Abstract Interpretation of Supermodular Games

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Abstract. Supermodular games are a well known class of noncooperative games which find significant applications in a variety of models, especially in operations research and economic applications. Supermodular games always have Nash equilibria which are characterized as fixed points of multivalued functions on complete lattices. Abstract interpretation is here applied to set up an approximation framework for Nash equilibria of supermodular games. This is achieved by extending the theory of abstract interpretation in order to cope with approximations of multivalued functions and by providing some methods for abstracting supermodular games, thus obtaining approximate Nash equilibria which are shown to be correct within the abstract interpretation framework.

1 Introduction

Supermodular Games. Games may have so-called strategic complementarities, which encode, roughly speaking, a complementarity relationship between own actions and rivals' actions, i.e., best responses of players have monotonic reactions. Games with strategic complementarities occur in a large array of models, especially in operations research and economic applications of noncooperative game theory—a significant sample of them is described in the book [16]. For example, strategic complementarities arise in economic game models where the players are competitive firms that must each decide how many goods to produce and an increase in the production of one firm increases the marginal revenues of the others, because this gives the other firms an incentive to produce more too. Pioneered by Topkis [15] in 1978, this class of games is formalized by so-called supermodular games, where the payoff functions of each player have the lattice-theoretical properties of supermodularity and increasing differences. In a supermodular game, the strategy space of every player is partially ordered and is assumed to be a complete lattice, while the utility in playing a higher strategy increases when the opponents also play higher strategies. It turns out that pure strategy Nash equilibria of supermodular games form a complete lattice w.r.t. the ordering relation of the strategy space, thus exhibiting the least and greatest Nash equilibria. Furthermore, since the best response correspondence of a supermodular game turns out to satisfy a monotonicity condition, its least and greatest equilibria can be characterized and calculated (under assumptions of finiteness) as least and greatest fixed points by Knaster-Tarski fixed point theorem, which provides the theoretical basis for the so-called Robinson-Topkis algorithm [16].

Battle of the sexes [18] is a popular and simple example of two-player (non)supermodular game. Assume that a couple, Alice and Bob, argues over what do on the weekend. Alice would prefer to go to the opera O, Bob would rather go to the football match F, both would prefer to go to the same place rather than different ones, in particular than the disliked ones. Where should they go? The following matrix with double-entry cells provides a model for this problem.

		Bob O	Bob F
Alice	O	$3, 2$	$1, 1$
Alice	F	$0, 0$	$2, 3$

Alice chooses a row (either O or F) while Bob chooses a column (either O or F). In each double-entry cell, the first and second numbers represent, resp., Alice's and Bob's utilities, i.e., preferences. Hence, $u_{\mathrm{A}}(O, O) = 3 = u_{\mathrm{B}}(F, F)$ is the greatest utility for both Alice and Bob, for two different strategies ((O, O) for Alice and (F, F) for Bob), while $u_{\mathrm{A}}(F, O) = 0 = u_{\mathrm{B}}(F, O)$ is the least utility for both Alice and Bob. This game has two pure strategy Nash equilibria: one (O, O) where both go to the opera and another (F, F) where both go to the football game. If the ordering between O and F is either $O < F$ or $F < O$ for both Alice and Bob then this game turns out to be supermodular. If $O < F$ then (O, O) and (F, F) are, resp., the least and greatest equilibria, while their roles are exchanged when $F < O$. Instead, if $F < O$ for Alice and $O < F$ for Bob then, as expected, this game is not supermodular: the two equilibria (O, O) and (F, F) are uncomparable so that least and greatest equilibria do not exist.

Motivation. Since the breakthrough on the PPAD-completeness of finding mixed Nash equilibria [7], the question of approximating Nash equilibria emerged as a key problem in algorithmic game theory [8,10]. In this context, approximate equilibrium refers to ϵ-approximation, with $\epsilon > 0$, meaning that, for each player, all the strategies have a payoff which is at most ϵ more (or less) than the precise payoff of the given strategy. On the other hand, the notion of (correct or sound) approximation is central in static program analysis. In particular, the abstract interpretation approach to static analysis relies on an order-theoretical model of the notion of approximation. Here, program properties are modelled by a (collecting) domain endowed with a partial order $\leq$ which plays the role of logical relation where $x \leq y$ means that the property x is logically stronger than y. Also, the fundamental principle of abstract interpretation is to provide an approximate interpretation of a program for a given abstraction of the properties of its concrete semantics. This leads to the key notion of abstract domain, defined as an ordered collection of abstract program properties which can be inferred by static analysis, where approximation is modeled by the ordering relation. Furthermore, program semantics are typically defined using fixed points and a basic result of abstract interpretation tells us that correctness of approximations is preserved from functions to their least/greatest fixed points.

Goal. The similarities between supermodular games and program semantics should be clear, since they both rely on order-theoretical models and on computing extremal fixed points of suitable functions on lattices. However, while static analysis of program semantics based on order-theoretical approximations is a well-established area since forty years, to the best of our knowledge, no attempt has been made to apply standard techniques used in static program analysis for defining a corresponding notion of approximation in supermodular games. The overall goal of this paper is to investigate whether and how abstract interpretation can be used to define and calculate approximate Nash equilibria of supermodular games, where the key notion of approximation will be modeled by a partial ordering relation similarly to what happens in static program analysis. This appears to be the first contribution to make use of an order-theoretical notion of approximation for equilibria of supermodular games, in particular by resorting to the abstract interpretation framework.

Contributions. Abstract interpretation essentially relies on: (1) abstract domains A which encode approximate program properties; (2) abstract functions $f^\sharp$ which must correctly approximate on A the behavior of some concrete operations f; (3) results of correctness for the abstract interpreter using A and $f^\sharp$, such as the correctness of extremal fixed points of abstract functions, e.g. $\mathrm{lfp}(f^\sharp)$ correctly approximates $\mathrm{lfp}(f)$; (4) widening/narrowing operators tailored for the abstract domains A to ensure and/or accelerate the convergence in iterative fixed point computations of abstract functions. We contribute to set up a general framework for designing abstract interpretations of supermodular games which encompasses the above points (1)-(3), while widening/narrowing operators are not taken into account since their definition is closely related to some specific abstract domain. Our main contributions can be summarized as follows.

Abstract interpretation is typically used for approximating single-valued functions on complete lattices. For supermodular games, best responses are multivalued functions of type $B : S_1 \times \cdots \times S_N \to \wp(S_1 \times \cdots \times S_N)$. A game strategy $s \in S_1 \times \cdots \times S_N$ is called a fixed point of B when $x \in B(x)$, and these fixed points turn out to characterize Nash equilibria of this game. As a preliminary step, we first show how abstractions of strategy spaces can be composed in order to define an abstraction of the product $S_1 \times \cdots \times S_N$, and, on the other hand, an abstraction of the product $S_1 \times \cdots \times S_N$ can be decomposed into abstract domains of the individual S_i's. Next, we provide a short and direct constructive proof ensuring the existence of fixed points for multivalued functions and we show how abstract interpretation can be generalized to cope with multivalued functions.

Then, we investigate how to define an "abstract interpreter" of supermodular games. The first approach consists in defining a supermodular game on an abstract strategy space. Given a supermodular game Γ with strategy spaces S_i and utility functions $u_i : S_1 \times \cdots \times S_N \to \mathbb{R}$, this means that we assume a family of abstractions A_i, one for each S_i, that gives rise to an abstract strategy space $A = A_1 \times \cdots \times A_N$, and a suitable abstract restriction of the utility functions $u_i^A : A_1 \times \cdots \times A_N \to \mathbb{R}$. This defines what we call an abstract game Γ^A, which, under some conditions, has abstract equilibria which correctly approximate the equilibria of Γ. This abstraction technique provides a generalization of an efficient algorithm by Echenique [9] for finding all the equilibria in a finite game with strategic complementarities. Moreover, we put forward a second notion of abstract game where the strategy spaces are subject to a kind of partial approximation, meaning that, for any utility function u_i for the player i, we consider approximations of the strategy spaces of the "other players", i.e., correct approximations of the functions $u_i(s_i, \cdot) : S_1 \times \cdots S_{i-1} \times S_{i+1} \times \cdots \times S_N \to \mathbb{R}$, for any given strategy $s_i \in S_i$. This abstraction technique gives rise to games having an abstract best response correspondence. This approach is inspired and somehow generalizes the implicit methodology of approximate computation of equilibria considered in Carl and Heikkilä's book [1, Chapter 8].

Our results are illustrated on some examples of supermodular games. In particular, a couple of examples of Bertrand oligopoly models are taken from the book [1].

2 Background on Games

2.1 Order-Theoretical Notions

Given a function $f : X \to Y$ and a subset $S \in \wp(X)$ then $f(S) \triangleq \{f(s) \in Y \mid s \in S\}$ while its powerset lifting $f^s : \wp(X) \to \wp(Y)$ is defined by $f^s(S) \triangleq f(S)$. A multi-valued function, also called correspondence in game theory terminology, is any mapping $f : X \to \wp(X)$. An element $x \in X$ is called a fixed point of a multivalued function f when $x \in f(x)$ while $\mathrm{Fix}(f) \triangleq \{x \in X \mid x \in f(x)\}$ denotes the corresponding set of fixed points. Let $\langle C, \leq, \wedge, \vee, \bot, \top\rangle$ be a complete lattice, compactly denoted by $\langle C, \leq\rangle$. Given a function $f : C \to C$, with a slight abuse of notation, $\mathrm{Fix}(f) \triangleq \{x \in C \mid x = f(x)\}$ denotes its set of fixed points of f, while $\mathrm{lfp}(f)$ and $\mathrm{gfp}(f)$ denote, resp., the least and greatest fixed points of f, when they exist (recall that least and greatest fixed points always exist for monotone functions). If $f : C \to C$ then for any ordinal $\alpha \in \mathbb{O}$, the α-power $f^\alpha : C \to C$ is defined by transfinite induction as usual: for any $x \in C$, (1) if $\alpha = 0$ then $f^0(x) \triangleq x$; (2) if $\alpha = \beta + 1$ then $f^{\beta+1}(x) \triangleq f(f^\beta(x))$; (3) if $\alpha = \vee\{\beta \in \mathbb{O} \mid \beta < \alpha\}$ then $f^\alpha(x) \triangleq \bigvee_{\beta<\alpha} f^\beta(x)$. If $f, g : X \to C$ then $f \sqsubseteq g$ denotes the standard pointwise ordering relation between functions, that is, $f \sqsubseteq g$ if for any $x \in X$, $f(x) \leq g(x)$.
Let us recall the following relations on the powerset $\wp(C)$: for any $X, Y \in \wp(C)$,

(Smyth)	$X \preceq_S Y \overset{\triangle}{\iff} \forall y \in Y. \exists x \in X.\, x \leq y$
(Hoare)	$X \preceq_H Y \overset{\triangle}{\iff} \forall x \in X. \exists y \in Y.\, x \leq y$
(Egli-Milner)	$X \preceq_{EM} Y \overset{\triangle}{\iff} X \preceq_S Y \;\&\; X \preceq_H Y$
(Veinott)	$X \preceq_V Y \overset{\triangle}{\iff} \forall x \in X. \forall y \in Y.\, x \wedge y \in X \;\&\; x \vee y \in Y$

Smyth $\preceq_S$, Hoare $\preceq_H$ and Egli-Milner $\preceq_{EM}$ relations are reflexive and transitive (i.e., preorders), while the Veinott relation $\preceq_V$ (also called strong set relation) is transitive and antisymmetric [17]. A multivalued function $f : C \to \wp(C')$ is S-monotone if for any $x, y \in C$, $x \leq y$ implies $f(x) \preceq_S f(y)$. H-, EM- and V-monotonicity are defined analogously. We also use the following notations:

$$\begin{aligned}
&\wp^\wedge(C) \triangleq \{X \in \wp(C) \mid \wedge X \in X\}\\
&\wp^\vee(C) \triangleq \{X \in \wp(C) \mid \vee X \in X\}\\
&\wp^\diamond(C) \triangleq \wp^\wedge(C) \cap \wp^\vee(C)\\
&\mathrm{SL}(C) \triangleq \{X \in \wp(C) \mid X \neq \varnothing,\ X \text{ subcomplete sublattice of } C\}
\end{aligned}$$

Observe that if $X, Y \in \wp^\wedge(C)$ then $X \preceq_S Y \Leftrightarrow \wedge X \leq \wedge Y$. Similarly, if $X, Y \in \wp^\vee(C)$ then $X \preceq_H Y \Leftrightarrow \vee X \leq \vee Y$ and if $X, Y \in \wp^\diamond(C)$ then $X \preceq_{EM} Y \Leftrightarrow \wedge X \leq \wedge Y \;\&\; \vee X \leq \vee Y$.

Supermodularity. Given a family of $N > 0$ sets $(S_i)_{i=1}^N$, $s \in \times_{i=1}^N S_i$ and $i \in [1, N]$ then $S_{-i} \triangleq S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots S_N$, while $s_{-i} \triangleq (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N) \in S_{-i}$. Let $\langle \mathbb{R}^N, \leq \rangle$ denote the product poset of real numbers, where for $s, t \in \mathbb{R}^N$, $s \leq t$ iff for any $i \in [1, N]$, $s_i \leq t_i$, while $s + t \triangleq (s_i + t_i)_{i=1}^N$ ($s - t$ is analogously defined).

Supermodular games rely on (quasi)supermodular functions (we refer to [1,16] for in-depth studies). Given a complete lattice C, a function $u : C \to \mathbb{R}^N$ is *supermodular* if for any $c_1, c_2 \in C$, $u(c_1 \vee c_2) + u(c_1 \wedge c_2) \geq u(c_1) + u(c_2)$, while u is *quasisupermodular* if for any $c_1, c_2 \in C$, $u(c_1 \wedge c_2) \leq u(c_1) \;\Rightarrow\; u(c_2) \leq u(c_1 \vee c_2)$ and $u(c_1 \wedge c_2) < u(c_1) \;\Rightarrow\; u(c_2) < u(c_1 \vee c_2)$. Note that supermodularity implies quasisupermodularity (the converse is not true). Let us recall that if $u : C \to \mathbb{R}^N$ is quasisupermodular then $\operatorname{argmax}(f) \triangleq \{x \in C \mid \forall y \in C.\ f(y) \leq f(x)\}$ turns out to be a sublattice of C.

A function $u : C_1 \times C_2 \to \mathbb{R}^N$ has *increasing differences* when for any $(x, y) \leq (x', y')$, we have that $u(x', y) - u(x, y) \leq u(x', y') - u(x, y')$, or, equivalently, the functions $u(x', \cdot) - u(x, \cdot)$ and $u(\cdot, y') - u(\cdot, y)$ are monotone. Moreover, a function $u : C_1 \times C_2 \to \mathbb{R}^N$ has the *single crossing property* when for any $(x, y) \leq (x', y')$, $u(x, y) \leq u(x', y) \;\Rightarrow\; u(x, y') \leq u(x', y')$ and $u(x, y) < u(x', y) \;\Rightarrow\; u(x, y') < u(x', y')$. Notice that if u has increasing differences then u has the single crossing property, while the converse does not hold.

Supermodularity on product complete lattices and increasing differences are related as follows: a function $u : C_1 \times C_2 \to \mathbb{R}^N$ is supermodular if and only if u has increasing differences and, for any $c_i \in C_i$, $u(c_1, \cdot) : C_2 \to \mathbb{R}^N$ and $u(\cdot, c_2) : C_1 \to \mathbb{R}^N$ are supermodular.

2.2 Noncooperative Games

Let us recall some basic notions on noncooperative games, which can be found, e.g., in the books [1,16].

A *noncooperative game* $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ for players $i = 1, \ldots, n$ (with $n \geq 2$) consists of a family of feasible strategy spaces $(S_i, \leq_i)_{i=1}^n$ which are assumed to be complete lattices, so that the strategy space $S \triangleq \times_{i=1}^n S_i$ is a complete lattice for the componentwise order $\leq$, and of a family of utility (or payoff) functions $u_i : S \to \mathbb{R}^{N_i}$, with $N_i \geq 1$. The i-th *best response correspondence* $B_i : S_{-i} \to \wp(S_i)$ is defined as

$$B_i(s_{-i}) \triangleq \operatorname{argmax}(u_i(\cdot, s_{-i})) = \{x_i \in S_i \mid \forall s_i \in S_i.\ u_i(s_i, s_{-i}) \leq u_i(x_i, s_{-i})\}$$

while the best response correspondence $B : S \to \wp(S)$ is defined by the product $B(s) \triangleq \times_{i=1}^n B_i(s_{-i})$. A strategy $s \in S$ is a *pure Nash equilibrium* for Γ when s is a fixed point of B, i.e., $s \in B(s)$, meaning that in s there is no feasible way for any player to strictly improve its utility if the strategies of all the other players remain unchanged. We denote by $\mathrm{Eq}(\Gamma) \in \wp(S)$ the set of Nash equilibria for Γ, so that $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B)$.

A noncooperative game $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ is *supermodular* when: (1) for any i, for any $s_{-i} \in S_{-i}$, $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^{N_i}$ is supermodular; (2) for any i, $u_i(\cdot, \cdot) : S_i \times S_{-i} \to \mathbb{R}^{N_i}$ has increasing differences. Also, Γ is *quasisupermodular* (or, with *strategic complementarities*) when: (1) for any i, for any $s_{-i} \in S_{-i}$, $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^{N_i}$ is quasisupermodular; (2) for any i, $u_i(\cdot, \cdot) : S_i \times S_{-i} \to \mathbb{R}^{N_i}$ has the single crossing property. In these cases, it turns out (cf. [16, Theorems 2.8.1 and 2.8.6]) that the i-th best response correspondence $B_i : S_{-i} \to \wp(S_i)$ is EM-monotone, as well as the best response correspondence $B : S \to \wp(S)$.

Let us recall that, given a complete lattice C, a function $f : C \to \mathbb{R}^N$ is order upper semicontinuous if for any chain $Y \subseteq C$,

$$\limsup_{x \in Y, x \to \vee Y} f(x) \leq f(\vee C) \quad \text{and} \quad \limsup_{x \in Y, x \to \wedge Y} f(x) \leq f(\wedge C).$$

It turns out (cf. [16, Lemma 4.2.2]) that if each $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^{N_i}$ is order upper semicontinuous then, for each $s \in S$, $B_i(s_{-i}) \in \mathrm{SL}(S_i)$, i.e., $B_i(s_{-i})$ is a nonempty subcomplete sublattice of S_i, so that $B(s) \in \mathrm{SL}(S)$ also holds. In particular, we have that $\wedge_i B_i(s_{-i}), \vee_i B_i(s_{-i}) \in B_i(s_{-i})$ as well as $\wedge B(s), \vee B(s) \in B(s)$, namely, $B_i(s_{-i}) \in \wp^\diamond(S_i)$ and $B(s) \in \wp^\diamond(S)$. It also turns out [19, Theorem 2] that $\langle \mathrm{Eq}(\Gamma), \leq \rangle$ is a complete lattice—although, in general, it is not a subcomplete sublattice of S—and therefore Γ admits the least and greatest Nash equilibria, which are denoted, respectively, by $\mathrm{leq}(\Gamma)$ and $\mathrm{geq}(\Gamma)$. It should be remarked that the hypothesis of upper semicontinuity for $u_i(\cdot, s_{-i})$ holds for any finite-strategy game, namely for those games where each strategy space S_i is finite. In the following, we will consider (quasi)supermodular games which satisfy this hypothesis of upper semicontinuity.

If, given any $s_i \in S_i$, the function $u_i(s_i, \cdot) : S_{-i} \to \mathbb{R}^{N_i}$ is monotone then it turns out [1, Propositions 8.23 and 8.51] that $\mathrm{geq}(\Gamma)$ majorizes all equilibria, i.e., for all i and $s \in \mathrm{Eq}(\Gamma)$, $u_i(\mathrm{geq}(\Gamma)) \geq u_i(s)$, while $\mathrm{leq}(\Gamma)$ minimizes all equilibria.

Computing Game Equilibria. Consider a (quasi)supermodular game $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ and let us define the functions $B_\wedge, B_\vee : S \to S$ as follows: $B_\wedge(s) \triangleq \wedge B(s)$ and $B_\vee(s) \triangleq \vee B(s)$. As recalled above, we have that $B_\wedge(s), B_\vee(s) \in B(s)$. When the image of the strategy space S for $B_\wedge$ turns out to be finite, the standard algorithm [16, Algorithm 4.3.2] for computing $\mathrm{leq}(\Gamma)$ consists in applying the constructive Knaster-Tarski fixed point theorem to the function $B_\wedge$ so that $\mathrm{leq}(\Gamma) = \bigvee_{k \geq 0} B_\wedge^k(\bot_S)$. Dually, we have that $\mathrm{geq}(\Gamma) = \bigwedge_{k \geq 0} B_\vee^k(\top_S)$. In particular, this procedure can be always used for finite games. The application of the chaotic iteration strategy in this fixed point computation yields the Robinson-Topkis (RT) algorithm [16, Algorithm 4.3.1] in Figure 1, also called round-robin optimization, which is presented in its version for least fixed points, while the statements in comments provide the version for calculating greatest fixed points.

Let us provide a running example of supermodular finite game.

Example 2.1. Consider a two players finite game Γ represented in so-called normal form by the following double-entry payoff matrix:

	1	**2**	**3**	**4**	**5**	**6**
6	-1, -3	-1, -1	2, 4	5, 6	6, 5	6, 5
5	0, 0	0, 2	3, 4	6, 6	7, 5	6, 5
4	3, 1	3, 3	3, 5	5, 6	5, 5	4, 4
3	2, 2	2, 4	2, 6	4, 5	4, 4	3, 2
2	6, 4	6, 6	6, 7	6, 4	5, 2	4, -1
1	6, 4	5, 6	5, 6	4, 2	3, 0	2, -3

$$
\begin{array}{l}
\langle s_1, ..., s_n\rangle := \langle \bot_1, ..., \bot_n\rangle; \quad // \langle s_1, ..., s_n\rangle := \langle \top_1, ..., \top_n\rangle; \\
\textbf{do} \; \{ \langle t_1, ..., t_n\rangle := \langle s_1, ..., s_n\rangle; \\
\quad s_1 := \wedge_1 B_1(s_{-1}); \qquad // \; s_1 := \vee_1 B_1(s_{-1}); \\
\quad \cdots \\
\quad s_n := \wedge_n B_n(s_{-n}); \qquad // \; s_n := \vee_n B_n(s_{-n}); \\
\} \; \textbf{while} \; \neg(\langle s_1, ..., s_n\rangle = \langle t_1, ..., t_n\rangle)
\end{array}
$$

Fig. 1. Robinson-Topkis (RT) algorithm.

Here, S_1 and S_2 are both the finite chain of integers $C = \langle\{1, 2, 3, 4, 5, 6\}, \leq\rangle$ and $u_1(x, y), u_2(x, y) : S_1 \times S_2 \to \mathbb{R}$ are, respectively, the first and second entry in the payoff matrix element determined by row x and column y. For example, $u_1(2, 6) = 4$ and $u_2(2, 6) = -1$. It turns out that both u_1 and u_2 have increasing differences, so that, since S_1 and S_2 are finite (chains), Γ is a finite supermodular game. The two best response correspondences $B_1, B_2 : C \to \mathrm{SL}(C)$ are as follows:

$$B_1(1) = \{1, 2\}, B_1(2) = B_1(3) = \{2\}, B_1(4) = \{2, 5\}, B_1(5) = \{5\}, B_1(6) = \{5, 6\};$$

$$B_2(1) = \{2, 3\}, B_2(2) = B_2(3) = \{3\}, B_2(4) = B_2(5) = B_2(6) = \{4\}.$$

Thus, $\mathrm{Eq}(\Gamma) = \{(2, 3), (5, 4)\}$, since this is the set $\mathrm{Fix}(B)$ of fixed points of the best response correspondence $B = B_1 \times B_2$: indeed, $(2, 3) \in B(2, 3) = \{(2, 3)\}$ and $(5, 4) \in B(5, 4) = \{(2, 4), (5, 4)\}$. We also notice that $u_1(\cdot, s_2), u_2(s_1, \cdot) : C \to \mathbb{R}$ are neither monotone nor antimonotone. The fixed point computations of the least and greatest equilibria through the RT algorithm in Figure 1 proceed as follows:

$$(\bot_1, \bot_2) = (1, 1) \mapsto \big(\wedge B_1(1, 1), 1\big) = (1, 1) \mapsto \big(1, \wedge B_2(1, 1)\big) = (1, 2) \mapsto \\ (2, 2) \mapsto (2, 3) \mapsto (2, 3) \mapsto (2, 3) \quad \text{(lfp)}$$

$$(\top_1, \top_2) = (6, 6) \mapsto (\vee B_1(6, 6), 6) = (6, 6) \mapsto (6, \vee B_2(6, 6)) = (6, 4) \mapsto \\ (5, 4) \mapsto (5, 4) \mapsto (5, 4) \quad \text{(gfp)} \quad \Box$$

3 Abstractions on Product Domains

Background on Abstract Interpretation. In standard abstract interpretation [2,3], abstract domains (also called abstractions), are specified by Galois connections/insertions (GCs/GIs for short). Concrete and abstract domains, $\langle C, \leq_C\rangle$ and $\langle A, \leq_A\rangle$, are assumed to be complete lattices which are related by abstraction and concretization maps $\alpha : C \to A$ and $\gamma : A \to C$ that give rise to a GC (α, C, A, γ), i.e., for all $a \in A$ and $c \in C$, $\alpha(c) \leq_A a \Leftrightarrow c \leq_C \gamma(a)$. Recall that a GC is a GI when $\alpha \circ \gamma = \mathrm{id}$. A GC is (finitely) disjunctive when γ preserves all (finite) lubs. We use $\mathrm{Abs}(C)$ to denote all the possible abstractions of C, where $A \in \mathrm{Abs}(C)$ means that A is an abstract domain of C specified by some GC/GI. Let us recall that a map $\rho : C \to C$ is a (upper) closure operator when: (i) $x \leq y \;\Rightarrow\; \rho(x) \leq \rho(y)$; (ii) $x \leq \rho(x)$; (iii) $\rho(\rho(x)) = \rho(x)$. We denote by $\mathrm{uco}(\langle C, \leq\rangle)$ the set of all closure operators on the complete lattice $\langle C, \leq_C\rangle$. We will make use of some well known properties of a GC (α, C, A, γ): (1) α is additive; (2) γ is co-additive; (3) $\gamma \circ \alpha : C \to C$ is a closure operator; (4) if $\rho : C \to C$ is a closure operator then $(\rho, C, \rho(C), \mathrm{id})$ is a GI; (5) (α, C, A, γ) is a GC iff $\gamma(A)$ is the image of a closure operator on C; (6) a GC (α, C, A, γ) is (finitely) disjunctive iff $\gamma(A)$ is meet- and (finitely) join-closed.

Example 3.1. Let us consider a concrete domain $\langle C, \leq\rangle$ which is a finite chain. Then, it turns out that (α, C, A, γ) is a GC iff $\gamma(A)$ is the image of a closure operator on C iff $\gamma(A)$ is any subset of C which contains the top element $\top_C$. Hence, for the game Γ in Example 2.1, where S_i is the chain of integers $\{1, 2, 3, 4, 5, 6\}$, we have that $A_1 = \{3, 5, 6\}$ and $A_2 = \{2, 6\}$ are two abstractions of C. □

Example 3.2. Let us consider the ceil function on real numbers $\lceil\cdot\rceil : \mathbb{R} \to \mathbb{R}$, that is, $\lceil x\rceil$ is the smallest integer not less than x. Let us observe that $\lceil\cdot\rceil$ is a closure operator on $\langle \mathbb{R}, \leq\rangle$ because: (1) $x \leq y \Rightarrow \lceil x\rceil \leq \lceil y\rceil$; (2) $x \leq \lceil x\rceil$; (3) $\lceil\lceil x\rceil\rceil = \lceil x\rceil$. Therefore, the ceil function allows us to view integer numbers $\mathbb{Z} = \lceil\mathbb{R}\rceil$ as an abstraction of real numbers. The ceil function can be generalized to any finite fractional part of real numbers: given any integer number $N \geq 0$, $\mathrm{cl}_N : \mathbb{R} \to \mathbb{R}$ is defined as follows: $\mathrm{cl}_N(x) = \frac{\lceil 10^N x\rceil}{10^N}$. For $N = 0$, $\mathrm{cl}_N(x) = \lceil x\rceil$, while for $N > 0$, $\mathrm{cl}_N(x)$ is the smallest rational number with at most N fractional digits not less than x. For example, if $x \in \mathbb{R}$ and $1 < x \leq 1.01$ then $\mathrm{cl}_2(x) = 1.01$. Clearly, it turns out that cl_N is a closure operator which permits to cast rational numbers with at most N fractional digits as an abstraction of real numbers. □

Let us show how abstractions of different concrete domains C_i can be composed in order to define an abstract domain of the product domain $\times_i C_i$, and, on the other hand, an abstraction of a product $\times_i C_i$ can be decomposed into abstract domains of the component domains C_i. In the following, we consider a finite family of complete lattices $\langle C_i, \leq_i\rangle_{i=1}^n$, while product domains are considered with the componentwise ordering relation.

Product Composition of Abstractions. As shown by Cousot and Cousot in [6, Section 4.4], given a family of GCs $(\alpha_i, C_i, A_i, \gamma_i)_{i=1}^n$, one can easily define a componentwise abstraction $(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$ of the product complete lattice $\times_{i=1}^n C_i$, where $\times_{i=1}^n C_i$ and $\times_{i=1}^n A_i$ are both complete lattices w.r.t. the componentwise partial order and for any $c \in \times_{i=1}^n C_i$ and $a \in \times_{i=1}^n A_i$,

$$\alpha(c) \triangleq (\alpha_i(c_i))_{i=1}^n, \qquad\qquad \gamma(a) \triangleq (\gamma_i(a_i))_{i=1}^n.$$

For any i, we also use the function $\gamma_{-i} : A_{-i} \to C_{-i}$ to denote $\gamma_{-i}(a_{-i}) = \gamma(a)_{-i} = (\gamma_j(a_j))_{j \neq i}$.

Lemma 3.3. *$(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$ is a GC. Moreover, if each $(\alpha_i, C_i, A_i, \gamma_i)$ is a (finitely) disjunctive GC then $(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$ is a (finitely) disjunctive GC.*

Let us observe that $(\alpha, \times_{i=1}^n C_i, \times_{i=1}^n A_i, \gamma)$ is a so-called nonrelational abstraction since the product abstraction $\times_{i=1}^n A_i$ does not take into account any relationship between the different concrete domains C_i.

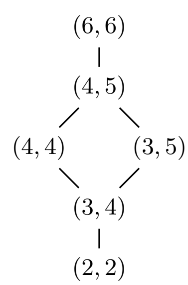
Decomposition of Product Abstractions. Let us show that any GC $(\alpha, \times_{i=1}^{n} C_i, A, \gamma)$ for the concrete product domain $\times_{i=1}^{n} C_i$ induces a family of corresponding abstractions $(\alpha_i, C_i, A_i, \gamma_i)$ of the components C_i as follows:

- $A_i \triangleq \{c_i \in C_i \mid \exists a \in A. \gamma(a)_i = c_i\} \subseteq C_i$, endowed with the partial order $\leq_i$ of C_i;
- for any $c_i \in C_i$, $\alpha_i(c_i) \triangleq \gamma(\alpha(c_i, \bot_{-i}))_i$; for any $x_i \in A_i$, $\gamma_i(x_i) \triangleq x_i$.

Lemma 3.4. $(\alpha_i, C_i, A_i, \gamma_i)$ *is a GC. Moreover, this GC is (finitely) disjunctive when* $(\alpha, \times_{i=1}^{n} C_i, A, \gamma)$ *is (finitely) disjunctive.*

A GC $(\alpha, \times_{i=1}^{n} C_i, A, \gamma)$ is defined to be *nonrelational* when it is isomorphic to the product composition, according to Lemma 3.3, of its component GCs as obtained by Lemma 3.4. Otherwise, $(\alpha, \times_{i=1}^{n} C_i, A, \gamma)$ is called *relational*. Of course, according to this definition, the product composition by Lemma 3.3 of abstract domains is trivially nonrelational. It is worth remarking that if A is relational then A cannot be obtained as a product of abstractions of C. As a consequence, the property of being relational for an abstraction A prevents the definition of a standard noncooperative game over the strategy space A since A cannot be obtained as a product domain.

Example 3.5. Let us consider the game Γ in Example 2.1 whose finite strategy space is $C \times C$, where $C = \{1, 2, 3, 4, 5, 6\}$ is a chain. Consider the subset $A \subseteq C \times C$ as depicted by the following diagram where the ordering is induced from $C \times C$:



Since A is meet- and join-closed and includes the greatest element $(6, 6)$ of $C \times C$, we have that A is a disjunctive abstraction of $C \times C$, where $\alpha : C \times C \to A$ is the closure operator induced by A and $\gamma : A \to C \times C$ is the identity. Observe that A is relational since its decomposition by Lemma 3.4 provides $A_1 = \{2, 3, 4, 6\}$ and $A_2 = \{2, 4, 5, 6\}$, and the product composition $A_1 \times A_2$ by Lemma 3.3 yields a more expressive abstraction than A, for example $(2, 4) \in (A_1 \times A_2) \smallsetminus A$. On the other hand, for the abstractions $A_1 = \{3, 5, 6\}$ and $A_2 = \{2, 6\}$ of Example 3.1, by Lemma 3.3, the product domain $A_1 \times A_2$ is a nonrelational abstraction of $C \times C$. □

4 Approximation of Multivalued Functions

Let $f : C \to C$ be some concrete monotone function and let $f^\sharp : A \to A$ be a corresponding monotone abstract function defined on some abstraction $A \in \mathrm{Abs}(C)$ specified by a GC (α, C, A, γ). Then, $f^\sharp$ is a correct (or sound) approximation of f on A

when $f \circ \gamma \sqsubseteq \gamma \circ f^\sharp$ holds. If $f^\sharp$ is a correct approximation of f then recall that we also have fixed point correctness, i.e., $\mathrm{lfp}(f) \leq_C \gamma(\mathrm{lfp}(f^\sharp))$ and $\mathrm{gfp}(f) \leq_C \gamma(\mathrm{gfp}(f^\sharp))$. Let us also recall that $f^A \triangleq \alpha \circ f \circ \gamma : A \to A$ is the best correct approximation of f on A, because it turns out that any abstract function $f^\sharp$ is correct iff $f^A \sqsubseteq f^\sharp$. Let us show how to lift these standard notions in order to approximate least/greatest fixpoints of multivalued functions.

4.1 Constructive Results for Fixed Points of Multivalued Functions

Let $f : C \to \wp(C)$ be a multivalued function and $f_\wedge, f_\vee : C \to C$ be the functions defined as: $f_\wedge(c) \triangleq \wedge f(c)$ and $f_\vee(c) \triangleq \vee f(c)$. The following constructive result ensuring the existence of least fixed points for a multivalued function is given by Straccia et al. in [14, Propositions 3.10 and 3.24]. We provide a shorter and more direct constructive proof than in [14] which is based on the constructive version of Tarski's fixed point theorem given by Cousot and Cousot [4].

Lemma 4.1. *If $f : C \to \wp^\wedge(C)$ is S-monotone then f has the least fixed point* $\mathrm{lfp}(f)$. *Moreover,* $\mathrm{lfp}(f) = \bigvee_{\alpha \in \mathbb{O}} f_\wedge^\alpha(\bot)$.

Proof. By hypothesis, $f(x) \in \wp^\wedge(C)$, so that $f_\wedge(x) \in f(x)$. If $x, y \in C$ and $x \leq y$ then, by hypothesis, $f(x) \preceq_S f(y)$. Therefore, since $f_\wedge(y) \in f(y)$, there exists some $z \in f(x)$ such that $z \leq f_\wedge(y)$, and, in turn, $f_\wedge(x) \leq z \leq f_\wedge(y)$. Hence, since $f_\wedge$ is a monotone function on a complete lattice, by Tarski's theorem, its least fixed point $\mathrm{lfp}(f_\wedge) \in C$ exists. Furthermore, by the constructive version of Tarski's theorem [4, Theorem 5.1], $\mathrm{lfp}(f_\wedge) = \bigvee_{\alpha \in \mathbb{O}} f_\wedge^\alpha(\bot)$. We have that $\mathrm{lfp}(f_\wedge) = f_\wedge(\mathrm{lfp}(f_\wedge)) \in f(\mathrm{lfp}(f_\wedge))$, hence $\mathrm{lfp}(f_\wedge) \in \mathrm{Fix}(f)$. Consider any $z \in \mathrm{Fix}(f)$. We prove by transfinite induction that for any $\alpha \in \mathbb{O}$, $f_\wedge^\alpha(\bot) \leq z$. If $\alpha = 0$ then $f_\wedge^0(\bot) = \bot \leq z$. If $\alpha = \beta + 1$ then $f_\wedge^\alpha(\bot) = f_\wedge(f_\wedge^\beta(\bot))$, and, since, by inductive hypothesis, $f_\wedge^\beta(\bot) \leq z$, then, by monotonicity of $f_\wedge$, $f_\wedge(f_\wedge^\beta(\bot)) \leq f_\wedge(z) = \wedge f(z) \leq z$. If $\alpha = \vee\{\beta \in \mathbb{O} \mid \beta < \alpha\}$ is a limit ordinal then $f_\wedge^\alpha(\bot) = \bigvee_{\beta<\alpha} f_\wedge^\beta(\bot)$; since, by inductive hypothesis, $f_\wedge^\beta(\bot) \leq z$ for any $\beta < \alpha$, we obtain that $f_\wedge^\alpha(\bot) \leq z$. This therefore shows that f has the least fixed point $\mathrm{lfp}(f) = \mathrm{lfp}(f_\wedge)$. ☐

By duality, as consequences of the above result, we obtain the following characterizations, where point (3) coincides with Zhou's theorem (see [19, Theorem 1] and [14, Proposition 3.15]), which is used for showing that pure Nash equilibria of a supermodular game form a complete lattice.

Corollary 4.2.
(1) *If $f : C \to \wp^\vee(C)$ is H-monotone then f has the greatest fixed point* $\mathrm{gfp}(f) = \bigwedge_{\alpha \in \mathbb{O}} f_\vee^\alpha(\top)$.
(2) *If $f : C \to \wp^\diamond(C)$ is EM-monotone then f has the least and greatest fixed points, where* $\mathrm{lfp}(f) = \bigvee_{\alpha \in \mathbb{O}} f_\wedge^\alpha(\bot)$ *and* $\mathrm{gfp}(f) = \bigwedge_{\alpha \in \mathbb{O}} f_\vee^\alpha(\top)$.
(3) *If $f : C \to \mathrm{SL}(C)$ is EM-monotone then $\langle \mathrm{Fix}(f), \leq\rangle$ is a complete lattice.*
(4) *If $f, g : C \to \mathrm{SL}(C)$ are EM-monotone and, for any $c \in C$, $f(c) \preceq_{EM} g(c)$ then* $\mathrm{Fix}(f) \preceq_{EM} \mathrm{Fix}(g)$.

4.2 Concretization-based Approximations

As argued by Cousot and Cousot in [5], a minimal requirement for defining an abstract domain consists in specifying the meaning of its abstract values through a concretization map. Let $\langle A, \leq_A \rangle$ be an abstraction of a concrete domain C specified by a monotone concretization map $\gamma : A \to C$. Let us observe that the powerset lifting $\gamma^s : \wp(A) \to \wp(C)$ is S-monotone, meaning that if $Y_1 \preceq_S Y_2$ then $\gamma^s(Y_1) \preceq_S \gamma^s(Y_2)$: in fact, if $\gamma(y_2) \in \gamma^s(Y_2)$ then there exists $y_1 \in Y_1$ such that $y_1 \leq_A y_2$, so that $\gamma(y_1) \in \gamma^s(Y_1)$ and $\gamma(y_1) \leq_C \gamma(y_2)$, i.e., $\gamma^s(Y_1) \preceq_S \gamma^s(Y_2)$. Analogously, γ^s is H- and EM-monotone. Then, consider a concrete S-monotone multivalued function $f : C \to \wp^\wedge(C)$, whose least fixed point exists by Lemma 4.1.

Definition 4.3 (Correct Approximation of Multivalued Functions). An abstract multivalued function $f^\sharp : A \to \wp(A)$ over A is a *S-correct approximation* of f when:

(1) $f^\sharp : A \to \wp^\wedge(A)$ and $f^\sharp$ is S-monotone (fixed point condition)
(2) for any $a \in A$, $f(\gamma(a)) \preceq_S \gamma^s(f^\sharp(a))$ (soundness condition)

H- and EM-correct approximations are defined by replacing in this definition S- with, respectively, H- and EM-, and $\wp^\wedge$ with, respectively, $\wp^\vee$ and $\wp^\diamond$. □

Let us point out that the soundness condition (2) in Definition 4.3 is close to the standard correctness requirement used in abstract interpretation: the main technical difference is that we deal with mere preorders $\langle \wp^\wedge(C), \preceq_S \rangle$ and $\langle \wp^\wedge(A), \preceq_S \rangle$ rather than posets. However, this is enough for guaranteeing a correct approximation of least fixed points.

Theorem 4.4 (Correct Least Fixed Point Approximation). *If $f^\sharp$ is a S-correct approximation of f then* $\mathrm{lfp}(f) \leq_C \gamma(\mathrm{lfp}(f^\sharp))$.

Dual results hold for H- and EM-correct approximations.

Corollary 4.5.
(1) *If $f^\sharp$ is a H-correct approximation of f then* $\mathrm{gfp}(f) \leq_C \gamma(\mathrm{gfp}(f^\sharp))$.
(2) *If $f^\sharp$ is a EM-correct approximation of f then* $\mathrm{Fix}(f) \preceq_{EM} \gamma^s(\mathrm{Fix}(f^\sharp))$*, in particular,* $\mathrm{lfp}(f) \leq_C \gamma(\mathrm{lfp}(f^\sharp))$ *and* $\mathrm{gfp}(f) \leq_C \gamma(\mathrm{gfp}(f^\sharp))$.

The approximation of least/greatest fixed points of multivalued functions can also be easily given for an abstraction map $\alpha : C \to A$. In this case, a S-monotone map $f^\sharp : A \to \wp^\wedge(A)$ is a called a correct approximation of a concrete S-monotone map $f : C \to \wp^\wedge(C)$ when, for any $c \in C$, $\alpha^s(f(c)) \preceq_S f^\sharp(\alpha(c))$, where $\alpha^s : \wp(C) \to \wp(A)$. Here, fixed point approximation states that $\alpha(\mathrm{lfp}(f)) \leq_A \mathrm{lfp}(f^\sharp)$.

4.3 Galois Connection-based Approximations

Let us now consider the ideal case where best approximations of concrete objects in an abstract domain A always exist, that is, A is specified by a GC (α, C, A, γ). However, recall that here $\preceq_S$ is a mere preorder and not a partial order. Then, given two preorders $\langle X, \preceq_X \rangle$ and $\langle Y, \preceq_Y \rangle$, we say that two functions $\beta : X \to Y$ and $\delta : Y \to X$

specify a *preorder-GC* (β, X, Y, δ) when δ and β are monotone (meaning, e.g. for β, that $x \preceq_X x' \Rightarrow \beta(x) \preceq_Y \beta(x')$) and the equivalence $\beta(x) \preceq_Y y \Leftrightarrow x \preceq_X \delta(y)$ holds. As expected, it turns out that GCs induce preorder-GCs for Smyth, Hoare and Egli-Milner preorders.

Lemma 4.6. *Let* (α, C, A, γ) *be a GC. Then,* $\big(\alpha^s, \langle \wp^\wedge(C), \preceq_S\rangle, \langle \wp^\wedge(A), \preceq_S\rangle, \gamma^s\big)$, $\big(\alpha^s, \langle \wp^\vee(C), \preceq_H\rangle, \langle \wp^\vee(A), \preceq_H\rangle, \gamma^s\big)$, $\big(\alpha^s, \langle \wp^\diamond(C), \preceq_{EM}\rangle, \langle \wp^\diamond(A), \preceq_{EM}\rangle, \gamma^s\big)$ *are all preorder-GCs.*

The Galois connection-based framework allows us to define best correct approximations of multivalued functions. If $f : C \to \wp(C)$ and (α, C, A, γ) is a GC then its *best correct approximation* on the abstract domain A is the multifunction $f^A : A \to \wp(A)$ defined as follows: $f^A(a) \triangleq \alpha^s(f(\gamma(a)))$. In particular, if $f : C \to \wp^\wedge(C)$ is S-monotone then $f^A : A \to \wp^\wedge(A)$ turns out to be S-monotone. Analogously for Hoare and Egli-Milner preorders. Similarly to standard abstract interpretation [3], f^A turns out to be the best S-correct approximation of f, as stated by the following result.

Lemma 4.7. *A S-monotone correspondence $f^\sharp : A \to \wp^\wedge(A)$ is a S-correct approximation of f iff for any $a \in A$, $f^A(a) \preceq_S f^\sharp(a)$. Also, analogous characterizations hold for H- and EM-correct approximations.*

Hence, it turns out that the fixed point approximations given by Theorem 4.4 and Corollary 4.5 apply to the best correct approximations f^A.

4.4 Approximations of Best Response Correspondences

The above abstract interpretation framework for multivalued functions can be then applied to (quasi)supermodular games by approximating their best response correspondences. In particular, one can abstract both the i-th best response correspondences $B_i : S_{-i} \to \mathrm{SL}(S_i)$ and the overall best response $B : S \to \mathrm{SL}(S)$.

Example 4.8. Let us consider the game Γ in Example 2.1 and the abstraction A of its strategy space $C \times C$ defined in Example 3.5. Then, one can define the best correct approximation B^A in A of the best response function $B : C \times C \to \mathrm{SL}(C \times C)$, that is, $B^A : A \to \wp(A)$ is defined as $B^A(a) \triangleq \alpha^s(B(\gamma(a)) = \alpha^s(B(a)) = \{\alpha(s_1, s_2) \in A \mid (s_1, s_2) \in B(a)\}$. We therefore have that:

$$B^A(2,2) = \alpha^s(\{(2,3)\}) = \{(3,4)\},\ B^A(3,4) = \alpha^s(\{(2,3),(5,3)\}) = \{(3,4),(6,6)\},$$
$$B^A(4,4) = \alpha^s(\{(2,4),(5,4)\}) = \{(3,4),(6,6)\},\ B^A(3,5) = \alpha^s(\{(5,3)\}) = \{(6,6)\},$$
$$B^A(4,5) = \alpha^s(\{((5,4)\}) = \{(6,6)\},\ B^A(6,6) = \alpha^s(\{(5,4),(6,4)\}) = \{(6,6)\}.$$

Hence, $\mathrm{Fix}(B^A) = \{(3,4),(6,6)\}$. Therefore, by Theorem 4.4 and Corollary 4.5, here we have that $\mathrm{leq}(\Gamma) = \mathrm{lfp}(B) = (2,3) \leq (3,4) = \mathrm{lfp}(B^A)$ and $\mathrm{geq}(\Gamma) = \mathrm{gfp}(B) = (5,4) \leq (6,6) = \mathrm{gfp}(B^A)$. □

5 Games with Abstract Strategy Spaces

Consider a game $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ and a corresponding family $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ of GCs of the strategy spaces S_i. By Lemma 3.3, $(\alpha, \times_{i=1}^n S_i, \times_{i=1}^n A_i, \gamma)$ defines a nonrelational product abstraction of the whole strategy space $\times_{i=1}^n S_i$. We define the i-th utility function $u_i^{\mathcal{G}} : \times_{i=1}^n A_i \to \mathbb{R}^{N_i}$ on the abstract strategy space $\times_{i=1}^n A_i$ simply by restricting u_i on $\gamma(\times_{i=1}^n A_i)$ as follows: $u_i^{\mathcal{G}}(a) \triangleq u_i(\gamma(a))$. We point out that this definition is a kind of generalization of the restricted games considered by Echenique [9, Section 2.3].

Lemma 5.1. *If $u_i(\cdot, s_{-i})$ is (quasi)supermodular and all the GCs in $\mathcal{G}$ are finitely disjunctive then $u_i^{\mathcal{G}}(\cdot, a_{-i}) : A_i \to \mathbb{R}^{N_i}$ is (quasi)supermodular. Also, if $u_i(s_i, \cdot)$ is monotone then $u_i^{\mathcal{G}}(a_i, \cdot) : A_{-i} \to \mathbb{R}^{N_i}$ is monotone.*

Let us also observe that if $u_i(s_i, s_{-i})$ has increasing differences (the single crossing property), $X \subseteq \times_{i=1}^n S_i$ is any subset of the strategy space and $u_{i/X} : X \to \mathbb{R}^{N_i}$ is the mere restriction of u_i to the subset X then $u_{i/X}$ still has increasing differences (the single crossing property). Hence, in particular, this holds for $u_i^{\mathcal{G}} : \times_{i=1}^n A_i \to \mathbb{R}$. As a consequence of this and of Lemma 5.1, we obtain the following class of *abstract (quasi)supermodular games*.

Corollary 5.2. *Let $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ be a (quasi)supermodular game and let $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ be a family of finitely disjunctive GCs. Then, $\Gamma^{\mathcal{G}} \triangleq \langle A_i, u_i^{\mathcal{G}} \rangle_{i=1}^n$ is a (quasi)supermodular game.*

Let us see an array of examples of abstract games.

Example 5.3. Consider the supermodular game Γ in Example 2.1 and the product abstraction $A_1 \times A_2 \in \mathrm{Abs}(S_1 \times S_2)$ as defined in Example 3.5. The restricted game $\Gamma^\sharp$ of Lemma 5.1 on the abstract space $\{3, 5, 6\} \times \{2, 6\}$ is thus specified by the following payoff matrix:

	2	**6**
6	-1, -1	6, 5
5	0, 2	6, 5
3	2, 4	3, 2

Since both A_1 and A_2 are trivially disjunctive abstractions, by Corollary 5.2, it turns out that $\Gamma^\sharp$ is supermodular. The best response correspondences $B_i^\sharp : A_{-i} \to \mathrm{SL}(A_i)$ for the supermodular game $\Gamma^\sharp$ are therefore as follows:

$$B_1^\sharp(2) = \{3\},\ B_1^\sharp(6) = \{5, 6\},\ B_2^\sharp(3) = \{2\};\ B_2^\sharp(5) = \{6\},\ B_2^\sharp(6) = \{6\}.$$

We observe that $B_2^\sharp$ is not a S-correct approximation of B_2 because: $B_2(3) = \{3\} \not\preceq_S \{2\} = B_2^\sharp(3)$. Indeed, it turns out that $\mathrm{Eq}(\Gamma^\sharp) = \{(3, 2), (5, 6), (6, 6)\}$, so that $\mathrm{leq}(\Gamma) = (2, 3) \not\leq (3, 2) = \mathrm{leq}(\Gamma^\sharp)$. Thus, in this case, the solutions of the abstract game $\Gamma^\sharp$ do not correctly approximate the solutions of Γ.

Instead, following the approach in Section 4.4 and analogously to Example 4.8, one can define the best correct approximation $B^A : A \to \mathrm{SL}(A)$ in $A \triangleq A_1 \times A_2$ of the best response correspondence B of Γ. Thus, $B^A(a_1, a_2) = \{(\alpha_1(s_1), \alpha_2(s_2)) \in A \mid (s_1, s_2) \in B(a_1, a_2)\}$ acts as follows:

$$B^A(3,2) = \{(3,6)\},\ B^A(3,6) = \{(5,6),(6,6)\},\ B^A(5,2) = \{(3,6)\},$$
$$B^A(5,6) = \{(5,6),(6,6)\},\ B^A(6,2) = \{(3,6)\},\ B^A(6,6) = \{(5,6),(6,6)\}.$$

Here, we have that $\mathrm{Fix}(B^A) = \{(5,6),(6,6)\}$, so that $\mathrm{leq}(\Gamma) = \mathrm{lfp}(B) = (2,3) \leq (5,6) = \mathrm{lfp}(B^A)$ and $\mathrm{geq}(\Gamma) = \mathrm{gfp}(B) = (5,4) \leq (6,6) = \mathrm{gfp}(B^A)$. □

Example 5.4. In Example 5.3, let us consider the abstraction $A_2 = \{4,6\} \in \mathrm{Abs}(S_2)$, so that the restricted supermodular game $\Gamma^\sharp$ is defined by the following payoff matrix:

	4	**6**
6	5, 6	6, 5
5	6, 6	6, 5
3	4, 5	3, 2

while the best response correspondences $B_i^\sharp$ turn out to be defined as:

$$B_1^\sharp(4) = \{5\},\ B_1^\sharp(6) = \{5,6\},\ B_2^\sharp(3) = \{4\};\ B_2^\sharp(5) = \{4\},\ B_2^\sharp(6) = \{4\}.$$

Thus, here we have that $\mathrm{Eq}(\Gamma^\sharp) = \{(5,4)\}$. In this case, it turns out that $B_i^\sharp$ is a EM-correct approximation of B_i, so that, by Corollary 4.5 (2), $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B) = \{(2,3),(5,4)\} \preceq_{EM} \{(5,4)\} = \mathrm{Fix}(B^\sharp) = \mathrm{Eq}(\Gamma^\sharp)$ holds. □

Example 5.5. Here, we consider the disjunctive abstractions $A_1 = \{4,5,6\} \in \mathrm{Abs}(S_1)$ and $A_2 = \{3,4,5,6\} \in \mathrm{Abs}(S_2)$, so that we have the following abstract supermodular game $\Gamma^\sharp$ over $A_1 \times A_2$:

	3	**4**	**5**	**6**
6	2, 4	5, 6	6, 5	6, 5
5	3, 4	6, 6	7, 5	6, 5
4	3, 5	5, 6	5, 5	4, 4

Best response functions $B_i^\sharp$ are therefore as follows:

$$B_1^\sharp(3) = \{4,5\},\ B_1^\sharp(4) = \{5\},\ B_1^\sharp(5) = \{5\},\ B_1^\sharp(6) = \{5,6\};$$
$$B_2^\sharp(4) = \{4\},\quad B_2^\sharp(5) = \{4\},\ B_2^\sharp(6) = \{4\}.$$

In this case, it turns out that $B_i^\sharp$ is a EM-correct approximation of B_i, so that the abstract best response $B^\sharp : A_1 \times A_2 \to \mathrm{SL}(A_1 \times A_2)$ is a EM-correct approximation of B. Then, by Corollary 4.5 (2), we have that $\mathrm{Eq}(\Gamma) = \mathrm{Fix}(B) = \{(2,3),(5,4)\} \preceq_{EM} \{(5,4)\} = \mathrm{Fix}(B^\sharp) = \mathrm{Eq}(\Gamma^\sharp)$. □

Thus, for the concrete supermodular game Γ of Example 2.1, the abstract games of Examples 5.4 and 5.5 can be viewed as correct approximations of the game Γ since

$$\mathrm{Eq}(\Gamma) \preceq_{EM} \gamma^s(\mathrm{Eq}(\Gamma^{\mathcal{G}}))$$

holds. This means that any Nash equilibrium of Γ is approximated by some Nash equilibrium of the abstract game $\Gamma^{\mathcal{G}}$ and, conversely, any Nash equilibrium of $\Gamma^{\mathcal{G}}$ approximates some Nash equilibrium of the concrete game Γ. In particular, $\mathrm{leq}(\Gamma) \leq \gamma^s(\mathrm{leq}(\Gamma^{\mathcal{G}}))$ and $\mathrm{geq}(\Gamma) \leq \gamma^s(\mathrm{geq}(\Gamma^{\mathcal{G}}))$. Instead, this approximation condition does not hold for the abstract game in Example 5.3. The following results provide conditions that justify these different behaviors.

Theorem 5.6 (Correctness of Games with Abstract Strategy Spaces). *Let* $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ *be a family of finitely disjunctive GIs,* $S = \times_{i=1}^n S_i$*,* $A = \times_{i=1}^n A_i$ *and* (α, S, A, γ) *be the nonrelational product composition of* $\mathcal{G}$*. Let* $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$ *be a (quasi)supermodular game, with best response* B*, and* $\Gamma^{\mathcal{G}} = \langle A_i, u_i^{\mathcal{G}} \rangle_{i=1}^n$ *be the corresponding abstract (quasi)supermodular game, with best response* $B^{\mathcal{G}}$*. Assume that for any* $a \in A$*,*

$$\bigvee_S B(\gamma(a)) \vee_S \gamma(\bigwedge_A B^{\mathcal{G}}(a)) \in \gamma(A) \qquad (*)$$

Then, $\mathrm{Eq}(\Gamma) \preceq_{EM} \gamma^s(\mathrm{Eq}(\Gamma^{\mathcal{G}}))$ *and, in particular,* $\mathrm{leq}(\Gamma) \leq \gamma^s(\mathrm{leq}(\Gamma^{\mathcal{G}}))$ *and* $\mathrm{geq}(\Gamma) \leq \gamma^s(\mathrm{geq}(\Gamma^{\mathcal{G}}))$*.*

This result depends on the condition $(*)$ which allows us to obtain a generalization of Echenique's result [9, Lemma 4] which is the basis for designing the efficient algorithm in [9, Section 4] that computes all the Nash equilibria in a finite game with strategic complementarities. Let us call (α, C, A, γ) a *principal filter GC* when the image $\gamma(A)$ is the principal filter at $\gamma(\bot_A)$, that is, $\gamma(A) = \{c \in C \mid \gamma(\bot_A) \leq c\}$ holds.

Corollary 5.7. *Let* $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ *be principal filter GCs. Then,* $\mathrm{Eq}(\Gamma) \preceq_{EM} \gamma^s(\mathrm{Eq}(\Gamma^{\mathcal{G}}))$*.*

Example 5.8. Let us consider the following finite supermodular game Δ taken from [1, Example 8.11], which is an example of the well known Bertrand oligopoly model [16]. Players $i \in \{1, 2, 3\}$ stand for firms which sell substitute products p_i (e.g., a can of beer), whose feasible selling prices (e.g., in euros) s_i range in $S_i \triangleq [a, b]$, where the smallest price shift is 5 cents. The payoff function $u_i : S_1 \times S_2 \times S_3 \to \mathbb{R}$ models the profit of the firm i:

$$u_i(s_1, s_2, s_3) \triangleq d_i(s_1, s_2, s_3)(s_i - c_i)$$

where $d_i(s_1, s_2, s_3)$ gives the demand of p_i, i.e., how many units of p_i the firm i sells in a given time frame, while c_i is the unit cost of p_i so that $(s_i - c_i)$ is the profit per unit. Following [1, Example 8.11], let us assume that:

$$u_1(s_1, s_2, s_3) = (370 + 213(s_2 + s_3) + 60s_1 - 230s_1^2)(s_1 - 1.10)$$
$$u_2(s_1, s_2, s_3) = (360 + 233(s_1 + s_3) + 55s_2 - 220s_2^2)(s_2 - 1.20)$$
$$u_3(s_1, s_2, s_3) = (375 + 226(s_1 + s_2) + 50s_3 - 200s_3^2)(s_3 - 1.25)$$

As shown in general in [1, Corollary 8.9], it turns out that each payoff function u_i has increasing differences and $u_i(s_i, \cdot)$ is monotone, so that the game Δ has the least and greatest price equilibria $\mathrm{leq}(\Delta)$ and $\mathrm{geq}(\Delta)$, and $\mathrm{geq}(\Delta)$ ($\mathrm{leq}(\Delta)$) provides the best (least) profits among all equilibria. It should be noted that [1, Example 8.11] considers as payoff functions the integer part of u_i, namely $\lfloor u_i(s_1, s_2, s_3) \rfloor$. However, we notice that this definition of payoff function does not have increasing differences, so that [1, Corollary 8.9], which assumes the hypothesis of increasing differences, cannot be correctly applied. Indeed, [1, Example 8.11] considers $S_i = \{x/20 \mid x \in [26, 42]_{\mathbb{Z}}\}$ and with $(1.3, 1.3, 1.8) \leq (1.35, 1.3, 1.85)$, we would have that

$$\lfloor u_1(1.35, 1.3, 1.8) \rfloor - \lfloor u_1(1.3, 1.3, 1.8) \rfloor = \lfloor 173.03125 \rfloor - \lfloor 143.92 \rfloor = 30 >$$
$$\lfloor u_1(1.35, 1.3, 1.85) \rfloor - \lfloor u_1(1.3, 1.3, 1.85) \rfloor = \lfloor 175.69375 \rfloor - \lfloor 146.05 \rfloor = 29$$

meaning that u_1 does not have increasing differences. By contrast, we consider here $S_i \triangleq \{x/20 \mid x \in [20, 46]_{\mathbb{Z}}\}$, namely the feasible prices range from 1 to 2.3 euros with a 0.05 shift. Using the standard RT algorithm in Figure 1 (we made a simple C++ implementation of RT), one obtains $\mathrm{leq}(\Delta) = (1.80, 1.90, 1.95) = \mathrm{geq}(\Delta)$, namely, the game Δ admits a unique Nash equilibrium. It turns out that the algorithm RT calculates $\mathrm{leq}(\Delta)$ starting from the bottom $(1.0, 1.0, 1.0) \in S_1 \times S_2 \times S_3$ through 12 calls to $\bigwedge B_i(s_{-i})$, while it may output the same equilibrium as $\mathrm{geq}(\Delta)$ beginning from the top $(2.3, 2.3, 2.3)$ through 9 calls to $\bigvee B_i(s_{-i})$.
Let us consider the following abstractions $A_i \in \mathrm{Abs}(S_i)$:

$$A_1 \triangleq \{x/20 \mid x \in [35, 38]_{\mathbb{Z}} \cup [42, 46]_{\mathbb{Z}}\},$$
$$A_2 \triangleq \{x/20 \mid x \in [36, 46]_{\mathbb{Z}}\},$$
$$A_3 \triangleq \{x/20 \mid x \in [38, 46]_{\mathbb{Z}}\}.$$

Notice that A_2 and A_3 are principal filter abstractions, while this is not the case for A_1, so that Corollary 5.7 cannot be applied. We observe that:

$$\{\textstyle\bigvee_1 B_1(a_{-1}) \in S_1 \mid a_{-1} \in A_2 \times A_3\} = \{36/20, 37/20, 38/20\},$$
$$\{\textstyle\bigvee_2 B_2(a_{-2}) \in S_2 \mid a_{-2} \in A_1 \times A_3\} = \{38/20, 39/20, 40/20\},$$
$$\{\textstyle\bigvee_3 B_3(a_{-3}) \in S_3 \mid a_{-3} \in A_1 \times A_2\} = \{39/20, 40/20, 41/20, 42/20\}.$$

The condition $(*)$ of Theorem 5.6 is therefore satisfied, because for any $a_{-i} \in A_{-i}$, we have that $\bigvee B_i(a_{-i}) \in A_i$. Hence, by Corollary 5.2, we consider the abstract supermodular game Δ^A on the abstract strategy spaces A_i. By exploiting the RT algorithm in Figure 1 for Δ^A, we still obtain a unique equilibrium $\mathrm{leq}(\Delta^A) = (1.80, 1.90, 1.95) = \mathrm{geq}(\Delta^A)$, so that in this case no approximation of equilibria occurs. Here, RT calculates $\mathrm{leq}(\Delta^A)$ starting from the bottom $(1.8, 1.8, 1.9)$ of $A_1 \times A_2 \times A_3$ through 6 calls to $\bigwedge B_i^A(a_{-i})$ and any call $\bigwedge B_i^A(a_{-i})$ scans the smaller abstract strategy space A_i instead of S_i. On the other hand, $(1.80, 1.90, 1.95) = \mathrm{geq}(\Delta)$ can be also calculated from the top $(2.3, 2.3, 2.3)$ still with 9 calls to $\bigvee B_i^A(a_{-i})$, each scanning the reduced abstract strategy spaces A_i. □

6 Games with Abstract Best Response

In the following, we put forward a notion of abstract game where the strategy spaces are subject to a form of partial approximation by abstract interpretation, meaning that we consider approximations of the strategy spaces of the "other players" for any utility function, i.e., correct approximations of the functions $u_i(s_i, \cdot)$, for any given s_i. This approach gives rise to games having an abstract best response correspondence. Here, we aim at providing a systematic abstraction framework for the implicit methodology of approximate computation of equilibria considered by Carl and Heikkilä [1] in their Examples 8.58, 8.63 and 8.64.

Given a game $\Gamma = \langle S_i, u_i \rangle_{i=1}^n$, we consider a family $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ of GCs and, by Lemma 3.3, their nonrelational product $(\alpha, \times_{i=1}^n S_i, \times_{i=1}^n A_i, \gamma)$, where we denote by $\rho \triangleq \gamma \circ \alpha \in \mathrm{uco}(\times_{i=1}^n S_i)$ the corresponding closure operator and, for any i, by $\rho_{-i} \in \mathrm{uco}(S_{-i})$ the closure operator corresponding to the $(n-i)$-th nonrelational product $(\alpha_{-i}, \times_{j \neq i} S_j, \times_{j \neq i} A_j, \gamma_{-i})$. The utility function $u_{i,\mathcal{G}} : \times_{i=1}^n S_i \to \mathbb{R}$ is then defined as follows: for any $s \in \times_{i=1}^n S_i$, $u_{i,\mathcal{G}}(s_i, s_{-i}) \triangleq u_i(s_i, \rho_{-i}(s_{-i}))$.

Lemma 6.1. *If $u_i(s_i, s_{-i})$ has increasing differences (the single crossing property) then $u_{i,\mathcal{G}}(s_i, s_{-i})$ has increasing differences (the single crossing property). Also, if $u_i(s_i, \cdot)$ is monotone then $u_{i,\mathcal{G}}(s_i, \cdot)$ is monotone.*

Let us also point out that if $u_i(\cdot, s_{-i})$ is (quasi)supermodular then $u_{i,\mathcal{G}}(\cdot, s_{-i})$ remains (quasi)supermodular as well, so that by defining the game $\Gamma_{\mathcal{G}} \triangleq \langle S_i, u_{i,\mathcal{G}} \rangle_{i=1}^n$ we obtain the following consequence.

Corollary 6.2. *If Γ is (quasi)supermodular then $\Gamma_{\mathcal{G}}$ is (quasi)supermodular.*

$\Gamma_{\mathcal{G}}$ is called a *game with abstract best response* because the i-th best response correspondence $B_{i,\mathcal{G}} : S_{-i} \to \mathrm{SL}(S_i)$ is such that $B_{i,\mathcal{G}}(s_{-i}) = \{s_i \in S_i \mid \forall x_i \in S_i . u_i(x_i, \rho_{-i}(s_{-i})) \leq u_i(x_i, \rho_{-i}(s_{-i}))\} = B_i(\rho_{-i}(s_{-i}))$, and, in turn, $B_{\mathcal{G}}(s) = B_{\mathcal{G}}(\rho(s)) = B(\rho(s))$ holds, namely, $B_{\mathcal{G}}$ can be viewed as the restriction of B to the abstract strategy space $\rho(S)$.

Corollary 6.3 (Correctness of Games with Abstract Best Response). *Let us consider a family $\mathcal{G} = (\alpha_i, S_i, A_i, \gamma_i)_{i=1}^n$ of GCs. Then, $\mathrm{Eq}(\Gamma) \preceq_{EM} \mathrm{Eq}(\Gamma_{\mathcal{G}})$ and, in particular, $\mathrm{leq}(\Gamma) \leq \mathrm{leq}(\Gamma_{\mathcal{G}})$ and $\mathrm{geq}(\Gamma) \leq \mathrm{geq}(\Gamma_{\mathcal{G}})$.*

Example 6.4. Let us consider the two-player game $\Gamma = \langle S_i, u_i \rangle_{i=1}^2$ in [1, Example 8.53], which is a further example of Bertrand oligopoly, where: $S_1 = S_2 = [\frac{3}{2}, \frac{5}{2}] \times [\frac{3}{2}, \frac{5}{2}]$ and the utility functions $u_i : S_1 \times S_2 \to \mathbb{R}^2$ are defined by $u_i((s_{i1}, s_{i2}), s_{-i}) = (u_{i1}(s_{i1}, s_{-i}), u_{i2}(s_{i2}, s_{-i})) \in \mathbb{R}^2$ with

$$
\begin{aligned}
&u_{11}(s_{11}, s_{21}, s_{22}) \triangleq \big(52 - 21 s_{11} + s_{21} + 4 s_{22} + 8\,\mathrm{sgn}(s_{21} s_{22} - 4)\big)(s_{11} - 1)\\
&u_{12}(s_{12}, s_{21}, s_{22}) \triangleq\\
&\quad \big(51 - 21 s_{12} - \mathrm{sgn}(s_{12} - \tfrac{11}{5}) + 2 s_{21} + 3 s_{22} + 4\,\mathrm{sgn}(s_{21} + s_{22} - 4)\big)(s_{12} - \tfrac{11}{10})\\
&u_{21}(s_{21}, s_{11}, s_{12}) \triangleq\\
&\quad \big(50 - 20 s_{21} - \mathrm{sgn}(s_{21} - \tfrac{11}{5}) + 3 s_{11} + 2 s_{12} + 2\,\mathrm{sgn}(s_{11} + s_{12} - 4)\big)(s_{21} - \tfrac{11}{10})\\
&u_{22}(s_{22}, s_{11}, s_{12}) \triangleq \big(49 - 20 s_{22} + 4 s_{11} + s_{12} + \mathrm{sgn}(s_{11} s_{12} - 4)\big)(s_{22} - 1)
\end{aligned}
$$

Since any utility function $u_{ij}(s_{ij}, s_{-i})$ does not depend on $s_{i,-j}$ (e.g., u_{11} and u_{12} do not depend, resp., on s_{12} and s_{11}), let us observe that $u_i(\cdot, s_{-i}) : S_i \to \mathbb{R}^2$ is supermodular. Moreover, by [1, Propositions 8.56, 8.57], we also have that $u_i(s_1, s_2)$ has the single crossing property, so that Γ is indeed quasisupermodular. Also, since S_i is a compact (for the standard topology) complete sublattice of $\mathbb{R}^2$, we also have that $u_i(\cdot, s_{-i})$ is order upper semicontinuous, so that, for any $s \in S_1 \times S_2$, the best response correspondence B satisfies $B(s) \in \mathrm{SL}(S_1 \times S_2)$. Indeed, as observed in [1, Example 8.53], it turns out that the utility functions $u_{ij}(\cdot, s_{-i}) : [\frac{3}{2}, \frac{5}{2}] \to \mathbb{R}$ have unique maximum points denoted by $f_{ij}(s_{-i})$ which are the solutions of the equations $\frac{d}{ds} u_{ij}(s, s_{-i}) = 0$. An easy computation then provides:

$$
\begin{aligned}
f_{11}(s_{21}, s_{22}) &\triangleq \tfrac{73}{42} + \tfrac{1}{42} s_{21} + \tfrac{2}{21} s_{22} + \tfrac{4}{21}\,\mathrm{sgn}(s_{21}s_{22} - 4)\\
f_{12}(s_{21}, s_{22}) &\triangleq \tfrac{247}{140} + \tfrac{1}{42} s_{21} + \tfrac{1}{14} s_{22} + \tfrac{2}{21}\,\mathrm{sgn}(s_{21} + s_{22} - 4)\\
f_{21}(s_{11}, s_{12}) &\triangleq \tfrac{9}{5} + \tfrac{3}{40} s_{11} + \tfrac{1}{20} s_{12} + \tfrac{1}{20}\,\mathrm{sgn}(s_{11} + s_{12} - 4)\\
f_{22}(s_{11}, s_{12}) &\triangleq \tfrac{69}{40} + \tfrac{1}{10} s_{11} + \tfrac{1}{40} s_{12} + \tfrac{1}{40}\,\mathrm{sgn}(s_{11}s_{12} - 4)
\end{aligned}
$$

so that the best response B can be simplified as follows:

$$B(s_{11}, s_{12}, s_{21}, s_{22}) = \big\{\big(f_{11}(s_{21}, s_{22}), f_{12}(s_{21}, s_{22}), f_{21}(s_{11}, s_{12}), f_{22}(s_{11}, s_{12})\big)\big\}.$$

As shown in [1, Example 8.53], least and greatest equilibria of Γ can be obtained by solving a linear system of four equations with four real variables:

$$
\begin{aligned}
\mathrm{leq}(\Gamma) &= \Big(\tfrac{4940854}{2778745}, \tfrac{5281784}{2778745}, \tfrac{5497457}{2778745}, \tfrac{10699993}{5557490}\Big),\\
\mathrm{geq}(\Gamma) &= \Big(\tfrac{6033654}{2778745}, \tfrac{5848294}{2778745}, \tfrac{5885617}{2778745}, \tfrac{11224753}{5557490}\Big).
\end{aligned}
$$

Carl and Heikkilä [1, Example 8.58] describe how to algorithmically derive approximate solutions of Γ by approximating the fractional part of real numbers through the floor function, namely, the greatest rational number with N fractional digits which is not more than a given real number. In this section we gave an abstract interpretation-based methodology for systematically designing this kind of approximate solutions which generalizes this approach by Carl and Heikkilä in [1, Example 8.58]. Here, we use the ceil abstraction of real numbers already described in Example 3.2. Thus, we consider the closure operator $\mathrm{cl}_3 : [\frac{3}{2}, \frac{5}{2}] \to [\frac{3}{2}, \frac{5}{2}]$, that is, $\mathrm{cl}_3(x)$ is the smallest rational number with at most 3 fractional digits not less than x. With a slight abuse of notation, cl_3 is also used to denote the corresponding componentwise function $\mathrm{cl}_3 : [\frac{3}{2}, \frac{5}{2}]^2 \to [\frac{3}{2}, \frac{5}{2}]^2$, namely, $\mathrm{cl}_3(s_{i1}, s_{i2}) = (\mathrm{cl}_3(s_{i1}), \mathrm{cl}_3(s_{i1}))$. Let A_{cl_3} be the following domain

$$A_{\mathrm{cl}_3} \triangleq \{\tfrac{y}{10^3} \in \mathbb{Q} \mid y \in [1500, 2500]_{\mathbb{Z}}\} = \{\mathrm{cl}_3(x) \mid x \in [\tfrac{3}{2}, \tfrac{5}{2}]\}$$

and $A \triangleq A_{\mathrm{cl}_3} \times A_{\mathrm{cl}_3}$. Then, $(\mathrm{cl}_3, [\frac{3}{2}, \frac{5}{2}], A_{\mathrm{cl}_3}, \mathrm{id})$ is a GC, so that, by Lemma 3.3, $\mathcal{G}_3 = (\mathrm{cl}_3, S_i, A, \mathrm{id})_{i=1}^2$ is a pair of GCs. Let us denote by $\Gamma_{\mathcal{G}_3}$ the corresponding game with abstract best response as defined in Corollary 6.2, so that $u_{i,\mathcal{G}_3}(s_i, s_{-i}) = u_i(s_i, \mathrm{cl}_3(s_{-i}))$. Thus, it turns out that the abstract best response correspondence $B_{\mathcal{G}_3}$ is defined as follows:

$$B(s_1, s_2) = \big\{\big(f_{11}(\mathrm{cl}_3(s_2)), f_{12}(\mathrm{cl}_3(s_2)), f_{21}(\mathrm{cl}_3(s_1)), f_{22}(\mathrm{cl}_3(s_1))\big)\big\}$$

so that $B_{\mathcal{G}_3}$ can be restricted to the finite domain $A \times A$ and therefore has a finite range. This allows us to compute the least and greatest equilibria of $\Gamma_{\mathcal{G}_3}$ by the standard RT algorithm in Figure 1. By relying on a simple C++ program, we obtained the following solutions:

$$\mathrm{leq}(\Gamma_{\mathcal{G}_3}) = \left(\tfrac{10669}{6000}, \tfrac{6653}{3500}, \tfrac{79139}{40000}, \tfrac{77017}{40000}\right), \quad \mathrm{geq}(\Gamma_{\mathcal{G}_3}) = \left(\tfrac{91199}{42000}, \tfrac{14733}{7000}, \tfrac{42363}{20000}, \tfrac{80793}{40000}\right).$$

By Corollary 6.3, we know that these are correct approximations, i.e., $\mathrm{leq}(\Gamma) \leq \mathrm{leq}(\Gamma_{\mathcal{G}_3})$ and $\mathrm{geq}(\Gamma) \leq \mathrm{geq}(\Gamma_{\mathcal{G}_3})$. Both fixed point calculations $\mathrm{leq}(\Gamma_{\mathcal{G}_3})$ and $\mathrm{geq}(\Gamma_{\mathcal{G}_3})$ need 16 calls to the abstract functions $f_{ij}(a_{-i})$, for some $a_{-i} \in A_{-i}$, which provide the unique maximum points for $u_{ij}(\cdot, a_{-i})$. It is worth noting that, even with the precision of 3 fractional digits of cl_3, the maximum approximation for these abstract solutions turns out to be quite small: $\mathrm{leq}(\Gamma_{\mathcal{G}_3})_{22} - \mathrm{leq}(\Gamma)_{22} = \frac{2148733}{22229960000} = 0.00009665932822$. $\square$

7 Further Work

We investigated whether and how the abstract interpretation technique can be applied to define and calculate approximate Nash equilibria of supermodular games, thus showing how a notion of approximation of equilibria can be modeled by an ordering relation analogously to what happens in static program analysis. To our knowledge, this is the first contribution towards the goal of approximating solutions of supermodular games by relying on an order-theoretical approach. We see a number of interesting avenues for further work on this subject. First, our notion of correct approximation of a multivalued function relies on a naive pointwise lifting of an abstract domain, as specified by a Galois connection, to Smyth, Hoare, Egli-Milner and Veinott preorder relations on the powerset, which is the range of best response correspondences in supermodular games. It is worth investigating whether abstract domains can be lifted through different and more sophisticated ways to this class of preordered powersets, in particular by taking into account that, for a particular class of complete lattices (that is, complete Heyting and co-Heyting algebras), the Veinott ordering gives rise to complete lattices [13]. Secondly, it could be interesting to investigate some further conditions which can guarantee the correctness of games with abstract strategy spaces (cf. Theorem 5.6). The goal here would be that of devising a notion of simulation between games whose strategy spaces are related by some form of abstraction, in order to prove that if Γ' simulates Γ then the equilibria of Γ are approximated by the equilibria of Γ'. Finally, while this paper set up the abstraction framework by using very simple abstract domains, the general task of designing useful and expressive abstract domains, possibly endowed with widening operators for efficient fixed point computations, for specific classes of supermodular games is left as an open issue.

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