Noname manuscript No.
(will be inserted by the editor)

Abstracting Nash Equilibria of Supermodular Games

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Abstract Supermodular games are a well known class of noncooperative games which find significant applications in a variety of models, especially in operations research and economic applications. Supermodular games always have Nash equilibria which are characterized as fixed points of multivalued functions on complete lattices. Abstract interpretation is here applied to set up an approximation framework for Nash equilibria of supermodular games. This is achieved by extending the theory of abstract interpretation in order to cope with approximations of multivalued functions and by providing some methods for abstracting supermodular games, thus obtaining approximate Nash equilibria which are shown to be correct within the abstract interpretation framework.

1 Introduction

Supermodular Games. Modern game theory is increasingly applied as a model of conflict and cooperation in a variety of fields ranging from economics to biology and computer science. Games may have so-called strategic complementarities, which encode, roughly speaking, a complementarity relationship between own actions and rivals' actions, i.e., best responses of any game player is increasing in actions of the opponents. Games with strategic complementarities occur in a large array of models, especially in operations research and economic applications of noncooperative game theory—a significant sample of them is described by Topkis' book [24]. For example, strategic complementarities arise in economic game models where the players are competitive firms that must each decide how many goods to produce and an increase in the production of one firm increases the marginal revenues of the others, because this gives the other firms an incentive to produce more too. Pionereed by Topkis [23] in 1978, this class of games is formalized by so-called supermodular games, where the payoff functions of each player have the lattice-theoretical properties of supermodularity and increasing differences. In a supermodular game, the strategy space of every player is partially ordered and is assumed to be a complete lattice, while the utility in playing a higher strategy increases when the opponents also play higher strategies. The well-known Nash equilibrium models the notion of solution of noncooperative games: each player is making the best possible strategy, taking into account the decisions of the rivals, and no player can benefit by modifying her/his strategy while the other players keep theirs

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unchanged. It turns out that so-called pure strategy Nash equilibria of supermodular games form a complete lattice w.r.t. the ordering relation of the strategy space, thus exhibiting the least and greatest Nash equilibria. Furthermore, since the best response correspondence of a supermodular game turns out to satisfy a monotonicity condition, its least and greatest equilibria can be characterized and calculated (under assumptions of finiteness) as least and greatest fixed points by Knaster-Tarski fixed point theorem, which provides the theoretical basis for the so-called Robinson-Topkis algorithm [24].

Battle of the sexes [26] is a popular and simple example of two-player (non)supermodular game. Assume that a couple, Alice and Bob, argues over what to do on the weekend. Alice would prefer to go to the opera O, Bob would rather go to the football match F, both would prefer to go to the same place rather than different ones, in particular than the disliked ones. Where should they go? The following matrix with double-entry cells provides a game model for this problem.

		Bob: O	Bob: F
Alice	O	$3, 2$	$1, 1$
	F	$0, 0$	$2, 3$

Alice chooses a row (either O or F) while Bob chooses a column (either O or F). In each double-entry cell, the first and second natural numbers represent, respectively, Alice's and Bob's utilities, i.e., preferences, where greater numbers mean higher preferences. Hence, $u_{\mathrm{A}}(O, O) = 3 = u_{\mathrm{B}}(F, F)$ is the greatest utility for both Alice and Bob, for two different strategies ((O, O) for Alice and (F, F) for Bob), while $u_{\mathrm{A}}(F, O) = 0 = u_{\mathrm{B}}(F, O)$ is the least utility for both Alice and Bob. This game has two pure strategy Nash equilibria: one (O, O) where both go to the opera and another (F, F) where both go to the football game. If the ordering between O and F is either $O < F$ or $F < O$ for both Alice and Bob then this game turns out to be supermodular. If $O < F$ then (O, O) and (F, F) are, respectively, the least and greatest equilibria, while their roles are exchanged when $F < O$. Instead, if $F < O$ for Alice and $O < F$ for Bob then, as intuitively expected, this game is not supermodular: the two equilibria (O, O) and (F, F) are uncomparable so that least and greatest equilibria do not exist.

Motivation. Since the breakthrough on the PPAD-completeness of finding mixed Nash equilibria [8], the question of approximating Nash equilibria emerged as a key problem in algorithmic game theory [9, 14]. In this context, approximate equilibrium refers to ϵ-approximation, with $\epsilon > 0$, meaning that, for each player, all the strategies have a payoff which is at most ϵ more (or less) than the precise payoff of the given strategy. On the other hand, the notion of (correct or sound) approximation is central in static program analysis [17]. In particular, the abstract interpretation approach to static analysis [3, 4] relies on an order-theoretical model of the notion of approximation. Here, program properties are modelled by a (collecting) domain endowed with a partial order $\leq$ which plays the role of logical relation where $x \leq y$ means that the property x is logically stronger than y. Also, the fundamental principle of abstract interpretation is to provide an approximate interpretation of a program for a given abstraction of the properties of its concrete semantics. This leads to the key notion of abstract domain, defined as an ordered collection of abstract program properties which can be inferred by static analysis, where approximation is modeled by the ordering relation. Furthermore, program semantics are typically defined using fixed points and a basic result of abstract interpretation tells us that correctness of approximations is preserved from functions to their least/greatest fixed points.

Goal. The similarities between supermodular games and program semantics should be clear, since they both rely on order-theoretical models and on computing extremal fixed points of suitable functions on lattices. However, while static analysis of program semantics based on order-theoretical approximations is a well-established area since forty years, to the best of our knowledge, no attempt has been made to apply standard techniques used in static program analysis for defining a corresponding notion of approximation in supermodular games. The overall goal of this paper is to investigate whether and how abstract interpretation can be used to define and calculate approximate Nash equilibria of supermodular games, where the key notion of approximation will be modeled by a partial ordering relation similarly to what happens in static program analysis. This appears to be the first contribution to make use of an order-theoretical notion of approximation for equilibria of supermodular games, in particular by resorting to the abstract interpretation framework.

Contributions. Static program analysis by abstract interpretation essentially relies on: (1) abstract domains A which encode approximate program properties; (2) abstract functions $f^\sharp$ which must correctly approximate on A the behavior of some concrete operations f; (3) results of correctness for the abstract interpreter using A and $f^\sharp$, such as the correctness of extremal fixed points of abstract functions, e.g., the abstract least fixed point $\mathrm{lfp}(f^\sharp)$ correctly approximates the concrete one $\mathrm{lfp}(f)$; (4) widening/narrowing operators tailored for the abstract domains A to ensure and/or accelerate the convergence in iterative fixed point computations of abstract functions. We contribute to set up a general framework for designing abstract interpretations of supermodular games which encompasses the above points (1)-(3), while widening/narrowing operators are not taken into account since their definition is closely related to some specific abstract domain. Our main contributions can be summarized as follows.

- Abstract interpretation is typically used for approximating single-valued functions on complete lattices. For N-players supermodular games, best responses B are modeled by multivalued functions of type $B : S_1 \times \cdots \times S_N \to \wp(S_1 \times \cdots \times S_N)$. A game strategy $s \in S_1 \times \cdots \times S_N$ is called a fixed point of B when $x \in B(x)$, and these fixed points turn out to characterize the Nash equilibria of this game. As a preliminary step, in Section 3, we first show how abstractions of strategy spaces can be composed in order to define an abstraction of the product $S_1 \times \cdots \times S_N$, and, on the other hand, an abstraction of the product $S_1 \times \cdots \times S_N$ can be decomposed into abstract domains of the individual S_i's. Next, in Section 4, we provide a short and direct constructive proof ensuring the existence of fixed points for multivalued functions and we show how abstract interpretation can be generalized to cope with multivalued functions.
- We investigate how to define an "abstract interpreter" of supermodular games. The first approach is described in Section 5 and consists in defining a supermodular game on an abstract strategy space. Given a supermodular game Γ with strategy spaces S_i and utility functions $u_i : S_1 \times \cdots \times S_N \to \mathbb{R}$, this means that we assume a family of abstractions A_i, one for each induvidual S_i, that gives rise to a product abstract strategy space $A = A_1 \times \cdots \times A_N$, and a suitable abstract restriction of the utility functions $u_i^A : A_1 \times \cdots \times A_N \to \mathbb{R}$. This defines what we call an abstract game Γ^A, which, under some conditions, has abstract equilibria which correctly approximate the equilibria of the original game Γ. The fixed point computations in the abstract domain A for the abstract game Γ^A will typically be more efficient than those in $S_1 \times \cdots \times S_N$ for Γ. This abstraction technique provides a generalization of an algorithm by Echenique [10] for finding all the Nash equilibria in a finite game with strategic complementarities.

- In Section 6 we put forward a second notion of abstract game where the strategy spaces are subject to a kind of partial approximation, meaning that, for any utility function u_i for the player i, we consider approximations of the strategy spaces of the "other players", i.e., correct approximations of the maps $u_i(s_i, \cdot) : S_1 \times \cdots S_{i-1} \times S_{i+1} \times \cdots \times S_N \to \mathbb{R}$, for any given strategy $s_i \in S_i$. This abstraction technique gives rise to games having an abstract best response correspondence. This approach is inspired and somehow generalizes the implicit methodology of approximate computation of equilibria considered in Carl and Heikkilä's book [2, Chapter 8].
- Our results are illustrated on a number of examples of supermodular games. In particular, a couple of examples of so-called Bertrand oligopoly models are taken from Carl and Heikkilä's book [2].

This is an expanded and revised version of the SAS article [20] including all the proofs.

2 Background on Games

2.1 Order-Theoretical Notions

Given a function $f : X \to Y$ and a subset $S \in \wp(X)$ then $f(S) \triangleq \{f(s) \in Y \mid s \in S\}$, while its powerset lifting $f^s : \wp(X) \to \wp(Y)$ is defined by $f^s \triangleq \lambda S.\, f(S)$. A multivalued function, also called correspondence in game theory terminology, is any mapping $f : X \to \wp(X)$. An element $x \in X$ is called a fixed point of a multivalued function $f : X \to \wp(X)$ when $x \in f(x)$, while $\mathrm{Fix}(f) \triangleq \{x \in X \mid x \in f(x)\}$ denotes the corresponding set of fixed points.

Let $\langle C, \leq, \wedge, \vee, \bot, \top\rangle$ be a complete lattice, with partial order $\leq$, glb $\wedge$, lub $\vee$, bottom element $\bot$ and top element $\top$, compactly denoted by $\langle C, \leq\rangle$. Given a function $f : C \to C$, with a slight abuse of notation, $\mathrm{Fix}(f) \triangleq \{x \in C \mid x = f(x)\}$ denotes its set of fixed points of f, while $\mathrm{lfp}(f)$ and $\mathrm{gfp}(f)$ denote, respectively, the least and greatest fixed points of f, when they exist (let us recall that least and greatest fixed points always exist for monotone functions). Let $\mathbb{O}$ denote the class of all ordinal numbers. If $f : C \to C$ then for any ordinal $\alpha \in \mathbb{O}$, the α-power $f^\alpha : C \to C$ is defined by transfinite induction as usual: for any $x \in C$, (1) if $\alpha = 0$ then $f^0(x) \triangleq x$; (2) if $\alpha = \beta + 1$ is a successor ordinal then $f^{\beta+1}(x) \triangleq f(f^\beta(x))$; (3) if $\alpha = \vee\{\beta \in \mathbb{O} \mid \beta < \alpha\}$ is a limit ordinal then $f^\alpha(x) \triangleq \bigvee_{\beta<\alpha} f^\beta(x)$. If $f, g : X \to C$ then $f \sqsubseteq g$ denotes the standard pointwise ordering relation between functions, that is, $f \sqsubseteq g$ if for any $x \in X$, $f(x) \leq g(x)$.

Let us recall the following relations on the powerset $\wp(C)$: for any $X, Y \in \wp(C)$,

$$
\begin{array}{ll}
\text{(Smyth)} & X \preceq_S Y \overset{\triangle}{\iff} \forall y \in Y. \exists x \in X.\, x \leq y \\
\text{(Hoare)} & X \preceq_H Y \overset{\triangle}{\iff} \forall x \in X. \exists y \in Y.\, x \leq y \\
\text{(Egli-Milner)} & X \preceq_{EM} Y \overset{\triangle}{\iff} X \preceq_S Y \;\&\; X \preceq_H Y \\
\text{(Veinott)} & X \preceq_V Y \overset{\triangle}{\iff} \forall x \in X. \forall y \in Y.\, x \wedge y \in X \;\&\; x \vee y \in Y
\end{array}
$$

Smyth $\preceq_S$, Hoare $\preceq_H$ and Egli-Milner $\preceq_{EM}$ relations are reflexive and transitive (i.e., preorders) and are typically used in powerdomain constructions [18,21]. The Veinott relation $\preceq_V$ is transitive and antisymmetric and is used for supermodular games [23,25], where is also called strong set relation. A multivalued function $f : C \to \wp(C')$ is S-monotone if for any $x, y \in C$, $x \leq y$ implies $f(x) \preceq_S f(y)$. H-, EM- and V-monotonicity are defined analogously. We also use the following notations:

$$\begin{aligned}
&\wp^{\wedge}(C) \triangleq \{X \in \wp(C) \mid \wedge X \in X\} \\
&\wp^{\vee}(C) \triangleq \{X \in \wp(C) \mid \vee X \in X\} \\
&\wp^{\diamond}(C) \triangleq \wp^{\wedge}(C) \cap \wp^{\vee}(C) \\
&\mathrm{SL}(C) \triangleq \{X \in \wp(C) \mid X \neq \varnothing,\ X \text{ subcomplete sublattice of } C\}
\end{aligned}$$

where $X \subseteq C$ is a subcomplete sublattice of C when for any $Y \subseteq X$, we have that $\wedge Y, \vee Y \in X$. Let us observe that:

$$\begin{aligned}
&\text{if } X, Y \in \wp^{\wedge}(C) \text{ then } X \preceq_S Y \ \Leftrightarrow\ \wedge X \leq \wedge Y; \\
&\text{if } X, Y \in \wp^{\vee}(C) \text{ then } X \preceq_H Y \ \Leftrightarrow\ \vee X \leq \vee Y; \\
&\text{if } X, Y \in \wp^{\diamond}(C) \text{ then } X \preceq_{EM} Y \ \Leftrightarrow\ \wedge X \leq \wedge Y \ \&\ \vee X \leq \vee Y.
\end{aligned}$$

Supermodularity. Given a family $(S_i)_{i=1}^N$ of $N > 0$ sets, an element $s \in \times_{i=1}^N S_i$ of their product and $i \in [1, N]$ then we use the following notations:

$$S_{-i} \triangleq S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots S_N, \quad s_{-i} \triangleq (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N) \in S_{-i}.$$

Let $\langle \mathbb{R}^N, \leq \rangle$ denote the product poset of real numbers, where for $s, t \in \mathbb{R}^N$, $s \leq t$ iff for any $i \in [1, N]$, $s_i \leq t_i$, while $s + t \triangleq (s_i + t_i)_{i=1}^N$ (and $s - t$ is analogously defined). If $f : X \times Y \to Z$ is any function defined on a product domain then, for any given $x \in X$, the notation $f(x, \cdot) : Y \to Z$ denotes the function $y \mapsto f(x, y)$; analogously for $f(\cdot, y) : X \to Z$.

Supermodular games rely on (quasi)supermodular functions (we refer to the books [2, 24] for in-depth studies). Given a complete lattice C, a function $u : C \to \mathbb{R}^N$ is *supermodular* if for any $c_1, c_2 \in C$,

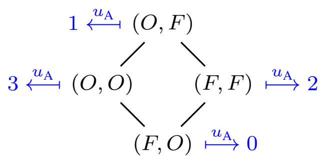
$$u(c_1 \vee c_2) + u(c_1 \wedge c_2) \geq u(c_1) + u(c_2)$$

while u is *quasisupermodular* if for any $c_1, c_2 \in C$,

$$\begin{aligned}
u(c_1 \wedge c_2) \leq u(c_1) \ &\Rightarrow\ u(c_2) \leq u(c_1 \vee c_2) \\
u(c_1 \wedge c_2) < u(c_1) \ &\Rightarrow\ u(c_2) < u(c_1 \vee c_2).
\end{aligned}$$

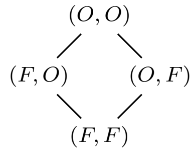
Let us remark that supermodularity implies quasisupermodularity (the converse is not true).

Example 2.1 Let us consider the battle of the sexes game described in Section 1. Let $C_{\mathrm{A}} = \{F, O\} = C_{\mathrm{B}}$ be the strategy spaces for Alice and Bob. If C_{A} is ordered by $F < O$ while C_{B} is ordered by $O < F$ then $C_{\mathrm{A}} \times C_{\mathrm{B}}$ turns out to be the lattice L_1 depicted by the following Hasse diagram, where, to help the reader, we also report (in blue) the output values of the function u_{A}:



In this case, $u_{\mathrm{A}}, u_{\mathrm{B}} : C_{\mathrm{A}} \times C_{\mathrm{B}} \to \mathbb{R}$ are both non-quasisupermodular. For example, we have that: $u_{\mathrm{A}}((O, O) \wedge (F, F)) = u_{\mathrm{A}}(F, O) = 0 \leq 3 = u_{\mathrm{A}}(O, O)$ whereas $u_{\mathrm{A}}(F, F) = 2 \not\leq 1 = u_{\mathrm{A}}(O, F) = u_{\mathrm{A}}((O, O) \vee (F, F))$. As a consequence, we also have that u_{A} and u_{B} are not supermodular.

On the other hand, if C_{A} and C_{B} are both ordered in the same way, let us say by $F < O$, then $C_{\mathrm{A}} \times C_{\mathrm{B}}$ is the following lattice L_2:



and u_{A} and u_{B} turn out to be supermodular. In fact, the only interesting case to check is for the pairs (F, O) and (O, F) which are uncomparable in the lattice $C_{\mathrm{A}} \times C_{\mathrm{B}}$:

$$u_{\mathrm{A}}((F,O) \vee (O,F)) + u_{\mathrm{A}}((F,O) \wedge (O,F)) = u_{\mathrm{A}}(O,O) + u_{\mathrm{A}}(F,F) = 3 + 2 \geq \\ 0 + 1 = u_{\mathrm{A}}(F,O) + u_{\mathrm{A}}(O,F)$$

so that u_{A} is supermodular and therefore quasisupermodular. $\square$

In general, the definitions of supermodular and quasisupermodular function do not require a complete lattice C which is defined as a product of some component complete lattices. If, instead, C is a product lattice $C_1 \times C_2$ then these notions are related to the so-called increasing differences and single crossing properties of functions. A function $u : C_1 \times C_2 \to \mathbb{R}^N$ has *increasing differences* when

$$(x,y) \leq (x',y') \;\Rightarrow\; u(x',y) - u(x,y) \leq u(x',y') - u(x,y')$$

or, equivalently, for any $(x,y) \leq (x',y')$, the functions $u(x',\cdot) - u(x,\cdot)$ and $u(\cdot,y') - u(\cdot,y)$ are monotone. The intuition is that this definition models a situation of "complementarity" of the input values for the components C_1 and C_2 of u: the incremental gain of the output value of $u(x,y)$ in choosing for the first component a higher x' rather than x is greater when the second component is higher, and analogously for the second component since the increasing differences condition can be equivalent stated as $u(x,y') - u(x,y) \leq u(x',y') - u(x',y)$.

Moreover, a function $u : C_1 \times C_2 \to \mathbb{R}^N$ has the *single crossing property* when for any $(x,y) \leq (x',y')$,

$$\begin{array}{l} u(x,y) \leq u(x',y) \;\Rightarrow\; u(x,y') \leq u(x',y') \\ u(x,y) < u(x',y) \;\Rightarrow\; u(x,y') < u(x',y'). \end{array}$$

Notice that if u has increasing differences then u has the single crossing property, while the converse does not hold. The intuition behind the single crossing property has its origins in a continuous function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$: here, the single crossing property for u means that, given any pair of values $x \leq x'$, the difference function $d(y) \triangleq u(x,y) - u(x',y) : \mathbb{R} \to \mathbb{R}$ crosses the horizontal axis at most once and from below, that is, if there exists some $\bar{y}$ such that $d(\bar{y}) = 0$ (i.e., d crosses the horizontal axis) then for any $y' \geq \bar{y}$, we have that $d(y') \geq 0$ (i.e., d never crosses the horizontal axis again).

Example 2.2 Let us consider again Example 2.1. In the case L_1, the function u_{A} does not have increasing differences and the single crossing property. In fact, for $(F,O) \leq (O,F)$, where (F,O) and (O,F) are the least and greatest pairs in L_1, we have that:

$$\begin{array}{rcl} u_{\mathrm{A}}(O,O) - u_{\mathrm{A}}(F,O) = 3 - 0 & \not\leq & 1 - 2 = u_{\mathrm{A}}(O,F) - u_{\mathrm{A}}(F,F) \\ u_{\mathrm{A}}(F,O) = 0 \leq 3 = u_{\mathrm{A}}(O,O) & \not\Rightarrow & u_{\mathrm{A}}(F,F) = 2 \leq 1 = u_{\mathrm{A}}(O,F) \end{array}$$

Similarly for $u_{\rm B}$. Instead, when considering the case L_2, $u_{\rm A}$ has increasing differences and the single crossing property. Here let us consider $(F,F) \leq (O,O)$, where (F,F) and (O,O) are the least and greatest pairs in L_2, so that:

$$\begin{array}{rcl} u_{\rm A}(O,F) - u_{\rm A}(F,F) = 1-2 & \leq & 3-0 = u_{\rm A}(O,O) - u_{\rm A}(F,O) \\ u_{\rm A}(F,F) = 2 \leq 1 = u_{\rm A}(O,F) & \Rightarrow & u_{\rm A}(F,O) = 0 \leq 3 = u_{\rm A}(O,O) \end{array}$$

and similarly for $u_{\rm B}$. $\square$

Supermodularity on product complete lattices and increasing differences are related as follows (see [24, Theorems 2.6.1 and 2.6.2]): a function $u : C_1 \times C_2 \to \mathbb{R}^N$ is supermodular if and only if u has increasing differences and, for any $c_i \in C_i$, $u(c_1, \cdot) : C_2 \to \mathbb{R}^N$ and $u(\cdot, c_2) : C_1 \to \mathbb{R}^N$ are supermodular. Analogously for quasisupermodularity and single crossing property: a function $u : C_1 \times C_2 \to \mathbb{R}^N$ is quasisupermodular if and only if u has the single crossing property and, given any $c_1 \in C_1$ and $c_2 \in C_2$, $u(c_1, \cdot) : C_2 \to \mathbb{R}^N$ and $u(\cdot, c_2) : C_1 \to \mathbb{R}^N$ are quasisupermodular.

2.2 Noncooperative Games

Let us recall some basic notions on noncooperative games, which can be found, e.g., in the books [2, 24].

A *noncooperative game* $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ for players $i = 1, ..., n$ (with $n \geq 2$) consists of a family of feasible strategy spaces $\langle S_i, \leq_i, \wedge_i, \vee_i \rangle_{i=1}^n$ which are assumed to be complete lattices and of a family of utility (or payoff) functions $u_i : \times_{i=1}^n S_i \to \mathbb{R}^{N_i}$, with $N_i \geq 1$. Hence, the product strategy space $S \triangleq \times_{i=1}^n S_i$ is a complete lattice for the componentwise partial order $\leq$, where $\wedge$ and $\vee$ denote its glb and lub. The i-th *best response correspondence* $B_i : S_{-i} \to \wp(S_i)$, with $i \in [1, n]$, is defined as

$$B_i(s_{-i}) \triangleq \mathrm{argmax}(u_i(\cdot, s_{-i})) = \{x_i \in S_i \mid \forall s_i \in S_i.\, u_i(s_i, s_{-i}) \leq u_i(x_i, s_{-i})\}.$$

The intuition is that $B_i(s_{-i})$ provides the set of strategies for the player i which produce the greatest utility for i while taking other players' strategies as given by s_{-i}. On the other hand, the best response correspondence $B : S \to \wp(S)$ on the whole strategy space S is defined by the product

$$B(s) \triangleq \times_{i=1}^n B_i(s_{-i}).$$

A strategy $s \in S$ is a *pure Nash equilibrium* for Γ when s is a fixed point of the best response correspondence B, i.e., $s \in B(s)$. Therefore, this means that in a pure Nash equilibrium s there is no feasible way for any player to strictly improve its utility if the strategies of all the other players remain unchanged. We denote by $\mathrm{Eq}(\Gamma) \in \wp(S)$ the set of Nash equilibria for Γ, so that $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B)$.

A noncooperative game $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ is *supermodular* when:

(1) for any i, for any $s_{-i} \in S_{-i}$, $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^{N_i}$ is supermodular;
(2) for any i, $u_i(\cdot, \cdot) : S_i \times S_{-i} \to \mathbb{R}^{N_i}$ has increasing differences.

Also, Γ is *quasisupermodular* (or, with *strategic complementarities*) when:

(1) for any i, for any $s_{-i} \in S_{-i}$, $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^{N_i}$ is quasisupermodular;
(2) for any i, $u_i(\cdot, \cdot) : S_i \times S_{-i} \to \mathbb{R}^{N_i}$ has the single crossing property.

In both these cases, it turns out (cf. [24, Theorems 2.8.1 and 2.8.6]) that any i-th best response correspondence $B_i : S_{-i} \to \wp(S_i)$ is EM-monotone, as well as the best response correspondence $B : S \to \wp(S)$.

Let us also recall [1, Section X.12] that the *interval topology* of a complete lattice $\langle C, \leq \rangle$ (actually, the interval topology can be defined for a mere poset) is defined by taking the closed intervals $[a, b] \triangleq \{x \in C \mid a \leq x \leq b\}$ as a subbasis of the closed sets and that a complete lattice is always compact for its interval topology. Also, a function $f : C \to \mathbb{R}^N$ is called *order upper semicontinuous* if f is upper semicontinuous (according to the standard definition) for the interval topology of C. It turns out (cf. [24, Lemma 4.2.2]) that if each $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^{N_i}$ is order upper semicontinuous then, for each $s \in S$, $B_i(s_{-i}) \in \mathrm{SL}(S_i)$, i.e., $B_i(s_{-i})$ is a nonempty subcomplete sublattice of S_i, so that $B(s) \in \mathrm{SL}(S)$ also holds. In particular, we have that $\wedge_i B_i(s_{-i}), \vee_i B_i(s_{-i}) \in B_i(s_{-i})$ as well as $\wedge B(s), \vee B(s) \in B(s)$, namely, $B_i(s_{-i}) \in \wp^{\diamond}(S_i)$ and $B(s) \in \wp^{\diamond}(S)$. With this hypothesis, it also turns out [27, Theorem 2] that the set of Nash equilibria gives rise to a complete lattice $\langle \mathrm{Eq}(\Gamma), \leq \rangle$—although, in general, it is not a subcomplete sublattice of S—and therefore Γ admits the least and greatest Nash equilibria, which are denoted, respectively, by $\mathrm{leq}(\Gamma)$ and $\mathrm{geq}(\Gamma)$. It should be remarked that the hypothesis of order upper semicontinuity for $u_i(\cdot, s_{-i})$ trivially holds for any finite-strategy game, namely for those games where each strategy space S_i is finite. In the following, we will consider (quasi)supermodular games which satisfy this hypothesis of order upper semicontinuity.

If, given any $s_i \in S_i$, the function $u_i(s_i, \cdot) : S_{-i} \to \mathbb{R}^{N_i}$ is monotone then it turns out [2, Propositions 8.23 and 8.51] that $\mathrm{geq}(\Gamma)$ majorizes all equilibria, i.e., for all i and $s \in \mathrm{Eq}(\Gamma)$, $u_i(\mathrm{geq}(\Gamma)) \geq u_i(s)$, while $\mathrm{leq}(\Gamma)$ minimizes all equilibria.

Example 2.3 Let us consider again the battle of the sexes supermodular game Γ described in Section 1 and in Example 2.1, where C_{A} and C_{B} are both ordered by $F < O$. We have that Alice and Bob best response correspondences, respectively, $B_{\mathrm{A}} : C_{\mathrm{B}} \to \wp(C_{\mathrm{A}})$ and $B_{\mathrm{B}} : C_{\mathrm{A}} \to \wp(C_{\mathrm{B}})$, are as follows:

$$B_{\mathrm{A}}(F) = \{F\} = B_{\mathrm{B}}(F) \qquad B_{\mathrm{A}}(O) = \{O\} = B_{\mathrm{B}}(O),$$

so that the whole best response correspondence $B : C_{\mathrm{A}} \times C_{\mathrm{B}} \to \wp(C_{\mathrm{A}} \times C_{\mathrm{B}})$ is defined as follows:

$$B(F,F) = \{(F,F)\} \quad B(F,O) = \{(O,F)\} \quad B(O,F) = \{(F,O)\} \quad B(O,O) = \{(O,O)\}$$

Thus, the fixed points of Γ, i.e. its Nash equilibria, are (F,F) and (O,O), so that $\mathrm{leq}(\Gamma) = (F,F)$ and $\mathrm{geq}(\Gamma) = (O,O)$. $\square$

Computing Game Equilibria. Consider a (quasi)supermodular game $\Gamma = \langle S_i, u_i \rangle_{i=1}^{n}$ and let us define the functions $B_{\wedge}, B_{\vee} : S \to S$ as follows:

$$B_{\wedge}(s) \triangleq \wedge B(s) \qquad B_{\vee}(s) \triangleq \vee B(s).$$

As recalled above, it turns out that $B_{\wedge}(s), B_{\vee}(s) \in B(s)$. When the image of the strategy space S for $B_{\wedge}$ turns out to be finite, the standard algorithm [24, Algorithm 4.3.2] for computing the least Nash equilibrium $\mathrm{leq}(\Gamma)$ consists in applying the constructive Knaster-Tarski fixed point theorem to the function $B_{\wedge}$ so that $\mathrm{leq}(\Gamma) = \bigvee_{k \geq 0} B_{\wedge}^{k}(\bot_S)$. Dually, we have that $\mathrm{geq}(\Gamma) = \bigwedge_{k \geq 0} B_{\vee}^{k}(\top_S)$. In particular, this procedure can be always used for finite games. The application of a standard chaotic iteration strategy in this fixed point computation [11] yields the Robinson-Topkis (RT) algorithm [24, Algorithm 4.3.1] in Figure 1,

$\langle s_1,...,s_n\rangle := \langle \bot_1,...,\bot_n\rangle$; // $\langle s_1,...,s_n\rangle := \langle \top_1,...,\top_n\rangle$;
do $\{\langle t_1,...,t_n\rangle := \langle s_1,...,s_n\rangle$;
  $s_1 := \wedge_1 B_1(s_{-1})$; // $s_1 := \vee_1 B_1(s_{-1})$;
  ...
  $s_n := \wedge_n B_n(s_{-n})$; // $s_n := \vee_n B_n(s_{-n})$;
$\}$ **while** $\neg(\langle s_1,...,s_n\rangle = \langle t_1,...,t_n\rangle)$

Fig. 1 Robinson-Topkis (RT) algorithm.

also called round-robin optimization, which is presented in its version for least fixed points, while the statements in comments provide the version for calculating greatest fixed points.

Let us provide a running example of supermodular finite game.

Example 2.4 Consider a two players finite game Γ represented in so-called normal form by the following double-entry payoff matrix M:

	1	**2**	**3**	**4**	**5**	**6**
6	-1, -3	-1, -1	2, 4	5, 6	6, 5	6, 5
5	0, 0	0, 2	3, 4	6, 6	7, 5	6, 5
4	3, 1	3, 3	3, 5	5, 6	5, 5	4, 4
3	2, 2	2, 4	2, 6	4, 5	4, 4	3, 2
2	6, 4	6, 6	6, 7	6, 4	5, 2	4, -1
1	6, 4	5, 6	5, 6	4, 2	3, 0	2, -3

The strategy spaces S_1 and S_2 are both the finite chain of integers $C = \langle\{1,2,3,4,5,6\},\leq\rangle$ and $u_1(x,y), u_2(x,y) : S_1 \times S_2 \to \mathbb{R}$ are, respectively, the first and second entry in the payoff matrix element determined by row x and column y, that is, if $M(x,y) = (a_1,a_2)$ then $u_i(x,y) = a_i$. For example, $u_1(2,6) = 4$ and $u_2(2,6) = -1$. It turns out that both u_1 and u_2 have increasing differences, so that, since S_1 and S_2 are finite (chains), Γ is a finite supermodular game. The two best response correspondences $B_1, B_2 : C \to \mathrm{SL}(C)$ are as follows:

$$B_1(1) = \{1,2\}, B_1(2) = B_1(3) = \{2\}, B_1(4) = \{2,5\}, B_1(5) = \{5\}, B_1(6) = \{5,6\};$$
$$B_2(1) = \{2,3\}, B_2(2) = B_2(3) = \{3\}, B_2(4) = B_2(5) = B_2(6) = \{4\}.$$

Thus, $\mathrm{Eq}(\Gamma) = \{(2,3),(5,4)\}$, since this is the set $\mathrm{Fix}(B)$ of fixed points of the best response correspondence $B = B_1 \times B_2$: indeed, $(2,3) \in B(2,3) = \{(2,3)\}$ and $(5,4) \in B(5,4) = \{(2,4),(5,4)\}$. We also notice that $u_1(\cdot,s_2), u_2(s_1,\cdot) : C \to \mathbb{R}$ are neither monotone nor antimonotone. The fixed point computations of the least and greatest equilibria through the RT algorithm in Figure 1 proceed as follows:

$$\begin{aligned}(\bot_1,\bot_2) = (1,1) &\mapsto \big(\wedge B_1(1),1\big) = (1,1)\\ &\mapsto \big(1,\wedge B_2(1)\big) = (1,2)\\ &\mapsto \big(\wedge B_1(2),2\big) = (2,2)\\ &\mapsto \big(2,\wedge B_2(2)\big) = (2,3)\\ &\mapsto \big(\wedge\, B_1(3),3\big) = (2,3)\end{aligned}$$

$$\begin{aligned}(\top_1, \top_2) = (6,6) &\mapsto \big(\vee B_1(6), 6\big) = (6,6)\\ &\mapsto \big(6, \vee B_2(6)\big) = (6,4)\\ &\mapsto \big(\vee B_1(4), 4\big) = (5,4)\\ &\mapsto \big(5, \vee B_2(5)\big) = (5,4)\end{aligned}$$

so that $\mathrm{leq}(\Gamma) = (2,3)$ and $\mathrm{geq}(\Gamma) = (5,4)$. □

3 Abstractions on Product Domains

3.1 Background on Abstract Interpretation

Static program analysis relies on correct (a.k.a. sound) and computable semantic approximations. A program P is modeled by some semantics $\mathrm{Sem}[\![P]\!]$ and a static analysis of P is designed as an approximate semantics $\mathrm{Sem}^\sharp[\![P]\!]$ which must be correct w.r.t. $\mathrm{Sem}[\![P]\!]$. This may be called global correctness of static analysis. Any (finite) program P is a suitable composition of a number of constituent expressions and subprograms c_i, e.g., Boolean and arithmetic expressions and assignments, and this is reflected on its global semantics $\mathrm{Sem}[\![P]\!]$ which is commonly defined by some combination of the semantics $\mathrm{Sem}[\![c_i]\!]$ of its components. Thus, global correctness of a static analysis of P relies on a local correctness for its components c_i. This global vs. local picture of static analysis correctness is very common, independently from the kind of programs (imperative, functional, reactive, etc.), of static analysis techniques (abstract interpretation, model checking, logical deductive systems, type systems, etc.), of program properties under analysis (safety, liveness, numerical properties, pointer aliasing, type safety, etc.). A basic general proof principle in static analysis is that global correctness is derived from local correctness. In particular this applies to static program analyses that are designed using some form of abstract interpretation [3, 4]. Let us consider a simplified but recurrent scenario, where $\mathrm{Sem}[\![P]\!]$ is defined as least (or greatest) fixed point $\mathrm{lfp}(f)$ of a monotone function f on some domain C of program properties, which is endowed with a partial order that encodes the relative precision of properties. In abstract interpretation, a static analysis is then specified as an abstract fixed point computation which must be correct for $\mathrm{lfp}(f)$. This is defined through an ordered abstract domain A of properties and an abstract semantic function $f^\sharp : A \to A$ on A that give rise to a fixed point-based static analysis $\mathrm{lfp}(f^\sharp)$ (whose decidability and/or practical scalability is usually ensured by chain conditions on A, widenings/narrowings operators, interpolations, etc.). Correctness relies on encoding approximation through a concretization map $\gamma : A \to C$ and/or an abstraction map $\alpha : C \to A$: the approximation of some value $c \in C$ through an abstract property $a \in A$ is encoded as $c \leq_C \gamma(a)$ or — equivalently, when α/γ form a Galois connection — $\alpha(c) \leq_A a$. In this scenario, global correctness translates to $\mathrm{lfp}(f) \leq \gamma(\mathrm{lfp}(f^\sharp))$, local correctness means $f \circ \gamma \sqsubseteq \gamma \circ f^\sharp$, and the well-known "fixed point approximation lemma" [3,4] tells us that local correctness implies global correctness.

In standard abstract interpretation, concrete and abstract domains (also called abstractions), $\langle C, \leq_C\rangle$ and $\langle A, \leq_A\rangle$, are assumed to be complete lattices which are related by abstraction and concretization maps $\alpha : C \to A$ and $\gamma : A \to C$ that give rise to a Galois connection (GC for short) (α, C, A, γ), i.e., for all $a \in A$ and $c \in C$, $\alpha(c) \leq_A a \Leftrightarrow c \leq_C \gamma(a)$. Recall that a GC is a Galois insertion (GI for short) when $\alpha\circ\gamma = \mathrm{id}$. A GC is called (finitely) disjunctive when γ preserves all (finite) lubs (also called γ is (finitely) disjunctive). We use $\mathrm{Abs}(C)$ to denote all the possible abstractions of C, where $A \in \mathrm{Abs}(C)$ means that A is an abstract domain of C specified by some GC/GI. Let us recall that a map $\rho : C \to C$ is a (upper) closure operator when: (i) ρ is monotone: $x \leq y \;\Rightarrow\; \rho(x) \leq \rho(y)$; (ii) ρ is increasing: $x \leq \rho(x)$; (iii) ρ is idempotent: $\rho(\rho(x)) = \rho(x)$. We denote by $\mathrm{uco}(\langle C, \leq\rangle)$ the set of all closure operators on the complete lattice $\langle C, \leq_C\rangle$. Throughout the paper, we will make use of some well known properties of a GC (α, C, A, γ) and of closure operators:

(1) α preserves arbitrary lubs;
(2) γ preserves arbitrary glbs;
(3) $\gamma \circ \alpha : C \to C$ is a closure operator
(4) if $\rho : C \to C$ is a closure operator then $(\rho, C, \rho(C), \mathrm{id})$ is a GI;
(5) (α, C, A, γ) is a GC iff $\gamma(A)$ is the image of a closure operator on C;
(6) $S \subseteq C$ is the image of a closure $\rho_S \in \mathrm{uco}(C)$ iff S is a Moore family (also called Moore-closed or meet-closed), i.e., closed under meets of arbitrary subsets, empty set included; in this case, $\rho_S(c) = \wedge_C\{x \in S \mid c \leq x\}$ (this is also called the closure operator induced by S);
(7) a GC (α, C, A, γ) is (finitely) disjunctive iff $\gamma(A)$ is a Moore family and closed under joins of (finite) subsets.

Also, if $\alpha : C \to A$ preserves arbitrary lubs then we can obtain a GC (α, C, A, α^+) by considering its so-called right-adjoint $\alpha^+ \triangleq \lambda a.\, \vee_C \{c \in C \mid \alpha(c) \leq_A a\}$. Dually, if $\gamma : C \to A$ preserves arbitrary glbs then (γ^-, C, A, γ) is a GC where $\gamma^- \triangleq \lambda c.\, \wedge_A \{a \in A \mid c \leq_C \gamma(a)\}$ is the left-adjoint of γ.

Example 3.1 Let us consider a concrete domain $\langle C, \leq\rangle$ which is a finite chain. Then, it turns out that (α, C, A, γ) is a GC iff, by point (5) above, $\gamma(A)$ is the image of a closure operator on C iff, by point (6) above, $\gamma(A)$ is any subset of C which contains the top element $\top_C$. As an example, for the game Γ in Example 2.4, where S_i is the chain of integers $\{1, 2, 3, 4, 5, 6\}$, we have that $A_1 = \{3, 5, 6\}$ and $A_2 = \{2, 6\}$ are two abstractions of C. □

Example 3.2 Let us consider the ceil function on real numbers $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{R}$, that is, $\lceil x\rceil$ is the smallest integer not less than x. Let us observe that $\lceil\cdot\rceil$ is a closure operator on $\langle \mathbb{R}, \leq\rangle$ because: (1) $x \leq y \;\Rightarrow\; \lceil x\rceil \leq \lceil y\rceil$; (2) $x \leq \lceil x\rceil$; (3) $\lceil\lceil x\rceil\rceil = \lceil x\rceil$. Therefore, the ceil function allows us to view integer numbers $\mathbb{Z} = \lceil\mathbb{R}\rceil$ as an abstraction of real numbers. The ceil function can be generalized to any finite fractional part of real numbers: given any integer number $N \geq 0$, $\mathrm{cl}_N : \mathbb{R} \to \mathbb{R}$ is defined as follows: $\mathrm{cl}_N(x) = \frac{\lceil 10^N x\rceil}{10^N}$. For $N = 0$, $\mathrm{cl}_N(x) = \lceil x\rceil$, while for $N > 0$, $\mathrm{cl}_N(x)$ is the smallest rational number with at most N fractional digits not less than x. For example, if $x \in \mathbb{R}$ and $1 < x \leq 1.01$ then $\mathrm{cl}_2(x) = 1.01$. Clearly, it turns out that cl_N is a closure operator which permits to cast rational numbers with at most N fractional digits as an abstraction of real numbers. □

In the following, we show how abstractions of different concrete domains C_i can be composed in order to define an abstract domain of the product domain $\times_i C_i$, and, on the other hand, an abstraction of a product $\times_i C_i$ can be decomposed into abstract domains of the component domains C_i.

3.2 Product Composition of Abstractions

As shown by Cousot and Cousot [7, Section 4.4], given a family of GCs $(\alpha_i, C_i, A_i, \gamma_i)_{i=1}^n$, one can easily define a componentwise GC $(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$, where $\times_{i=1}^n C_i$ and

$\times_{i=1}^n A_i$ are both complete lattices w.r.t. the componentwise partial order and for any $c \in \times_{i=1}^n C_i$ and $a \in \times_{i=1}^n A_i$,

$$\alpha(c) \triangleq (\alpha_i(c_i))_{i=1}^n \qquad\qquad \gamma(a) \triangleq (\gamma_i(a_i))_{i=1}^n.$$

For any $i \in [1, n]$, we also use the function $\gamma_{-i} : A_{-i} \to C_{-i}$ to denote $\gamma_{-i}(a_{-i}) \triangleq \gamma(a)_{-i} = (\gamma_j(a_j))_{j\neq i}$.

Lemma 3.3 *$(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$ is a GC. Moreover, if each $(\alpha_i, C_i, A_i, \gamma_i)$ is a (finitely) disjunctive GC then $(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$ is a (finitely) disjunctive GC.*

Proof Easy, since, for any $c \in \times_{i=1}^n C_i$ and $a \in \times_{i=1}^n A_i$, we have that $\alpha(c) \le a \Leftrightarrow \forall i.\, \alpha_i(c_i) \le_i a_i \Leftrightarrow \forall i.\, c_i \le_i \gamma_i(a_i) \Leftrightarrow c \le \gamma(a)$. Moreover, if $X \subseteq \times_{i=1}^n A_i$ and γ preserves arbitrary (finite when X is finite) lubs then $\gamma(\vee X) = \gamma((\vee_i X_i)_{i=1}^n) = (\gamma_i(\vee_i X_i))_{i=1}^n = (\vee_i \gamma_i(X_i))_{i=1}^n = (\vee_i \gamma(X)_i)_{i=1}^n = \vee\gamma(X)$. ☐

Let us observe that $(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$ is a so-called nonrelational abstraction since the product abstraction $\times_{i=1}^n A_i$ does not take into account any relationship between the different concrete domains C_i.

3.3 Decomposition of Product Abstractions

Let us show that any GC $(\alpha, \times_{i=1}^n C_i, A, \gamma)$ for the concrete product domain $\times_{i=1}^n C_i$ induces a family of corresponding abstractions $(\alpha_i, C_i, A_i, \gamma_i)$ of the component concrete domains C_i as follows:

- $A_i \triangleq \{c_i \in C_i \mid \exists a \in A. \gamma(a)_i = c_i\} \subseteq C_i$, endowed with the partial order $\le_i$ of C_i
- for any $c_i \in C_i$, $\alpha_i(c_i) \triangleq \gamma(\alpha(c_i, \bot_{-i}))_i$
- for any $x_i \in A_i$, $\gamma_i(x_i) \triangleq x_i$

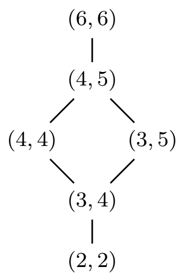
where for any $c_i \in C_i$, $(c_i, \bot_{-i})$ is used to denote $(\bot_1, ..., \bot_{i-1}, c_i, \bot_{i+1}, ..., \bot_n) \in \times_{i=1}^n C_i$.

Lemma 3.4 *For any $i \in [1, n]$, $(\alpha_i, C_i, A_i, \gamma_i)$ is a GC. Moreover, this GC is (finitely) disjunctive when $(\alpha, \times_{i=1}^n C_i, A, \gamma)$ is (finitely) disjunctive.*

Proof Let us show that $A_i \subseteq C_i$ is a Moore family. If $X \subseteq A_i$ then for any $x \in X$ there exists some $a_x \in A$ such that $\gamma(a_x)_i = x$. Then, let $a \triangleq \wedge_A \{a_x \in A \mid x \in X\} \in A$. Since γ preserves arbitrary meets, we have that $\gamma(a) = \wedge_C \{\gamma(a_x) \in C \mid x \in X\}$, so that $\gamma(a)_i = \wedge_{C_i} X$, that is, $\wedge_{C_i} X \in A_i$. Hence, since A_i is meet-closed, we have that the identical function $\gamma_i = \mathrm{id} : A_i \to C_i$ preserves arbitrary meets and therefore is a concretization map. Let us check that α_i is the left adjoint of γ_i, i.e., for any $c_i \in C_i$, $\alpha_i(c_i) = \gamma(\alpha(c_i, \bot_{-i}))_i = \wedge_{C_i}\{x_i \in A_i \mid c_i \le_i x_i\}$. On the one hand, since $(c_i, \bot_{-i}) \le \gamma(\alpha(c_i, \bot_{-i}))$, we have that $c_i \le_i \gamma(\alpha(c_i, \bot_{-i}))_i$, so that since $\gamma(\alpha(c_i, \bot_{-i}))_i \in A_i$, we conclude that $\wedge_{C_i}\{x_i \in A_i \mid c_i \le_i x_i\} \le_i \gamma(\alpha(c_i, \bot_{-i}))_i$. On the other hand, if $x_i \in A_i$ and $c_i \le_i x_i$ then $x_i = \gamma(a)_i$ for some $a \in A$, so that we have that $(c_i, \bot_{-i}) \le \gamma(a)$, therefore $\gamma(\alpha(c_i, \bot_{-i})) \le \gamma(\alpha(\gamma(a))) = \gamma(a)$, and, in turn, $\gamma(\alpha(c_i, \bot_{-i}))_i \le_i \gamma(a)_i = x_i$, which implies that $\gamma(\alpha(c_i, \bot_{-i}))_i \le_i \wedge_{C_i}\{x_i \in A_i \mid c_i \le_i x_i\}$. Finally, let us observe that if γ is (finitely) disjunctive and $X \subseteq A_i$, so that for any $x \in X$ there exists some $a_x \in A$ such that $\gamma(a_x)_i = x$, then $\gamma(\vee_A \{a_x \in A \mid x \in X\}) = \vee\{\gamma(a_x) \in \times_{i=1}^n C_i \mid x \in X\}$, so that $\gamma(\vee_A \{a_x \in A \mid x \in X\})_i = \vee_i \gamma(a_x)_i = \vee_i X$, namely, $\vee_i X \in A_i$, meaning that $\gamma_i = \mathrm{id}$ is (finitely) disjunctive. ☐

Let us recall that two GCs $(\alpha', C, A', \gamma')$ and $(\alpha'', C, A'', \gamma'')$ are isomorphic when $\gamma' \circ \alpha' = \gamma'' \circ \alpha''$. We define a GC $(\alpha, \times_{i=1}^{n} C_i, A, \gamma)$ as *nonrelational* when it is isomorphic to the product composition, according to Lemma 3.3, of its component GCs as obtained by Lemma 3.4. Otherwise, $(\alpha, \times_{i=1}^{n} C_i, A, \gamma)$ is called *relational*. Of course, according to this definition, the product composition by Lemma 3.3 of abstract domains is trivially nonrelational. It is worth remarking that if A is relational then A cannot be obtained as a product of abstractions of C. As a consequence, the property of being relational for an abstraction A prevents the definition of a standard noncooperative game over the strategy space A since A cannot be obtained as a product domain.

Example 3.5 Let us consider the game Γ in Example 2.4 whose finite strategy space is $C \times C$, where $C = \{1, 2, 3, 4, 5, 6\}$ is a chain. Consider the subset $A \subseteq C \times C$ as depicted by the following diagram where the ordering is induced from $C \times C$:



Since A is meet- and join-closed and includes the greatest element $(6, 6)$ of $C \times C$, we have that A is a disjunctive abstraction of $C \times C$, where $\alpha : C \times C \to A$ is the closure operator induced by A (cf. point (6) in Section 3.1) and $\gamma : A \to C \times C$ is the identity. Observe that A is relational since its decomposition by Lemma 3.4 provides $A_1 = \{2, 3, 4, 6\}$ and $A_2 = \{2, 4, 5, 6\}$, and the product composition $A_1 \times A_2$ by Lemma 3.3 yields a more expressive abstraction than A, for example $(2, 4) \in (A_1 \times A_2) \smallsetminus A$. On the other hand, for the abstractions $A_1 = \{3, 5, 6\}$ and $A_2 = \{2, 6\}$ of Example 3.1, the product domain $A_1 \times A_2$ defined according to Lemma 3.3 is a nonrelational abstraction of $C \times C$. $\square$

4 Approximation of Multivalued Functions

Let $f : C \to C$ be some concrete monotone function, $A \in \mathrm{Abs}(C)$ be an abstraction specified by a GC (α, C, A, γ) and $f^\sharp : A \to A$ be a corresponding monotone abstract function which is a correct (also called sound) approximation of f, that is, $f \circ \gamma \sqsubseteq \gamma \circ f^\sharp$ holds. Let us recall that fixed point correctness for $f^\sharp$ holds, i.e., $\mathrm{lfp}(f) \leq_C \gamma(\mathrm{lfp}(f^\sharp))$ and $\mathrm{gfp}(f) \leq_C \gamma(\mathrm{gfp}(f^\sharp))$. Also, let us recall that $f^A \triangleq \alpha \circ f \circ \gamma : A \to A$ is the best correct approximation of f on A, because it turns out that any abstract function $f^\sharp$ is a correct approximation of f iff $f^A \sqsubseteq f^\sharp$ holds. In the following, we show how to lift these standard notions in order to approximate least/greatest fixpoints of multivalued functions.

4.1 Constructive Results for Fixed Points of Multivalued Functions

Let $f : C \to \wp(C)$ be a multivalued function and $f_\wedge, f_\vee : C \to C$ be the functions defined as:

$$f_\wedge(c) \triangleq \wedge f(c) \qquad f_\vee(c) \triangleq \vee f(c).$$

The following constructive result ensuring the existence of least fixed points for a multivalued function is given by Straccia et al. in [22, Propositions 3.10 and 3.24]. We provide here a shorter and more direct constructive proof than in [22] which is based on the constructive version of Tarski's fixed point theorem given by Cousot and Cousot [5, Theorem 5.1].

Lemma 4.1 *If* $f : C \to \wp^{\wedge}(C)$ *is S-monotone then* f *has the least fixed point* $\mathrm{lfp}(f)$. *Moreover,* $\mathrm{lfp}(f) = \bigvee_{\alpha\in\mathbb{O}} f_{\wedge}^{\alpha}(\bot)$.

Proof By hypothesis, for any $x \in C$, $f(x) \in \wp^{\wedge}(C)$, so that $f_{\wedge}(x) \in f(x)$. If $x, y \in C$ and $x \leq y$ then, by hypothesis, $f(x) \preceq_S f(y)$. Therefore, since $f_{\wedge}(y) \in f(y)$, there exists some $z \in f(x)$ such that $z \leq f_{\wedge}(y)$, and, in turn, $f_{\wedge}(x) \leq z \leq f_{\wedge}(y)$. Hence, since $f_{\wedge}$ is a monotone function on a complete lattice, by Knaster-Tarski's theorem, its least fixed point $\mathrm{lfp}(f_{\wedge}) \in C$ exists. Furthermore, by the constructive version of Tarski's theorem [5, Theorem 5.1], $\mathrm{lfp}(f_{\wedge}) = \bigvee_{\alpha\in\mathbb{O}} f_{\wedge}^{\alpha}(\bot)$. We have that $\mathrm{lfp}(f_{\wedge}) = f_{\wedge}(\mathrm{lfp}(f_{\wedge})) \in f(\mathrm{lfp}(f_{\wedge}))$, hence $\mathrm{lfp}(f_{\wedge}) \in \mathrm{Fix}(f)$. Consider any $z \in \mathrm{Fix}(f)$. We prove by transfinite induction that for any $\alpha \in \mathbb{O}$, $f_{\wedge}^{\alpha}(\bot) \leq z$. If $\alpha = 0$ then $f_{\wedge}^{0}(\bot) = \bot \leq z$. If $\alpha = \beta + 1$ then $f_{\wedge}^{\alpha}(\bot) = f_{\wedge}(f_{\wedge}^{\beta}(\bot))$, and, since, by inductive hypothesis, $f_{\wedge}^{\beta}(\bot) \leq z$, then, by monotonicity of $f_{\wedge}$, $f_{\wedge}(f_{\wedge}^{\beta}(\bot)) \leq f_{\wedge}(z) = \wedge f(z) \leq z$. If $\alpha = \vee\{\beta \in \mathbb{O} \mid \beta < \alpha\}$ is a limit ordinal then $f_{\wedge}^{\alpha}(\bot) = \bigvee_{\beta<\alpha} f_{\wedge}^{\beta}(\bot)$; since, by inductive hypothesis, $f_{\wedge}^{\beta}(\bot) \leq z$ for any $\beta < \alpha$, we obtain that $f_{\wedge}^{\alpha}(\bot) \leq z$. This therefore shows that f has the least fixed point $\mathrm{lfp}(f) = \mathrm{lfp}(f_{\wedge})$. □

By duality, as consequences of the above result, we obtain the following characterizations, where point (3) coincides with Zhou's theorem (see [27, Theorem 1] and [22, Proposition 3.15]), which is used for showing that pure Nash equilibria of a supermodular game form a complete lattice.

Corollary 4.2
(1) *If* $f : C \to \wp^{\vee}(C)$ *is H-monotone then* f *has the greatest fixed point* $\mathrm{gfp}(f) = \bigwedge_{\alpha\in\mathbb{O}} f_{\vee}^{\alpha}(\top)$.
(2) *If* $f : C \to \wp^{\diamond}(C)$ *is EM-monotone then* f *has the least and greatest fixed points, where* $\mathrm{lfp}(f) = \bigvee_{\alpha\in\mathbb{O}} f_{\wedge}^{\alpha}(\bot)$ *and* $\mathrm{gfp}(f) = \bigwedge_{\alpha\in\mathbb{O}} f_{\vee}^{\alpha}(\top)$.
(3) *If* $f : C \to \mathrm{SL}(C)$ *is EM-monotone then* $\langle \mathrm{Fix}(f), \leq \rangle$ *is a complete lattice.*
(4) *If* $f, g : C \to \mathrm{SL}(C)$ *are EM-monotone and, for any* $c \in C$, $f(c) \preceq_{EM} g(c)$ *then* $\mathrm{Fix}(f) \preceq_{EM} \mathrm{Fix}(g)$.

Proof Point (1) is dual to Lemma 4.1, and together imply Point (2). Point (3) is proved in [27, Theorem 1]. Let us prove point (4). By Point (3), both $\mathrm{Fix}(f)$ and $\mathrm{Fix}(g)$ are complete lattices for $\leq$. Thus, $\mathrm{Fix}(f) \preceq_{EM} \mathrm{Fix}(g)$ holds iff $\wedge \mathrm{Fix}(f) = \mathrm{lfp}(f) \leq \mathrm{lfp}(g) = \wedge \mathrm{Fix}(g)$ and $\vee \mathrm{Fix}(f) = \mathrm{gfp}(f) \leq \mathrm{gfp}(g) = \vee \mathrm{Fix}(g)$. Moreover, since, for any $c \in C$, $f(c) \preceq_{EM} g(c)$, we also have that $f_{\wedge}(c) = \wedge f(c) \leq \wedge g(c) = g_{\wedge}(c)$, thus, as a consequence, $\mathrm{lfp}(f_{\wedge}) \leq \mathrm{lfp}(g_{\wedge})$. The proof of Lemma 4.1 shows that $\mathrm{lfp}(f) = \mathrm{lfp}(f_{\wedge})$ and $\mathrm{lfp}(g) = \mathrm{lfp}(g_{\wedge})$, so that we obtain $\mathrm{lfp}(f) \leq \mathrm{lfp}(g)$. The proof for $\mathrm{gfp}(f) \leq \mathrm{gfp}(g)$ is dual. □

4.2 Concretization-based Approximations

As argued by Cousot and Cousot in [6], a minimal requirement for defining an abstract domain consists in specifying the meaning of its abstract values through a concretization map. Let $\langle A, \leq_A \rangle$ be an abstraction of a concrete domain C specified by a monotone concretization map $\gamma : A \to C$. Let us observe that its powerset lifting $\gamma^s : \wp(A) \to \wp(C)$ is S-monotone, meaning that if $Y_1 \preceq_S Y_2$ then $\gamma^s(Y_1) \preceq_S \gamma^s(Y_2)$: in fact, if $\gamma(y_2) \in \gamma^s(Y_2)$ then there exists $y_1 \in Y_1$ such that $y_1 \leq_A y_2$, so that $\gamma(y_1) \in \gamma^s(Y_1)$ and $\gamma(y_1) \leq_C \gamma(y_2)$, i.e., $\gamma^s(Y_1) \preceq_S \gamma^s(Y_2)$. Analogously, γ^s is H- and EM-monotone. Then, consider a concrete S-monotone multivalued function $f : C \to \wp^\wedge(C)$, whose least fixed point exists by Lemma 4.1.

Definition 4.3 (Correct Approximation of Multivalued Functions) An abstract multivalued function $f^\sharp : A \to \wp(A)$ over A is a *S-correct approximation* of f when:

(1) $f^\sharp : A \to \wp^\wedge(A)$ and $f^\sharp$ is S-monotone (fixed point condition)
(2) for any $a \in A$, $f(\gamma(a)) \preceq_S \gamma^s(f^\sharp(a))$ (soundness condition)

H- and EM-correct approximations are defined by replacing in this definition S- with, respectively, H- and EM-, and $\wp^\wedge$ with, respectively, $\wp^\vee$ and $\wp^\diamond$. □

Let us point out that the soundness condition (2) in Definition 4.3 is close to the standard correctness requirement used in abstract interpretation: the main technical difference is that we deal with mere preorders $\langle \wp^\wedge(C), \preceq_S \rangle$ and $\langle \wp^\wedge(A), \preceq_S \rangle$ rather than posets. However, this is enough for guaranteeing a correct approximation of least fixed points.

Theorem 4.4 (Correct Least Fixed Point Approximation) *If $f^\sharp$ is a S-correct approximation of f then* $\mathrm{lfp}(f) \leq_C \gamma(\mathrm{lfp}(f^\sharp))$.

Proof Let us consider $f_\wedge : C \to C$ and $f^\sharp_\wedge : A \to A$. By Lemma 4.1, $\mathrm{lfp}(f) = \mathrm{lfp}(f_\wedge)$ and $\mathrm{lfp}(f^\sharp) = \mathrm{lfp}(f^\sharp_\wedge)$. Let us check that $f^\sharp_\wedge$ is a standard correct approximation of $f_\wedge$. For any $a \in A$, $\gamma(f^\sharp_\wedge(a)) \in \gamma^s(f^\sharp(a))$, hence, since $f(\gamma(a)) \preceq_S \gamma^s(f^\sharp(a))$, we have that there exists some $z \in f(\gamma(a))$ such that $z \leq \gamma(f^\sharp_\wedge(a))$, so that $f_\wedge(\gamma(a)) = \wedge f(\gamma(a)) \leq z \leq \gamma(f^\sharp_\wedge(a))$. Hence, by the concretization-based fixed point transfer (see [16, Theorem 2.2.4]), it turns out that $\mathrm{lfp}(f_\wedge) \leq_C \gamma(\mathrm{lfp}(f^\sharp_\wedge))$, therefore showing that $\mathrm{lfp}(f) \leq \gamma(\mathrm{lfp}(f^\sharp))$. □

Analogous results hold for H- and EM-correct approximations.

Corollary 4.5
(1) *If $f^\sharp$ is a H-correct approximation of f then* $\mathrm{gfp}(f) \leq_C \gamma(\mathrm{gfp}(f^\sharp))$.
(2) *If $f^\sharp$ is a EM-correct approximation of f then* $\mathrm{Fix}(f) \preceq_{EM} \gamma^s(\mathrm{Fix}(f^\sharp))$, *in particular,* $\mathrm{lfp}(f) \leq_C \gamma(\mathrm{lfp}(f^\sharp))$ *and* $\mathrm{gfp}(f) \leq_C \gamma(\mathrm{gfp}(f^\sharp))$.

Proof Point (1) follows by duality from Theorem 4.4. By Theorem 4.4 and point (1), we obtain $\mathrm{lfp}(f) \leq \gamma(\mathrm{lfp}(f^\sharp))$ and $\mathrm{gfp}(f) \leq \gamma(\mathrm{gfp}(f^\sharp))$. Hence, if $x \in \mathrm{Fix}(f)$ then $x \leq_C \mathrm{gfp}(f) \leq \gamma(\mathrm{gfp}(f^\sharp)) \in \gamma^s(\mathrm{Fix}(f^\sharp))$, so that $\mathrm{Fix}(f) \preceq_H \gamma^s(\mathrm{Fix}(f^\sharp))$. Dually, $\mathrm{Fix}(f) \preceq_S \gamma^s(\mathrm{Fix}(f^\sharp))$, so that $\mathrm{Fix}(f) \preceq_{EM} \gamma^s(\mathrm{Fix}(f^\sharp))$ holds. □

The approximation of least/greatest fixed points of multivalued functions can also be easily given for a monotone abstraction map $\alpha : C \to A$. In this case, a S-monotone map $f^\sharp : A \to \wp^\wedge(A)$ is called a correct approximation of a concrete S-monotone map $f : C \to \wp^\wedge(C)$ when, for any $c \in C$, $\alpha^s(f(c)) \preceq_S f^\sharp(\alpha(c))$, where $\alpha^s : \wp(C) \to \wp(A)$. Here, fixed point approximation states that $\alpha(\mathrm{lfp}(f)) \leq_A \mathrm{lfp}(f^\sharp)$.

4.3 Galois Connection-based Approximations

Let us now consider the ideal case where best approximations of concrete objects in an abstract domain A always exist, that is, A is specified by a GC (α, C, A, γ). However, recall that here we deal with mere preorders such as $\preceq_S$ and $\preceq_H$ rather than standard partial orders.

Definition 4.6 (preorder-GC) Given two preorders $\langle X, \preceq_X\rangle$ and $\langle Y, \preceq_Y\rangle$ and two functions $\beta : X \to Y$ and $\delta : Y \to X$, (β, X, Y, δ) is a *preorder-GC* when:

- δ and β are monotone;
- $\beta(x) \preceq_Y y \Leftrightarrow x \preceq_X \delta(y)$. □

It turns out that standard GCs induce preorder-GCs for Smyth, Hoare and Egli-Milner preorders as follows.

Lemma 4.7 *Let* (α, C, A, γ) *be a GC. Then,* $\big(\alpha^s, \langle\wp^\wedge(C), \preceq_S\rangle, \langle\wp^\wedge(A), \preceq_S\rangle, \gamma^s\big)$, $\big(\alpha^s, \langle\wp^\vee(C), \preceq_H\rangle, \langle\wp^\vee(A), \preceq_H\rangle, \gamma^s\big)$, $\big(\alpha^s, \langle\wp^\diamond(C), \preceq_{EM}\rangle, \langle\wp^\diamond(A), \preceq_{EM}\rangle, \gamma^s\big)$ *are all preorder-GCs.*

Proof Let us check that α^s is S-monotone: if $X \preceq_S Y$ and $\alpha(y) \in \alpha^s(Y)$ then there exists $x \in X$ such that $x \leq_C y$, so that, by monotonicity of α, $\alpha(x) \leq_A \alpha(y)$, and therefore $\alpha^s(X) \preceq_S \alpha^s(Y)$. Analogously, γ^s is S-monotone. Let us check that $\alpha^s(X) \preceq_S Y \Rightarrow X \preceq_S \gamma^s(Y)$: if $\gamma(y) \in \gamma^s(Y)$ then there exists $\alpha(x) \in \alpha^s(X)$ such that $\alpha(x) \leq_A y$, and, since (α, C, A, γ) is a GC, this implies that $x \leq_C \gamma(y)$, so that $X \preceq_S \gamma^s(Y)$. Analogously, it turns out that $X \preceq_S \gamma^s(Y) \Rightarrow \alpha^s(X) \preceq_S Y$. Hence, this shows that $\big(\alpha^s, \langle\wp^\wedge(C), \preceq_S\rangle, \langle\wp^\wedge(A), \preceq_S\rangle, \gamma^s\big)$ is a preorder-GC. The proofs for Hoare and Egli-Milner preorders are analogous. □

The Galois connection-based framework allows us to define best correct approximations of multivalued functions. If $f : C \to \wp(C)$ and (α, C, A, γ) is a GC then its *best correct approximation* f^A is defined as follows:

$$f^A : A \to \wp(A) \qquad f^A(a) \triangleq \alpha^s(f(\gamma(a))).$$

In particular, if $f : C \to \wp^\wedge(C)$ is S-monotone then $f^A : A \to \wp^\wedge(A)$ turns out to be S-monotone. Analogously for Hoare and Egli-Milner preorders. Similarly to standard abstract interpretation [4], it turns out that f^A is the best possible S-correct approximation of f on the abstract domain A, as stated by the following result.

Lemma 4.8 *A* S*-monotone correspondence* $f^\sharp : A \to \wp^\wedge(A)$ *is a* S*-correct approximation of* f *iff for any* $a \in A$, $f^A(a) \preceq_S f^\sharp(a)$. *Also, analogous characterizations hold for* H*- and* EM*-correct approximations.*

Proof An easy consequence of Lemma 4.7, since for any $a \in A$, $f^A(a) = \alpha^s(f(\gamma(a)) \preceq_S f^\sharp(a)$ iff for any $a \in A$, $f(\gamma(a)) \preceq_S \gamma^s(f^\sharp(a))$. □

Hence, it turns out that the fixed point approximations given by Theorem 4.4 and Corollary 4.5 apply to the best correct approximations f^A as well.

Completeness. In abstract interpretation, completeness [4, 12] formalizes an ideal situation where the abstract function $f^\sharp$ on A is capable of not losing information w.r.t. the abstraction in A of the concrete function f, that is, the equality $\alpha(f(c)) = f^\sharp(\alpha(c))$ always holds. As a key consequence, completeness lifts to fixed points, meaning that $\alpha(\mathrm{lfp}(f)) = \mathrm{lfp}(f^\sharp)$ holds. Let us show that this property also holds for multivalued functions.

Given a GC (α, C, A, γ), an abstract S-monotone function $f^\sharp : A \to \wp^\wedge(A)$ is called a *complete approximation* of a S-monotone function $f : C \to \wp^\wedge(C)$ when for any $c \in C$, $\alpha^s(f(c)) = f^\sharp(\alpha(c))$.

Lemma 4.9 (Complete Least Fixed Point Approximation) *If $f^\sharp$ is a complete approximation of f then $\alpha(\mathrm{lfp}(f)) = \mathrm{lfp}(f^\sharp)$.*

Proof By Lemma 4.1, $\mathrm{lfp}(f) = \mathrm{lfp}(f_\wedge)$ and $\mathrm{lfp}(f^\sharp) = \mathrm{lfp}(f^\sharp_\wedge)$. Since $f_\wedge(c) \in f(c)$, we have that $\alpha(f_\wedge(c)) \in \alpha^s(f(c))$, so that $\alpha(f_\wedge(c)) = \wedge\alpha^s(f(c))$. By hypothesis, $\wedge\alpha^s(f(c)) = \wedge f^\sharp(\alpha(c)) = f^\sharp_\wedge(\alpha(c))$, so that $\alpha \circ f_\wedge = f^\sharp_\wedge \circ \alpha$ holds. Thus, by complete fixed point transfer [4, Theorem 7.1.0.4], $\alpha(\mathrm{lfp}(f_\wedge)) = \mathrm{lfp}(f^\sharp_\wedge)$. □

4.4 Approximations of Best Response Correspondences

This abstract interpretation framework for multivalued functions can be then applied to (quasi)supermodular games by approximating their best response correspondences. In particular, one can abstract both the i-th best response correspondences $B_i : S_{-i} \to \mathrm{SL}(S_i)$ and the overall best response $B : S \to \mathrm{SL}(S)$.

Example 4.10 Let us consider the game Γ in Example 2.4 and the abstraction A of its strategy space $C \times C$ defined in Example 3.5. Then, one can define the best correct approximation B^A in A of the best response function $B : C \times C \to \mathrm{SL}(C \times C)$, that is, $B^A : A \to \wp(A)$ is defined as $B^A(a) \triangleq \alpha^s(B(\gamma(a))) = \alpha^s(B(a)) = \{\alpha(s_1, s_2) \in A \mid (s_1, s_2) \in B(a)\}$. We therefore have that:

$$B^A(2,2) = \alpha^s(\{(2,3)\}) = \{(3,4)\},\ B^A(3,4) = \alpha^s(\{(2,3),(5,3)\}) = \{(3,4),(6,6)\},$$
$$B^A(4,4) = \alpha^s(\{(2,4),(5,4)\}) = \{(3,4),(6,6)\},\ B^A(3,5) = \alpha^s(\{(5,3)\}) = \{(6,6)\},$$
$$B^A(4,5) = \alpha^s(\{((5,4)\}) = \{(6,6)\},\ B^A(6,6) = \alpha^s(\{(5,4),(6,4)\}) = \{(6,6)\}.$$

Hence, $\mathrm{Fix}(B^A) = \{(3,4),(6,6)\}$. Recall from Example 2.4 that $\mathrm{Fix}(B) = \{(2,3),(5,4)\}$. By Theorem 4.4 and Corollary 4.5, here we have that $\mathrm{leq}(\Gamma) = \mathrm{lfp}(B) = (2,3) \le (3,4) = \mathrm{lfp}(B^A)$ and $\mathrm{geq}(\Gamma) = \mathrm{gfp}(B) = (5,4) \le (6,6) = \mathrm{gfp}(B^A)$. □

5 Games with Abstract Strategy Space

Consider a game $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ and a corresponding family $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ of Galois connections for the strategy spaces S_i. By Lemma 3.3, $(\alpha, \times_{i=1}^n S_i, \times_{i=1}^n A_i, \gamma)$ defines a nonrelational product abstraction of the whole strategy space $\times_{i=1}^n S_i$.

Definition 5.1 (Game with Abstract Strategy Space) The abstraction of the game Γ induced by $\mathcal{G}$ is defined by the game $\Gamma^{\mathcal{G}} = \langle A_i, u_i^{\mathcal{G}} \rangle_{i=1}^n$ where the i-th utility function $u_i^{\mathcal{G}}$ on the abstract strategy space $\times_{i=1}^n A_i$ is obtained by restricting u_i on $\gamma(\times_{i=1}^n A_i)$ as follows:

$$u_i^{\mathcal{G}} : \times_{i=1}^n A_i \to \mathbb{R}^{N_i} \qquad u_i^{\mathcal{G}}(a) \triangleq u_i(\gamma(a)). \qquad □$$

We point out that this definition of abstract game is a kind of generalization of the restricted games considered by Echenique [10, Section 2.3].

Lemma 5.2 *If $u_i(\cdot, s_{-i})$ is (quasi)supermodular and all the GCs in $\mathcal{G}$ are finitely disjunctive then $u_i^{\mathcal{G}}(\cdot, a_{-i}) : A_i \to \mathbb{R}^{N_i}$ is (quasi)supermodular. Also, if $u_i(s_i, \cdot)$ is monotone then $u_i^{\mathcal{G}}(a_i, \cdot) : A_{-i} \to \mathbb{R}^{N_i}$ is monotone.*

Proof Let us check that $u_i^{\mathcal{G}}(\cdot, a_{-i})$ is supermodular:

$$\begin{aligned}
u_i^{\mathcal{G}}(a_i \vee_{A_i} b_i, a_{-i}) + u_i^{\mathcal{G}}(a_i \wedge_{A_i} b_i, a_{-i}) = &\quad \text{[by definition]}\\
u_i(\gamma_i(a_i \vee_{A_i} b_i), \gamma_{-i}(a_{-i})) + u_i(\gamma_i(a_i \wedge_{A_i} b_i), \gamma_{-i}(a_{-i})) = &\quad [\gamma_i \text{ is disjunctive}]\\
u_i(\gamma_i(a_i) \vee_i \gamma_i(b_i), \gamma_{-i}(a_{-i})) + u_i(\gamma_i(a_i) \wedge_i \gamma_i(b_i), \gamma_{-i}(a_{-i})) \geq &\quad [u_i \text{ is supermodular}]\\
u_i(\gamma_i(a_i), \gamma_{-i}(a_{-i})) + u_i(\gamma_i(b_i), \gamma_{-i}(a_{-i})) = &\quad \text{[by definition]}\\
u_i^{\mathcal{G}}(a_i, a_{-i}) + u_i^{\mathcal{G}}(b_i, a_{-i})&
\end{aligned}$$

The proof of quasisupermodularity for $u_i^{\mathcal{G}}(\cdot, a_{-i})$ is analogous. Let us check that $u_i^{\mathcal{G}}(a_i, \cdot)$ is monotone. Consider $a_{-i} \leq b_{-i}$, so that, by monotonicity of γ_{-i}, we have that $\gamma_{-i}(a_{-i}) \leq \gamma_{-i}(b_{-i})$. Therefore, by monotonicity of $u_i(\gamma_i(a_i), \cdot)$, it turns out that: $u_i^{\mathcal{G}}(a_i, a_{-i}) = u_i(\gamma_i(a_i), \gamma_{-i}(a_{-i})) \leq u_i(\gamma_i(a_i), \gamma_{-i}(b_{-i})) = u_i^{\mathcal{G}}(a_i, b_{-i})$. □

Let us also observe that if $u_i(s_i, s_{-i})$ has increasing differences (the single crossing property), $X \subseteq \times_{i=1}^n S_i$ is any subset of the strategy space and $u_{i_{/X}} : X \to \mathbb{R}^{N_i}$ is the mere restriction of u_i to the subset X then $u_{i_{/X}}$ still has increasing differences (the single crossing property). Hence, in particular, this holds for $u_i^{\mathcal{G}} : \times_{i=1}^n A_i \to \mathbb{R}$. As a consequence of this observation and of Lemma 5.2, we obtain the following class of abstract (quasi)supermodular games.

Corollary 5.3 *Let $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ be (quasi)supermodular. If $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ is a family of finitely disjunctive GCs then $\Gamma^{\mathcal{G}} \triangleq \langle A_i, u_i^{\mathcal{G}} \rangle_{i=1}^n$ is a (quasi)supermodular game.*

Let us see an array of examples of abstract games.

Example 5.4 Consider the supermodular game Γ in Example 2.4 and the product abstraction $A_1 \times A_2 \in \mathrm{Abs}(S_1 \times S_2)$, where $A_1 = \{3, 5, 6\}$ and $A_2 = \{2, 6\}$, as defined in Example 3.5. The abstract game $\Gamma^\sharp$ of Definition 5.1 on the abstract space $A_1 \times A_2$ is thus specified by the following payoff matrix:

		A_2	
		2	**6**
	6	-1, -1	6, 5
A_1	**5**	0, 2	6, 5
	3	2, 4	3, 2

Since both A_1 and A_2 are trivially disjunctive abstractions, by Corollary 5.3, it turns out that $\Gamma^\sharp$ is supermodular. The best response correspondences $B_i^\sharp : A_{-i} \to \mathrm{SL}(A_i)$ for the supermodular game $\Gamma^\sharp$ are therefore as follows:

$$B_1^\sharp(2) = \{3\},\ B_1^\sharp(6) = \{5, 6\}; \quad B_2^\sharp(3) = \{2\},\ B_2^\sharp(5) = \{6\},\ B_2^\sharp(6) = \{6\}.$$

We observe that $B_2^\sharp$ is not a S-correct approximation of B_2 because: $B_2(3) = \{3\} \not\preceq_S \{2\} = B_2^\sharp(3)$. Indeed, it turns out that $\mathrm{Eq}(\Gamma^\sharp) = \{(3, 2), (5, 6), (6, 6)\}$, so that $\mathrm{leq}(\Gamma) =$

$(2,3) \not\leq (3,2) = \mathrm{leq}(\Gamma^\sharp)$. Thus, in this case, the solutions of the abstract game $\Gamma^\sharp$ do not correctly approximate the solutions of Γ.
Instead, following the approach in Section 4.4 and analogously to Example 4.10, one can define the best correct approximation $B^A : A \to \mathrm{SL}(A)$ in $A \triangleq A_1 \times A_2$ of the best response correspondence B of Γ. Thus, $B^A(a_1, a_2) = \{(\alpha_1(s_1), \alpha_2(s_2)) \in A \mid (s_1, s_2) \in B(a_1, a_2)\}$ acts as follows:

$$
\begin{aligned}
B^A(3,2) &= \{(\alpha_1(2), \alpha_2(3))\} = \{(3,6)\},\\
B^A(3,6) &= \{(\alpha_1(5), \alpha_2(3)), (\alpha_1(6), \alpha_2(3))\} = \{(5,6),(6,6)\},\\
B^A(5,2) &= \{(\alpha_1(2), \alpha_2(4))\} = \{(3,6)\},\\
B^A(5,6) &= \{(\alpha_1(5), \alpha_2(4)), (\alpha_1(6), \alpha_2(4))\} = \{(5,6),(6,6)\},\\
B^A(6,2) &= \{(\alpha_1(2), \alpha_2(4))\} = \{(3,6)\},\\
B^A(6,6) &= \{(\alpha_1(5), \alpha_2(4)), (\alpha_1(6), \alpha_2(4))\} = \{(5,6),(6,6)\}.
\end{aligned}
$$

Here, we have that $\mathrm{Fix}(B^A) = \{(5,6),(6,6)\}$, so that $\mathrm{leq}(\Gamma) = \mathrm{lfp}(B) = (2,3) \leq (5,6) = \mathrm{lfp}(B^A)$ and $\mathrm{geq}(\Gamma) = \mathrm{gfp}(B) = (5,4) \leq (6,6) = \mathrm{gfp}(B^A)$. □

Example 5.5 In Example 5.4, let us consider the abstraction $A_2 = \{4,6\} \in \mathrm{Abs}(S_2)$, so that the abstract supermodular game $\Gamma^\sharp$ of Definition 5.1 is determined by the following payoff matrix:

	4	**6**
6	5, 6	6, 5
5	6, 6	6, 5
3	4, 5	3, 2

while the best response correspondences $B_i^\sharp$ turn out to be defined as:

$$B_1^\sharp(4) = \{5\},\ B_1^\sharp(6) = \{5,6\}; \quad B_2^\sharp(3) = \{4\};\ B_2^\sharp(5) = \{4\},\ B_2^\sharp(6) = \{4\}.$$

Thus, here we have that $\mathrm{Eq}(\Gamma^\sharp) = \{(5,4)\}$. In this case, it turns out that $B_i^\sharp$ is a *EM*-correct approximation of B_i, so that, by Corollary 4.5 (2), $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B) = \{(2,3),(5,4)\} \preceq_{EM} \{(5,4)\} = \mathrm{Fix}(B^\sharp) = \mathrm{Eq}(\Gamma^\sharp)$ holds. □

Example 5.6 In this case we consider the disjunctive abstractions $A_1 = \{4,5,6\} \in \mathrm{Abs}(S_1)$ and $A_2 = \{3,4,5,6\} \in \mathrm{Abs}(S_2)$, so that we have the following abstract supermodular game $\Gamma^\sharp$ over $A_1 \times A_2$:

	3	**4**	**5**	**6**
6	2, 4	5, 6	6, 5	6, 5
5	3, 4	6, 6	7, 5	6, 5
4	3, 5	5, 6	5, 5	4, 4

The best response functions $B_i^\sharp$ are therefore as follows:

$$
\begin{aligned}
&B_1^\sharp(3) = \{4,5\},\ B_1^\sharp(4) = \{5\},\ B_1^\sharp(5) = \{5\},\ B_1^\sharp(6) = \{5,6\};\\
&B_2^\sharp(4) = \{4\}, \quad B_2^\sharp(5) = \{4\},\ B_2^\sharp(6) = \{4\}.
\end{aligned}
$$

In this case, it turns out that $B_i^\sharp$ is a *EM*-correct approximation of B_i, so that the abstract best response $B^\sharp : A_1 \times A_2 \to \mathrm{SL}(A_1 \times A_2)$ is a *EM*-correct approximation of B. Then, by Corollary 4.5 (2), we have that $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B) = \{(2,3),(5,4)\} \preceq_{EM} \{(5,4)\} = \mathrm{Fix}(B^\sharp) = \mathrm{Eq}(\Gamma^\sharp)$. □

Thus, for the concrete supermodular game Γ of Example 2.4, the abstract games $\Gamma^{\mathcal{G}}$ of Examples 5.5 and 5.6 can be viewed as *correct approximations* of the game Γ since

$$\mathrm{Eq}(\Gamma) \preceq_{EM} \gamma^s(\mathrm{Eq}(\Gamma^{\mathcal{G}}))$$

holds. This means that any Nash equilibrium of Γ is approximated by some Nash equilibrium of the abstract game $\Gamma^{\mathcal{G}}$ and, conversely, any Nash equilibrium of $\Gamma^{\mathcal{G}}$ approximates some Nash equilibrium of the concrete game Γ. In particular, $\mathrm{leq}(\Gamma) \leq \gamma^s(\mathrm{leq}(\Gamma^{\mathcal{G}}))$ and $\mathrm{geq}(\Gamma) \leq \gamma^s(\mathrm{geq}(\Gamma^{\mathcal{G}}))$. Instead, this approximation condition does not hold for the abstract game in Example 5.4. The following results provide conditions that justify these different behaviors.

Theorem 5.7 (Correctness of Games with Abstract Strategy Space) *Let us consider a family $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ of finitely disjunctive GIs, $S = \times_{i=1}^n S_i$, $A = \times_{i=1}^n A_i$ and let (α, S, A, γ) be the nonrelational product composition of $\mathcal{G}$. Let $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ be a (quasi)supermodular game, with best response B, and $\Gamma^{\mathcal{G}} = \langle A_i, u_i^{\mathcal{G}} \rangle_{i=1}^n$ be the corresponding abstract (quasi)supermodular game, with best response $B^{\mathcal{G}}$. Assume that for any $a \in A$,*

$$\textstyle\bigvee_S B(\gamma(a)) \vee_S \gamma(\bigwedge_A B^{\mathcal{G}}(a)) \in \gamma(A) \qquad (*)$$

Then, $\mathrm{Eq}(\Gamma) \preceq_{EM} \gamma^s(\mathrm{Eq}(\Gamma^{\mathcal{G}}))$ *and, in particular,* $\mathrm{leq}(\Gamma) \leq \gamma^s(\mathrm{leq}(\Gamma^{\mathcal{G}}))$ *and* $\mathrm{geq}(\Gamma) \leq \gamma^s(\mathrm{geq}(\Gamma^{\mathcal{G}}))$.

Proof We have that $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B)$ and $\mathrm{Eq}(\Gamma^{\mathcal{G}}) = \mathrm{Fix}(B^{\mathcal{G}})$, where $B : S \to \wp^{\diamond}(S)$ and $B^{\mathcal{G}} : A \to \wp^{\diamond}(A)$ are EM-monotone. Thus, by Corollary 4.5 (2), in order to prove that $\mathrm{Eq}(\Gamma) \preceq_{EM} \gamma^s(\mathrm{Eq}(\Gamma^{\mathcal{G}}))$ it is enough to prove that for any $a \in A$, $B(\gamma(a)) \preceq_{EM} \gamma^s(B^{\mathcal{G}}(a))$. Let $h \triangleq \bigvee_S B(\gamma(a)) \in S$, so that $h \in B(\gamma(a))$, and $k \triangleq \bigwedge_A B^{\mathcal{G}}(a) \in A$, so that, by Corollary 5.3, $k \in B^{\mathcal{G}}(a)$. By condition $(*)$, we have that $h \vee_S \gamma(k) \in \gamma(A)$. Let us consider some $i \in [1, n]$. Therefore, $h_i \vee_i \gamma_i(k_i) \in \gamma_i(A_i)$, that is, $h_i \vee_i \gamma_i(k_i) = \gamma_i(b_i)$, for some $b_i \in A_i$. Hence, since $k_i \in B_i^{\mathcal{G}}(a_{-i})$, we have that

$$u_i(h_i \vee_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) = u_i(\gamma_i(b_i), \gamma_{-i}(a_{-i})) = u_i^{\mathcal{G}}(b_i, a_{-i}) \leq \\ u_i^{\mathcal{G}}(k_i, a_{-i}) = u_i(\gamma_i(k_i), \gamma_{-i}(a_{-i})).$$

Also, since $h_i \in B_i(\gamma(a)_{-i}) = B_i(\gamma_{-i}(a_{-i}))$, it turns out that $u_i(h_i \wedge_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) \leq u_i(h_i, \gamma_{-i}(a_{-i}))$. Furthermore, since u_i is supermodular, we also have that

$$u_i(h_i \wedge_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) + u_i(h_i \vee_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) \geq \\ u_i(h_i, \gamma_{-i}(a_{-i})) + u_i(\gamma_i(k_i), \gamma_{-i}(a_{-i})).$$

We therefore obtain:

$$\begin{aligned} u_i(h_i, \gamma_{-i}(a_{-i})) + u_i(\gamma_i(k_i), \gamma_{-i}(a_{-i})) &\geq \\ u_i(h_i \wedge_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) + u_i(h_i \vee_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) &\geq \\ u_i(h_i, \gamma_{-i}(a_{-i})) + u_i(\gamma_i(k_i), \gamma_{-i}(a_{-i})) & \end{aligned}$$

so that

$$u_i(h_i, \gamma_{-i}(a_{-i})) + u_i(\gamma_i(k_i), \gamma_{-i}(a_{-i})) = \\ u_i(h_i \wedge_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) + u_i(h_i \vee_i \gamma_i(k_i), \gamma_{-i}(a_{-i}))$$

and, in turn, $u_i(h_i \wedge_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) = u_i(h_i, \gamma_{-i}(a_{-i}))$ and

$$u_i^{\mathcal{G}}(b_i, a_{-i}) = u_i(h_i \vee_i \gamma_i(k_i), \gamma_{-i}(a_{-i})) = u_i(\gamma_i(k_i), \gamma_{-i}(a_{-i})) = u_i^{\mathcal{G}}(k_i, a_{-i}).$$

Thus, $h_i \wedge_i \gamma_i(k_i) \in B_i(\gamma_{-i}(a_{-i}))$ and $h_i \vee_i \gamma_i(k_i) \in \gamma_i(B_i^{\mathcal{G}}(a_{-i}))$. Therefore, it turns out that $h \wedge \gamma(k) \in B(\gamma(a))$ and $h \vee \gamma(k) \in \gamma^s(B^{\mathcal{G}}(a))$. Hence, if $s \in B(\gamma(a))$ then $s \leq h \leq h \vee \gamma(k) \in \gamma^s(B^{\mathcal{G}}(a))$, while if $t \in \gamma^s(B^{\mathcal{G}}(a))$ then $t = \gamma^s(d)$, for some $d \in B^{\mathcal{G}}(a)$, so that $k \leq_A d$ and, in turn, $t = \gamma^s(d) \geq \gamma(k) \geq h \wedge \gamma(k) \in B(\gamma(a))$, thus showing that $B(\gamma(a)) \preceq_{EM} \gamma^s(B^{\mathcal{G}}(a))$. The proof for quasisupermodular games is analogous. □

This result depends on the condition $(*)$ which allows us to obtain a generalization of Echenique's result [10, Lemma 4] which is the basis for designing the efficient algorithm in [10, Section 4] that computes all the Nash equilibria in a finite game with strategic complementarities. Let us call (α, C, A, γ) a *principal filter Galois connection* when the image $\gamma(A)$ is the principal filter at $\gamma(\bot_A)$, that is, $\gamma(A) = \{c \in C \mid \gamma(\bot_A) \leq c\}$ holds.

Corollary 5.8 *Let $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ be a family of principal filter Galois connections. Then,* $\mathrm{Eq}(\Gamma) \preceq_{EM} \gamma^s(\mathrm{Eq}(\Gamma^{\mathcal{G}}))$.

Proof Observe that the product $(\alpha, \times_{i=1}^n S_i, \times_{i=1}^n A_i, \gamma)$ is a principal filter GC. Then, this comes as a straight consequence of Theorem 5.7, since $\bigvee_S B(\gamma(a)) \vee_S \gamma(\bigwedge_A B^{\mathcal{G}}(a)) \geq \gamma(\bigwedge_A B^{\mathcal{G}}(a)) \geq \gamma((\bot_{A_i})_{i=1}^n)$, so that $\bigvee_S B(\gamma(a)) \vee_S \gamma(\bigwedge_A B^{\mathcal{G}}(a)) \in \gamma(A)$ holds. □

Let us consider the following finite supermodular game taken from the book [2, Example 8.11], which is an example of so-called Bertrand oligopoly model [24].

Example 5.9 In this game Δ, players $i \in \{1, 2, 3\}$ stand for firms which sell substitute products p_i (e.g., a can of beer), whose feasible selling prices (e.g., in euros) s_i range in $S_i \triangleq [a, b]$, where the smallest price shift is 5 cents. The payoff function $u_i : S_1 \times S_2 \times S_3 \to \mathbb{R}$ models the profit of the firm i:

$$u_i(s_1, s_2, s_3) \triangleq d_i(s_1, s_2, s_3)(s_i - c_i)$$

where $d_i(s_1, s_2, s_3)$ gives the demand of p_i, i.e., how many units of p_i the firm i sells in a given time frame (e.g., one year), while c_i is the unit cost of p_i so that $(s_i - c_i)$ is the profit per unit. Following [2, Example 8.11], let us consider the following specific payoff functions:

$$\begin{aligned}
u_1(s_1, s_2, s_3) &= (370 + 213(s_2 + s_3) + 60s_1 - 230s_1^2)(s_1 - 1.10)\\
u_2(s_1, s_2, s_3) &= (360 + 233(s_1 + s_3) + 55s_2 - 220s_2^2)(s_2 - 1.20)\\
u_3(s_1, s_2, s_3) &= (375 + 226(s_1 + s_2) + 50s_3 - 200s_3^2)(s_3 - 1.25)
\end{aligned}$$

As shown in general in [2, Corollary 8.9], it turns out that each payoff function u_i has increasing differences and $u_i(s_i, \cdot)$ is monotone, so that the game Δ has the least and greatest price equilibria $\mathrm{leq}(\Delta)$ and $\mathrm{geq}(\Delta)$, and $\mathrm{geq}(\Delta)$ ($\mathrm{leq}(\Delta)$) provides the best (least) profits among all equilibria. It should be noted that [2, Example 8.11] considers as payoff functions the integer part of u_i, namely $\lfloor u_i(s_1, s_2, s_3) \rfloor$. However, we notice that this definition of payoff function does not have increasing differences, so that [2, Corollary 8.9], which assumes the hypothesis of increasing differences, cannot be correctly applied. Indeed, [2, Example 8.11] considers $S_i = \{x/20 \mid x \in [26,42]_{\mathbb{Z}}\}$ and with $(1.3,1.3,1.8) \leq (1.35,1.3,1.85)$, we would obtain that

$$\lfloor u_1(1.35,1.3,1.8)\rfloor - \lfloor u_1(1.3,1.3,1.8)\rfloor = \lfloor 173.03125\rfloor - \lfloor 143.92\rfloor = 30 >$$
$$\lfloor u_1(1.35,1.3,1.85)\rfloor - \lfloor u_1(1.3,1.3,1.85)\rfloor = \lfloor 175.69375\rfloor - \lfloor 146.05\rfloor = 29$$

meaning that u_1 does not have increasing differences. By contrast, let us consider in this example

$$S_i \triangleq \{x/20 \mid x \in [20,46]_{\mathbb{Z}}\},$$

namely the feasible prices range from 1 to 2.3 euros with a 0.05 shift. Using the standard RT algorithm in Figure 1 (we made a simple C++ implementation of RT to obtain the results reported here), one obtains $\mathrm{leq}(\Delta) = (1.80,1.90,1.95) = \mathrm{geq}(\Delta)$, namely, the game Δ admits a unique Nash equilibrium. It turns out that the algorithm RT calculates $\mathrm{leq}(\Delta)$ starting from the bottom $(1.0,1.0,1.0) \in S_1 \times S_2 \times S_3$ through 12 calls to $\bigwedge B_i(s_{-i})$, while it may output the same equilibrium as $\mathrm{geq}(\Delta)$ beginning from the top $(2.3,2.3,2.3)$ through 9 calls to $\bigvee B_i(s_{-i})$.
Let us consider the following abstractions $A_i \in \mathrm{Abs}(S_i)$:

$$A_1 \triangleq \{x/20 \mid x \in [35,38]_{\mathbb{Z}} \cup [42,46]_{\mathbb{Z}}\},$$
$$A_2 \triangleq \{x/20 \mid x \in [36,46]_{\mathbb{Z}}\},$$
$$A_3 \triangleq \{x/20 \mid x \in [38,46]_{\mathbb{Z}}\}.$$

Notice that A_2 and A_3 are principal filter abstractions, while this is not the case for A_1 because $A_1 \subsetneq \{x \in S_i \mid 35/20 \leq x\}$, so that Corollary 5.8 cannot be applied. We observe that:

$$\{\textstyle\bigvee_1 B_1(a_{-1}) \in S_1 \mid a_{-1} \in A_2 \times A_3\} = \{36/20, 37/20, 38/20\},$$
$$\{\textstyle\bigvee_2 B_2(a_{-2}) \in S_2 \mid a_{-2} \in A_1 \times A_3\} = \{38/20, 39/20, 40/20\},$$
$$\{\textstyle\bigvee_3 B_3(a_{-3}) \in S_3 \mid a_{-3} \in A_1 \times A_2\} = \{39/20, 40/20, 41/20, 42/20\}.$$

The condition $(*)$ of Theorem 5.7 is therefore satisfied, because for any $a_{-i} \in A_{-i}$, we have that $\bigvee B_i(a_{-i}) \in A_i$. Hence, by Corollary 5.3, we consider the abstract supermodular game Δ^A on the abstract strategy spaces A_i. By exploiting the RT algorithm in Figure 1 for Δ^A, we still obtain a unique equilibrium $\mathrm{leq}(\Delta^A) = (1.80,1.90,1.95) = \mathrm{geq}(\Delta^A)$, so that in this case no approximation of equilibria happens. Here, RT calculates $\mathrm{leq}(\Delta^A)$ starting from the bottom $(1.8,1.8,1.9)$ of $A_1 \times A_2 \times A_3$ through 6 calls to $\bigwedge B_i^A(a_{-i})$ and any call $\bigwedge B_i^A(a_{-i})$ scans the smaller abstract strategy space A_i instead of S_i. On the other hand, $(1.80,1.90,1.95) = \mathrm{geq}(\Delta)$ can be also calculated from the top $(2.3,2.3,2.3)$ still with 9 calls to $\bigvee B_i^A(a_{-i})$, each scanning the reduced abstract strategy spaces A_i. □

6 Games with Abstract Best Response

In the following, we put forward a notion of abstract game where the strategy spaces are subject to a form of partial approximation by abstract interpretation, meaning that we consider approximations of the strategy spaces of the "other players" for any utility function, i.e., correct approximations of the functions $u_i(s_i,\cdot)$, for any given s_i. This approach gives rise to games having an abstract best response correspondence. Here, we aim at providing a systematic abstraction framework for an implicit methodology of approximate computation of equilibria considered in Carl and Heikkilä's book [2] in Examples 8.58, 8.63 and 8.64.

Given a game $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$, we consider a family $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ of GCs and, by Lemma 3.3, their nonrelational product $(\alpha, \times_{i=1}^n S_i, \times_{i=1}^n A_i, \gamma)$, where we denote by $\rho \triangleq \gamma \circ \alpha \in \mathrm{uco}(\times_{i=1}^n S_i)$ the corresponding closure operator and, for any $i \in [1, n]$, by $\rho_{-i} \in \mathrm{uco}(S_{-i})$ the closure operator corresponding to the i-th nonrelational product $(\alpha_{-i}, \times_{j \neq i} S_j, \times_{j \neq i} A_j, \gamma_{-i})$. The i-th utility function $u_{i,\mathcal{G}}$ is then defined as follows:

$$u_{i,\mathcal{G}} : \times_{i=1}^n S_i \to \mathbb{R} \qquad u_{i,\mathcal{G}}(s_i, s_{-i}) \triangleq u_i(s_i, \rho_{-i}(s_{-i})).$$

Lemma 6.1 *If $u_i(s_i, s_{-i})$ has increasing differences (the single crossing property) then $u_{i,\mathcal{G}}(s_i, s_{-i})$ has increasing differences (the single crossing property). Also, if $u_i(s_i, \cdot)$ is monotone then $u_{i,\mathcal{G}}(s_i, \cdot)$ is monotone.*

Proof Assume that $(s_i, s_{-i}) \leq (t_i, t_{-i})$. Hence, $s_{-i} \leq_{-i} t_{-i}$, so that, by monotonicity of ρ_{-i}, $\rho_{-i}(s_{-i}) \leq_{-i} \rho_{-i}(t_{-i})$, and, in turn, $(s_i, \rho_{-i}(s_{-i})) \leq (t_i, \rho_{-i}(t_{-i}))$. Then:

$$\begin{aligned} u_{i,\mathcal{G}}(t_i, s_{-i}) - u_{i,\mathcal{G}}(s_i, s_{-i}) = &\quad \text{[by definition]} \\ u_i(t_i, \rho_{-i}(s_{-i})) - u_i(s_i, \rho_{-i}(s_{-i})) \leq &\quad \text{[since } u_i \text{ has increasing differences]} \\ u_i(t_i, \rho_{-i}(t_{-i})) - u_i(s_i, \rho_{-i}(t_{-i})) = &\quad \text{[by definition]} \\ u_{i,\mathcal{G}}(t_i, t_{-i}) - u_{i,\mathcal{G}}(s_i, t_{-i}). & \end{aligned}$$

The single crossing property for $u_{i,\mathcal{G}}(s_i, s_{-i})$ can be proved similarly. Let $s_{-i} \leq_{-i} t_{-i}$, so that, by monotonicity of ρ_{-i}, $\rho_{-i}(s_{-i}) \leq_{-i} \rho_{-i}(t_{-i})$. Then, by monotonicity of $u_i(s_i, \cdot)$, we obtain: $u_{i,\mathcal{G}}(s_i, s_{-i}) = u_i(s_i, \rho_{-i}(s_{-i})) = u_i(s_i, \rho_{-i}(t_{-i})) = u_{i,\mathcal{G}}(s_i, t_{-i})$, thus proving the monotonicity of $u_{i,\mathcal{G}}(s_i, \cdot)$. □

Let us also point out that if $u_i(\cdot, s_{-i})$ is (quasi)supermodular then $u_{i,\mathcal{G}}(\cdot, s_{-i})$ remains (quasi)supermodular as well. Hence, if $\Gamma_{\mathcal{G}} \triangleq \langle S_i, u_{i,\mathcal{G}} \rangle_{i=1}^n$ denotes the corresponding game then we obtain the following consequence.

Corollary 6.2 *If Γ is (quasi)supermodular then $\Gamma_{\mathcal{G}}$ is (quasi)supermodular.*

$\Gamma_{\mathcal{G}}$ is called a *game with abstract best response* because the i-th best response correspondence $B_{i,\mathcal{G}} : S_{-i} \to \mathrm{SL}(S_i)$ is such that

$$B_{i,\mathcal{G}}(s_{-i}) = \{s_i \in S_i \mid \forall x_i \in S_i . u_i(x_i, \rho_{-i}(s_{-i})) \leq u_i(s_i, \rho_{-i}(s_{-i}))\} = B_i(\rho_{-i}(s_{-i}))$$

and, in turn, $B_{\mathcal{G}}(s) = B_{\mathcal{G}}(\rho(s)) = B(\rho(s))$ holds, namely, $B_{\mathcal{G}}$ can be viewed as the restriction of B to the abstract strategy space $\rho(S)$.

Corollary 6.3 (Correctness of Games with Abstract Best Response) *Let us consider a family $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ of GCs. Then, $\mathrm{Eq}(\Gamma) \preceq_{EM} \mathrm{Eq}(\Gamma_{\mathcal{G}})$ and, in particular, $\mathrm{leq}(\Gamma) \leq \mathrm{leq}(\Gamma_{\mathcal{G}})$ and $\mathrm{geq}(\Gamma) \leq \mathrm{geq}(\Gamma_{\mathcal{G}})$.*

Proof Since, by Corollary 6.2, $\Gamma_{\mathcal{G}}$ is (quasi)supermodular, we have that $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B)$ and $\mathrm{Eq}(\Gamma_{\mathcal{G}}) = \mathrm{Fix}(B_{\mathcal{G}})$. We have that for any $s \in \times_{i=1}^n S_i$, by extensiveness of ρ, $s \leq \rho(s)$, so that, since B is monotone, we obtain $B(s) \preceq_{EM} B(\rho(s)) = B_{\mathcal{G}}(s)$. Hence, by Corollary 4.2 (4), we obtain that $\mathrm{Fix}(B) \preceq_{EM} \mathrm{Fix}(B_{\mathcal{G}})$. □

Example 6.4 Let us consider the two-player game $\Gamma = \langle S_i, u_i\rangle_{i=1}^2$ in [2, Example 8.53], which is a further example of Bertrand oligopoly, where: $S_1 = S_2 = [\frac{3}{2}, \frac{5}{2}] \times [\frac{3}{2}, \frac{5}{2}]$ and the utility functions $u_i : S_1 \times S_2 \to \mathbb{R}^2$ are defined by

$$u_i((s_{i1}, s_{i2}), s_{-i}) \triangleq (u_{i1}(s_{i1}, s_{-i}), u_{i2}(s_{i2}, s_{-i})) \in \mathbb{R}^2$$

with (sgn here denotes the standard sign function):

$$\begin{aligned}
&u_{11}(s_{11}, s_{21}, s_{22}) \triangleq \big(52 - 21s_{11} + s_{21} + 4s_{22} + 8\,\mathrm{sgn}(s_{21}s_{22} - 4)\big)(s_{11} - 1)\\
&u_{12}(s_{12}, s_{21}, s_{22}) \triangleq\\
&\quad \big(51 - 21s_{12} - \mathrm{sgn}(s_{12} - \tfrac{11}{5}) + 2s_{21} + 3s_{22} + 4\,\mathrm{sgn}(s_{21} + s_{22} - 4)\big)(s_{12} - \tfrac{11}{10})\\
&u_{21}(s_{21}, s_{11}, s_{12}) \triangleq\\
&\quad \big(50 - 20s_{21} - \mathrm{sgn}(s_{21} - \tfrac{11}{5}) + 3s_{11} + 2s_{12} + 2\,\mathrm{sgn}(s_{11} + s_{12} - 4)\big)(s_{21} - \tfrac{11}{10})\\
&u_{22}(s_{22}, s_{11}, s_{12}) \triangleq \big(49 - 20s_{22} + 4s_{11} + s_{12} + \mathrm{sgn}(s_{11}s_{12} - 4)\big)(s_{22} - 1)
\end{aligned}$$

Since any utility function $u_{ij}(s_{ij}, s_{-i})$ does not depend on $s_{i,-j}$ (e.g., u_{11} and u_{12} do not depend, respectively, on s_{12} and s_{11}), let us observe that $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^2$ is supermodular. Moreover, by [2, Propositions 8.56, 8.57], we also have that $u_i(s_1, s_2)$ has the single crossing property, so that Γ is indeed quasisupermodular. Also, since S_i is a compact (for the standard topology) complete sublattice of $\mathbb{R}^2$, we also have that $u_i(\cdot, s_{-i})$ is order upper semicontinuous, so that, for any $s \in S_1 \times S_2$, the best response correspondence B satisfies $B(s) \in \mathrm{SL}(S_1 \times S_2)$. Indeed, as observed in [2, Example 8.53], it turns out that the utility functions $u_{ij}(\cdot, s_{-i}) : [\frac{3}{2}, \frac{5}{2}] \to \mathbb{R}$ have unique maximum points denoted by $f_{ij}(s_{-i})$ which are the solutions of the equations $\frac{d}{ds}u_{ij}(s, s_{-i}) = 0$. An easy computation then provides:

$$\begin{aligned}
f_{11}(s_{21}, s_{22}) &\triangleq \tfrac{73}{42} + \tfrac{1}{42}s_{21} + \tfrac{2}{21}s_{22} + \tfrac{4}{21}\,\mathrm{sgn}(s_{21}s_{22} - 4)\\
f_{12}(s_{21}, s_{22}) &\triangleq \tfrac{247}{140} + \tfrac{1}{42}s_{21} + \tfrac{1}{14}s_{22} + \tfrac{2}{21}\,\mathrm{sgn}(s_{21} + s_{22} - 4)\\
f_{21}(s_{11}, s_{12}) &\triangleq \tfrac{9}{5} + \tfrac{3}{40}s_{11} + \tfrac{1}{20}s_{12} + \tfrac{1}{20}\,\mathrm{sgn}(s_{11} + s_{12} - 4)\\
f_{22}(s_{11}, s_{12}) &\triangleq \tfrac{69}{40} + \tfrac{1}{10}s_{11} + \tfrac{1}{40}s_{12} + \tfrac{1}{40}\,\mathrm{sgn}(s_{11}s_{12} - 4)
\end{aligned}$$

Hence, the best response B can be simplified as follows:

$$B(s_{11}, s_{12}, s_{21}, s_{22}) = \big\{\big(f_{11}(s_{21}, s_{22}), f_{12}(s_{21}, s_{22}), f_{21}(s_{11}, s_{12}), f_{22}(s_{11}, s_{12})\big)\big\}.$$

As shown in [2, Example 8.53], least and greatest equilibria of Γ can be obtained by solving a linear system of four equations with four real variables:

$$\begin{aligned}
\mathrm{leq}(\Gamma) &= \Big(\tfrac{4940854}{2778745}, \tfrac{5281784}{2778745}, \tfrac{5497457}{2778745}, \tfrac{10699993}{5557490}\Big),\\
\mathrm{geq}(\Gamma) &= \Big(\tfrac{6033654}{2778745}, \tfrac{5848294}{2778745}, \tfrac{5885617}{2778745}, \tfrac{11224753}{5557490}\Big).
\end{aligned}$$

Carl and Heikkilä [2, Example 8.58] describe how to algorithmically derive approximate solutions of Γ by approximating the fractional part of real numbers through the floor function, namely, the greatest rational number with N fractional digits which is not more than a given real number. In this section we give an abstract interpretation-based methodology for systematically designing this kind of approximate solutions which generalizes Carl and Heikkilä's approach in [2, Example 8.58]. Here, we use the ceil abstraction cl_N of real numbers already described in Example 3.2. In detail, we consider the closure operator

$\mathrm{cl}_3 : [\frac{3}{2}, \frac{5}{2}] \to [\frac{3}{2}, \frac{5}{2}]$, that is, $\mathrm{cl}_3(x)$ is the smallest rational number with at most 3 fractional digits not less than x. With a slight abuse of notation, cl_3 is also used to denote the corresponding componentwise function $\mathrm{cl}_3 : [\frac{3}{2}, \frac{5}{2}]^2 \to [\frac{3}{2}, \frac{5}{2}]^2$, namely, $\mathrm{cl}_3(s_{i1}, s_{i2}) = (\mathrm{cl}_3(s_{i1}), \mathrm{cl}_3(s_{i2}))$. Let A_{cl_3} be the following domain

$$A_{\mathrm{cl}_3} \triangleq \{\tfrac{y}{10^3} \in \mathbb{Q} \mid y \in [1500, 2500]_{\mathbb{Z}}\} = \{\mathrm{cl}_3(x) \mid x \in [\tfrac{3}{2}, \tfrac{5}{2}]\}$$

and $A \triangleq A_{\mathrm{cl}_3} \times A_{\mathrm{cl}_3}$. Then, $(\mathrm{cl}_3, [\frac{3}{2}, \frac{5}{2}], A_{\mathrm{cl}_3}, \mathrm{id})$ is a GC, so that, by Lemma 3.3, $\mathcal{G}_3 = (\mathrm{cl}_3, S_i, A, \mathrm{id})_{i=1}^2$ is a pair of GCs. Let us denote by $\Gamma_{\mathcal{G}_3}$ the corresponding game with abstract best response as defined in Corollary 6.2, so that $u_{i,\mathcal{G}_3}(s_i, s_{-i}) = u_i(s_i, \mathrm{cl}_3(s_{-i}))$. Thus, it turns out that the abstract best response correspondence $B_{\mathcal{G}_3}$ is defined as follows:

$$B(s_1, s_2) = \big\{\big(f_{11}(\mathrm{cl}_3(s_2)), f_{12}(\mathrm{cl}_3(s_2)), f_{21}(\mathrm{cl}_3(s_1)), f_{22}(\mathrm{cl}_3(s_1))\big)\big\}$$

so that $B_{\mathcal{G}_3}$ can be restricted to the finite domain $A \times A$ and therefore has a finite range. This allows us to compute the least and greatest equilibria of $\Gamma_{\mathcal{G}_3}$ by the standard RT algorithm in Figure 1. By relying on a simple C++ program to obtain the output of RT, we derive the following solutions:

$$\mathrm{leq}(\Gamma_{\mathcal{G}_3}) = \Big(\tfrac{10669}{6000}, \tfrac{6653}{3500}, \tfrac{79139}{40000}, \tfrac{77017}{40000}\Big),$$
$$\mathrm{geq}(\Gamma_{\mathcal{G}_3}) = \Big(\tfrac{91199}{42000}, \tfrac{14733}{7000}, \tfrac{42363}{20000}, \tfrac{80793}{40000}\Big).$$

By Corollary 6.3, we know that these are correct approximations, i.e., $\mathrm{leq}(\Gamma) \le \mathrm{leq}(\Gamma_{\mathcal{G}_3})$ and $\mathrm{geq}(\Gamma) \le \mathrm{geq}(\Gamma_{\mathcal{G}_3})$. Both fixed point calculations $\mathrm{leq}(\Gamma_{\mathcal{G}_3})$ and $\mathrm{geq}(\Gamma_{\mathcal{G}_3})$ need 16 calls to the abstract functions $f_{ij}(a_{-i})$, for some $a_{-i} \in A_{-i}$, which provide the unique maximum points for $u_{ij}(\cdot, a_{-i})$. It is worth noting that, even with the precision of 3 fractional digits of cl_3, the maximum approximation for these abstract solutions turns out to be quite small: $\mathrm{leq}(\Gamma_{\mathcal{G}_3})_{22} - \mathrm{leq}(\Gamma)_{22} = \frac{2148733}{22229960000} = 0.00009665932822$. □

7 Further Work

This work investigated whether and how the abstract interpretation technique can be applied to define and calculate approximate Nash equilibria of supermodular games, thus showing how a notion of approximation of equilibria can be modeled by an ordering relation analogously to what happens in static program analysis. To our knowledge, this is the first contribution towards the goal of approximating solutions of supermodular games by relying on an order-theoretic approach. We see a number of interesting avenues for further work on this subject. First, our notion of correct approximation of multivalued functions relies on a naive pointwise lifting of an abstract domain, as specified by a Galois connection, to Smyth, Hoare, Egli-Milner and Veinott preorder relations on the powerset, which is the range of best response correspondences in supermodular games. It is worth investigating whether abstract domains can be lifted through different and more sophisticated ways to this class of preordered powersets, in particular by taking into account that, for a particular class of complete lattices (that is, complete Heyting and co-Heyting algebras), the Veinott ordering gives rise to complete lattices [19]. Secondly, it could be interesting to investigate some further conditions which can guarantee the correctness of games with abstract strategy spaces (cf. Theorem 5.7). The goal here would be that of devising a notion of simulation between supermodular games whose strategy spaces are related by some form of abstraction, in order to prove that if Γ' simulates Γ then the equilibria of Γ are approximated by the equilibria of Γ'. Also, while this paper set up the abstraction framework by using very simple abstract domains, the general task of designing useful and expressive abstract domains, possibly endowed with widening operators for efficient fixed point computations, for specific classes of supermodular games is left as an open issue. Finally, it is worth mentioning that the technique of eliminating the so-called dominated strategies in a normal form game [13, Chapter 4] appears to have some similarities with the abstract strategy spaces considered in Section 5. Therefore, it could be worth investigating whether this form of game reduction can be viewed and studied under the lens of abstract interpretation.

Acknowledgements The author thanks the anonymous reviewers for their useful suggestions. The author has been partially supported by the University of Padova under the PRAT project "ANCORE" no. CPDA148418.

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