$$

Then

$$s^+_{\mathrm{set}}(r) = \bigl\{(r, 0), (r, 1)\bigr\}, \qquad s_{\mathrm{set}}(r) = \{r\} \times [0, 1], \quad \forall r \in \mathbb{R},$$

while

$$(\hat{s}|_{\mathbb{R}\setminus\mathbb{Q}})^+_{\mathrm{set}}(r) = (s|_{\mathbb{R}\setminus\mathbb{Q}})_{\mathrm{set}}(r) = \{1\}, \quad \forall r \in \mathbb{R}.$$

Of course, *continuous sections are robust*. Moreover, it is well known—see e.g. [5]—that the section Df in Example 2.9 is robust. Namely, for every full subset $\mathcal{F} \subset DIFF(f)$ one has

$$(Df|_{\mathcal{F}})_{\mathrm{set}} = Df_{\mathrm{set}}.$$

In other words, *differentials of locally Lipschitz maps are robust*—and their convex, upper semi-continuous, envelopes coincide with their Clarke's generalized gradient.

The *Lie bracket* $[f,g]$ *of two Lipschitz vector fields* f *and* g—i.e., the section considered in Example 2.10—is further instance of *robust* section. That is, for every full subset $\mathcal{F} \subset DIFF(f) \cap DIFF(g)$, one has

$$\left([f,g]\big|_{\mathcal{F}}\right)_{\mathrm{set}} = [f,g]_{\mathrm{set}}.$$

This was already proved in [13]. Moreover it is a consequence of a more general result which is stated in Proposition 2.14 below.

By Proposition 2.6 we obtain

Proposition 2.13. *Let M' be a full subset of M, and let $s : M' \to E$ be a robust, locally bounded, section of E on M'. Moreover, let $G : M \rightsquigarrow E$ be a convex, upper semi-continuous, set-valued, section with closed values. Then the following conditions are equivalent*:

(i) *$s(q) \in G(q)$, for almost every $q \in M'$;*[9]
(ii) *$s_{\mathrm{set}}(q) \subseteq G(q)$, for all $q \in M$.*

In particular,

$$s(q') = 0$$

for almost every $q' \in M'$ if and only if

$$s_{\mathrm{set}}(q) = \{0\} \quad \forall q \in M.$$

Proof. In view of Proposition 2.6, condition (i) follows from (ii). Conversely, let us assume that there exists a full subset $M^{\#} \subset M$ such that $s(q) \in G(q)$ holds true for all $q \in M' \cap M^{\#}$. Since s is robust, (ii) follows from (i) by replacing M' with $M' \cap M^{\#}$ in Proposition 2.6. The last statement is a consequence of Corollary 2.8. □

In Proposition 2.14 below we give a sufficient condition for a section to be robust. Next we will exploit this result in order to define Lie derivatives and exterior derivatives for Lipschitz tensors and Lipschitz k-forms, respectively.

Let M be a n-dimensional manifold, and let E, E_1, and E_2 be M-based vector bundles of class C^1, having dimension m, m_1, and m_2, respectively.

Proposition 2.14. *Let s_1, s_2 be locally Lipschitz sections of E_1 and E_2, respectively, and let $s : M' \to E$ be a locally bounded, measurable section of E such that*:

(i) *M' is a full subset of $DIFF(s_1) \cap DIFF(s_2)$;*
(ii) *if S, S_1, and S_2 are the arrays representing s, s_1, and s_2, respectively, on a coordinate chart (U,x), then there exist continuous functions*

$$\Psi : \mathbb{R}^{m_1+m_2} \to \mathbb{R}^m, \qquad \Lambda : \mathbb{R}^{m_1+m_2} \to \mathrm{Hom}\left(\mathbb{R}^{n(m_1+m_2)}, \mathbb{R}^m\right)^{10}$$

[9] This means that there exist a full subset $M^{\#} \subset M$ such that $s(q) \in G(q)$ holds true for all $q \in M' \cap M^{\#}$.

[10] We use $Hom(\mathbb{R}^{n(m_1+m_2)}, \mathbb{R}^m)$ to denote the vector space of homomorphisms from $\mathbb{R}^{n(m_1+m_2)}$ into $\mathbb{R}^m$.

such that

$$S(x) = \Psi\big(S_1(x), S_2(x)\big) + \Lambda\big(S_1(x), S_2(x)\big) \cdot \big(DS_1(x), DS_2(x)\big), \tag{2}$$

where we have used $\Lambda(v) \cdot w$ to denote the image of w through the map $\Lambda(v)$.

Then the section s is robust.

Remark 2.15. It is straightforward to check that in fact conditions (i) and (ii) in Proposition 2.14 are *chart-independent*.

Proof of Proposition 2.14. The statement is of local character, so we can assume M to be an open subset of $\mathbb{R}^n$ and identify s, s_1, and s_2 with the arrays S, S_1, and S_2, respectively. Let us fix $x \in M$. We have to show that, for every full subset $\mathcal{F} \subset M'$, one has

$$S_{\mathrm{set}}(x) = (S|_{\mathcal{F}})_{\mathrm{set}}(x).$$

Let us consider the section

$$(DS_1, DS_2) : \mathit{DIFF}(S_1) \cap \mathit{DIFF}(S_2) \to \big(\mathit{DIFF}(S_1) \cap \mathit{DIFF}(S_2)\big) \times \mathbb{R}^{n(m_1+m_2)}.$$

As it has been recalled in Example 2.9, (DS_1, DS_2) is robust (and its convex u.s. envelope is nothing but the Clarke gradient of the map (S_1, S_2)). Hence, in particular,

$$\big((DS_1, DS_2)|_{\mathcal{F}}\big)_{\mathrm{set}} = (DS_1, DS_2)_{\mathrm{set}}.$$

Therefore, by the continuity of Ψ and Λ (and the linearity of each map $w \mapsto \Lambda(v) \cdot w$), for every $v \in \mathbb{R}^{m_1+m_2}$ we have

$$\begin{aligned}
S_{\mathrm{set}}(x) &= \mathrm{co}\big[S^+_{\mathrm{set}}(x)\big] \\
&= \mathrm{co}\big[\big(\Psi(S_1, S_2) + \Lambda(S_1, S_2) \cdot (DS_1, DS_2)\big)^+_{\mathrm{set}}(x)\big] \\
&= \Psi\big(S_1(x), S_2(x)\big) + \Lambda\big(S_1(x), S_2(x)\big) \cdot \mathrm{co}\big[(DS_1, DS_2)^+_{\mathrm{set}}(x)\big] \\
&= \Psi\big(S_1(x), S_2(x)\big) + \Lambda\big(S_1(x), S_2(x)\big) \cdot (DS_1, DS_2)_{\mathrm{set}}(x) \\
&= \Psi\big(S_1(x), S_2(x)\big) + \Lambda\big(S_1(x), S_2(x)\big) \cdot \big((DS_1, DS_2)|_{\mathcal{F}}\big)_{\mathrm{set}}(x) \\
&= \Psi\big(S_1(x), S_2(x)\big) + \Lambda\big(S_1(x), S_2(x)\big) \cdot \mathrm{co}\big[\big((DS_1, DS_2)|_{\mathcal{F}}\big)^+_{\mathrm{set}}(x)\big] \\
&= \mathrm{co}\big[\Psi\big(S_1(x), S_2(x)\big) + \Lambda\big(S_1(x), S_2(x)\big) \cdot \big((DS_1, DS_2)|_{\mathcal{F}}\big)^+_{\mathrm{set}}(x)\big] \\
&= \mathrm{co}\big[(S|_{\mathcal{F}})^+_{\mathrm{set}}(x)\big] = (S|_{\mathcal{F}})_{\mathrm{set}}(x). \qquad \square
\end{aligned}$$

3. Lie and exterior derivatives in the Lipschitz case

3.1. Lie and exterior derivatives in the smooth case

Let us begin by recalling the classical notions of Lie derivative and exterior derivative. Next, we will extend these notions to the case when the involved functions are just locally Lipschitz.

3.1.1. Lie derivatives

Let M be a manifold, let r, s be non-negative integers and let E be the (r,s)-type tensor bundle on M[11]. If $T : M \to E$ is a section of E (i.e. a (r,s)*-type tensor field*) of class C^1 and f is a vector field on M of class C^1, then the *Lie derivative* $L_f T$ *of* T *along* f is the (continuous) section of E defined as follows.

Let (U, x) be a coordinate chart, and let

$$T(q) = T^{i_1,\dots,i_r}_{j_1,\dots,j_s}(q)\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} \quad \forall q \in U,$$

where: (i) the multi-indexes $i_1, \dots, i_r$ and $j_1, \dots, j_s$ range, respectively, over all r-tuples and s-tuples of elements of $\{1, \dots, n\}$; (ii) the real functions $T^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ are of class C^1; and (iii) the summation convention is adopted. Then the Lie derivative $L_f T$ is expressed on U by

$$L_f T = W^{i_1,\dots,i_r}_{j_1,\dots,j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}, \tag{3}$$

where, for every value of the multi-indexes $(i_1, \dots, i_r)$ and $(j_1, \dots, j_s)$, the map $W^{i_1,\dots,i_r}_{j_1,\dots,j_s}$: $U \to \mathbb{R}$ is defined by

$$\begin{aligned} W^{i_1,\dots,i_r}_{j_1,\dots,j_s} \doteq{}& f^c \frac{\partial T^{i_1\dots i_r}_{j_1\dots j_s}}{\partial x^c} - \frac{\partial f^{i_1}}{\partial x^c} T^{c\dots i_r}_{j_1\dots j_s} - \cdots - \frac{\partial f^{i_r}}{\partial x^c} T^{i_1\dots i_{r-1}c}_{j_1\dots j_s} \\ &+ \frac{\partial f^c}{\partial x^{j_1}} T^{i_1\dots i_r}_{c\dots j_s} + \cdots + \frac{\partial f^c}{\partial x^{j_s}} T^{i_1\dots i_r}_{j_1\dots j_{s-1}c}. \end{aligned}$$

As is well known, this definition is in fact chart-independent. Finally, let us point out that if g is a vector field, then

$$L_f g = [f, g].$$

3.1.2. Exterior derivatives

Let n be a positive integer, let M be a n-dimensional manifold, and let $U \subset M$ be open. For every integer h such that $0 \leqslant h \leqslant n$ and every $q \in U$, let Λ^h_q denote the space of skew-symmetric, h-linear forms, on $(T_q M)^h$. Let $\Lambda^h(U)$ be the (U-based) corresponding vector bundle, and, for every $r = 0, 1$ let us use

$$\Omega^h_r(U)$$

to denote the set of sections of $\Lambda^h(U)$ that are of class C^r. Namely, $\Omega^h_r(U)$ is the set of *h-forms* on U that are of class C^r. In addition, we use

$$\Omega^h_{0,1}(U) \quad \big(\subset \Omega^h_0(U)\big)$$

[11] This means that, for each $q \in M$,

$$E_q = \underbrace{T_q M \otimes \cdots \otimes T_q M}_{r \text{ times}} \otimes \underbrace{T^*_q M \otimes \cdots \otimes T^*_q M}_{s \text{ times}}$$

to denote the set of h-forms defined on U that are of class $C^{0,1}$, also called *locally Lipschitz h-forms*. In particular, $\Omega_1^0(U)$ and $\Omega_{0,1}^0(U)$ denote the set of real functions defined on U that are, respectively, continuously differentiable and locally Lipschitz.

Let us recall the definition of exterior derivative for a h-form of class C^1.

Definition 3.1. Let h be an integer such $0 \leqslant h \leqslant n-1$, and let $\omega \in \Omega_1^h(U)$. The *exterior derivative $d\omega$* of ω is a $(h+1)$-form of class C^0 defined as follows: if

$$\omega(q) = \sum_\sigma c_{\sigma_1,\dots,\sigma_h}\, dx^{i_1} \wedge \cdots \wedge dx^{i_h}\,[12]$$

is the local expression of ω on a coordinate chart (U', x), then

$$d\omega(q) = \sum_\sigma \sum_{r=1}^n \frac{\partial c_{i_1,\dots,i_h}}{\partial x^r}\, dx^r \wedge dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_h}, \tag{4}$$

for all $q \in U'$.[13]

3.2. Lie and exterior derivatives in the Lipschitz case

3.2.1. Lie derivatives

Let M be a manifold and let E be a tensor bundle based on M. If T is a locally Lipschitz section of E and f is a locally Lipschitz vector field on M, then for all $q \in \mathit{DIFF}(T) \cap \mathit{DIFF}(f)$, $L_f T(q)$ can be defined as in (3). In other words, $L_f T$ is a section of E on the full subset $\mathit{DIFF}(T) \cap \mathit{DIFF}(f)$. Lemma 3.2 below establishes that, as in the case of Clarke's generalized gradient, the convex u.s. envelope of $L_f T$ coincides with the convex u.s. envelope of any restriction of $L_f T$ to a full subset of $\mathit{DIFF}(T) \cap \mathit{DIFF}(f)$.

Lemma 3.2. *The section $L_f T : \mathit{DIFF}(T) \cap \mathit{DIFF}(f) \to E$ is robust. That is, for every full subset $\mathcal{F} \subset \mathit{DIFF}(T) \cap \mathit{DIFF}(f)$, one has*

$$(L_f T|_{\mathcal{F}})_{\mathrm{set}} = (L_f T)_{\mathrm{set}}.$$

Proof. This result follows directly from Proposition 2.14, because of the local form (3) of $L_f T$. $\square$

Definition 3.3 *(Set-valued Lie derivative).* The convex upper semi-continuous envelope

$$(L_f T)_{\mathrm{set}} : M \rightsquigarrow E$$

—which, in view of Lemma 3.2, coincides with $(L_f T|_{\mathcal{F}})_{\mathrm{set}}$, for every full subset $\mathcal{F}$—will be called the (*set-valued*) *Lie derivative of T along f*.

---

[12] Of course the coefficients $c_{\sigma_1,\dots,\sigma_h}$ are functions of class C^r, and the summation is performed over all strictly increasing h-tuples $\sigma = (\sigma_1, \dots, \sigma_h)$ with values in $\{1, \dots, n\}$.

[13] As is well known, this definition is in fact independent of the system of coordinates x, so that the exterior derivative d is a well-defined map from $\Omega_r^h(U)$ into $\Omega_{r-1}^{h+1}(U)$.

By Proposition 2.5 we obtain

Proposition 3.4. *The Lie derivative* $(L_f T)_{\mathrm{set}}$ *is an upper semi-continuous, set-valued, section of* E *having non-empty, convex, compact values.*

Remark 3.5. In particular, if T coincides with a vector field g, at every $q \in DIFF(f) \cap DIFF(g)$ one has

$$L_f g(q) = [f, g](q).$$

Hence, we obtain

$$(L_f g)_{\mathrm{set}}(q) = [f, g]_{\mathrm{set}}(q) \quad \forall q \in M,$$

so recovering, as a particular case, the set-valued Lie bracket introduced in [12]—see Remark 2.10.

3.2.2. Exterior derivatives

Let M be a manifold of dimension n and let h an integer such that $0 \leqslant h \leqslant n-1$. Let ω be a locally Lipschitz h-form. Then, by Rademacher's theorem, $d\omega$ is a measurable, locally bounded, section of $\Gamma^{h+1}(DIFF(\omega))$—namely $d\omega$ is a measurable, locally bounded, $(h+1)$-form on $DIFF(\omega)$. Therefore, we can consider its convex u.s. envelope. This envelope turns out to be robust, as stated in Lemma 3.6 below.

Lemma 3.6. *The section* $d\omega : DIFF(\omega) \to \Gamma^{h+1}(M)$ *is robust. That is, for every full subset* $\mathcal{F} \subset DIFF(\omega)$*, one has*

$$(d\omega|_{\mathcal{F}})_{\mathrm{set}} = (d\omega)_{\mathrm{set}}.$$

Proof. This result follows directly from Proposition 2.14, because of the local form (4) of $d\omega$. $\square$

In view of Lemma 3.6, it is meaningful to give the following definition:

Definition 3.7 *(Set-valued exterior derivative).* The convex upper semi-continuous set-valued envelope

$$(d\omega)_{\mathrm{set}} : M \rightsquigarrow \Gamma^{h+1}(M)$$

will be called the (*set-valued*) *exterior derivative of* ω.

As in the case of the Lie derivative, by Proposition 2.5 we obtain

Proposition 3.8. *The exterior derivative* $d\omega_{\mathrm{set}}$ *is an upper semi-continuous, set-valued, section of* $\Gamma^{h+1}(M)$ *with non-empty, convex, compact values.*

4. A Frobenius-type theorem

4.1. Distributions and codistributions

For any finite subset $\{v_1, \ldots, v_r\}$ of a real vector space V, let us use $\operatorname{span}\{v_1, \ldots, v_r\}$ to denote the linear subspace generated by $\{v_1, \ldots, v_r\}$.

Definition 4.1. Let n, k be non-negative integers such that $k \leqslant n$, and let M be a n-dimensional manifold. By a k-dimensional *distribution* of class C^1 [respectively $C^{0,1}$] we mean a subset $\Delta \subseteq TM$ such that, for every $\bar{q} \in M$,

(i) $\Delta_{\bar{q}} \doteq \Delta \cap T_{\bar{q}}M$ is a linear subspace of $T_{\bar{q}}M$ of dimension k,
(ii) there is a neighborhood U of $\bar{q}$ and vector fields $f_1, \ldots, f_k$ of class C^1 [respectively $C^{0,1}$], defined on U verifying

$$\Delta_q = \operatorname{span}\big\{f_1(q), \ldots, f_k(q)\big\}$$

for all $q \in U$. Any such set of vector fields is called a *local frame* of class C^1 [respectively $C^{0,1}$] for Δ.

Definition 4.2. Let n, h be non-negative integers such that $h \leqslant n$, and let M be a n-dimensional manifold. By a h-dimensional *codistribution* of class C^1 [respectively $C^{0,1}$] we mean a subset $\Theta \subseteq T^*M$ such that, for every $\bar{q} \in M$,

(i) $\Theta_{\bar{q}} \doteq \Delta \cap T^*_{\bar{q}}M$ is a linear subspace of $T^*_{\bar{q}}M$ of dimension h,
(ii) there is a neighborhood U of $\bar{q}$ and 1-forms $\omega^1, \ldots, \omega^h$ of class C^1 [respectively $C^{0,1}$], defined on U verifying

$$\Theta_q = \operatorname{span}\big\{\omega^1(q), \ldots, \omega^h(q)\big\}$$

for all $q \in U$. Any such set of 1-forms is called a *local frame* of class C^1 [respectively $C^{0,1}$] for Θ.

We shall be mainly concerned with distributions [respectively codistributions] of class $C^{0,1}$, which we also call *Lipschitz* distributions [respectively codistributions].

Definition 4.3. If Δ is a distribution on a manifold M, and f is a vector field defined on a subset $M' \subseteq M$, we shall say that f *belongs to* Δ if, for every $q \in M'$, $f(q) \in \Delta_q$. In a similar way, we define the notion of *a 1-form belonging to a codistribution*.

Definition 4.4. Let M and Δ as in Definition 4.1. The $(n-k)$-dimensional codistribution $\Delta^\dagger$ defined by

$$\Delta^\dagger_q = \big\{\omega \in T^*M \;\big|\; \langle \omega, v\rangle = 0 \ \forall v \in \Delta_q\big\} \quad \forall q \in M,$$

is called *the annihilating codistribution of* Δ.

Remark 4.5. By the Implicit Function Theorem,[14] a distribution Δ is of class C^1 [respectively $C^{0,1}$] if and only if the codistribution $\Delta^\dagger$ is of class C^1 [respectively $C^{0,1}$].

4.2. Involutivity, commutativity, and integrability

Let us extend the notion of involutivity to Lipschitz distributions. Actually, two kinds of extensions of this concept will be given. Indeed, one can either assume involutivity in a classical sense on a full measure subset—so adopting a "weak derivative" approach, as in [15]—or consider an everywhere defined notion of involutivity by relying on the afore introduced set-valued Lie bracket. The two notions are in fact equivalent, as proved in Theorem 4.11 below. Let us remark that in the case of vector fields of class C^1, the extended involutivity notion turn out coincide with the classical one.

Definition 4.6 *(Involutivity of distributions).* Let n, k be non-negative integers such that $k \leqslant n$, let M be a n-dimensional manifold, and let Δ be a Lipschitz, k-dimensional distribution on M. We say that Δ is *set-involutive* if, for every pair of locally Lipschitz vector fields f and g (on M) belonging to Δ, one has

$$[f,g]_{\mathrm{set}}(q) \subset \Delta_q \quad \forall q \in M. \tag{5}$$

In addition, we say that Δ is *involutive almost everywhere* if, for every pair of locally Lipschitz vector fields f and g (on M) belonging to Δ and for almost every $q \in M$, one has

$$[f,g](q) \in \Delta_q.$$

Definition 4.7 *(Involutivity of families of vector fields).* Let n be a non-negative integer, and let M be a n-dimensional manifold. Let $U \subseteq M$ be an open subset and let $\mathcal{V}$ be a family of Lipschitz vector fields on U. We say that $\mathcal{V}$ is *set-involutive* if, for every pair $f, g \in \mathcal{V}$ and every $q \in M$, one has

$$[f,g]_{\mathrm{set}}(q) \subset \operatorname{span}\{h(q) \mid h \in \mathcal{V}\}. \tag{6}$$

In addition, we say that $\mathcal{V}$ is *involutive almost everywhere* if, for every pair $f, g \in \mathcal{V}$ and for almost every $q \in \mathit{DIFF}(f) \cap \mathit{DIFF}(g)$, one has

$$[f,g](q) \in \operatorname{span}\{h(q) \mid h \in \mathcal{V}\}.$$

Let us give a notion of *commutativity* for locally Lipschitz vector fields.

Definition 4.8 *(Commutativity of vector fields).* Let n be a non-negative integer let M be a n-dimensional manifold. Let $U \subseteq M$ be an open subset and let f and g be locally Lipschitz vector fields on U. We say that f and g *set-commute* if

$$[f,g]_{\mathrm{set}}(q) = \{0\} \quad \forall q \in U. \tag{7}$$

[14] In some version fit for locally Lipschitz maps—see e.g.[5].

In addition, we say that f and g *commute almost everywhere* if, for almost every $q \in DIFF(f) \cap DIFF(g)$, one has

$$[f, g](q) = 0. \tag{8}$$

Finally, let us give a notion of integrability for a Lipschitz distributions.

Definition 4.9 *($C^{1,1}$-integrability).* Let n, k be non-negative integers such that $k \leqslant n$. Let Δ be a k-dimensional Lipschitz distribution on a n-dimensional manifold M. We say that Δ is *completely $C^{1,1}$-integrable* if for each $q \in M$ there exist a neighborhood U of q, open subsets $X \subseteq \mathbb{R}^k$, $Y \subseteq \mathbb{R}^{n-k}$, and a Lipschitz coordinate chart $(U, (x, y))$ verifying $(x, y)(U) = X \times Y$, such that the following condition is satisfied:

If $\bar{q} \in U$ and $(\bar{x}, \bar{y}) \doteq (x, y)(\bar{q})$ then:

(i) *the inverse image $(x, y)^{-1}(X \times \{\bar{y}\})$ is a submanifold of class $C^{1,1}$*; *and*
(ii) $T_{\bar{q}}((x, y)^{-1}(X \times \{\bar{y}\})) = \Delta_{\bar{q}}$.

Remark 4.10. As we pointed out in the Introduction, in the case when Δ is of class C^1 one does not require the submanifold $(x, y)^{-1}(X \times \{\bar{y}\})$ *to be of class C^2*: actually, this would be the natural analogue of the regularity property we are assuming in (i). In fact, when Δ is of class C^1, the coordinates (x, y) turn out to be of class C^1, so the C^1-regularity of $(x, y)^{-1}(X \times \{\bar{y}\})$ is guaranteed by the Implicit Function Theorem. And this is enough to make (ii) meaningful. One the contrary, our definition of $C^{1,1}$-integrability *includes a non-trivial regularity requirement*, namely condition (i), without which (ii) would be meaningless. The noticeable fact—see Theorem 4.11 below—is that Lipschitz distributions that are almost everywhere involutive *are, in fact, $C^{1,1}$-integrable*. That is, in the integration process one gains regularity—see also Remark 4.12.

4.3. A Frobenius-type theorem

We are going to present a Frobenius-type result for locally Lipschitz distributions. Let us point out that in the case when the involved distributions are of class C^1, there are three types of conditions that are equivalent to complete integrability. In fact, some of them involve only vector fields and their Lie brackets. These are the commutativity and the involutivity conditions. Other versions of the Frobenius Theorem deal only with the forms spanning the annihilating distribution and their exterior derivatives. Finally, Frobenius Theorem can be formulated by means of a condition involving both forms and vector fields (and the Lie derivative of the former along the latter). A similar classification can be applied, respectively, to the subsets (2)–(7), (8)–(11), and (12)–(13) of the set of twelve conditions of Theorem 4.11 below, which turn out to be equivalent to complete integrability. Furthermore, within each group, the same condition is given in a *a.e. version* and in a *set-valued version*.

Theorem 4.11. *Let n, k be non-negative integers such that $k \leqslant n$, and let Δ be a k-dimensional Lipschitz distribution on a n-dimensional manifold M. Then the following conditions are equivalent*:

(I) *Δ is completely $C^{1,1}$-integrable.*
(II) *Δ is set-involutive.*

(III) *Δ is involutive almost everywhere.*
(IV) *Every Lipschitz local frame of Δ is set-involutive.*
(V) *Every Lipschitz local frame of Δ is involutive almost everywhere.*
(VI) *For every $\bar{q} \in M$ there exists an open neighborhood $U \subseteq M$ of $\bar{q}$ and a Lipschitz local frame $\{g_1, \dots, g_k\}$ of Δ such that for every $i, j = 1, \dots, k$, g_i and g_j set-commute.*
(VII) *For every $\bar{q} \in M$ there exists an open neighborhood $U \subseteq M$ of $\bar{q}$ and a Lipschitz local frame $\{g_1, \dots, g_k\}$ of Δ such that for every $i, j = 1, \dots, k$, g_i and g_j commute almost everywhere.*
(VIII) *If $U \subseteq M$ and $\{\omega^1, \dots, \omega^{n-k}\}$ are an open subset and a frame of $\Delta^\dagger$ on U, respectively, then, for every $\alpha = 1, \dots, n-k$ and for every $q \in U$, one has*

$$\left(d\omega^\alpha\right)_{\mathrm{set}}(q) \subset \Lambda^1_q(M) \wedge \omega^1(q) + \cdots + \Lambda^1_q(M) \wedge \omega^{n-k}(q).$$ [15]

(IX) *If U and $\{\omega^1, \dots, \omega^{n-k}\}$ are as in condition* (VIII)*, then, for every $\alpha = 1, \dots, n-k$, and for almost every $q \in \bigcap_{\beta=1}^{n-k} (DIFF(\omega^\beta) \cap U)$, one has*

$$d\omega^\alpha(q) \in \Lambda^1_q(M) \wedge \omega^1(q) + \cdots + \Lambda^1_q(M) \wedge \omega^{n-k}(q). \tag{9}$$

(X) *If U and $\{\omega^1, \dots, \omega^{n-k}\}$ are as in condition* (VIII)*, and we let $\Gamma \doteq \omega^1 \wedge \cdots \wedge \omega^{n-k}$, then, for every $\alpha = 1, \dots, n-k$ and every $q \in U$, one has*

$$\left(d\omega^\alpha\right)_{\mathrm{set}}(q) \wedge \Gamma(q) = \{0\}.$$ [16]

(XI) *If U, Γ, and $\{\omega^1, \dots, \omega^{n-k}\}$, are as in condition* (X)*, then, for every $\alpha = 1, \dots, n-k$ and almost every $q \in \bigcap_{\beta=1}^{n-k} (DIFF(\omega^\beta) \cap U)$, one has*

$$d\omega^\alpha(q) \wedge \Gamma(q) = 0. \tag{10}$$

(XII) *If f is a locally Lipschitz vector field belonging to Δ and ω is a locally Lipschitz* 1*-form belonging to $\Delta^\dagger$, then, for every $q \in M$, one has*

$$(L_f \omega)_{\mathrm{set}}(q) \subseteq \Delta^\dagger(q).$$

(XIII) *If f and ω are as in* (XII)*, then, for almost every $q \in M$, one has*

$$L_f \omega(q) \in \Delta^\dagger(q).$$

[15] The right-hand side denotes the subset of elements of $\Lambda^2_q(M)$ having the form

$$\theta_1 \wedge \omega^1(q) + \cdots + \theta_{n-k} \wedge \omega^{n-k}(q),$$

where $\theta_\alpha \in \Lambda^1_q(M)$ for all $\alpha = 1, \dots, n-k$.

[16] The left-hand side denotes the subset of all elements of $\Lambda^{n-k+2}_q(M)$ having the form

$$\beta \wedge \Gamma(q),$$

where $\beta \in (d\omega^\alpha)_{\mathrm{set}}(q)$. Moreover, both conditions (X) and (XI) are meaningful only if $k \geqslant 2$.

Remark 4.12. Let us point out that passing through commutativity—which is assumed in conditions (VI) and (VII)—is essential in order to establish $C^{1,1}$ regularity. Indeed the flow of a locally Lipschitz vector field is of class $C^{1,1}$ in the time-variable but it is only locally Lipschitz in the initial state. The main point is that commutativity allows us to avoid the uncomfortable dependence on the initial state when one differentiates compositions of several flows.[17] Notice that this is not possible when commutativity is replaced with the involutivity assumption. The remarkable fact, however, is that involutivity is still sufficient for establishing $C^{1,1}$ regularity. Similarly, a simplified version of the here exploited arguments would show that, when $r \geqslant 1$, the local integral manifolds of a distribution of class C^r are, in fact, of class C^{r+1}. Perhaps, the fact that this is not usually noticed derives from the misleading circumstance that integral manifolds are *level sets of functions of class* C^r (and not C^{r+1}). However, this does not prevent these level sets from being of class C^s, with $s > r$. This is, in fact what happens in our case: the level sets of the maps y_j are submanifold of class $C^{1,1}$, though the y_j are just Lipschitz.

5. A PDEs counterpart

Let n, k be integers such that $0 < k \leqslant n$, let M be a n-dimensional manifold, and let $f_1, \dots, f_k$ be locally Lipschitz vector fields on M. Let us assume that, for every $q \in \mathcal{M}$, the vectors $f(q), \dots, f_k(q)$ are linearly independent, and let us consider the system of $k \times (n-k)$ differential equations

$$\begin{cases} \langle du_1(q), f_1(q)\rangle = 0, \\ \quad\vdots \\ \langle du_1(q), f_k(q)\rangle = 0, \\ \quad\vdots \\ \langle du_{n-k}(q), f_1(q)\rangle = 0, \\ \quad\vdots \\ \langle du_{n-k}(q), f_k(q)\rangle = 0 \end{cases} \tag{11}$$

where we mean that $u = (u_1, \dots, u_{n-k})$ is a map from M into $\mathbb{R}^{n-k}$.

Definition 5.1. Let $U \subseteq M$ be an open subset. We shall say that a map $u : U \to \mathbb{R}^{n-k}$ is a *solution of* (11) if:

(i) u is locally Lipschitz;
(ii) (11) is verified at almost every $q \in U$.

5.1. *Almost everywhere complete systems*

Definition 5.2. System (11) is said to be *almost everywhere complete* if, for every $\bar{q} \in M$, there is a neighborhood U of $\bar{q}$ and a solution

$$u = (u_1, \dots, u_{n-k}) : U \to \mathbb{R}^{n-k}$$

[17] See the beginning of the proof of Theorem 4.11.

of (11) such that, for almost every

$$q \in U \cap DIFF(u_1) \cap \cdots \cap DIFF(u_{n-k}),$$

the subset

$$\operatorname{span}\left\{du_1(q), \ldots, du_{n-k}(q)\right\}$$

is a $(n-k)$-dimensional subspace of $T_q^* M$.

This definition is a natural extension of the classical concept of *complete system*—see e.g. [9]—where the functions $u_1, \ldots, u_{n-k}$ are required to be of class C^1 and to solve (11) *at every* $q \in U$. Actually, in the case when the vector fields are of class C^1, the Frobenius theorem can be rephrased by saying that system (11) is complete if and only if, for all $q \in M$ and $i, j = 1, \ldots, k$, one has

$$[f_i, f_j](q) \in \operatorname{span}\left\{f_1(q), \ldots, f_k(q)\right\}.$$

The next theorem extends this result to the case of locally Lipschitz solutions.

Theorem 5.3. *The following conditions are equivalent*:

(I) *System* (11) *is almost everywhere complete.*
(II) *The family* $\{f_1, \ldots, f_k\}$ *is set-involutive.*
(III) *The family* $\{f_1, \ldots, f_k\}$ *is almost everywhere involutive.*

Proof. By Theorem 4.11, conditions (II) and (III) are equivalent. Let us prove that (II) implies (I). By Theorem 4.11, for each $\bar{q} \in M$, there exist a neighborhood U of $\bar{q}$, open subsets $X \subseteq \mathbb{R}^k$, $Y \subseteq \mathbb{R}^{n-k}$, and a Lipschitz chart $(U, (x, y))$ verifying $(x, y)(U) = X \times Y$, such that the following holds true:

If $\bar{q}$ is a point of U and we let $(\bar{x}, \bar{y}) \doteq (x, y)(\bar{q})$, then

(i) $(x, y)^{-1}(X \times \{\bar{y}\})$ *is a submanifold of class* $C^{1,1}$, *and*
(ii) $T_q((x, y)^{-1}(X \times \{\bar{y}\})) = \operatorname{span}\{f_1(\bar{q}), \ldots, f_k(\bar{q})\}$.

Hence condition (I) follows from (II) as soon as one lets $u_i(\cdot) \doteq y_i(\cdot)$ for all $i = 1, \ldots, n-k$. Indeed, if $\bar{q} \in DIFF(u)$ and $i = 1, \ldots, n-k$, one clearly has

$$\operatorname{span}\left\{f_1(\bar{q}), \ldots, f_k(\bar{q})\right\} = T_q\left((x, y)^{-1}\left(X \times \{\bar{y}\}\right)\right) \in \operatorname{Ker}\left(du_i(\bar{q})\right).$$

Therefore, for every $j = 1, \ldots, k$,

$$f_j(\bar{q}) \in \operatorname{Ker}(du_1)(\bar{q}) \cap \cdots \cap \operatorname{Ker}(du_{n-k})(\bar{q}).$$

This means that (11) is verified at $\bar{q}$. Since $DIFF(u)$ is a full subset of U, (11) is almost everywhere complete.

In order to prove that (I) implies (III) we will rely on the following elementary result.

Lemma 5.4. *Let $U \subseteq M$ be an open subset and let $u : U \to \mathbb{R}^{n-k}$ be a solution to* (11). *For every $j = 1, \dots, k$, let $\phi_t^{f_j}$ denote the* (*local*) *flow of the vector field f_j. Then, for every $i = 1, \dots, k$, every $\bar{q} \in U$, the map*

$$t \to u \circ \phi_t^{f_i}(\bar{q}),$$

which is defined on a open interval containing $t = 0$, is constant.

Proof. The proof is standard and based on mollifications. Yet, we do not omit it for the sake of completeness. Obviously, it is sufficient to prove the result in the case when M is an open subset of $\mathbb{R}^n$ and $t \in [-\epsilon, \epsilon]$, for a suitably small ϵ. Let r be a real number such that $\bar{q} + 2r\bar{B} \subset U$. By standard arguments we can construct a Lipschitz map $\tilde{u} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that $u = \tilde{u}$ on $q + r\bar{B} \subset U$ and $\tilde{u}(q) = 0$ for all $q \in \mathbb{R}^n \backslash (\bar{q} + 2r\bar{B} \subset U)$. Let us fix a non-negative, real-valued, function φ on $\mathbb{R}^n$, such that $\varphi \in C^\infty$, $\int_{\mathbb{R}^n} \varphi(q)\,dq = 1$ and $\varphi(x) = 0$ whenever $\|q\| > 1$. For every $j = 1, \dots, n-k$ and any $\rho > 0$, let us consider the *ρ-regularization* of $\tilde{u}^j$ as the function $\tilde{u}_\rho : \mathbb{R}^n \to \mathbb{R}^{n-k}$ obtained by setting, for every $q \in \mathbb{R}^n$,

$$\tilde{u}_\rho^j(q) = \int_{\mathbb{R}^n} \varphi(h)\tilde{u}^j(q + \rho h)\,dh. \tag{12}$$

It is well known that $\tilde{u}_\rho$ is function of class C^∞ and that, for all $q \in \mathbb{R}^n$,

$$D\tilde{u}_\rho^j(q) = \int_{\mathbb{R}^n} \varphi(h) D\tilde{u}^j(q + \rho h)\,dh.$$

Let $\bar{q} \in \mathbb{R}^n$ and let $\epsilon > 0$ be such that $t \to \phi_t^{f_i}(\bar{q})$ turns out to be defined on $[-\epsilon, \epsilon]$. Then, as soon as $\rho \in [0, 1]$ there exists a compact subset $K \subset \mathbb{R}^n$ such that, for every $t \in [\epsilon, \epsilon]$ one has

$$\begin{aligned}
&\tilde{u}_\rho^j \circ \phi_t^{f_i}(\bar{q}) - \tilde{u}_\rho^j(\bar{q}) \\
&\quad = \int_0^t \frac{d}{d\tau}\big(\tilde{u}_\rho^j \circ \phi_\tau^{f_i}(\bar{q})\big)\,d\tau = \int_0^t \big\langle D\big(\tilde{u}_\rho^j \circ \phi_\tau^{f_i}(\bar{q})\big), f_i\big(\phi_\tau^{f_i}(\bar{q})\big)\big\rangle\,d\tau \\
&\quad = \int_0^t \int_K \varphi(h)\big\langle D\big(\tilde{u}^j \circ \phi_\tau^{f_i}(\bar{q} + \rho h)\big), f_i\big(\phi_\tau^{f_i}(\bar{q} + \rho h)\big)\big\rangle\,d\tau \\
&\qquad + \int_0^t \int_K \varphi(h)\big\langle D\big(\tilde{u}^j \circ \phi_\tau^{f_i}(\bar{q} + \rho h)\big), f_i\big(\phi_\tau^{f_i}(\bar{q})\big) - f_i\big(\phi_\tau^{f_i}(\bar{q} + \rho h)\big)\big\rangle\,dh\,d\tau.
\end{aligned} \tag{13}$$

Since the spatial integral inside the last integral is a convex combination of the values of the map

$$h \to \big\langle D\big(\tilde{u}^j \circ \phi_\tau^{f_i}(\bar{q} + \rho h)\big), f_i\big(\phi_\tau^{f_i}(\bar{q})\big) - f_i\big(\phi_\tau^{f_i}(\bar{q} + \rho h)\big)\big\rangle,$$

which is essentially bounded on K, uniformly for $t \in [-\epsilon, \epsilon]$, it follows that the last integral is infinitesimal when ρ tends to zero. By hypothesis the map $\langle Du^j, f^i \rangle$ is equal to zero almost everywhere. Hence

$$\int_0^t \int_K \varphi(h) \big\langle D\big(\tilde{u}^j \circ \phi_\tau^{f_i}(\bar{q}+\rho h)\big), f_i\big(\phi_\tau^{f_i}(\bar{q}+\rho h)\big)\big\rangle \, d\tau = 0,$$

for every sufficiently small ρ. Therefore the right-hand side of (13) converges to zero as ρ goes to zero. Since $\tilde{u}_\rho^j$ converges to $\tilde{u}$, uniformly on compact sets, it follows that

$$\tilde{u}^j \circ \phi_t^{f_i}(\bar{q}) - \tilde{u}^j(\bar{q}) = 0$$

for all $t \in [-\epsilon, \epsilon]$. Let us notice that $u(t) = \tilde{u}(t)$ for every $t \in [-\epsilon, \epsilon]$, provided ϵ is sufficiently small. Hence the lemma is proved. □

Conclusion of the proof of Theorem 5.3. Let us choose $j_1, j_2 = 1, \dots, k$, $\bar{q} \in \mathit{DIFF}(u) \cap \mathit{DIFF}(f_1) \cap \cdots \cap \mathit{DIFF}(f_k)$, and let u be a almost everywhere solution of (11). In particular there exists an open neighborhood U of $\bar{q}$ such that $\operatorname{span}\{du_1(q), \dots, du_{n-k}(q)\}$ has dimension $n-k$ at almost every $q \in U$. Let us define the map $\xi : \mathbb{R} \to \mathbb{R}$ by letting

$$\xi(0) \doteq 0, \qquad \xi(t) \doteq \frac{t}{\sqrt{|t|}} \quad \forall t \neq 0.$$

Furthermore, for sufficiently small t, let us set

$$F(t) \doteq \phi_{-\xi(t)}^{f_{j_2}} \circ \phi_{-\xi(t)}^{f_{j_1}} \circ \phi_{\xi(t)}^{f_{j_2}} \circ \phi_{\xi(t)}^{f_{j_1}}.$$

Finally, let us consider the function

$$t \to u \circ F(t).$$

Clearly there is a positive number ϵ such that the map $F(t)$ is well defined for every $t \in [\epsilon, \epsilon]$ and takes values in U. Moreover, by the previous lemma, the function $u \circ F$ is constant on $[\epsilon, \epsilon]$. In addition, in [13] it has been proved—as a special case of an asymptotic formula valid for *semi-differentiable* vector fields—that F is differentiable at $t = 0$ and

$$\frac{dF}{dt}(0) = [f_{j_1}, f_{j_2}](\bar{q}).$$

Therefore, for every $i = 1, \dots, n-k$,

$$0 = \frac{d(u_i \circ F)}{dt}(0) = \big\langle Du_i(\bar{q}), [f_{j_1}, f_{j_2}](\bar{q})\big\rangle.$$

Hence

$$[f_{j_1}, f_{j_2}](\bar{q}) \in \bigcap_{i=1,\dots,n-k} \operatorname{Ker}\big(du_i(\bar{q})\big) = \operatorname{span}\big\{f_1(\bar{q}), \dots, f_k(\bar{q})\big\}.$$

Since the set $DIFF(u) \cap DIFF(f_1) \cap \cdots \cap DIFF(f_k)$ has full measure, this concludes the proof. $\square$

5.2. Cauchy problems

In order to formulate a "Cauchy problem" for system (11), let us consider a $(n-k)$-dimensional submanifold $N \subset M$ of class C^1. Let us assume that N is *non-characteristic* for (11). This means that for any $q \in N$ and any $i = 1, \dots, k$, the vector $f_i(q) \in T_q M$ is not tangent to N.

An *initial value* on N is a locally Lipschitz map $u_0 : N \to \mathbb{R}^{n-k}$.

Definition 5.5. A pair (U, u) is called a *local solution of* (11) *with initial value* u_0 provided $U \subset M$ is an open subset containing N and u is a map from U into $\mathbb{R}^{n-k}$ which solves (11) (almost everywhere on U) and verifies the *initial value condition*

$$u(q) = u_0(q) \quad \forall q \in N. \tag{14}$$

Theorem 5.6. *There exists a local solution (U, u) of* (11) *with initial value u_0. Moreover, if $(\tilde{U}, \tilde{u})$ is any such solution, then*

$$u(q) = \tilde{u}(q) \quad \forall q \in U \cap \tilde{U}.$$

Finally, if (U, u) is a local solution with initial condition u_0, for every $\bar{q} \in U$, the level set

$$L_{u(\bar{q})} \doteq \{q \in U \mid u(q) = u(\bar{q})\}$$

is the union of sub-manifolds of U of class $C^{1,1}$. In particular, if u_0 is injective then every such level set is a submanifold of class $C^{1,1}$.

Proof. Let us consider the map

$$F(\eta, t_1, \dots, t_k) \doteq \phi_{f_1}^{t_1} \circ \cdots \circ \phi_{f_k}^{t_k}(\eta).$$

By the Inverse Map Theorem for Lipschitz functions, there exists an open subset $A \subset N \times \mathbb{R}^k$ verifying $N \times \{0\} \subseteq A$ and such that F is a Lipschitz, invertible, map from A onto the (open) subset $U = F(A) \subset M$. Moreover, the inverse of $F_{|A}$, which is here denoted by F^{-1}, is Lipschitz. Of course one has $N \in U$, since for every $\eta \in N$, one has $F(\eta, 0, \dots, 0) = \eta$. Let π denote the canonical projection of $N \times \mathbb{R}^k$ onto N, and let us define the map

$$u : U \to \mathbb{R}^{n-k}, \qquad u(q) \doteq u_0 \circ \pi \circ F^{-1}(q).$$

The map u is locally Lipschitz and verifies the initial condition (14). Moreover, for every $q \in N$, $i = 1, \dots, k$, and any s such that $\phi_s^{f_i}(q) \in U$, one has $u \circ \phi_s^{f_i}(q) = u(q)$, i.e., u is constant along the trajectories of the vector fields $f_1, \dots, f_k$. In particular, for every $q \in DIFF(u) \cap DIFF(f_1) \cap \cdots \cap DIFF(f_k)$, $i = 1, \dots, n-k$, $j = 1, \dots, k$, one has

$$\frac{d}{ds}\bigg|_{s=0} \big(u_i\big(\phi_s^{f_j}(q)\big)\big) = \big\langle Du_i(q), f_j\big\rangle = 0.$$

Since $DIFF(u) \cap DIFF(f_1) \cap \cdots \cap DIFF(f_k)$ has full measure, it follows that (U, u) is a local solution of (11) with initial condition equal to u_0. Since by Lemma 5.4 every such solution is constant along the trajectories of the vector fields $f_1, \ldots, f_k$, the uniqueness stated in thesis of the theorem follows.

Still by Lemma 5.4, one has

$$L_{u(\bar{q})} = \bigcup_{\{q \in N \mid u(q) = u(\bar{q})\}} S_q,$$

where

$$S_q \doteq \left\{ F\left(\pi \circ F^{-1}(q), t\right) \;\middle|\; \left(\pi \circ F^{-1}(q), t\right) \in A \right\}.$$

On the other hand, by Proposition 1.3, each S_q is a submanifold of class $C^{1,1}$. Hence the final statement is proved both in the general case and in the case when u_0 is injective. Actually, in this event, one has

$$L_{u(q)} = S_q,$$

for all $q \in U$. $\square$

Remark 5.7. Even in the case when u_0 is injective and, say, of class C^2, there is no hope of improving the general result about the regularity of the a solution to the Cauchy problem (11)–(14). In fact, *though the level sets are sub-manifolds of class* $C^{1,1}$, this solution is just locally Lipschitz, as shown in the following example. Let us set

$$k = 1, \qquad M = \mathbb{R}^2, \qquad N = \{0\} \times \mathbb{R}, \qquad f(x, y) \doteq \begin{pmatrix} 1 \\ \varphi(x, y) \end{pmatrix}, \qquad u_0(y) = y,$$

where

$$\varphi(x, y) = \begin{cases} y & \text{if } y \geqslant 0, \\ \frac{2xy}{1+x^2} & \text{if } y < 0. \end{cases}$$

Clearly f is locally Lipschitz, but it is not differentiable. Moreover u_0 is of class C^∞. The unique solution to the Cauchy problem

$$\big(\langle Du, f\rangle =\big) \quad \frac{\partial u}{\partial x} + \varphi(x, y)\frac{\partial u}{\partial y} = 0, \qquad u(0, y) = u_0(y)$$

is the function u defined by

$$u(x, y) = \begin{cases} ye^{-x} & \text{if } y \geqslant 0, \\ \frac{y}{1+x^2} & \text{if } y < 0, \end{cases}$$

which is not differentiable at $(x, 0)$ as soon as $x \neq 0$. Notice, however, that the level sets are submanifolds of class C^∞. Indeed, if $c \geqslant 0$ the level set L_c is the graph of the map $y = ce^x$, while, as soon as $c < 0$, L_c coincides with the graph of $y = c(1 + x^2)$.

6. Proof of Theorem 4.11

The proof of Theorem 4.11 will be based on a combination of the properties of set-valued Lie brackets, Lie derivatives, and exterior derivatives, and the fact—established in [13]—that the orbits[18] of commutative vector fields are $C^{1,1}$ submanifolds.

6.0.1. Some preliminaries

Let n, k be integers such that $1 \leqslant k \leqslant n$, and let (U, z) be a chart on M. Let us consider k locally Lipschitz vector fields $f_1, \dots, f_k$ on U such that, for every $q \in U$, the vectors $f_1(q), \dots, f_k(q)$ are linearly independent.

For $j = 1, \dots, k$ and $l = 1, \dots, n$, let $f_j^l : U \to \mathbb{R}$ denote the lth component of f_j with respect to the coordinates z. That is, f_j^l is (the locally Lipschitz map) defined by

$$f_j^l(q) \doteq \langle dz^l, f_j(q) \rangle \quad \forall q \in U.$$

Then

$$F(q) = \begin{pmatrix} f_1^1 & \cdots & f_k^1 \\ \vdots & & \vdots \\ f_1^k & \cdots & f_k^k \\ \vdots & & \vdots \\ f_1^n & \cdots & f_k^n \end{pmatrix}$$

is a $(n \times k)$-matrix valued function on U of class $C^{0,1}$. Moreover, for every $q \in U$ the rank of $F(q)$ is equal to k.

Hence, up to reordering the coordinates $\{z^r\}$—and, if necessary, taking a smaller domain U—we can assume that the $(k \times k)$-submatrix

$$A(q) = \begin{pmatrix} f_1^1 & \cdots & f_k^1 \\ \vdots & & \vdots \\ f_1^k & \cdots & f_k^k \end{pmatrix} \tag{15}$$

is non-singular, for every $q \in U$.

Let $GL(k)$ denote the space of $k \times k$, non-singular, real matrixes. If, for every $q \in U$, we use $(\beta_j^l(q))_{j,l=1,\dots,k}$ to denote the inverse of the matrix $A(q)$, we obtain

Lemma 6.1. *The map $q \mapsto (\beta_j^l(q))_{j=1,\dots,k}^{l=1,\dots,k}$ from U into $GL(k)$ is locally Lipschitz. Moreover, letting*

$$g_j(q) \doteq \sum_{l=1}^{k} \beta_j^l(q) f_l(q), \quad \forall q \in U, \ \forall j + 1, \dots, k, \tag{16}$$

[18] The *orbit* through a point $\bar{q}$ of a family $\mathcal{F}$ of vector fields is the subset which can be reached from $\bar{q}$ by means of finite concatenations of trajectories of the fields belonging to $\mathcal{F}$.

the set $\{g_1, \ldots, g_k\}$ *turns out to be a local frame of* Δ *verifying*

$$\langle dz^l, g_j(q)\rangle = \delta^l_j \tag{17}$$

for all $j, l = 1, \ldots, k$ *and* $q \in U$*, where* δ^l_j *is the Kronecker symbol.*

This allows us to show that:

Lemma 6.2. *Let* f *be a vector field belonging to a Lipschitz distribution* Δ*. Then* f *is locally Lipschitz if and only if, for every coordinate chart* (U, z) *and every local frame* $\{f_1, \ldots, f_k\}$ *defined on* U*, there are locally Lipschitz maps*

$$\alpha^1, \ldots, \alpha^k : U \to \mathbb{R}$$

such that

$$f = \sum_{j=1}^{k} \alpha^j f_j. \tag{18}$$

Proof. Let f, (U, z), and $\{f_1, \ldots, f_k\}$ as in the above statement. By standard arguments, we can also assume that (U, z) is such that the matrix $A(q)$ defined in (15) is non-singular for every $q \in U$.[19] Moreover, let $(\beta^l_j)^{l=1,\ldots,k}_{j=1,\ldots,k}$ and $\{g_1, \ldots, g_k\}$ be as in Lemma 6.1. In particular, for every $q \in U$, there exist real numbers $\gamma^1(q), \ldots, \gamma^k(q)$ such that

$$f(q) = \sum_{l=1}^{k} \gamma^l(q) g_l(q).$$

Since

$$\langle dz^l, f(q)\rangle = \gamma^l(q), \quad \forall l = 1, \ldots, k,\ q \in U,$$

the maps $\gamma^1, \ldots, \gamma^k$ turn out to be locally Lipschitz continuous. Therefore, since for every $q \in U$ one has

$$f(q) = \sum_{l=1}^{k} \gamma^l(q) g_l(q) = \sum_{j,l=1}^{k} \gamma^l(q) \beta^j_l(q) f_j(q),$$

the thesis follows, with $\alpha_j = \sum_{l=1}^{k} \gamma^l \beta^j_l$, for all $j = 1, \ldots, k$. □

[19] The more general case may be recovered in a standard way, by means of partition of unity.

6.0.2. Proof of Theorem 4.11

As pointed out in Remark 4.12, the fact that the commutativity condition (VI) implies condition (I)—that is, $C^{1,1}$-integrability—was, to a great extent, established in [13]. Let us just point out that the main argument consists in the fact that, under hypothesis (VI), the map

$$\Phi(q, t_1, \dots, t_k) \doteq \phi_{g_1}^{t^1} \circ \dots \circ \phi_{g_k}^{t^k}(q)$$

verifies

$$\frac{\partial \Phi}{\partial t_j} = g_j \circ \Phi. \tag{19}$$

This is a consequence of the commutativity result established in [13]. Notice that, for every $q \in U$, $\Phi(q, \cdot)$ can be defined on a ball B_q centered in the origin of $\mathbb{R}^k$ and having a suitable radius r_q. Moreover, (19) implies, in particular, that, for every $q \in U$, (i) the map $\Phi(q, \cdot)$ is of class $C^{1,1}$, and (ii) the tangent space to the submanifold $\Phi(q, B_q)$[20] at a point q' coincides with $\Delta_{q'}$.

Let us show that (I) implies (III). Let Δ be a locally Lipschitz, completely $C^{1,1}$-integrable, distribution on M. Let f and g be locally Lipschitz vector fields belonging to Δ, and let $\bar{q} \in \mathit{DIFF}(f) \cap \mathit{DIFF}(g)$. Let $(U, (x, y))$ be a coordinate chart near $\bar{q}$ as in Definition 4.9. In particular the vector fields $h_j \doteq \frac{\partial}{\partial x_j}$, $j = 1, \dots, k$, form a local frame defined on U. So, by Lemma 6.2, there exist locally Lipschitz functions $a^1, \dots, a^k, b^1, \dots, b^k : U \to \mathbb{R}$ such that

$$f(\bar{q}) = \sum_{j=1}^{k} a^j \frac{\partial}{\partial x_j}, \qquad g(\bar{q}) = \sum_{j=1}^{k} b^j \frac{\partial}{\partial x_j}$$

for all $\bar{q} \in U$. Therefore

$$[f, g](\bar{q}) = \sum_{j,l=1}^{k} \left(\frac{\partial b^j \circ x^{-1}}{\partial x^l}\big(x(\bar{q})\big) a^l(\bar{q}) - \frac{\partial a^j \circ x^{-1}}{\partial x^l}\big(x(\bar{q})\big) b^l(\bar{q}) \right) \frac{\partial}{\partial x^j},$$

so, in particular, $[f, g](\bar{q}) \in \Delta_{\bar{q}}$. Since $\mathit{DIFF}(f) \cap \mathit{DIFF}(g)$ is a full subset of M, (III) is proved.

The fact that (II) is equivalent to (III) is a straightforward consequence of Proposition 2.13.

Moreover, it is clear that (III) implies (V). Let us prove that (V) implies (III). Let f, g be vector fields belonging to Δ. Clearly, it is sufficient to show that for every open subset $U \subset M$ belonging to an open covering of M and for almost every $q \in \mathit{DIFF}(f) \cap \mathit{DIFF}(g) \cap U$, one has

$$[f, g](q) \in \Delta_q.$$

Let (U, z), $\{f_1, \dots, f_k\}$ be a coordinate chart and a local frame of Δ on U, respectively. By Lemma 6.2 there exist locally Lipschitz real maps $\alpha^1, \dots, \alpha^k, \beta^1, \dots, \beta^k$ defined on U such that

$$f = \sum_{j=1}^{k} \alpha^j f_j, \qquad g = \sum_{j=1}^{k} \beta^j f_j.$$

[20] In view of Proposition 1.3, this submanifold is of class $C^{1,1}$.

For every $q \in \bigcap_{j=1}^{k}(DIFF(f_j) \cap DIFF(\alpha^j) \cap DIFF(\beta^j))$ one has

$$[f,g](q) = \sum_{j,l=1}^{k} \Big(\big(\alpha^j(q)\beta^l(q)\big) \cdot [f_j, f_l](q) + \big(\alpha^l(q)\langle d\beta^j(q), f_l(q)\rangle\big) \cdot f_j(q) - \big(\beta^j(q)\langle d\alpha^l(q), f_j(q)\rangle\big) \cdot f_l(q)\Big).$$

By (V) we have $[f_j, f_l](q) \in \Delta_q$, for all $i, j = 1, \dots, k$. Therefore, we obtain

$$[f,g](q) \in \Delta_q,$$

which yields the thesis, since $\bigcap_{j=1}^{k}(DIFF(f_j) \cap DIFF(\alpha^j) \cap DIFF(\beta^j))$ is a full subset of U.

Let us prove that (V) implies (VII). Let $\bar{q}$, (U, z), and $\{f_1, \dots, f_k\}$ be, respectively, a point of M, a coordinate chart near $\bar{q}$, and a local frame for Δ defined on U.[21] By Lemma 6.1 there exists a locally Lipschitz map $q \mapsto (\beta_j^l(q))_{j,l=1,\dots,k}$ from U into $GL(k)$ such that, letting

$$g_j \doteq \sum_{l=1}^{k} \beta_j^l f_l,$$

one has that the set $\{g_1, \dots, g_k\}$ is a local frame of Δ verifying

$$\langle dz^l, g_j(q)\rangle = \delta_j^l \tag{20}$$

for all $j, l = 1, \dots, k$ and $q \in U$. In particular, for every

$$q \in U' \doteq \bigcap_{j,l,r=1}^{k} \big(DIFF(f_j) \cap DIFF\big(\beta_r^l\big)\big)$$

one has

$$\langle dz^l, [g_j, g_r](q)\rangle = 0 \quad \forall l, j, r = 1, \dots, k. \tag{21}$$

On the other hand, since $\{f_1, \dots, f_k\}$ is almost everywhere involutive, there exists a full subset $U'' \subseteq U'$ such that, for every $q \in U''$ one has

$$\begin{aligned}
[g_j, g_r](q) &= \left[\sum_{l=1}^{k} \beta_j^l f_l, \sum_{m=1}^{k} \beta_r^m f_m\right] \\
&= \sum_{l,j,m,r=1}^{k} \big(\beta_j^l(q)\beta_r^m(q) \cdot [f_l, f_m](q) + \beta_r^m(q)\langle d\beta_j^l(q), f_m(q)\rangle \cdot f_l(q) \\
&\quad - \beta_j^l(q)\langle d\beta_r^m(q), f_l(q)\rangle \cdot f_m(q)\big) \in \operatorname{span}\big\{f_1(q), \dots, f_k(q)\big\} \\
&= \operatorname{span}\big\{g_1(q), \dots, g_k(q)\big\}.
\end{aligned} \tag{22}$$

[21] If necessary, U have to be reduced to a smaller neighborhood of $\bar{q}$.

By (20)–(22), we get

$$[g_j, g_r](q) = 0 \quad \forall q \in U''.$$

Since U'' is a full subset of U, (VII) is proved.

In order to prove that (VII) implies (V) it is sufficient to write the elements of a given local frame $\{f_1, \dots, f_k\}$ as linear combinations of the elements of a frame $\{g_1, \dots, g_k\}$ whose elements commute pairwise. Indeed, by Lemma 6.2, the corresponding coefficients are locally Lipschitz function, so (V) follows by direct computation.

The equivalence of (IV) and (V) is a consequence of Proposition 2.13. The latter also implies that (VI) is equivalent to (VII).

In order to prove the equivalence of conditions (VIII), (IX), (X), and (XI), let us begin by recalling a basic result on the exterior algebra of a finite-dimensional vector space.

Lemma 6.3. *Let h, k, n be non-negative integers such that $1 \leqslant h \leqslant k \leqslant n$. Let W and $V \subseteq W$ be a n-dimensional, real, vector space, and a k-dimensional subspace, respectively. Let $V^\dagger$ denote the annihilator of V—i.e. the subspace of W^* made of all forms ω such that $V \subseteq \operatorname{Ker}\omega$—, and let $\omega^1, \dots, \omega^{n-k}$ be a basis for $V^\dagger$. Let us choose $\theta \in \Lambda^h$, where Λ^h denotes the vector space of alternating h-linear forms on W. Then the following conditions are equivalent*:

(i) $\theta(v_1, \dots, v_h) = 0$ *for all h-tuples* $(v_1, \dots, v_h) \in V^h$.
(ii) *There exist forms $\theta_1, \dots, \theta_{n-k} \in \Lambda^{h-1}$ such that*

$$\theta = \theta_1 \wedge \omega^1 + \cdots + \theta_{n-k} \wedge \omega^{n-k}.$$

(iii) *If we set $\gamma \doteq \omega^1 \wedge \cdots \wedge \omega^{n-k}$, then*

$$\theta \wedge \gamma = 0.$$

The equivalence of conditions (IX) and (XI) is a straightforward consequence of Lemma 6.3. Indeed, for $\alpha = 1, \dots, n-k$ and $q \in \bigcap_{\beta=1}^{n-k} (DIFF(\omega^\beta) \cap U)$, it is sufficient to set $\theta \doteq d\omega^\alpha(q)$ to obtain that (9) holds true if and only if (10) is verified.

The equivalence of conditions (IX) and (VIII) is a consequence of Proposition 2.13.[22]

In order to establish that (XI) is equivalent to (X), let us observe that obvious arguments including compactness and linearity imply that

$$\left(d\omega^\alpha \wedge \Gamma\right)_{\text{set}} = \left(d\omega^\alpha\right)_{\text{set}} \wedge \Gamma.$$

Then the thesis follows from Proposition 2.13.

Let us prove that condition (IX) is equivalent to condition (V). For this purpose, it is clearly enough to prove the following fact:

[22] Notice that the set-valued section

$$q \rightsquigarrow \Lambda_q^1(M) \wedge \omega^1(q) + \cdots + \Lambda_q^1(M) \wedge \omega^{n-k}(q)$$

is continuous and convex.

Claim A. *For every open subset $U \subseteq M$, every frame $\{f_1, \dots, f_k\}$ of Δ on U, every frame $\{w_1, \dots, \omega^{n-k}\}$ of $\Delta^\dagger$ on U, there exists a full subset U' such that for each*

$$q \in U'' \doteq \bigcap_{r=1}^{k} \bigcap_{\beta=1}^{n-k} \big(U' \cap DIFF(f_r) \cap DIFF(\omega^\beta)\big)$$

the condition

$$[f_i, f_j](q) \in \Delta_q \quad \forall i, j = 1, \dots, k, \tag{23}$$

is equivalent to

$$d\omega^\alpha(q) \in \Lambda^1_q(M) \wedge \omega^1(q) + \cdots + \Lambda^1_q(M) \wedge \omega^{n-k}(q) \quad \forall \alpha = 1, \dots, n-k. \tag{24}$$

In order to prove Claim A let us recall a basic result connecting Lie bracketing with exterior differentiation:

Lemma 6.4. *Let n, h be non-negative integers such that $1 \leqslant h \leqslant n$. Let M be a n-dimensional manifold and let ω be a locally Lipschitz h-form on M. Moreover, let $f_1, \dots, f_{h+1}$ be locally Lipschitz vector fields. Then, for each point $q \in \bigcap_{i=1}^{h+1}(DIFF(\omega) \cap DIFF(f_i))$ one has,*

$$\begin{aligned} &d\omega(f_1, \dots, f_{h+1})(q) \\ &\quad = -\sum_{i=1}^{h+1} (-1)^i \big\langle d\big(\omega(f_1, \dots, \hat{f}_i, \dots f_{h+1})\big)(q), f_i(q)\big\rangle \\ &\qquad - \sum_{1 \leqslant i < j \leqslant h+1} (-1)^{i+j} \omega\big([f_i, f_j], f_1, \dots, \hat{f}_i, \dots f_{h+1}\big)(q).^{23} \end{aligned} \tag{25}$$

In particular, for $h = 1$ one has

$$d\omega(f_1, f_2)(q) = \big\langle d\big(\omega(f_2)\big)(q), f_1(q)\big\rangle - \big\langle d\big(\omega(f_1)\big)(q), f_2(q)\big\rangle - \omega\big([f_1, f_2]\big)(q).$$

Remark 6.5. This result is usually formulated when all objects are of class C^1. However, a direct computation in a system of local coordinates shows that the above version—which concerns points of common differentiability—holds true as well.

In view of Lemma 6.4, for every $q \in U''$, $\alpha = 1, \dots, n-k$, and $i, j = 1, \dots, k$, one has

$$d\omega^\alpha(f_i, f_j)(q) = \big\langle d\big(\omega^\alpha(f_j)\big)(q), f_i(q)\big\rangle - \big\langle d\big(\omega^\alpha(f_i)\big)(q), f_j(q)\big\rangle - \omega^\alpha\big([f_i, f_j]\big)(q). \tag{26}$$

Since $\omega^\alpha(f_i) = \omega^\alpha(f_j) = 0$ identically on U, this yields

$$d\omega^\alpha(f_i, f_j)(q) = -\omega^\alpha\big([f_i, f_j]\big)(q). \tag{27}$$

[23] Here, the hat over an argument means that that argument is omitted.

Hence (23) holds if and only if

$$d\omega^\alpha(f_i, f_j)(q) = 0.$$

Since this holds true for all $\alpha = 1, \dots, n-k$ and $i, j = 1, \dots, k$, by Lemma 6.3 we obtain Claim A. This ends the proof that (IX) implies (V).

As a consequence of Proposition 2.13, conditions (XII) and (XIII) are equivalent. In order to conclude the proof of Theorem 4.11 it is sufficient to establish that (XIII) is equivalent to condition (IX). In turn, for this purpose it is clearly enough to prove the following claim:

Claim B. *Let $U \subseteq M$, $\{f_1, \dots, f_k\}$, and $\{\omega^1, \dots, \omega^{n-k}\}$ be, respectively, an open subset, a frame of Δ on U, and a frame of $\Delta^\dagger$ on U. Then there exists a full subset $U' \subseteq U$ such that, for every $i, j = 1, \dots, k$, $\alpha = 1, \dots, n-k$, and every*

$$q \in U'' \doteq \bigcap_{r=1}^{k} \bigcap_{\beta=1}^{n-k} \big(U' \cap DIFF(f_r) \cap DIFF\big(\omega^\beta\big)\big),$$

the condition

$$d\omega^\alpha(q) \in \Lambda^1_q(M) \wedge \omega^1(q) + \cdots + \Lambda^1_q(M) \wedge \omega^{n-k}(q) \tag{28}$$

is equivalent to

$$L_{f_i}\omega^\alpha(q) \subseteq \Delta^\dagger(q). \tag{29}$$

Similarly to what is usually done in the smooth case, we will exploit Lemma 6.7 below, which establishes a relation between Lie derivatives of forms and exterior derivatives. In order to state this result, let us recall the notion of *interior product* of a vector field and a differential form.

Definition 6.6. Let n, h be integers such that $h \leqslant n$, and let M be a n-dimensional manifold. Let $M' \subset M$ be any subset, and let θ and g be a h-form and a vector field, respectively, defined on M'. The *interior product* of g and θ is defined as the $(h-1)$-form on M' obtained by setting, for any $(h-1)$-tuple $(X_1, \dots, X_{h-1})$ of vector fields (defined on M'),

$$i_g\theta(X_1, \dots, X_{h-1})(q) \doteq h\theta(g, X_1, \dots, X_{h-1})(q), \quad \forall q \in M'.$$

Lemma 6.7. *Let n, h, M, M', θ, and g be as in Definition 6.6. Let us assume that $M' = M$ and that θ and g are locally Lipschitz. Then, for every $q \in DIFF(g) \cap DIFF(\theta)$, one has*[24]

$$L_g\theta(q) = i_g d\theta(q) + d(i_g\theta)(q).$$

In particular, if $h = 1$, one has, for every $q \in DIFF(g) \cap DIFF(\theta)$ and $v \in T_qM$,

$$\big\langle L_g\theta(q), v\big\rangle = d\theta\big(g(q), v\big) + \big\langle d\langle \theta, g\rangle(q), v\big\rangle.$$

[24] Like in the case of Lemma 6.4, the result stated in Lemma 6.7 is non-standard, for it concerns nonsmooth objects. However no extra proof is required to establish it, as it can be checked by direct computation in a system of coordinates.

In order to prove Claim B, it is now sufficient to identify θ and g with ω^α and f_j, respectively. Since f_j belongs to Δ and ω^α belongs to $\Delta^\dagger$, this yields

$$\big\langle L_{f_j}\omega^\alpha(q), v\big\rangle = d\omega^\alpha\big(f_j(q), v\big)$$

for all $q \in U''$. Therefore (29) holds for all $j = 1, \dots, k$ if and only if, for any pair $(w, v) \in (\Delta_q)^2$, one has

$$d\omega^\alpha(w, v) = 0.$$

By Lemma 6.3 this is equivalent to (28), so Claim B is proved. The proof of Theorem 4.11 is concluded.

Acknowledgment

The author wishes to thank his former student Daniel Gessutti, for interesting conversations on the subject of this paper.

References

[1] J. Aubin, A. Cellina, Differential Inclusions, Springer-Verlag, Berlin, 1984.
[2] J. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, MA, 1990.
[3] C. Calcaterra, Foliating metric spaces: A generalization of Frobenius' Theorem, arXiv: math.MG/0608416v1.
[4] W.L. Chow, Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung, Math. Ann. 117 (1940) 98–105.
[5] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[6] A. Clebsh, Über die simultane Integration linearer partieller Differentialgleichungen, J. Reine Angew. Math. 65 (1866) 257–268.
[7] C. Deahana, Über die Bedingungen der Integrabilität linearer Differentialgleichungen erster Ordnung zwischen einer beliebigen Anzahl veränderlicher Größen, J. Reine Angew. Math. 20 (1840) 340–349.
[8] F.G. Frobenius, Über das Pfaffsche Problem, J. Reine Angew. Math. 82 (1877) 267–282.
[9] P. Hartman, Ordinary Differential Equations, second ed., Birkhäuser, Basel, 1982.
[10] V. Jurdjevic, Geometric Control Theory, Cambridge Stud. Adv. Math., vol. 51, Cambridge Univ. Press, Cambridge, 1997.
[11] W. Michor, Topics in differential geometry, http://www.mat.univie.ac.at/~michor/~dgbook.ps.
[12] F. Rampazzo, H.J. Sussmann, Set-valued differentials and a nonsmooth version of Chow's theorem, in: Proc. 40th IEEE Conference on Decision and Control, vol. 3, Orlando, FL, December 2001, IEEE, New York, 2001, pp. 2613–2618.
[13] F. Rampazzo, H.J. Sussmann, Commutators of flow maps of nonsmooth vector fields, J. Differential Equations 232 (1) (2007) 134–175.
[14] B.L. Reinhart, Differential Geometry of Foliations, Springer-Verlag, Berlin, 1983.
[15] S. Simić, Lipschitz distributions and Anosov flows, Proc. Amer. Math. Soc. 124 (1996) 1869–1887.