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A density approach to Hamilton-Jacobi equations with t-measurable Hamiltonians

Ariela BRIANI
Dipartimento di Matematica
Università di Pisa
via Buonarroti 2
56127 Pisa, Italy
e-mail: briani@dm.unipi.it

Franco RAMPAZZO
Dipartimento di Matematica
Pura ed Applicata
Università di Padova
Via Belzoni 7
Padova, Italy
e-mail: rampazzo@math.unipd.it

Abstract. In 1985 H. Ishii [Is85] proposed a generalization of the notion of (continuous) viscosity solution for an Hamilton-Jacobi equation with a *t-measurable* Hamiltonian---that is, a Hamiltonian which is measurable in time and continuous in the other variables. This notion turned out to agree with natural applications, like Control and Differential Games Theory. Since then, several improvements have been achieved for the standard situation when the Hamiltonian is continuous. It is someway an accepted general idea that parallel improvements are likely for t-measurable Hamiltonians as well, though such a job might appear a bit tedious because of the necessarily involved technicalities.

In this paper we show that Ishii's definition of viscosity solution coincides with the one which would arise by extending by density *the standard definition.* Namely, we regard a t-measurable Hamiltonian H as an element of the closure (for suitable topologies) of a class of continuous Hamiltonians. On the other hand, we show that the set of Ishii's (sub-, super-) solutions for H is nothing but the limit set of the (sub-, super-) solutions corresponding to continuous Hamiltonians approaching H. This put us in the condition of establishing *comparison*, *existence*, and *regularity* results by deriving them from the analogous results for the case of continuous Hamiltonians.

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1 Introduction

After the introduction of the concept of continuous viscosity solution for a first order partial differential equation, the problem of giving a notion of viscosity solution in the "discontinuous case"—meaning a situation where both the Hamiltonian and the solution are allowed to be discontinuous—was systematically investigated since the publication of H. Ishii's papers [Is85] in 1985 and [Is87] in 1987 (see e.g. [Ba92] and [BCD97] for introductory material on the subject).

In the present paper we consider the particular case when the equation has the form

$$\frac{\partial u}{\partial t} + H(t, x, u, \nabla u) = 0 \text{ in }]0, T[\times \mathbb{R}^N, \tag{1.1}$$

with the Hamiltonian H measurable in t and continuous in the remaining variables. We shall refer to this situation as the *t-measurable case*. Since the pioneering paper by H. Ishii it was clear that, due to the special position of the t-derivative in the equation, this case deserves an ad hoc treatment. In fact, H. Ishii himself studied this case separately from other situations involving discontinuities. Incidentally, let us notice that it is natural to assume that solutions to (1.1) are *continuous* (in both variables), for, roughly speaking, one has to integrate H in time.

Despite its complicated appearance, the definition proposed by H. Ishii has been shown to be quite satisfactory. To begin with, it is rapidly seen that when H is continuous this definition reduces to the standard one. Moreover, it agrees with some classical applications: for example, the upper and lower value of a zero-sum differential game involving a dynamics and a Lagrangian measurable in time are solutions (according to Ishii's definition) of the associated Isaac's equations.

What is perhaps lacking is a set of results showing that Ishii's solution *is the natural extension* of the usual one (in a sense that will be clarified below). Actually, this is among the aims of the present paper.

Let us remark that H. Ishii's paper [Is85] contained results which at the time were almost as general as those devoted to the "continuous case". In 1987, P.L. Lions and B. Perthame ([LP87]) provided three equivalent formulations of the notion of solution given by H. Ishii. More or less at the same time, relying on the so-called blow up method, N. Barron and R. Jensen (see [BJ87]) introduced a new concept of (Lipschitz continuous) solution. However, the question of the equivalence with H. Ishii's definitions was left open by the authors. More recently, a definition of solution for t-measurable Hamiltonians for a special class of Hamilton-Jacobi equations has been proposed by P. Cardaliaguet and S. Plaskacz

([CP00]). Under some regularity assumptions this definition turns out to be equivalent to Ishii's one, the equivalence being a result of uniqueness and representation properties.

After the quoted papers, (to our knowledge) not much attention has been paid to the question, at least in the case of first order equations (see e.g. [Nu90], [Nu92] for extensions to second order equations). In the meantime results concerning the continuous case have been more and more sharpened, and no extensions to the t-measurable case of such improvements have been provided. Notice that, on one hand, the task of actually performing such extensions would require a quite tedious technical effort. On the other hand, these extensions would be quite reasonable from a theoretical point of view—notice, for instance, that in the trivial case when H is independent of x and ∇u, (1.1) reduces to an o.d.e. with Caratheodory type conditions. Furthermore, the concern about t-measurable Hamiltonians is motivated by quite natural applications (e.g. to optimal control and differential game theories).

These arguments have pushed us to look for rigorous contents to a general principle which, in a vague form, reads as follows:

If a (*comparison, existence, regularity, ...*) *result holds true when the Hamiltonian is continuous, it is* still *valid when one weakens the hypothesis of continuity in t to mere measurability.*

Such a fact would allow one to avoid the direct exploitation of the involved notion of solution for the t-measurable case: indeed it would be enough to establish a certain result in the continuous case, in that the extension to the general case would be guaranteed by the criterion stated above. It must be noticed that this would be quite similar to what happens in the theory of Caratheodory solutions of ordinary differential equations.

Our leading idea consists in proving semi-continuity properties for the two (multivalued) maps S^- and S^+ which, to each Hamiltonian, associate the corresponding set of subsolutions and supersolutions, respectively (see Section 3). Eventually we wish to exploit these properties by regarding a given t-measurable Hamiltonian as an element in the L^1-closure of a suitable class of *continuous* Hamiltonians. Let us remark that the so-called *stability results* establish nothing but upper semicontinuity properties for these maps. However, for our purposes it is essential to establish also lower semicontinuity properties of the multivalued maps S^- and S^+. Here these properties are referred to as *approximability* properties. This means that if u is a subsolution [resp. supersolution] corresponding to H and H_n is a sequence of Hamiltonians converging to H, then there exists a sequence of corresponding subsolutions [resp. supersolutions] u_n which converge to u. Actually, these are the contents of Theorem 3.3 below.

Relying on such properties of S^- and S^+, in Section 4 we derive a comparison result (in particular, a uniqueness result) for t-measurable Hamiltonians which, loosely speaking, holds as soon as the analogous result is valid for the approximating continuous Hamiltonians (see Theorem 4.1). Successively, the exploitation of a local comparison result for continuous Hamiltonians allows us to deal with cases which require weaker assumptions in the approximation of the given Hamiltonian H. Moreover, in Subsection 4.1 we prove a result on the approximation of t-measurable Hamiltonians. In turn, this implies a comparison result (see Theorem 4.10) which *does not* involve an explicit assumption on how the Hamiltonian has to be approximated.

In Section 5, we prove uniform continuity properties for the Solution Map, namely the univalued map that to each element H in a suitable class of continuous Hamiltonians associates the (unique) viscosity solution (for fixed initial data). This allows us to extend (by density) the Solution Map to *t-measurable* Hamiltonians. Notice that, in view of the stability property, the images of these Hamiltonians via the (so extended) Solution Map coincide with the Ishii's type solutions. In particular, this is a way to prove existence results starting by analogous results for Hamiltonians approaching a given (t-measurable) one.

Notation. For a given normed space, we will denote the open ball of center x and radius δ by $B(x,\delta)$. $L^1(0,T)$ will denote the usual Lebesgue space, while, for every compact subset K in an Euclidean space $\mathbb{R}^q$, $L^1(0,T;C(K))$ will be the set of L^1 functions from $[0,T]$ into the complete space $C(K)$. Moreover, for every natural number q, $L^1(0,T;C(\mathbb{R}^q))$ will denote the family of L^1 functions $t \to H(t,\cdot,\cdot,\cdot)$ that take values in the set of real continuous functions on $\mathbb{R}^q$.

2 The Cauchy problem

Let us consider the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + H(t,x,u,\nabla u) = 0 \quad \text{in }]0,T[\times\mathbb{R}^N \tag{2.1}$$

with the initial condition

$$u(0,x) = \varphi(x) \text{ in } \mathbb{R}^N. \tag{2.2}$$

Throughout this paper we shall assume the following:

(1) The function φ is continuous on $\mathbb{R}^N$, and $0 < T < \infty$.

(2) The Hamiltonian $H(t,x,u,p) : [0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is measurable in t and continuous in (x,u,p). Moreover, $H(t,0,0,0) \in L^1(0,T)$.

(3) For any $R > 0$, there exists a map $\omega_R(t,k)$, continuous, nonnegative, nondecreasing, subadditive in k and measurable in t, such that $\omega_R(t,0) = 0$ and

$$|H(t,x,s,p) - H(t,y,r,q)| \le \omega_R(t, |x-y| + |s-r| + |p-q|)$$

for all $|x|,|y|,|s|,|r|,|p|,|q| \le R$.

(4) For each $(t, x, p) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$ the function $u \to H(t, x, u, p)$ is monotone increasing.

The definition of viscosity solution in this framework was given by H. Ishii in [Is85]. We do not recall H. Ishii's definition, for we prefer to exploit the following, equivalent, definition, provided by P.L. Lions and B. Perthame in [LP87]:

Definition 2.1 Let $O \subset \mathbb{R}^N$ be an open subset and let $u(t, x)$ be a continuous function on $Q \doteq [0, T] \times O$.

$u(t, x)$ is a *viscosity subsolution* of (2.1) at $(t_0, x_0) \in Q$ if for every $\phi \in C^1(O)$ and $b \in L^1(0, T)$ such that (t_0, x_0) is a local maximum for

$$u(t, x) + \int_0^t b(s)ds - \phi(x)$$

one has

$$\lim_{\delta \downarrow 0^+} \operatorname*{ess\,inf}_{|t - t_0| < \delta} \inf \{H(t, x, s, p) - b(t) \colon |x - x_0| \le \delta,\ \ |p - \nabla\phi(x_0)| \le \delta,$$
$$|s - u(t_0, x_0)| \le \delta\} \le 0.$$

$u(t, x)$ is a *viscosity supersolution* of (2.1) at $(t_0, x_0) \in Q$ if for every $\phi \in C^1(O)$ and $b \in L^1(0, T)$ such that (t_0, x_0) is a local minimum for

$$u(t, x) + \int_0^t b(s)ds - \phi(x)$$

one has

$$\lim_{\delta \downarrow 0^+} \operatorname*{ess\,sup}_{|t - t_0| < \delta} \sup \{H(t, x, s, p) - b(t) \colon |x - x_0| \le \delta, |p - \nabla\phi(x_0)| \le \delta,$$
$$|s - u(t_0, x_0)| \le \delta\} \ge 0.$$

$u(t, x)$ is a *viscosity solution* of (2.1) at $(t_0, x_0) \in Q$ if it is both a viscosity subsolution and a viscosity supersolution of (2.1) at (t_0, x_0).

Remark 2.2 It is easy to see that when H is continuous these notions coincide with the standard ones, for which we refer to the monographs [Ba94] and [BCD97].

3 Stability and approximability of solutions

If one considers the set-valued function that maps a given Hamiltonian H into the set of the corresponding subsolutions [resp. supersolutions], two kinds of continuity issues defined below arise quite naturally.

The "stability" issue. The first continuity question is usually refereed to as the *stability* issue. It can be roughly expressed as follows:

if (H_n) *is a sequence of Hamiltonians converging to H, and* u_n *is a sequence of subsolutions* [*resp. supersolutions*] *corresponding to the* H_n *which converge to a map* u*, is that true that* u *is a subsolution* [*resp. supersolution*] *corresponding to* H*?*

The "approximability" issue. The second continuity question we are going to consider will be here referred to as the *approximability* issue. It can be roughly expressed as follows:

if (H_n) *is a sequence of Hamiltonians converging to* H*, and* u *is a subsolution* [*resp. supersolution*] *corresponding to* H*, does a sequence* u_n *of subsolutions* [*resp. supersolutions*] *corresponding to the* H_n *exist such that the maps* u_n *converge to* u*?*

Remark 3.1 In the language of set-valued analysis stability and approximability are, respectively, *upper and lower semicontinuity properties* of the functions which map a Hamiltonian H in the corresponding set of subsolutions [resp. supersolutions].

A result addressing the stability issue was proved by Ishii in [Is85] (Proposition 7.1). Let us recall it.

Theorem 3.2 (Stability) *Let* O *be an open subset of* $\mathbb{R}^N$ *and let* $Q \doteq]0,T[\times O$. *Let* H *and* H_n *be functions on* $Q \times \mathbb{R} \times \mathbb{R}^N$ *for all* $n \in N$. *Let* $u_n \in C(Q)$ *be a viscosity subsolution* [*resp. supersolution*] *of* (2.1) *in* Q *for all* $n \in N$. *Assume that* $H_n - H \to 0$ *in* $L^1(0,T;C(K))$ *for any compact* $K \subset \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ *and* $u_n(t,x) \to u(t,x)$ *uniformly for any compact subset* K *of* Q *for some* $u \in C(Q)$ *as* $n \to \infty$. *Then,* $u(t,x)$ *is a viscosity subsolution* [*resp. supersolution*] *of* (2.1) *in* Q.

On the other hand, Theorem 3.3 below states some approximability results. It concerns subsolutions, but the obvious counterpart about supersolutions is also valid.

Theorem 3.3 (Approximability) *Let* $(H_n)_{n\in\mathbb{N}}, H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *be Hamiltonians satisfying hypotheses* (2)-(3)-(4). *The following hold.*

i) *If* u *is a subsolution of* (2.1) *in* $]0,T[\times\mathbb{R}^N$*, and*

$$\lim_{n\to\infty} \int_0^T \sup_{(x,u,p)\in(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)} |H_n(t,x,u,p) - H(t,x,u,p)|dt = 0,$$

then the functions

$$u_n(t,x) \doteq u(t,x) - \int_0^t \sup_{(x,u,p)\in(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)} |H_n(s,x,u,p) - H(s,x,u,p)|ds$$

are viscosity subsolutions in $]0,T[\times\mathbb{R}^N$ *of*

$$\frac{\partial u}{\partial t}(t,x) + H_n(t,x,u,\nabla u) = 0, \tag{3.1}$$

and verify

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^N} |u_n(t,x) - u(t,x)| = 0. \tag{3.2}$$

ii) *Let* O *be an open bounded subset in* $\mathbb{R}^N$, *and let* $u : [0,T]\times O \to \mathbb{R}$ *be a subsolution of* (2.1) *in* $]0,T[\times O$, *and let* $\Lambda \geq 0$ *be a Lipschitz constant and a bound for the function* $x \mapsto u(t,x)$, *for any* $t \in [0,T]$. *Moreover, let us assume that*

$$\lim_{n\to\infty} \int_0^T \sup_{(x,u,p)\in K} |H_n(t,x,u,p) - H(t,x,u,p)|dt = 0$$

for every compact subset $K \subset (\overline{O}\times\mathbb{R}\times\mathbb{R}^N)$. *If*

$$E \doteq \{(x,u,p) \in O\times\mathbb{R}\times\mathbb{R}^N \ : \ (x,|u|,|p|) \in \overline{O}\times[0,\Lambda+1]^2\}$$

then, for n *sufficiently large, the functions*

$$u_n(t,x) \doteq u(t,x) - \int_0^t \sup_{(x,u,p)\in E} |H_n(s,x,u,p) - H(s,x,u,p)|ds$$

are viscosity subsolutions of (3.1) *in* $]0,T[\times O$ *and verify*

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\overline{O}} |u_n(t,x) - u(t,x)| = 0. \tag{3.3}$$

Remark 3.4 We omit the analogous statement for supersolutions, which can be obtained from Theorem 3.3 just by adding the integrals instead of subtracting them in the definitions of the maps u_n. Accordingly, the proof of such a statement would be completely similar to Theorem 3.3's proof.

Proof of Theorem 3.3 The proofs of the two cases are akin, so we limit ourselves to provide the details for i) only.

Proof of i). Let us set

$$k_n(t) \doteq \sup_{(x,u,p)\in(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)} |H_n(t,x,u,p) - H(t,x,u,p)|.$$

By definition of $u_n(t,x)$ we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^N} |u_n(t,x)-u(t,x)| \le \int_0^T k_n(s)ds,$$

which, by hypothesis i), implies (3.2).

Let us prove now that, for every fixed $n\in\mathbb{N}$, $u_n(t,x)$ is a viscosity subsolution of (3.1) in $]0,T[\times\mathbb{R}^N$. Let $\phi\in C^1(\mathbb{R}^N)$, $b\in L^1(0,T)$, and (t_0,x_0) be a local maximum of

$$u_n(t,x)+\int_0^t b(s)\,ds-\phi(x).$$

By the definition of $u_n(t,x)$, the point (t_0,x_0) is a local maximum of

$$u(t,x)+\int_0^t (b(s)-k_n(s))\,ds-\phi(x).$$

Since u is a subsolution of (2.1), this implies

$$\lim_{\delta\downarrow 0^+}\operatorname*{ess\,inf}_{|t-t_0|<\delta}\inf_{A_\delta}\{H(t,x,s,p)-(b(t)-k_n(t))\}\le 0, \tag{3.4}$$

where $A_\delta \doteq \{(x,s,p)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N : |x-x_0|\le\delta, |s-u(t_0,x_0)|\le\delta, |p-\nabla\phi(x_0)| \le\delta\}$. By the definition of k_n and the monotonicity of u, one has

$$\begin{aligned}
&H_n(t,x,s,p)-b(t)\\
&\quad = H(t,x,s+u(t_0,x_0)-u_n(t_0,x_0),p)-(b(t)-k_n(t))\\
&\qquad -k_n(t)+(H_n(t,x,s,p)-H(t,x,s,p))\\
&\qquad +H(t,x,s,p))-H(t,x,s+u(t_0,x_0)-u_n(t_0,x_0),p)\\
&\quad\le H(t,x,s+u(t_0,x_0)-u_n(t_0,x_0),p)-(b(t)-k_n(t)).
\end{aligned}\tag{3.5}$$

For each $n\in\mathbb{N}$ let us set

$$A_\delta^n \doteq \{(x,s,p)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N : |x-x_0|\le\delta, |s-u_n(t_0,x_0)|\le\delta, |p-\nabla\phi(x_0)|\le\delta\}.$$

Since $(x,s,p)\in A_\delta^n$ if and only if $(x,q,p)\in A_\delta$, with $q\doteq s+u(t_0,x_0)-u_n(t_0,x_0)$, we obtain

$$\inf_{A_\delta^n}\{H_n(t,x,s,p)-b(t)\}\le\inf_{A_\delta}\{H(t,x,q,p)-(b(t)-k_n(t))\}.$$

Then, by (3.4),

$$\begin{aligned}
&\lim_{\delta\downarrow 0^+}\operatorname*{ess\,inf}_{|t-t_0|<\delta}\inf_{A_\delta^n}\{H_n(t,x,s,p)-b(t)\}\\
&\quad\le\lim_{\delta\downarrow 0^+}\operatorname*{ess\,inf}_{|t-t_0|<\delta}\inf_{A_\delta}\{H(t,x,q,p)-(b(t)-k_n(t))\}\le 0,
\end{aligned}$$

that is, u_n is a subsolution.

Remark 3.5 With almost unchanged proof one can easily obtain the following similar approximability results.

Let $(H_n)_{n\in\mathbb{N}}, H : [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be Hamiltonians satisfying hypotheses (2)-(3)-(4). The following hold.

iii) Let O be an open bounded subset in $\mathbb{R}^N$, and let $u : [0,T] \times O \to \mathbb{R}$ be a subsolution of (2.1) in $]0,T[\times O$. Moreover, let us assume that

$$\lim_{n\to\infty} \int_0^T \sup_{(x,u,p)\in K_1\times\mathbb{R}^N} |H_n(t,x,u,p) - H(t,x,u,p)|dt = 0$$

for every compact $K_1 \subset \mathbb{R}^N \times \mathbb{R}$. Let us set

$$E \doteq \{(x,u,p) \in \overline{O} \times [m-1, M+1] \times \mathbb{R}^N\},$$

where $M \doteq \max_{\{(t,x)\in[0,T]\times\overline{O}\}} u(t,x)$, and $m \doteq \min_{\{(t,x)\in[0,T]\times\overline{O}\}} u(t,x)$.

Then, for n sufficiently large, the functions

$$u_n(t,x) \doteq u(t,x) - \int_0^t \sup_{(x,u,p)\in E} |H_n(s,x,u,p) - H(s,x,u,p)|ds$$

are viscosity subsolutions of (3.1) in $]0,T[\times O$ and verify

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\overline{O}} |u_n(t,x) - u(t,x)| = 0. \tag{3.6}$$

iv) Let u be a subsolution of (2.1) in $]0,T[\times\mathbb{R}^N$, and let $\Lambda \geq 0$ be a Lipschitz constant and a bound for the function $x \mapsto u(t,x)$, for any $t \in [0,T]$. Moreover, let us assume

$$\lim_{n\to\infty} \int_0^T \sup_{(x,u,p)\in(\mathbb{R}^N\times\mathbb{R}\times K_2)} |H_n(t,x,u,p) - H(t,x,u,p)|dt = 0$$

for every compact $K_2 \subset \mathbb{R}^N$. Let us set

$$E \doteq \{(x,s,p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \ : \ (x,s,|p|) \in \mathbb{R}^N \times \mathbb{R} \times [0, \Lambda+1]\}.$$

Then, the functions

$$u_n(t,x) \doteq u(t,x) - \int_0^t \sup_{(x,u,p)\in E} |H_n(s,x,u,p) - H(s,x,u,p)|ds$$

are viscosity subsolutions of (3.1) in $]0,T[\times\mathbb{R}^N$ and verify

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^N} |u_n(t,x) - u(t,x)| = 0. \tag{3.7}$$

4 Comparison via approximation

In the present section we address the main object of the paper, that is, we exploit Theorem 3.3 (and the analogous result for supersolutions) for deducing comparison results from akin results concerning continuous Hamiltonians.

The first result in this direction is Theorem 4.1 below, where one assumes the same hypotheses as in Theorem 3.3 (or Remark 3.5).

Theorem 4.1 (and its proof) is more an instance on how to exploit the *approximability* results of the previous section than a result to be utilized in concrete situations. As a matter of fact, in order to cover some actual applications (in particular, some control and differential games problems), we present a comparison result, namely Theorem 4.7, which holds under a set of hypotheses which is not contained in the hypotheses of Theorem 4.1. Moreover this theorem covers several situations where the analogous result in [Is85] cannot be applied (see Remark 4.9). Theorem 4.7 can be proved thanks to a localization argument, which in turn relies on the crucial exploitation of a *local* comparison result for continuous Hamiltonians (see Theorem 4.6 below and [Is84], [Le01]).

One can naturally object that both Theorem 4.1 and Theorem 4.7 assume the existence of suitable continuous Hamiltonians converging to the given one, which might be a serious drawback for the actual exploitation of such results. For this reason, in Subsection 4.1 we address the question of the existence of such approximating Hamiltonians. This leads to a comparison result involving (besides hypotheses (**A**)) just a weak assumption on the regularity of the given Hamiltonian (see Definition 4.4 below).

In what follows, by saying that a "comparison result holds for the equation

$$\frac{\partial u}{\partial t}(t,x) + H(t,x,u,\nabla u) = 0 \quad \text{in }]0,T[\times O\text{"} \tag{4.1}$$

(where $O \subseteq \mathbb{R}^N$ is an open set), we mean that the following holds true:

If u_1 is a subsolution of (4.1) *and u_2 is a supersolution of* (4.1) (*in a class of continuous maps to be specified*) *verifying*

$$u_1(t,x) \le u_2(t,x) \text{ on } ([0,T]\times\partial O)\cup(\{0\}\times\overline{O}),$$

then

$$u_1(t,x) \le u_2(t,x) \text{ for all } (t,x)\in[0,T]\times O.$$

Theorem 4.1 *Let O be an open subset of $\mathbb{R}^N$. Select one among the hypotheses* i)-iv) *of Theorem* 3.3 *or Remark* 3.5. *Let H be an Hamiltonian verifying the basic hypotheses* (2)-(4) *and let u, v, and $(H_n)_{n\in\mathbb{N}}$ be a subsolution of* (2.1), *a supersolution of* (2.1) *and a sequence of Hamiltonians, respectively, verifying the selected hypothesis among* i)-iv). *Moreover let u, v satisfy*

$$u(t,x) \le v(t,x) \text{ on } ([0,T]\times\partial O)\cup(\{0\}\times\overline{O}).$$

Finally for each H_n, let a comparison result hold for the equation

$$\frac{\partial u}{\partial t}(t,x) + H_n(t,x,u,\nabla u) = 0 \text{ in }]0,T[\times O, \tag{4.2}$$

(*in the class of maps singled out by the selected hypothesis among* i)-iv)).

Then,

$$u(t,x) \le v(t,x) \text{ for all } (t,x) \in [0,T] \times O.$$

Proof. By contradiction let us assume that there exists a $(t_0,x_0) \in]0,T[\times O$ such that

$$u(t_0,x_0) > v(t_0,x_0).$$

By Theorem 3.3 (or Remark 3.5) there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of subsolution of (4.2) defined on $[0,T]\times O$ such that

$$\lim_{n\to\infty} u_n(t,x) = u(t,x)$$

uniformly on $[0,T]\times\overline{O}$, and $u_n(t,x) \le u(t,x)$ for every $(t,x)\in[0,T]\times\overline{O}$. Analogously, there exists a sequence of supersolutions $(v_n)_{n\in\mathbb{N}}$ of (4.2), $(v_n)_{n\in\mathbb{N}}$ defined on $[0,T]\times O$ such that

$$\lim_{n\to\infty} v_n(t,x) = v(t,x)$$

uniformly on $[0,T]\times\overline{O}$, and $v_n(t,x) \ge v(t,x)$ for every $(t,x)\in[0,T]\times\overline{O}$. Hence, there exists a sufficiently large $n\in\mathbb{R}$ such that

$$u_n(t_0,x_0) > v_n(t_0,x_0),$$

against the fact that a comparison result holds for the equation (4.2). □

Remark 4.2 With an almost unchanged proof one could easily prove a comparison result for the so-called *state-constraint boundary conditions*. By that, one means that

i) $u : [0,T]\times O \to \mathbb{R}$ is a viscosity solution on $]0,T]\times O$.

ii) $u(x,0) = \varphi(x)$ for all $x\in\overline{O}$.

iii) u is a supersolution of (2.1) on $]0,T[\times\partial O$.

Remark 4.3 The uniqueness of the solution of the Cauchy problem (2.1)-(2.2) is a straightforward consequence of the previous theorem, at least in the cases where the approximation of H is as in i) and iv) (where $O = \mathbb{R}^N$). Notice, however, that this comparison result can be naturally exploited for problems in $]0,T[\times O$ as well, where O is an open subset of $\mathbb{R}^N$.

4.1 On approximating Hamiltonians

First, we wish to focus the actual possibility of approximating H in a way as prescribed in the hypotheses of Theorem 4.7. This will allows us to state Theorem 4.10 and Corollary 4.11 below, whose hypotheses can be certainly easily checked.

Let us begin by considering a general hypothesis which later will be assumed on our Hamiltonians.

Definition 4.4 Let $F : [0,T] \times \mathbb{R}^q$ be a map. We say that F verifies the *approximation hypothesis* (AP) if the following holds: for every compact subset $Q \subset \mathbb{R}^q$ there exist a L^∞ map $v : [0,T] \to \mathbb{R}$ and a t-integrable modulus ω such that

$$|F(t,x) - F(s,y)| \leq |v(t) - v(s)| + \omega(t, |x-y|) + \omega(s, |x-y|)$$

where, by saying that ω is a t-integrable modulus we mean that:

i) for every $t \in [0,T]$ the map $\omega(t,\cdot) : [0,+\infty[\to [0,+\infty[$, is a modulus, i.e. an increasing map vanishing and continuous at zero,

and

ii)
$$\lim_{\delta \to 0} \int_0^T \omega(t,\delta) dt = 0.$$

Let $\alpha : \mathbb{R} \to [0,+\infty[$, $\beta : \mathbb{R}^q \to [0,+\infty[$ be non-negative C^∞ maps having L^1-norms equal to 1 and supported in the balls $B^1 = \{h \in \mathbb{R} : |h| \leq 1\}$, $B^q = \{k \in \mathbb{R}^q : |k| \leq 1\}$, respectively.

Let us extend F to $\mathbb{R}^{q+1}$ by setting $F(t,x) = 0$ whenever $t \in \mathbb{R} \backslash [0,T]$. For every natural number ν, let us consider the mollification

$$F_\nu(t,x) \doteq \int_{\mathbb{R}^{q+1}} F\left(t + \frac{h}{\nu}, x + \frac{k}{\nu}\right) \alpha(h)\beta(k) dh dk.$$

Proposition 4.5 *Let us assume that the function F verifies hypothesis* (AP)*. Then, for all compact subsets $Q \subset \mathbb{R}^q$ one has*

$$\lim_{\nu \to \infty} \int_0^T \sup_{x \in Q} |F_\nu(t,x) - F(t,x)| dt = 0.$$

Proof. Let $Q \subset \mathbb{R}^q$ be a compact subset, and let us extend the corresponding maps v and ω in hypothesis (AP) to $\mathbb{R}$ and $\mathbb{R} \times [0,+\infty[$ by setting

$$v(s) = \omega(s,\delta) = 0 \quad \forall (s,\delta) \in (\mathbb{R} \backslash [0,T]) \times [0,+\infty[.$$

Set $M \doteq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}$. Then, by applying Fubini-Tonelli's theorem one obtains,

$$\begin{aligned}
&\int_0^T \sup_{x\in Q} |F_\nu(t,x) - F(t,x)| dt \\
&= \int_0^T \sup_{x\in Q} \left| \int_{\mathbb{R}^{q+1}} \left(F\left(t + \frac{h}{\nu}, x + \frac{k}{\nu}\right) - F(t,x) \right) \alpha(h)\beta(k) dh\, dk \right| dt \\
&\leq \int_0^T \left[\int_{\mathbb{R}} M^2 \left(\omega\left(t + \frac{h}{\nu}, \frac{1}{\nu}\right) + \omega\left(t, \frac{1}{\nu}\right) \right) + M \left| v\left(t + \frac{h}{\nu}\right) - v(t) \right| \alpha(h) dh \right] dt \\
&= 2M^2 \int_{B_1} \left[\int_0^T \omega\left(t, \frac{1}{\nu}\right) dt \right] dh + \int_0^T M \left[\int_{B_1} \alpha(h) \left| v\left(t + \frac{1}{\nu}\right) - v(t) \right| dh \right] dt \\
&\leq 2M^2 \int_{B_1} \left[\int_0^T \omega\left(t, \frac{1}{\nu}\right) dt \right] dh + \int_0^T M^2 \left[\nu \int_{-1/\nu}^{1/\nu} |v(t+s) - v(t)| ds \right] dt
\end{aligned}$$

and the last row converges to zero by the hypotheses on ω and by the fact that the maps

$$t \mapsto \nu \left(\int_{-1/\nu}^{1/\nu} |v(t+s) - v(t)| ds \right)$$

are pointwise bounded by $\|v\|_\infty$ and tend to zero at all Lebesgue points t of v (i.e. almost everywhere). □

4.2 Weaker hypotheses

The approximation considered in i) and iv) of Theorem 3.3, Remark 3.5 respectively, do not cover some natural applications, e.g. in control theory. As a possible answer to such an objection, let us show how the ideas underlying Theorem 4.1 can be adjusted in order to prove a comparison result (Theorem 4.7) under the hypotheses (A) below (which, on one hand, in general do not allow for an as strong approximation as the one prescribed in i) and iv), and, on the other hand, are quite common in the case of continuous Hamiltonians).

Hypotheses (A)**.**

(1A) *The Hamiltonian $H(t,x,u,p)$ is measurable in t and continuous in (x,u,p). Moreover, $H(t,0,0,0) \in L^1(0,T)$.*

(2A) *For each $(t,x,p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R}^N$ the function $u \to H(t,x,u,p)$ is monotone increasing.*

(3A) *The Hamiltonian $H(t,x,u,p)$ is Lipschitz continuous w.r.t. x and there exist $C > 0$ and $\beta \geq 0$ such that*

$$\left| \frac{\partial H}{\partial x}(t,x,u,p) \right| \leq C(\beta + |p|)$$

for a.e. $(t,x,u,p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$.

(4A) *The Hamiltonian* $H(t,x,u,p)$ *is Lipschitz continuous w.r.t.* p *and there exist* $A, B > 0$ *such that*

$$\left|\frac{\partial H}{\partial p}(t,x,u,p)\right| \leq A|x| + B$$

for a.e. $(t,x,u,p) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$.

First, we recall the local comparison theorem Theorem A.1 in [Le01].

Theorem 4.6 (Theorem A.1 in [Le01]) *Let* H *be a continuous Hamiltonian on* $[0,T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$. *Assume* (3A) *and* (4A) *and let* $f, g \in C([0,T] \times \mathbb{R}^N)$. *Let* $x_0 \in \mathbb{R}^N$ *and* $r > 0$ *and define* $L = B + A(1 + |x_0|)$.

If $u \in C([0,T] \times \overline{B}(x_0,r))$ *is a viscosity subsolution of*

$$\frac{\partial w}{\partial t} + H(t,x,w,\nabla w) = f \text{ in }]0,T[\times \overline{B}(x_0,r) \text{ and } w(0,x) = u_0(x) \text{ in } B(x_0,r)$$

and v *is a viscosity supersolution of*

$$\frac{\partial w}{\partial t} + H(t,x,w,\nabla w) = g \text{ in }]0,T[\times \overline{B}(x_0,r) \text{ and } u(0,x) = v_0(x) \text{ in } B(x_0,r)$$

then

$$\begin{aligned} &u(t,x) - v(t,x) \\ &\quad \leq \sup_{y \in \overline{B}(x_0,r)} \{u_0(y) - v_0(y)\} + \int_0^t \sup_{y \in \overline{B}(x_0,r)} \{f(y,s) - g(y,s)\}\, ds \end{aligned} \tag{4.3}$$

for every $(x,t) \in \overline{D}(x_0,r)$ *where*

$$\overline{D}(x_o,r) = \{(t,x) \in (0,T) \times B(x_0,r) \;:\; e^{LT}(1 + |x - x_0|) - 1 \leq r\}.$$

In particular, if $r \geq 2e^{LT}$ *then* (4.3) *holds in* $[0,T] \times B(x_0, e^{-LT}\frac{r}{2})$.

Theorem 4.7 *Let* H *verify hypotheses* (A), *and assume that there exists a sequence of continuous Hamiltonians* $(H_n)_{n\in\mathbb{N}}$ *verifying* (A) *and*

$$\lim_{n\to\infty} \int_0^T \sup_{(x,u,p)\in K} |H_n(t,x,u,p) - H(t,x,u,p)| dt = 0 \tag{4.4}$$

for every compact $K \subset (\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$.

Let u *and* v *be a subsolution and a supersolution of* (2.1), *respectively. Let us assume that they are locally Lipschitz continuous in* x, *uniformly w.r.t.* $t \in [0,T]$, *and that*

$$u(0,x) \leq v(0,x) \quad \forall x \in \mathbb{R}^N.$$

Then,

$$u(t,x) \leq v(t,x) \text{ for all } (t,x) \in [0,T] \times \mathbb{R}^N.$$

Proof of Theorem 4.7 Let us fix $(t_0, x_0) \in]0, T[\times\mathbb{R}^N$ and $r > 2e^{LT}$ where $L = B + A(1 + |x_0|)$, and let us consider the bounded set $O = B(x_0, r)$. Let u_n and v_n be the maps constructed in ii) of Theorem 3.3. Hence the u_n and the v_n are subsolutions ad supersolutions of (3.1), respectively, and satisfy

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\overline{O}} |u_n(t, x) - u(t, x)| = 0,$$

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\overline{O}} |v_n(t, x) - v(t, x)| = 0.$$

Moreover one has

$$v_n(0, x) = v(0, x) \quad \text{and} \quad u_n(0, x) = u(0, x).$$

Hence by the comparison Theorem A.1 in [Le01] (Theorem 4.6), we have that

$$u_n(t, x) \leq v_n(t, x)$$

for all $(t, x) \in [0, T] \times B(x_0, e^{-LT}\frac{r}{2})$. Hence one has

$$u(t, x) \leq v(t, x) \text{ for all } (t, x) \in [0, T] \times B\left(x_0, e^{-LT}\frac{r}{2}\right)$$

as well. One achieves the conclusion by arguing as in the proof of Theorem 4.1. □

As a straightforward consequence, one obtains the following uniqueness result.

Corollary 4.8 *Let us assume the same hypotheses as in Theorem 4.7. Among the maps that are locally Lipschitz continuous in x uniformly w.r.t. t there exists at most one viscosity solution of*

$$\frac{\partial u}{\partial t}(t, x) + H(t, x, u, \nabla u) = 0 \text{ in }]0, T[\times\mathbb{R}^N \tag{4.5}$$

$$u(0, x) = \varphi(x) \text{ in } \mathbb{R}^N. \tag{4.6}$$

Remark 4.9 With Theorem 4.7 we aim to give just an instance of application of Theorem 3.3. As a simple example which is covered by Theorem 4.7 while is not covered by the comparison theorem in [Is85], let us consider a Hamiltonian of the form

$$H(t, x, u, p) = \alpha(t)(p \cdot x)$$

where α is a bounded measurable map defined on $[0, T]$. If α_n is a sequence of continuous maps approaching α in the L^1-norm, then the Hamiltonians

$$H_n(t, x, u, p) = \alpha_n(t)(p \cdot x)$$

agree with the hypotheses of Theorem 4.7.

Finally we are in the position of giving more exploitable versions of both Theorem 4.7 and Corollary 4.8. They follow from the elementary fact that once H verifies hypotheses (A) and hypothesis (AP) then the mollifications provided by Proposition 4.5 are continuous in t and verify hypotheses (A) as well.

Theorem 4.10 *Let H verify hypotheses* (A)*, and assume further that H verifies hypothesis* (AP)*, with $q = 2N+1$. Let u and v be a subsolution and a supersolution of* (2.1)*, respectively. Let us assume that they are locally Lipschitz continuous in x, uniformly w.r.t. $t \in [0,T]$, and that*

$$u(0,x) \leq v(0,x) \quad \forall x \in \mathbb{R}^N.$$

Then,

$$u(t,x) \leq v(t,x) \text{ for all } (t,x) \in [0,T] \times \mathbb{R}^N.$$

Corollary 4.11 *Let us assume the same hypotheses as in Theorem* 4.10*. Among the maps that are locally Lipschitz continuous in x uniformly w.r.t. t there exists at most one viscosity solution of*

$$\frac{\partial u}{\partial t}(t,x) + H(t,x,u,\nabla u) = 0 \text{ in }]0,T[\times\mathbb{R}^N \tag{4.7}$$

$$u(0,x) = \varphi(x) \text{ in } \mathbb{R}^N. \tag{4.8}$$

5 Uniform continuity for the Solution Map and the existence question

Let us fix the initial data φ in (2.2). We will prove here a uniform continuity property (see Theorem 5.1 below) for the map that associates the unique solution of (2.1)-(2.2) to each *continuous* Hamiltonian verifying hypotheses (1A)-(4A) and hypothesis (5A) below. We shall refer to this map as to the *Solution Map*. In particular, the Solution Map can be extended to $t-$measurable Hamiltonians. In view of the stability theorem (Theorem 3.2), such an extension will map a given (t-measurable) Hamiltonian into the corresponding solution of (2.1)-(2.2). Hence, in particular, this is a way to establish existence results (see Corollary 5.2). As in the previous section we shall assume hypothesis (AP) (see Definition 4.4) to obtain a result (Corollary 5.3) with *no explicit approximation assumptions* on H.

Together with (1A)-(4A) we will assume the following hypothesis:

(5A) For any $R > 0$, there exists modulus m_R such that

$$|H(t,x,s,p) - H(t,y,r,q)| \leq m_R(|x-y| + |s-r| + |p-q|)$$

for each $(t,x,s,p),(t,y,r,q) \in [0,T] \times \mathbb{R}^N \times \mathbb{R} \times B(0,R)$.

Here is the result concerning a uniform continuity property of the Solution Map.

Theorem 5.1 *Let us assume that the initial data φ is Lipschitz continuous on $\mathbb{R}^N$. Let $\tilde{H}_1$ and $\tilde{H}_2$ be continuous Hamiltonians verifying hypotheses* (1A)–(5A). *If $\tilde{u}_1$ and $\tilde{u}_2$ are the (unique) corresponding viscosity solutions of* (2.1)–(2.2)*, then, fix $R > 0$, one has*

$$\begin{aligned} &\sup_{(t,x)\in[0,T]\times B(0,R)} |\tilde{u}_1(t,x) - \tilde{u}_2(t,x)| \\ &\leq \int_0^T \sup_{(x,u,p)\in B(0,\bar{M})} |\tilde{H}_1(t,x,u,p) - \tilde{H}_2(t,x,u,p)|dt, \end{aligned} \tag{5.1}$$

where $\bar{M} = \max\{R, M_R\}$, M_R being the constant appearing in Lemma 5.5 *below.*

Corollary 5.2 *Let us assume that the initial data φ is Lipschitz continuous on $\mathbb{R}^N$. Let H verify hypotheses* (1A)–(5A) *and assume that there exists a sequence $(\tilde{H}_n)_{n\in\mathbb{N}}$ of* continuous *Hamiltonians satisfying hypothesis* (1A)–(5A) *and such that*

$$\lim_{n\to\infty} \int_0^T \sup_{(x,u,p)\in K} |\tilde{H}_n(t,x,u,p) - H(t,x,u,p)|dt = 0$$

for every compact $K \subset (\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$.

Then, the Cauchy problem $(2.1) - (2.2)$ *admits a (unique) solution. Moreover, this solution is locally Lipschitz continuous in the x-variable, uniformly w.r.t. $t \in [0,T]$.*

In view of Proposition 4.5 we easily get the following more concrete result:

Corollary 5.3 *Let us assume that the initial data φ is Lipschitz continuous on $\mathbb{R}^N$. Let H verify hypotheses* (1A)-(5A) *and hypothesis* (AP)*. Then, the Cauchy problem* (2.1)-(2.2) *admits a (unique) solution. Moreover, this solution is locally Lipschitz continuous in the x-variable, uniformly w.r.t. $t \in [0,T]$.*

Remark 5.4 Let us observe that as a byproduct of the approach considered here, one gets a regularity result having the same strength as the one valid for the continuous case.

5.1 Proofs of Theorem 5.1 and Corollary 5.2

Proof of Theorem 5.1 Fix $R > 0$, for every $(t,x) \in [0,T] \times \mathbb{R}^N$, let us set

$$\underline{u}_2(t,x) \doteq \tilde{u}_1(t,x) - \int_0^t \psi(s)ds, \tag{5.2}$$

and

$$\overline{u}_2(t,x) \doteq \tilde{u}_1(t,x) + \int_0^t \psi(s)ds, \tag{5.3}$$

where $\psi(t) \doteq \sup_{B(0,\bar{M})} |\tilde{H}_1(t,x,u,p) - \tilde{H}_2(t,x,u,p)|$. Let us show that $\underline{u}_2$ is a viscosity subsolution of (2.1) when $H = \tilde{H}_2$. (Again, let us recall that it is trivial to show that for a continuous Hamiltonian the concept of viscosity solution provided in Definition 2.1 reduces to the standard one.)

Let $\phi \in C^1(]0,T[\times\mathbb{R}^N)$, and let $(t_0,x_0) \in]0,T[\times B(0,R)$ be a local maximum of

$$\underline{u}_2(t,x) - \phi(t,x),$$

which implies that (t_0,x_0) is a local maximum of

$$\tilde{u}_1(t,x) - \int_0^t \psi(s)ds - \phi(t,x).$$

Since $\tilde{u}_1$ is a subsolution of (2.1) when $H = \tilde{H}_1$, this implies

$$\frac{\partial}{\partial t}\phi(t_0,x_0) + \psi(t_0) + \tilde{H}_1(t_0,x_0,\tilde{u}_1(t_0,x_0),\nabla\phi(t_0,x_0)) \le 0. \tag{5.4}$$

Moreover, in view of Lemma 5.5 below, one has

$$|\nabla\phi(t_0,x_0)|, |\tilde{u}_1(t_0,x_0)| \le R.$$

Hence,

$$\begin{aligned}
&\frac{\partial}{\partial t}\phi(t_0,x_0) + \tilde{H}_2(t_0,x_0,\underline{u}_2(t_0,x_0),\nabla\phi(t_0,x_0)) \\
&= \frac{\partial}{\partial t}\phi(t_0,x_0) + \tilde{H}_1(t_0,x_0,\tilde{u}_1(t_0,x_0),\nabla\phi(t_0,x_0)) + \psi(t_0) \\
&\quad - \psi(t_0) + \tilde{H}_2(t_0,x_0,\tilde{u}_1(t_0,x_0),\nabla\phi(t_0,x_0)) - \tilde{H}_1(t_0,x_0,\tilde{u}_1(t_0,x_0),\nabla\phi(t_0,x_0)) \\
&\quad + \tilde{H}_2(t_0,x_0,\underline{u}_2(t_0,x_0),\nabla\phi(t_0,x_0)) - \tilde{H}_2(t_0,x_0,\tilde{u}_1(t_0,x_0),\nabla\phi(t_0,x_0)) \le 0.
\end{aligned}$$

This implies that $\underline{u}_2$ is a subsolution. Similarly, one can prove that $\overline{u}_2$ is a viscosity supersolution of (2.1) with $H = \tilde{H}_2$.

Let us consider now the solution $\tilde{u}_2(t,x)$ of (2.1) corresponding to the Hamiltonian $\tilde{H}_2$. By Theorem VI.1 in [CL87], we know that

$$\underline{u}_2(t,x) \le \tilde{u}_2(t,x) \le \overline{u}_2(t,x) \quad \forall (t,x) \in [0,T]\times\mathbb{R}^N. \tag{5.5}$$

Moreover, by construction,

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times B(0,R)} |\tilde{u}_1(t,x) - \underline{u}_2(t,x)| \\
&\qquad \le \int_0^T \sup_{(x,u,p)\in B(0,\bar{M})} |\tilde{H}_1(t,x,u,p) - \tilde{H}_2(t,x,u,p)|dt,
\end{aligned} \tag{5.6}$$

and

$$\sup_{(t,x)\in[0,T]\times B(0,R)} |\tilde{u}_1(t,x) - \overline{u}_2(t,x)| \leq \int_0^T \sup_{(x,u,p)\in B(0,\bar{M})} |\tilde{H}_1(t,x,u,p) - \tilde{H}_2(t,x,u,p)|dt. \quad (5.7)$$

Then (5.1) follows from (5.5), (5.6), and (5.7). □

Proof of Corollary 5.2 By Lemma 5.5 below, for each $n \in \mathbb{N}$ and $R > 0$, there exists a unique $\tilde{u}_n(t,x)$ viscosity solution of (2.1)-(2.2) (when $H = \tilde{H}_n$). Moreover

$$\max\{|\tilde{u}_n(t,x)|, |\nabla\tilde{u}_n(t,x)|\} \leq M_R \text{ a.e. } (t,x) \in [0,T] \times B(0,R). \quad (5.8)$$

In view of Theorem 5.1 one has

$$\sup_{(t,x)\in[0,T]\times B(0,R)} |\tilde{u}_{n+p}(t,x) - \tilde{u}_n(t,x)| \leq \int_0^T \sup_{(x,u,p)\in B(0,\bar{M})} |\tilde{H}_{n+p}(t,x,u,p) - \tilde{H}_n(t,x,u,p)|dt$$

for all natural numbers n, p. This implies that there exists a continuous map $u : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \tilde{u}_n(t,x) = u(t,x) \quad (5.9)$$

uniformly on compact subsets of $[0,T] \times \mathbb{R}^N$. Hence, by Theorem 3.2, $u(t,x)$ is the viscosity solution of (2.1) in $]0,T[\times\mathbb{R}^N$. Moreover, by (5.8) and (5.9) we obtain that u is Lipschitz continuous in the x-variable uniformly w.r.t. t $\in [0,T]$, which completes the proof. □

Lemma 5.5 *Let us assume that the initial data φ is Lipschitz continuous on $\mathbb{R}^N$. If $H(t,x,u,p)$ is continuous and satisfies assumptions* (1A)–(4A) *then:*

(i) *there exists at most one viscosity solution $\hat{u}$ of* (2.1)–(2.2)*, and for every $R > 0$ one has*

$$\max\{|\hat{u}(t,x)|, |\nabla\hat{u}(t,x)|\} \leq M_R \text{ a.e. } (t,x) \in [0,T] \times B(0,R), \quad (5.10)$$

for a suitable constant M_R. In particular $\hat{u}$ is Lipschitz continuous in the x-variable (*uniformly w.r.t. $t \in [0,T]$*)*, with constant M_R on each ball $B(0,R)$.*

(ii) *If hypothesis* (5A) *is verified as well, then there exists a* (*unique*) *viscosity solution of* (2.1)–(2.2)*.*

The proof of the existence assertion can be found e.g. in [CL87], where uniqueness and regularity results are proved as well (see Theorem VI.1 and Theorem VII.1). The regularity claim and the estimate (5.10) are proved in [Le01] (Theorem 4.1).

Remark 5.6 Actually, the result in [Le01] concerns Hamiltonians that are independent of u. However, in view of the monotonicity assumption (2A), the proof of the more general version stated above can be obtained straightforwardly by the original one.

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