Tracing just-in-time compilation is a popular compilation technique for the efficient implementation of dynamic languages, which is commonly used for JavaScript, Python and PHP. It relies on two key ideas. First, it monitors program execution in order to detect so-called hot paths, i.e., the most frequently executed program paths. Then, hot paths are optimized by exploiting some information on program stores which is available and therefore gathered at runtime. The result is a residual program where the optimized hot paths are guarded by sufficient conditions ensuring some form of equivalence with the original program. The residual program is persistently mutated during its execution, e.g., to add new optimized hot paths or to merge existing paths. Tracing compilation is thus fundamentally different from traditional static compilation. Nevertheless, despite the practical success of tracing compilation, very little is known about its theoretical foundations. We provide a formal model of tracing compilation of programs using abstract interpretation. The monitoring phase (viz., hot path detection) corresponds to an abstraction of the trace semantics of the program that captures the most frequent occurrences of sequences of program points together with an abstraction of their corresponding stores, e.g., a type environment. The optimization phase (viz., residual program generation) corresponds to a transform of the original program that preserves its trace semantics up to a given observation as modeled by some abstraction. We provide a generic framework to express dynamic optimizations along hot paths and to prove them correct. We instantiate it to prove the correctness of dynamic type specialization and constant variable folding. We show that our framework is more general than the model of tracing compilation introduced by Guo and Palsberg [2011] which is based on operational bisimulations. In our model we can naturally express hot path reentrance and common optimizations like dead-store elimination, which are either excluded or unsound in Guo and Palsberg’s framework.

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1. INTRODUCTION

Efficient traditional static compilation of popular dynamic languages like JavaScript, Python and PHP is very hard if not impossible. In particular, these languages present so many dynamic features which make all traditional static analyses used for program optimization very imprecise. Therefore, practical implementations of dynamic languages should rely on dynamic information in

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order to produce an optimized version of the program. Tracing just-in-time (JIT) compilation (TJITC) [Bala et al. 2000; Bauman et al. 2015; Bebenita et al. 2010; Böhm et al. 2011; Bolz et al. 2009; Bolz et al. 2011; Gal et al. 2006; Gal et al. 2009; Pall 2005; Schilling 2013] has emerged as a valuable implementation and optimization technique for dynamic languages (and not only, e.g. Java [Haubl and Mossenböck 2011; Haubl et al. 2014; Inoue et al. 2011]). For instance, the Facebook HipHop virtual machine for PHP and the V8 JavaScript engine of Google Chrome use some form of tracing compilation [Adams et al. 2014; Facebook Inc. 2013; Google Inc. 2010]. The Mozilla Firefox JavaScript engine used to have a tracing engine, called TraceMonkey, which has been later substituted by whole-method just-in-time compilation engines (initially JägerMonkey and then IonMonkey) [Mozilla Foundation 2010; Mozilla Foundation 2013].

**The Problem.** Tracing JIT compilers leverage runtime profiling of programs to detect and record often executed paths, called hot paths, and then they optimize and compile only these paths at runtime. A path is a linear sequence (i.e., no loops or join points are allowed) of instructions through the program. Profiling may also collect information about the values that the program variables may assume during the execution of that path, which is then used to specialize/optimize the code of the hot path. Of course, this information is not guaranteed to hold for all the subsequent executions of the hot path. Since optimizations rely on that information, the hot path is augmented with guards that check the profiled conditions, such as, for example, variable types and constant variables. When a guard fails, execution jumps back to the old, non-optimized code. The main hypotheses of tracing compilers, confirmed by the practice, are: (i) loop bodies are the most interesting code to optimize, so they only consider paths inside program loops; and (ii) optimizing straight-line code is easier than a whole-method analysis (involving loops, goto, etc.).

Hence, tracing JIT compilers look quite different than traditional compilers. These differences raise some natural questions on trace compilation: (i) what is a viable formal model, which is generic yet realistic enough to capture the behavior of real optimizers? (ii) which optimizations are sound? (iii) how can one prove their soundness? In this paper we answer these questions.

Our formal model is based on program trace semantics [Cousot 2002] and abstract interpretation [Cousot and Cousot 1977; Cousot and Cousot 2002]. Hot path detection is modeled just as an abstraction of the trace semantics of the program, which only retains: (i) the sequences of program points which are repeated more than some threshold; (ii) an abstraction of the possible program stores, e.g., the type of the variables instead of their concrete values. As a consequence, a hot path does not contain loops nor join points. Furthermore, in the hot path, all the correctness conditions (i.e., guards) are explicit, for instance before performing integer addition, we should check that the operands are integers. If the guard condition is not satisfied then the execution leaves the hot path, reverting to the non-optimized code. Guards are essentially elements of some abstract domain, which is then left as a parameter in our model. The hot path is then optimized using standard compilation techniques—we only require the optimization to be sound.

We define the correctness of the residual (or extracted) program in terms of an abstraction of its trace semantics: the residual program is correct if it is indistinguishable, up to some abstraction of its trace semantics, from the original program. Examples of abstractions are the program store at the exit of a method, or the stores at loop entry and loop exit points.

**Main Contributions.** This paper puts forward a formal model of TJITC whose key features are as follows:
An Abstract Interpretation-based Model of Tracing Just-In-Time Compilation

– We provide the first model of tracing compilation based on abstract interpretation of trace semantics of programs.
– We provide a more general and realistic framework than the model of TJITC by Guo and Palsberg [2011] based on program bisimulations: we employ a less restrictive correctness criterion that enables the correctness proof of practically implemented optimizations; hot paths can be annotated with runtime information on the stores, notably type information; optimized hot loops can be re-entered.
– We formalize and prove the correctness of type specialization of hot paths.

Our model focuses on source-to-source program transformations and optimizations of a low level imperative language with untyped global variables, which may play the role of intermediate language of some virtual machine. Our starting point is that program optimizations can be seen as transformations that lose some information on the original program, so that optimizations can be viewed as approximations and in turn can be formalized as abstract interpretations. More precisely, we rely on the insight by Cousot and Cousot [2002] that a program source can be seen as an abstraction of its trace semantics, i.e. the set of all possible execution sequences, so that a source-to-source optimization can be viewed as an abstraction of a transform of the program trace semantics. In our model, soundness of program optimizations is defined as program equivalence w.r.t. an observational abstract interpretation of the program trace semantics. Here, an observational abstraction induces a correctness criterion by describing what is observable about program executions, so that program equivalence means that two programs are indistinguishable by looking only at their observable behaviors.

A crucial part of tracing compilation is the selection of the hot path(s) to optimize. This choice is made at runtime based on program executions, so it can be seen once again as an abstraction of trace semantics. Here, a simple trace abstraction selects cyclic instruction sequences, i.e. loop paths, that appear at least \( N \) times within a single execution trace. These instruction sequences are recorded together with some property of the values assumed by program variables at that point, which is represented as an abstract store belonging to a suitable store abstraction, which in general depends on the successive optimizations to perform.

A program optimization can be seen as an abstraction of a semantic transformation of program execution traces, as described by Cousot and Cousot [2002]. The advantage of this approach is that optimization properties, such as their correctness, are easier to prove at a semantic level. The optimization itself can be defined on the whole program or, as in the case of real tracing JIT compilers, can be restricted to the hot path. This latter restriction is achieved by transforming the original program so that the hot path is extracted, i.e. made explicit: the hot path is added to the program as a path with no join points that jumps back to the original code when execution leaves it. A guard is placed before each command in this hot path that checks if the necessary conditions, as selected by the store abstraction, are satisfied. A program optimization can then be confined to the hot path only, making it linear, by ignoring the parts of the program outside it. The guards added to the hot path allows us to retain precision.

We apply our TJITC model to type specialization. Type specialization is definitely the key optimization for dynamic languages such as Javascript [Gal et al. 2009], as they make available generic operations whose execution depends on the type of runtime values of their operands. Moreover, as a further application of our model, we consider the constant variable folding optimization along hot paths, which relies on the standard constant propagation abstract domain [Wegman and Zadeck 1991].

**Related Work.** A formal model for tracing JIT compilation has been put forward by Guo and Palsberg [2011] at POPL symposium. It is based on operational bisimulation.
[Milner 1995] to describe the equivalence between source and optimized programs. We show how this model can be expressed within our framework through the following steps: Guo and Palsberg’s language is compiled into ours; we then exhibit an observational abstraction which is equivalent to Guo and Palsberg’s correctness criterion; finally, after some minor changes that address a few differences in path selection, the transformations performed on the source program turn out to be the same. Our framework overcomes some significant limitations in Guo and Palsberg’s model. The bisimulation equivalence model used in [Guo and Palsberg 2011] implies that the optimized program has to match every change to the store made by the original program, whereas in practice we only need this match to hold in certain program points and for some variables, such as in output instructions. This limits the number of real optimizations that can be modeled in this framework. For instance, dead store elimination is proven unsound in [Guo and Palsberg 2011], while it is implemented in actual tracing compilers [Gal et al. 2009, Section 5.1]. Furthermore, their formalization fails to model some important features of actual TJITC implementation: (i) traces are mere linear paths of instructions, i.e., they cannot be annotated with store properties; (ii) hot path selection is completely non-deterministic, since they do not model a selection criterion; and, (iii) once execution leaves an optimized hot path the program will not be able to re-enter it.

It is also worth citing that abstract interpretation of program trace semantics roots at the foundational work by Cousot [1997, 2002] and has been widely used as a technique for defining a range of static program analyses [Barbuti et al. 1999; Colby and Lee 1996; Handjieva and Tzolovski 1998; Logozzo 2009; Rival and Mauborgne 2007; Schmidt 1998; Spoto and Jensen 2003]. Also, [Rival 2004] describes various program optimizations as the trace abstractions they preserve. In the Cousot and Cousot terminology [Cousot and Cousot 2002], Rival’s approach corresponds to offline transformations whereas tracing compilation is an online transformation.

**Structure.** The rest of the paper is organized as follows. Sections 2 and 3 contain some necessary background: the language considered in the paper and its operational trace semantics are defined in Section 2, while Section 3 recalls some basic notions of abstract interpretation, in particular for defining abstract domains of program stores. Hot paths are formally defined in Section 4 as a suitable abstract interpretation of program traces, while Section 5 defines the program transform for extracting a given hot path. The correctness of the hot path extraction transform is defined and proved correct in Section 6, which also introduces in Subsection 6.2 program optimizations along hot paths together with a methodology for proving their correctness. Section 7 applies our model of hot path optimization to type specialization of untyped program commands, while Section 8 describes an application to constant variable folding along hot paths. Nested hot paths and the corresponding program transform for their extraction are the subject of Section 9. Section 10 provides a thorough formal comparison of our model with [Guo and Palsberg 2011]’s framework for tracing compilation. Finally, Section 11 concludes, also discussing some directions for future work.

This is an expanded and revised version of the POPL symposium article [Dissegna et al. 2014] including all the proofs.

### 2. LANGUAGE AND CONCRETE SEMANTICS

#### 2.1. Notation

Given a finite set $X$ of objects, we will use the following notation concerning sequences: $\epsilon$ is the empty sequence; $X^+$ is the set of nonempty finite sequences of objects of $X$; $X^* \triangleq X^+ \cup \{\epsilon\}$; if $\sigma \in X^*$ then $|\sigma|$ denotes the length of $\sigma$; indices of objects in a
nonempty sequence $\sigma \in X^+$ start from 0 and thus range in the interval $[0,|\sigma| - 1]$; if $\sigma \in X^+$ and $i \in [0,|\sigma|]$ then $\sigma_i \in X$ (or $\sigma(i)$) denotes the $i$-th object in $\sigma$; if $\sigma \in X^*$ and $i,j \in [0,|\sigma|]$ then $\sigma_{[i,j]} \in X^*$ denotes the subsequence $\sigma_i \sigma_{i+1} \ldots \sigma_j$, which is therefore the empty sequence if $j < i$, while if $k \in \mathbb{N}$ then $\sigma_{[-k]} \in X^*$ denotes the suffix $\sigma_{k+1} \ldots \sigma_{|\sigma|-1}$, which is the empty sequence when $k \geq |\sigma|$.

If $f : X \to Y$ is any function then its collecting version $f^c : \varphi(X) \to \varphi(Y)$ is defined pointwise by $f^c(S) \triangleq \{f(x) \in Y \mid x \in S\}$, and when clear from the context, by a slight abuse of notation, it is sometimes denoted by $f$ itself.

### 2.2. Syntax

We consider a basic low level language with untyped global variables, a kind of elementary dynamic language, which is defined through the notation used in [Cousot and Cousot 2002]. Program commands range in $C$ and consist of a labeled action which specifies a next label ($L$ is the undefined label, where the execution becomes stuck: it is used for defining final commands).

- **Labels:** $L \in \mathbb{L}$, $L \notin \mathbb{L}$
- **Values:** $v \in \mathbb{Value}$
- **Variables:** $x \in Var$
- **Expressions:** $E \vartriangleright E ::= v \mid x \mid E_1 + E_2$
- **Boolean Expressions:** $B \vartriangleright B ::= tt \mid ff \mid E_1 \leq E_2 \mid \neg B \mid B_1 \land B_2$
- **Actions:** $A \vartriangleright A ::= x := E \mid B \mid skip$
- **Commands:** $C \vartriangleright C ::= L : A \to L' \; (\text{with } L' \in \mathbb{L} \cup \{L\})$

For any command $L : A \to L'$, we use the following notation:

$$\text{lbl}(L : A \to L') \triangleq L, \; \; \text{act}(L : A \to L') \triangleq A, \; \; \text{suc}(L : A \to L') \triangleq L'.$$

Commands $L : B \to L'$ whose action is a Boolean expression are called conditionals. A program $P \in \varphi(C)$ is a (possibly infinite, at least in theory) set of commands. In order to be well-formed, if a program $P$ includes a conditional $C \equiv L : B \to L'$ then $P$ must also include a unique complement conditional $L : \neg B \to L''$, which is denoted by $\text{compl}(C)$ or $C^c$, where $\neg B$ is taken to be equal to $B$, so that $\text{compl}(\text{compl}(C)) = C$. The set of well-formed programs is denoted by Program. In our examples, programs $P$ will be deterministic, i.e., for any $C_1, C_2 \in P$ such that $\text{lbl}(C_1) = \text{lbl}(C_2)$: (1) if $\text{act}(C_1) \neq \text{act}(C_2)$ then $C_1 = \text{compl}(C_2)$; (2) if $\text{act}(C_1) = \text{act}(C_2)$ then $C_1 = C_2$. We say that two programs $P_1$ and $P_2$ are equal up to label renaming, denoted by $P_1 \equiv P_2$, when there exists a suitable renaming for the labels of $P_1$ that makes $P_1$ equal to $P_2$.

### 2.3. Transition Semantics

The language semantics relies on values ranging in $\mathbb{Value}$, possibly undefined values ranging in $\mathbb{Value}_u$, truth values in $\mathbb{Bool}$, possibly undefined truth values ranging in $\mathbb{Bool}_u$, and type names ranging in $\mathbb{Types}$, which are defined as follows:

- $\mathbb{Value} \triangleq \mathbb{Z} \cup \mathbb{Char}^*$
- $\mathbb{Value}_u \triangleq \mathbb{Z} \cup \mathbb{Char}^* \cup \{\text{undef}\}$
- $\mathbb{Bool} \triangleq \{\text{true}, \text{false}\}$
- $\mathbb{Bool}_u \triangleq \{\text{true}, \text{false}, \text{undef}\}$
- $\mathbb{Types} \triangleq \{\text{Int}, \text{String}, \text{Undef}, \top_T, \bot_T\}$
\[\textbf{E} : \text{Exp} \rightarrow \text{Store} \rightarrow \text{Value}_\text{a}\]
\[\text{E}[v]_\rho \triangleq v \quad \text{E}[x]_\rho \triangleq \rho(x)\]
\[\text{E}[E_1 + E_2]_\rho \triangleq \begin{cases} \text{E}[E_1]_\rho + \text{Z}\text{E}[E_2]_\rho & \text{if type}(\text{E}[E_1]_\rho) = \text{Int} \\ \text{E}[E_1]_\rho \cdot \text{E}[E_2]_\rho & \text{if type}(\text{E}[E_1]_\rho) = \text{String} \\ \text{undef} & \text{otherwise} \end{cases}\]

\[\textbf{B} : \text{BExp} \rightarrow \text{Store} \rightarrow \text{Bool}_\text{a}\]
\[\text{B}[\text{tt}]_\rho \triangleq \text{true} \quad \text{B}[\text{ff}]_\rho \triangleq \text{false}\]
\[\text{B}[E_1 \leq E_2]_\rho \triangleq \begin{cases} \text{E}[E_1]_\rho \leq \text{Z}\text{E}[E_2]_\rho & \text{if type}(\text{E}[E_1]_\rho) = \text{Int} \\ \exists \sigma \in \text{String} \cdot \text{E}[E_2]_\rho = (\text{E}[E_1]_\rho) \cdot \sigma & \text{if type}(\text{E}[E_1]_\rho) = \text{String} \\ \text{undef} & \text{otherwise} \end{cases}\]
\[\text{B}[\neg B]_\rho \triangleq \neg\text{B}[B]_\rho \quad \text{B}[B_1 \land B_2]_\rho \triangleq \text{B}[B_1]_\rho \land \text{B}[B_2]_\rho\]

\[\textbf{A} : \text{\&} \rightarrow \text{Store} \rightarrow \text{Store} \cup \{\perp\}\]
\[\text{A}[\text{skip}]_\rho \triangleq \rho\]
\[\text{A}[[x := E]]_\rho \triangleq \begin{cases} \rho[x/E]_\rho & \text{if } \text{E}[E]_\rho \neq \text{undef} \\ \perp & \text{if } \text{E}[E]_\rho = \text{undef} \end{cases}\]
\[\text{A}[[B]]_\rho \triangleq \begin{cases} \rho & \text{if } \text{B}[B]_\rho = \text{true} \\ \perp & \text{if } \text{B}[B]_\rho \in \{\text{false, undef}\} \end{cases}\]

Fig. 1. Semantics of program expressions and actions.

where \text{Char} is a nonempty finite set of characters and \text{undef} is a distinct symbol. The mapping \text{type} : \text{Value}_\text{a} \rightarrow \text{Types} provides the type of any possibly undefined value:

\[\text{type}(v) \triangleq \begin{cases} \text{Int} & \text{if } v \in \mathbb{Z} \\ \text{String} & \text{if } v \in \text{Char}^* \\ \text{Undef} & \text{if } v = \text{undef} \end{cases}\]

The type names \text{\perp}_\text{T} and \text{T}_\text{⊥} will be used in Section[7] as, respectively, top and bottom type, that is, subtype and supertype of all types.

Let \text{Store} \triangleq \text{Var} \rightarrow \text{Value}_\text{a} denote the set of possible stores on variables in \text{Var}, where \rho(x) = \text{undef} means that the store \rho is not defined on a program variable \(x \in \text{Var}\). Hence, let us point out that the symbol \text{undef} will be used to represent both store undefinedness and a generic error when evaluating an expression (e.g., additions and comparisons between integers and strings), two situations which are not distinguished in our semantics. A store \rho \in \text{Store} will be denoted by \([x/\rho(x)]_\rho \neq \text{undef}\) thus omitting undefined variables, while \([\_]\) will denote the totally undefined store. If \(P \in \text{Program}\) then \text{vars}(P) denotes the set of variables in \text{Var} that occur in \(P\), so that \text{Store}_\rho \triangleq \text{vars}(P) \rightarrow \text{Value}_\text{a} is the set of possible stores for \(P\).

The semantics of expressions \text{E}, Boolean expressions \text{B} and program actions \text{A} is standard and goes as defined in Fig. 1. Let us remark that:

(i) the binary function \(+_{\mathbb{Z}}\) denotes integer addition, \(\leq_{\mathbb{Z}}\) denotes integer comparison, while \(\cdot\) is string concatenation;
(ii) logical negation and conjunction in $\text{Bool}$ are extended in order to handle $\text{undef}$ as follows: $\neg \text{undef} = \text{undef}$ and $\text{undef} \land b = \text{undef} = b \land \text{undef}$;

(iii) $\rho[x/v]$ denotes a store update for the variable $x$ with $v \in \text{Value}$;

(iv) the distinct symbol $\perp \notin \text{Value}$ is used to denote the result of: $A[x := E]_\rho$ when the evaluation of the expression $E$ for $\rho$ generates an error; $A[B]_\rho$ when the evaluation of the Boolean expression $B$ for $\rho$ is either false or generates an error.

With a slight abuse of notation we also consider the collecting versions of the semantic functions in Fig.1 which are defined as follows:

$$
E : \text{Exp} \rightarrow \phi(\text{Store}) \rightarrow \phi(\text{Value})_\rho
$$

$$
E[E]_\rho \triangleq \{ E[E]_\rho \in \text{Value} \mid \rho \in S \}
$$

$$
B : \text{BExp} \rightarrow \phi(\text{Store}) \rightarrow \phi(\text{Store})_\rho
$$

$$
B[B]_\rho \triangleq \{ \rho \in S \mid B[B]_\rho = \text{true} \}
$$

$$
A : \mathcal{A} \rightarrow \phi(\text{Store}) \rightarrow \phi(\text{Store})_\rho
$$

$$
A[A]_\rho \triangleq \{ A[A]_\rho \mid \rho \in S, A[A]_\rho \in \text{Store} \}
$$

Let us point out that, in the above collecting versions, if $E[E]_\rho = \text{undef}$ then $E[E]_\rho = \{ \text{undef} \}$ and $A[x := E]_\rho = \emptyset$, while if $B[B]_\rho \in \{ \text{false}, \text{undef} \}$ then $B[B]_\rho = \emptyset$ and $A[B]_\rho = \emptyset$.

Generic program states are pairs of stores and commands; $\text{State} \triangleq \text{Store} \times C$. We extend the previous functions $\text{lbl}, \text{act}$ and $\text{suc}$ to be defined on states, meaning that they are defined on the command component of a state. Also, $\text{store}(s)$ and $\text{cmd}(s)$ return, respectively, the store and the command of a state $s$. The transition semantics $\text{S} : \text{State} \rightarrow \phi(\text{State})$ is a relation between generic states defined as follows:

$$
\text{S}(\rho, C) \triangleq \{ (\rho', C') \in \text{State} \mid \rho' \in A[\text{act}(C)]_\rho, \text{suc}(C) = \text{lbl}(C') \}
$$

If $P$ is a program then $\text{State}_P \triangleq \text{State} \times P$ is the set of possible states of $P$. Given $P \in \text{Program}$, the program transition relation $\text{S}[P] : \text{State}_P \rightarrow \phi(\text{State}_P)$ between states of $P$ is defined as:

$$
\text{S}[P](\rho, C) \triangleq \{ (\rho', C') \in \text{State}_P \mid \rho' \in A[\text{act}(C)]_\rho, C' \in P, \text{suc}(C) = \text{lbl}(C') \}
$$

Let us remark that, according to the above definition, if $C \equiv L : A \rightarrow L', C_1 \equiv L' : B \rightarrow L''$ and $C_2 \equiv L'' : \neg B \rightarrow L'''$ are all commands in $P$ and $\rho' \in A[A]_\rho$ then we have that $\text{S}[P](\rho, C) = \{ (\rho', C_1), (\rho', C_2) \}$.

A state $s \in \text{State}_P$ is stuck for $P$ when $\text{S}[P] s = \emptyset$. Let us point that:

(i) If the conditional command of a state $s = (\rho, L : B \rightarrow L') \in \text{State}_P$ is such that $B[B]_\rho = \text{false}$ then $s$ is stuck for $P$ because there exists no store $\rho' \in A[B]_\rho = \emptyset$.

(ii) If the command of a state $s = (\rho, L : A \rightarrow L) \in \text{State}_P$ has the undefined label $L$ as next label then $s$ is stuck for $P$.

(iii) We have a stuck state $s$ when an error happens. E.g., this is the case for an undefined evaluation of an addition as in $s = (\langle y/3, z/\text{foo} \rangle, L : x := y + z \rightarrow L')$ and for an undefined evaluation of a Boolean expression as in $s = (\langle y/3, z/\text{foo} \rangle, L : y \leq x \rightarrow L')$.

Programs typically have an entry point, and this is modeled through a distinct initial label $L_0 \in L$ from which execution starts. $\text{State}_{P_0} \triangleq \{ (\rho, C) \mid \text{lbl}(C) = L_0 \}$ denotes the set of possible initial states for $P$.

2.3.1. Trace Semantics. A partial trace is any nonempty finite sequence of generic program states which are related by the transition relation $\text{S}$. Hence, the set Trace of
partial traces is defined as follows:

\[
\text{Trace} \triangleq \{ \sigma \in \text{State}^+ \mid \forall i \in [1, |\sigma|). \sigma_i \in \text{S} \sigma_{i-1} \}.
\]

The partial trace semantics of \( P \in \text{Program} \) is in turn defined as follows:

\[
\mathbf{T}[P] = \text{Trace}_P \triangleq \{ \sigma \in (\text{State}_P)^+ \mid \forall i \in [1, |\sigma|). \sigma_i \in \text{S}[P] \sigma_{i-1} \}.
\]

A trace \( \sigma \in \text{Trace}_P \) is complete if for any state \( s \in \text{State}_P \), \( \sigma s \notin \text{Trace}_P \) and \( s \sigma \notin \text{Trace}_P \). Observe that \( \text{Trace}_P \) contains all the possible partial traces of \( P \), complete traces included. Let us remark that a trace \( \sigma \in \text{Trace}_P \) does not necessarily begin with an initial state, namely it may happen that \( \sigma_0 \notin \text{State}_P \). Traces of \( P \) starting from initial states are denoted by

\[
\mathbf{T}^\top \mathbf{[P]} = \text{Trace}_P \triangleq \{ \sigma \in \text{Trace}_P \mid \sigma_0 \in \text{State}_P \}.
\]

Also, a complete trace \( \sigma \in \text{Trace}_P \) such that \( \text{succ}(\sigma_{|\sigma|-1}) = \text{L} \) corresponds to a terminating run of the program \( P \).

**Example 2.1.** Let us consider the program \( Q \) below written in some while-language:

\[
x := 0;
\]
\[
\text{while} \ (x \leq 20) \ do
\]
\[
\quad x := x + 1;
\]
\[
\quad \text{if} \ (x \% 3 = 0) \ then \ x := x + 3;
\]

Its translation as a program \( P \) in our language is given below, with \( L_4 = L_0 \), where, with a little abuse, we assume an extended syntax that allows expressions like \( x \% 3 = 0 \).

\[
P = \{ C_0 \equiv L_0 : x := 0 \rightarrow L_1,
\]
\[
C_1 \equiv L_1 : x \leq 20 \rightarrow L_2, C_1^c \equiv L_1 : \neg (x \leq 20) \rightarrow L_5,
\]
\[
C_2 \equiv L_2 : x := x + 1 \rightarrow L_3,
\]
\[
C_3 \equiv L_3 : (x \% 3 = 0) \rightarrow L_4, C_3^c \equiv L_3 : \neg (x \% 3 = 0) \rightarrow L_1
\]
\[
C_4 \equiv L_4 : x := x + 3 \rightarrow L_1, C_5 \equiv L_5 : \text{skip} \rightarrow L_1
\]

Its trace semantics from initial states \( \text{Trace}_P \) includes the following complete traces, where \( [\ ] \) is the initial totally undefined store.

\[
\langle [\ ], C_0 \rangle ([x/0], C_1^c)
\]
\[
\langle [\ ], C_0 \rangle ([x/0], C_1) ([x/0], C_2) ([x/1], C_3)
\]
\[
\langle [\ ], C_0 \rangle ([x/0], C_1) ([x/0], C_2) ([x/1], C_3^c) ([x/1], C_1)
\]
\[
\cdots
\]
\[
\langle [\ ], C_0 \rangle ([x/0], C_1) ([x/0], C_2) ([x/1], C_3^c) ([x/1], C_1) \cdots ([x/21], C_4) ([x/24], C_1)
\]
\[
\langle [\ ], C_0 \rangle ([x/0], C_1) ([x/0], C_2) ([x/1], C_3^c) ([x/1], C_1) \cdots ([x/21], C_4) ([x/24], C_1) ([x/24], C_5)
\]

Observe that the last trace corresponds to a terminating run of \( P \). □

### 3. ABSTRACTIONS

#### 3.1. Abstract Interpretation Background

In standard abstract interpretation [Cousot and Cousot 1977; Cousot and Cousot 1979], abstract domains, also called abstractions, are specified by Galois connections/insertions (GCs/GIs for short) or, equivalently, adjunctions. Concrete and abstract domains, \( \langle C, \leq_C \rangle \) and \( \langle A, \leq_A \rangle \), are assumed to be complete lattices which are related by abstraction and concretization maps \( \alpha : C \rightarrow A \) and \( \iota : A \rightarrow C \).
γ : A → C such that, for all a and c, α(c) ≤_A a ⇔ c ≤_C γ(a). A GC is a GI when
\( \alpha \circ \gamma = \lambda x. x \). It is well known that a join-preserving α uniquely determines, by
adjunction, γ as follows: γ(a) = \( \forall c \in C \mid \alpha(c) \leq_A a \); conversely, a meet-preserving γ
uniquely determines, by adjunction, α as follows: α(c) = \( \land \{ a \in A \mid c \leq_C γ(a) \} \).

Let \( f : C \to C \) be some concrete monotone function—-for simplicity, we consider 1-ary
functions—and let \( f^2 : A \to A \) be a corresponding monotone abstract function defined
on some abstraction \( A \) related to \( C \) by a GC. Then, \( f^2 \) is a correct abstract interpretation
of \( f \) on \( A \) when \( \alpha \circ f \subseteq f^2 \circ \alpha \) holds, where \( \subseteq \) denotes the pointwise ordering
between functions. Moreover, the abstract function \( f^A \triangleq \alpha \circ f \circ \gamma : A \to A \) is called the
best correct approximation of \( f \) on \( A \) because any abstract function \( f^\sharp \) is correct iff
\( f^A \subseteq f^\sharp \). Hence, for any \( A \), \( f^A \) plays the role of the best possible approximation of \( f \) on
the abstraction \( A \).

### 3.2. Store Abstractions

As usual in abstract interpretation [Cousot and Cousot 1977], a store property is modeled by some abstraction Store^\sharp of \( \wp(\text{Store}) \) which is formalized through a Galois connection:

\[
(\alpha_{\text{store}}, \wp(\wp(\text{Store})), \subseteq, (\text{Store}^\sharp, \leq), γ_{\text{store}}).
\]

Given a program \( P \), when Store^\sharp is viewed as an abstraction of \( \wp(\wp(\text{Store}_P)), \subseteq \) we emphasize it by adopting the notation Store^\sharp_P. A store abstraction Store^\sharp_P also induces a state abstraction State^\sharp_P \triangleq \text{Store}^\sharp_P \times P and, in turn, a trace abstraction defined by Trace^\sharp_P \triangleq (\text{State}^\sharp_P)^\star.

#### 3.2.1. Nonrelational Abstractions

Nonrelational store abstractions (i.e., relationships between program variables are not taken into account) can be easily designed by a standard pointwise lifting of some value abstraction. Let Value^\sharp be an abstraction of sets of possibly undefined values in \( \wp(\text{Value}_u) \) as formalized by a Galois connection

\[
(\alpha_{\text{value}}, \wp(\wp(\text{Value}_u)), \subseteq, (\text{Value}^\sharp, \leq_{\text{Value}}), γ_{\text{value}}).
\]

The abstract domain Value^\sharp induces a nonrelational store abstraction

\[
ρ^\sharp ∈ \text{Store}^\sharp_{\text{value}} \triangleq \wp(\text{Var} \to \text{Value}^\sharp, \subseteq)
\]

where \( \subseteq \) is the pointwise ordering induced by \( \leq_{\text{Value}} \): \( ρ^\sharp_1 \subseteq ρ^\sharp_2 \) iff for all \( x \in \text{Var} \), \( ρ^\sharp_1(x) \leq_{\text{Value}} ρ^\sharp_2(x) \). Hence, the bottom and top abstract stores are, respectively, \( λx.⊥_{\text{Value}} \) and \( λx.\top_{\text{Value}} \). The abstraction map \( α_{\text{value}}^\sharp : \wp(\text{Store}) → \text{Store}^\sharp_{\text{value}} \) is defined as follows:

\[
α_{\text{value}}^\sharp(S) ≜ \lambda x. α_{\text{value}}(\{ ρ(x) ∈ \text{Value}_u \mid ρ ∈ S \})
\]

The corresponding concretization map \( γ_{\text{value}}^\sharp : \text{Store}^\sharp_{\text{value}} → \wp(\wp(\text{Store})) \) is defined, as recalled in Section 3.1, by adjunction from the abstraction map \( α_{\text{value}}^\sharp \) and it is easy to check that it can be given as follows:

\[
γ_{\text{value}}^\sharp(ρ^\sharp) = \{ ρ ∈ \text{Store} \mid \forall x ∈ \text{Var} . ρ(x) ∈ γ_{\text{value}}(ρ^\sharp(x)) \}.
\]

Let us observe that:

(i) \( α_{\text{value}}^\sharp(∅) = λx. α_{\text{value}}(∅) = λx. ⊥_{\text{Value}} \) because \( α_{\text{value}}(∅) = ⊥_{\text{Value}} \) always holds in a GC;

(ii) \( α_{\text{value}}^\sharp([[]]) = λx. α_{\text{value}}(\{ \text{undef} \}) \);
(iii) if \( \gamma_{\text{value}}(\bot_{\text{Value}}) = \emptyset, \rho^\prime \in \text{Store}_{\text{value}}^\prime \) and \( \rho^\prime(x) = \bot_{\text{Value}} \) then \( \gamma_{\text{value}}(\rho^\prime) = \emptyset; \)
(iv) if \( \gamma_{\text{value}}(\bot_{\text{Value}}) = \{\text{undef}\} \) then \( \gamma_{\text{value}}(\lambda x.\bot_{\text{Value}}) = \{[\]}. \)

**Example 3.1 (The constant propagation abstraction).** The constant propagation (see [Wegman and Zadeck 1991]) lattice \((\text{CP}, \preceq)\) is depicted below.

![Lattice Diagram]

where \( \{v_i\}_{i \in \mathbb{Z}} \) is any enumeration of \( \text{Value}_n \), thus \( \text{undef} \) is included. Abstraction \( \alpha_{\text{cp}} : \wp(\text{Value}_n) \to \text{CP} \) and concretization \( \gamma_{\text{cp}} : \text{CP} \to \wp(\text{Value}_n) \) functions are defined as follows:

\[
\alpha_{\text{cp}}(S) \triangleq \begin{cases} 
\bot & \text{if } S = \emptyset \\
S & \text{otherwise} 
\end{cases}
\]

\[
\gamma_{\text{cp}}(a) \triangleq \begin{cases} 
\emptyset & \text{if } a = \bot \\
\{a\} & \text{if } a = v_i \\
\text{Value}_n & \text{if } a = \top 
\end{cases}
\]

and give rise to a GI \((\alpha_{\text{cp}}, (\wp(\text{Value}_n), \subseteq), (\text{CP}, \preceq), \gamma_{\text{cp}})\). The corresponding nonrelational store abstraction is denoted by \( \text{CP}_{\text{st}} \triangleq (\wp(\text{Var} \to \text{CP}), \preceq) \), where \( \alpha_{\text{CP}} : \wp(\text{Store}) \to \text{CP}_{\text{st}} \) and \( \gamma_{\text{CP}} : \text{CP}_{\text{st}} \to \wp(\text{Store}) \) denote the abstraction and concretization maps. For example, for \( \text{Var} = \{x, y, z, w\} \) and omitting the bindings \( v/\text{undef} \) also in abstract stores, we have that:

\[
\alpha_{\text{CP}}([x/2, y/\text{foo}, z/1], [x/2, y/\text{bar}]) = [x/2, y/\top, z/\top],
\]

\[
\gamma_{\text{CP}}([x/2, y/\top, w/\text{foo}]) = \{\rho \in \text{Store} : \rho(x) = 2, \rho(y) \in \text{Value}_n, \rho(z) = \text{undef}, \rho(w) = \text{foo}\},
\]

\( \gamma_{\text{CP}}([x/2, y/\top, w/\bot]) = \emptyset. \)

4. HOT PATH SELECTION

A loop path is a sequence of program commands which is repeated in some execution of a program loop, together with a store property which is valid at the entry of each command in the path. A loop path becomes hot when, during the execution, it is repeated at least a fixed number \( N \) of times. In a TJITC, hot path selection is performed by a loop path monitor that also records store properties (see, e.g., [Gal et al. 2009]). Here, hot path selection is not operationally defined, it is instead semantically modeled as an abstraction map over program traces, i.e., program executions.

Given a program \( P \) and therefore its trace semantics \( \text{Trace}_P \), we first define a mapping \( \text{loop} : \text{Trace}_P \to \wp(\text{Trace}_P) \) that returns all the loop paths in some execution trace of \( P \). More precisely, a loop path is a proper substring (i.e., a segment) \( \tau \) of a program trace \( \sigma \) such that:

1. the successor command in \( \sigma \) of the last state in \( \tau \) exists and coincides with the command – or its complement, when this is the last loop iteration – of the first state in \( \tau \);
2. there is no other such command within \( \tau \) (otherwise the sequence \( \tau \) would contain multiple iterations);
3. the last state of \( \tau \) performs a backward jump in the program \( P \).
To recognize backward jumps, we consider a topological order on the control flow graph of commands in $P$, denoted by $\prec$. This leads to the following formal definition:

$$\text{loop}(\langle p_0, C_0 \rangle \cdots \langle p_n, C_n \rangle) \triangleq \{ \langle p_i, C_i \rangle \langle p_{i+1}, C_{i+1} \rangle \cdots \langle p_j, C_j \rangle \mid 0 \leq i \leq j < n, C_i \prec C_j, \text{suc}(C_j) = \text{lbl}(C_i), \forall k \in (i, j), C_k \notin \{ C_i, \text{cmpl}(C_i) \} \}.$$  

Let us remark that a loop path

$$\langle p_i, C_i \rangle \cdots \langle p_j, C_j \rangle \in \text{loop}(\langle p_0, C_0 \rangle \cdots \langle p_n, C_n \rangle)$$

may contain some sub-loop path, namely it may happen that $\text{loop}(\langle p_i, C_i \rangle \cdots \langle p_j, C_j \rangle) \neq \emptyset$ so that some commands $C_k$, with $k \in [i, j]$, may occur more than once in $\langle p_i, C_i \rangle \cdots \langle p_j, C_j \rangle$; for example, this could be the case of a while loop whose body includes a nested while loop.

We abuse notation by using $\alpha_{\text{store}}$ to denote a map $\alpha_{\text{store}} : \text{Trace}_P \to \text{Trace}^s_P$, which “abstracts” a program trace $\tau$ into $\text{Trace}^s_P$, by abstracting the sequence of stores occurring in $\tau$:

$$\alpha_{\text{store}}(\langle p_0, C_0 \rangle \cdots \langle p_n, C_n \rangle) \triangleq \langle \alpha_{\text{store}}(\{ p_0 \}), C_0 \rangle \cdots \langle \alpha_{\text{store}}(\{ p_n \}), C_n \rangle.$$  

Given a static integer parameter $N > 0$, we define a function

$$\text{hot}^N : \text{Trace}_P \to \wp(\text{Trace}^s_P)$$

which returns the set of Store$^s$-abstracted loop paths appearing at least $N$ times in some program trace. In order to count the number of times a loop path appears within a trace we need an auxiliary function $\text{count} : \text{Trace}^s_P \times \text{Trace}^s_P \to \mathbb{N}$ such that $\text{count}(\sigma, \tau)$ yields the number of times an abstract trace $\tau$ occurs in an abstract trace $\sigma$:

$$\text{count}(\langle a_0, C_0 \rangle \cdots \langle a_n, C_n \rangle, \langle b_0, C'_0 \rangle \cdots \langle b_m, C'_m \rangle) \triangleq \sum_{i=0}^{n-m} \begin{cases} 1 & \text{if } \langle a_i, C_i \rangle \cdots \langle a_{i+m}, C_{i+m} \rangle = \langle b_0, C'_0 \rangle \cdots \langle b_m, C'_m \rangle \\ 0 & \text{otherwise} \end{cases}.$$  

Hence, $\text{hot}^N$ can be defined as follows:

$$\text{hot}^N(\sigma = \langle p_0, C_0 \rangle \cdots \langle p_n, C_n \rangle) \triangleq \{ \langle a_0, C_0 \rangle \cdots \langle a_n, C_n \rangle \mid \exists (p_i, C_i) \cdots (p_j, C_j) \in \text{loop}(\sigma) \text{ s.t.} \alpha_{\text{store}}(\langle p_i, C_i \rangle \cdots (p_j, C_j)) = \langle a_0, C_0 \rangle \cdots \langle a_n, C_n \rangle, \text{count}(\alpha_{\text{store}}(\sigma), \langle a_0, C_0 \rangle \cdots \langle a_n, C_n \rangle) \geq N \}.$$  

Finally, an abstraction map $\alpha_{\text{hot}}^N : \wp(\text{Trace}_P) \to \wp(\text{Trace}^s_P)$ collects the results of applying $\text{hot}^N$ to a set of traces:

$$\alpha_{\text{hot}}^N(T) \triangleq \bigcup_{\sigma \in T} \text{hot}^N(\sigma).$$  

A $N$-hot path $hp$ in a program $P$ is therefore any $hp \in \alpha_{\text{hot}}^N(\text{Trace}_P)$ and is compactly denoted as $hp = \langle a_0, C_0, \ldots, a_n, C_n \rangle$. Let us observe that if the hot path corresponds to the body of some while loop then its first command $C_0$ is a conditional, namely $C_0$ is the Boolean guard of the while loop. We define the successor function $\text{next}$ for indices in a hot path $\langle a_0, C_0, \ldots, a_n, C_n \rangle$ as follows: $\text{next} \triangleq \lambda i. i = n ? 0 \ddot{\vdash} i + 1$. For a $N$-hot path $\langle a_0, C_0, \ldots, a_n, C_n \rangle \in \alpha_{\text{hot}}(\text{Trace}_P)$, for any $i \in [0, n]$, if $C_i$ is a conditional command $L_i : B_i \to L_{\text{next}(i)}$ then throughout the paper its complement $C_i^c = \text{cmpl}(C_i)$ will be also denoted by $L_i : \neg B_i \to L_{\text{next}(i)}$.  

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Example 4.1. Let us consider the program $P$ in Example 2.1 and a trivial one-point store abstraction $\text{Store}^0 = \{\top\}$, where all the stores are abstracted to the same abstract store $\top$, i.e., $\alpha_{\text{store}} = \lambda S.\top$. Here, we have two 2-hot paths in $P$, that is, it turns out that $\alpha^0_{\text{hot}}(\text{Trace}_P) = \{hp_1, hp_2\}$ where:

$$hp_1 = (\top, C_1 \equiv L_1 : x \leq 20 \rightarrow L_2, \top, C_2 \equiv L_2 : x := x + 1 \rightarrow L_3,$$

$$\top, C_3 \equiv L_3 : (x \% 3 = 0) \rightarrow L_1);$$

$$hp_2 = (\top, C_1 \equiv L_1 : x \leq 20 \rightarrow L_2, \top, C_2 \equiv L_2 : x := x + 1 \rightarrow L_3,$$

$$\top, C_3 \equiv L_3 : (x \% 3 = 0) \rightarrow L_4, \top, C_4 \equiv x := x + 3 \rightarrow L_1).$$

Therefore, the hot paths $hp_1$ and $hp_2$ correspond, respectively, to the cases where the Boolean test $(x \% 3 = 0)$ fails and succeeds. Observe that the maximal sequence of different values assumed by the program variable $x$ is as follows:

$$? \mapsto 0 \mapsto 1 \mapsto 2 \mapsto 3 \mapsto 6 \mapsto 7 \mapsto 8 \mapsto 9 \mapsto 12 \mapsto 13 \mapsto 14 \mapsto 15 \mapsto 18 \mapsto 19 \mapsto 20 \mapsto 21 \mapsto 24$$

Hence, if $\sigma$ is the complete terminating trace of $P$ in Example 2.1 then it turns out that $\text{count}(\alpha_{\text{store}}(\sigma), hp_1) = 8$ and $\text{count}(\alpha_{\text{store}}(\sigma), hp_2) = 4$. 

5. TRACE EXTRACTION

For any abstract store $a \in \text{Store}^0$, a corresponding Boolean expression denoted by $\text{guard } E_a \in \text{BExp}$ is defined (where the notation $E_a$ should hint at an expression which is induced by the abstract store $a$), whose semantics is as follows: for any $\rho \in \text{Store}$,

$$\text{B}[\text{guard } E_a] \rho \triangleq \begin{cases} \text{true} & \text{if } \rho \in \gamma_{\text{store}}(a) \\ \text{false} & \text{if } \rho \notin \gamma_{\text{store}}(a) \end{cases}$$

In turn, we also have program actions $\text{guard } E_a \in \mathcal{A}$ such that:

$$\text{A}[\text{guard } E_a] \rho \triangleq \begin{cases} \rho & \text{if } \rho \in \gamma_{\text{store}}(a) \\ \bot & \text{if } \rho \notin \gamma_{\text{store}}(a) \end{cases}$$

Let $P$ be a program and $hp = (a_0, C_0, \ldots, a_n, C_n) \in \alpha^N_{\text{hot}}(\text{Trace}_P)$ be a hot path on some store abstraction $\text{Store}^0$. We define a syntactic transform of $P$ where the hot path $hp$ is explicitly extracted from $P$. This is achieved by a suitable relabeling of each command $C_i$ in $hp$ which is in turn preceded by the conditional $\text{guard } E_a$, induced by the corresponding store property $a_i$. To this aim, we consider three injective relabeling functions

$$\ell : [0, n] \rightarrow L_1 \quad \mathbf{1} : [1, n] \rightarrow L_2 \quad (\cdot) : \mathbb{L} \rightarrow \mathbb{L}$$

where $L_1, L_2$ and $\mathbb{L}$ are pairwise disjoint sets of fresh labels, so that $\text{labels}(P) \cap (L_1 \cup L_2 \cup \mathbb{L}) = \emptyset$. The transformed program $\text{extr}_{hp}(P)$ for the hot path $hp$ is defined as follows and a graphical example of this transform is depicted in Fig. 2.

Definition 5.1 (Trace extraction transform). The trace extraction transform of $P$ for the hot path $hp = (a_0, C_0, \ldots, a_n, C_n)$ is:

$$\text{extr}_{hp}(P) \triangleq P \setminus \{\{C_0\} \cup \{\text{cmpl}(C_0) \mid \text{cmpl}(C_0) \in P\}\}$$

$$\cup \{L_0 : \text{act}(C_0) \rightarrow L_1\} \cup \{L_0 : \neg \text{act}(C_0) \rightarrow L_1^\ast \mid \text{cmpl}(C_0) \in P\} \cup \text{stitch}_P(hp)$$
where the stitch of $hp$ into $P$ is defined as follows:

$$\text{stitch}_P(hp) \triangleq \{ L_0 : \text{guard } E_{a_0} \rightarrow \ell_0, L_0 : \neg \text{guard } E_{a_0} \rightarrow L_0 \}$$

$$\cup \{ \ell_i : \text{act}(C_i) \rightarrow \ell_{i+1} \mid i \in [0, n-1] \} \cup \{ \ell_n : \text{act}(C_n) \rightarrow L_0 \}$$

$$\cup \{ \ell_i : \neg \text{act}(C_i) \rightarrow L_{\text{next}(i)} \mid i \in [0, n], \text{compl}(C_i) \in P \}$$

$$\cup \{ L_i : \text{guard } E_{a_i} \rightarrow \ell_i, L_i : \neg \text{guard } E_{a_i} \rightarrow L_i \mid i \in [1, n] \}. \quad \square$$

The new command $L_0 : \text{guard } E_{a_0} \rightarrow \ell_0$ is therefore the entry conditional of the stitched hot path $\text{stitch}_P(hp)$, while any command $C \in \text{stitch}_P(hp)$ such that $\text{suc}(C) \in \text{labels}(P) \cup \top$ is a potential exit (or bail out) command of $\text{stitch}_P(hp)$.

**Lemma 5.2.** If $P$ is well-formed then, for any hot path $hp$, $\text{extr}_{hp}(P)$ is well-formed.

**Proof.** Recall that a program is well-formed when for any its conditional command it also includes a unique complement conditional. It turns out that $\text{extr}_{hp}(P)$ is well-formed because: (1) $P$ is well-formed; (2) for each conditional in $P_{\new} = \text{extr}_{hp}(P) \setminus P = ...$
Let us remark that the stitch of the hot path \(hp\) into \(P\) is always a linear sequence of different commands, namely, \(\text{stitch}_P(hp)\) does not contain loops nor join points. Furthermore, this happens even if the hot path \(hp\) does contain some inner sub-loop. Technically, this is achieved as a consequence of the fact that the above relabeling functions \(\ell\) and \(i\) are required to be injective. Hence, even if some command \(C\) occurs more than once inside \(hp\), e.g., \(C_i = C_j = C_k\) for some \(i, j, k \in [0, n-1]\) with \(i \neq j\), then these multiple occurrences of \(C\) in \(hp\) are transformed into differently labeled commands in \(\text{stitch}_P(hp)\), e.g., because \(\ell_i \neq \ell_j\) and \(i+1 \neq j+1\).

Let us now illustrate the trace extraction transform on a first simple example.

**Example 5.3.** Let us consider the program \(P\) in Example 2.1 and the hot path \(hp = (\top, C_1, \top, C_2, \top, C_3)\) in Example 4.1 (denoted there by \(hp_1\)), where stores are abstracted to the trivial one-point abstraction \(\text{Store}^\top = \{\top\}\). Here, for any \(\rho \in \text{Store}\), we have that \(B[\text{guard } E_T\rho = \top]\). The trace extraction transform of \(P\) w.r.t. \(hp\) is therefore as follows:

\[
\text{extr}_{hp}(P) = P \setminus \{C_1, C_2, C_3\} \cup \{L_1: x \leq 20 \rightarrow L_2, L_2: \neg(x \leq 20) \rightarrow L_5\} \cup \text{stitch}_P(hp)
\]

where

\[
\text{stitch}_P(hp) = \{H_0 : L_1 : \text{guard } E_T \rightarrow \ell_0, H_0^c : L_1 : \neg \text{guard } E_T \rightarrow \top_1\}
\]

\[
\setminus \{H_1 \equiv \ell_0 : x \leq 20 \rightarrow L_1, H_1^c \equiv \ell_0 : \neg(x \leq 20) \rightarrow L_0\}
\]

\[
\setminus \{H_2 \equiv \ell_1 : \text{guard } E_T \rightarrow \ell_1, H_2^c \equiv \ell_1 : \neg \text{guard } E_T \rightarrow L_2\}
\]

\[
\setminus \{H_3 \equiv \ell_1 : x := x + 1 \rightarrow L_2\}
\]

\[
\setminus \{H_4 \equiv \ell_2 : \text{guard } E_T \rightarrow \ell_2, H_4^c \equiv \ell_2 : \neg \text{guard } E_T \rightarrow L_3\}
\]

\[
\setminus \{H_5 \equiv \ell_2 : \neg(x \% 3 = 0) \rightarrow L_1, H_5^c \equiv \ell_2 : (x \% 3 = 0) \rightarrow L_4\}.
\]

The flow graph of \(\text{extr}_{hp}(P)\) is depicted in Figure 3 while a higher level representation using while-loops and gotos is as follows:

\[
x := 0;
L_1: \text{while } \text{guard } E_T \text{ do}
\]

\[
\text{if } \neg(x \leq 20) \text{ then goto } L_5
\]

\[
\text{if } \neg \text{guard } E_T \text{ then goto } L_2
\]

\[
x := x + 1;
\]

\[
\text{if } \neg \text{guard } E_T \text{ then goto } L_3
\]

\[
\text{if } (x \% 3 = 0) \text{ then goto } L_4
\]

\[
\text{if } \neg(x \leq 20) \text{ then goto } L_5
\]

\[
L_2: x := x + 1;
\]

\[
L_3: \text{if } \neg(x \% 3 = 0) \text{ then goto } L_1
\]

\[
L_4: x := x + 3;
\]

\[
\text{goto } L_1;
\]

\[
L_5: \text{skip}; 
\]

6. **CORRECTNESS**

As advocated by Cousot and Cousot [2002] par. 3.8, correctness of dynamic program transformations and optimizations should be defined with respect to some observational abstraction of program trace semantics: a dynamic program transform is correct.
when, at some level of abstraction, the observation of the execution of the subject program is equivalent to the observation of the execution of the transformed/optimized program.

**Store Changes Abstraction.** The approach by [Guo and Palsberg [2011]] to tracing compilation basically relies on a notion of correctness that requires the same store changes to happen in both the transformed/optimized program and the original program. This can be easily encoded by an observational abstraction $\alpha_{sc} : \wp(\text{Trace}) \rightarrow \wp(\text{Store}^*)$ of partial traces that observes store changes in execution traces:

$$sc : \text{Trace} \rightarrow \text{Store}^*$$

$$sc(\sigma) \triangleq \begin{cases} \\
\varepsilon & \text{if } \sigma = \varepsilon \\
\rho & \text{if } \sigma = \langle \rho, C \rangle \\
sc(\langle \rho, C_{1} \rangle \sigma') & \text{if } \sigma = \langle \rho, C_{0} \rangle \langle \rho, C_{1} \rangle \sigma' \\
\rho_{0} sc(\langle \rho_{1}, C_{1} \rangle \sigma') & \text{if } \sigma = \langle \rho_{0}, C_{0} \rangle \langle \rho_{1}, C_{1} \rangle \sigma', \rho_{0} \neq \rho_{1} \\
\end{cases}$$

$$\alpha_{sc}(T) \triangleq \{ sc(\sigma) | \sigma \in T \}$$

Since the function $\alpha_{sc}$ obviously preserves arbitrary set unions, as recalled in Section [3.1] it admits a right adjoint $\gamma_{sc} : \wp(\text{Store}^*) \rightarrow \wp(\text{Trace})$ defined as $\gamma_{sc}(S) \triangleq \bigcup \{ T | T \in}$

**Fig. 3.** The flow graph of the trace extraction transform $\text{extr}_{hp}(P)$ in Example [5.3] where commands of $\text{stitch}_{hp}(P)$ are in blue (in black/white: densely dotted) shapes, while commands of the source program $P$ are in pink (in black/white: loosely dotted) shapes.
where we have a GC \((\alpha_{sc}, (\langle \text{Trace} \rangle, \subseteq), (\langle \text{Store}^* \rangle, \subseteq), \gamma_{sc})\).

By a slight abuse of notation, \(\alpha_{sc}\) is also used as an abstraction of the partial trace semantics of a given program \(P\), that is, \(\alpha_{sc} : \langle \text{Trace}_P \rangle \rightarrow \langle \text{Store}_P^* \rangle\), which, clearly, gives rise to a corresponding GC \((\alpha_{sc}, (\langle \text{Trace}_P \rangle, \subseteq), (\langle \text{Store}_P^* \rangle, \subseteq), \gamma_{sc})\).

**Output Abstraction.** The store changes abstraction \(\alpha_{sc}\) may be too strong in practice. This can be generalized to any observational abstraction of execution traces \(\alpha_o : (\langle \text{Trace} \rangle, \subseteq) \rightarrow (A, \leq A)\) (which gives rise to a GC). As a significant example, one may consider an output abstraction that demands to have the same stores (possibly restricted to some subset of program variables) only at some specific output points. For example, in a language with no explicit output primitives, as that considered by Guo and Palsberg [2011], one could be interested just in the final store of the program (when it terminates), or in the entry and exit stores of any loop containing an extracted hot path. If we consider a language including a distinct primitive command “put \(X\)” that “outputs” the value of program variables ranging in some set \(X\) then we may want to have the same stores for variables in \(X\) at each output point put \(X\). In this case, optimizations should preserve the same sequence of outputs, i.e. optimizations should not modify the order of output commands. More formally, this can be achieved by adding a further sort of actions: put \(X \in k\), where \(X \subseteq \text{Var}\) is a set of program variables. The semantics of put \(X\) obviously does not affect program stores, i.e., \(A[\text{put } X]\rho \equiv \rho\). Correspondingly, if \(\text{Store}_X\) denotes stores on variables ranging in \(X\) then the following output abstraction \(\alpha_{out} : \langle \text{Trace} \rangle \rightarrow \langle \text{Store}_X^* \rangle\) of partial traces observes program stores at output program points only:

\[
\alpha_{out}(T) \triangleq \{ out(\sigma) \mid \sigma \in T \}
\]

where \(\rho_X\) denotes the restriction of the store \(\rho\) to variables in \(X\). Similarly to \(\alpha_{sc}\), here again we have a GC \((\alpha_o, (\langle \text{Trace} \rangle, \subseteq), (\langle \text{Store}_X^* \rangle, \subseteq), \gamma_o)\).

**Example 6.1 (Dead store elimination).** This approach based on a generic observational abstraction enables to prove the correctness of program optimizations that are unsound in Guo and Palsberg [2011]’s framework based on the store changes abstraction, such as dead store elimination. For example, in a program fragment such as

```
while (x ≤ 0) do
    z := 0;
    x := x + 1;
    z := 1;
```

one can extract the hot path \(hp = (x ≤ 0, z := 0, x := x + 1, z := 1)\) (here we ignore store abstractions) and perform dead store elimination of the command \(z := 0\) by optimizing \(hp\) to \(hp' = (x ≤ 0, x := x + 1, z := 1)\). As observed by Guo and Palsberg [2011], this is clearly unsound in bisimulation-based correctness because this hot path optimization does not output bisimilar code. By contrast, this optimization can be made sound by choosing and then formalizing an observational abstraction of program traces which requires to have the same stores at the beginning and at the exit of loops containing an extracted hot path, while outside of hot paths one could still consider the store changes abstraction. □
**Observational Abstraction.** One can generalize the store changes abstraction $\alpha_{sc}$ by considering any observational abstraction $\alpha_o : \langle \wp(\text{Trace}), \subseteq \rangle \rightarrow \langle A, \leq_A \rangle$ which is less precise (i.e., more approximate) than $\alpha_{sc}$: this means that for any $T_1, T_2 \in \wp(\text{Trace})$, if $\alpha_{sc}(T_1) = \alpha_{sc}(T_2)$ then $\alpha_o(T_1) = \alpha_o(T_2)$, or, equivalently, for any $T \in \wp(\text{Trace})$, $\gamma_{sc}(\alpha_{sc}(T)) \subseteq \gamma_o(\alpha_o(T))$. Informally, this means that $\alpha_o$ abstracts more information than $\alpha_{sc}$. As an example, when considering programs with output actions, the following abstraction $\alpha_{osc} : \wp(\text{Trace}) \rightarrow \wp(\text{Store}_X^\alpha)$ observes store changes at output program points only:

$$osc : \text{Trace} \rightarrow \text{Store}_X^\alpha$$

$$osc(\sigma) \triangleq \begin{cases} 
\varepsilon & \text{if } \sigma = \varepsilon \text{ or } \sigma = (\rho, C), \text{ act}(C) \neq \text{put } X \\
\rho_{|X} & \text{if } \sigma = (\rho, C), \text{ act}(C) = \text{put } X \\
osc((\rho, L_1 : \text{put } X \rightarrow L'_1)\sigma') & \text{if } \sigma = (\rho, C_0)(\rho, L_1 : A_1 \rightarrow L'_1)\sigma', \text{ act}(C_0) = \text{put } X \\
osc((\rho, L_1 : A_1 \rightarrow L'_1)\sigma') & \text{if } \sigma = (\rho, C_0)(\rho, L_1 : A_1 \rightarrow L'_1)\sigma', \text{ act}(C_0) \neq \text{put } X \\
osc((\rho_0, C_1)\sigma') & \text{if } \sigma = (\rho_0, C_0)(\rho_1, C_1)\sigma', \rho_0 \neq \rho_1, \text{ act}(C_0) = \text{put } X \\
osc((\rho_1, C_1)\sigma') & \text{if } \sigma = (\rho_0, C_0)(\rho_1, C_1)\sigma', \rho_0 \neq \rho_1, \text{ act}(C_0) \neq \text{put } X 
\end{cases}$$

$$\alpha_{osc}(T) \triangleq \{ osc(\sigma) | \sigma \in T \}$$

Clearly, it turns out that $\alpha_{osc}$ is more approximate than $\alpha_{sc}$ since $osc(\sigma)$ records a store change $\rho_0\rho_1$ only when the two contiguous subsequences of commands whose common stores are $\rho_0$ and $\rho_1$ contain among them at least a put command.

### 6.1. Correctness of Trace Extraction

It turns out that the observational correctness of the hot path extraction transform in Definition 5.1 can be proved w.r.t. the observational abstraction $\alpha_{sc}$ of store changes.

**Theorem 6.2** (Correctness of Trace Extraction). For any $P \in \text{Program}$ and $hp \in \alpha_{osc}(\text{Trace}_P)$, we have that $\alpha_{sc}([\text{extr}_{hp}(P)]) = \alpha_{sc}([P])$.

This is the crucial result concerning the correctness of our hot path extraction transform. We will show in Section 10.5 (see Theorem 10.12) that the correctness of the hot path extraction strategy defined in [Guo and Palsberg 2011] can be proved by a simple adaptation of the proof technique that we will use here.

In order to prove Theorem 6.2 we need to define some suitable "dynamic" transformations of execution traces. Let us fix a hot path $hp = (a_0, C_0, \ldots, a_n, C_n) \in \alpha_{osc}(\text{Trace}_P)$ (w.r.t. some store abstraction) and let $P_{hp} \triangleq \text{extr}_{hp}(P)$ denote the corresponding transform of $P$ given by Definition 5.1. We first define a mapping $\text{tr}_{hp}^{out}$ of the execution traces of the program $P$ into execution traces of the transformed program $P_{hp}$ that unfolds the hot path $hp$ (or any prefix of it) according to the hot path extraction strategy given by Definition 5.1. A function application $\text{tr}_{hp}^{out}(\tau)$ should replace any occurrence of the hot path $hp$ in the execution trace $\tau \in \text{Trace}_P$ with its corresponding guarded and suitably relabeled path obtained through Definition 5.1. More precisely, Fig. 4 provides the definitions for the following two functions:

$$\text{tr}_{hp}^{out} : \text{Trace}_P \rightarrow \text{Trace}_{P_{hp}} \quad \text{tr}_{hp}^{in} : \text{Trace}_P \rightarrow (\text{State}_P \cup \text{State}_{P_{hp}})^*$$

Let us first describe how the trace transform $\text{tr}_{hp}^{out}$ works. A function application $\text{tr}_{hp}^{out}(s\sigma)$ on a trace $s\sigma$ of $P$—the superscript $out$ hints that the first state $s$ of the trace $s\sigma$ is still outside of the hot path $hp$ so that $\text{tr}_{hp}^{out}(s\sigma)$ could either enter into the transform of $hp$ or remain outside of $hp$—triggers the unfolding of the hot path $hp$ in $P_{hp}$ when the first state $s$ is such that:
(i) $s = (\rho, C_0)$, where $C_0$ is the first command of $hp$;
(ii) the entry conditional guard $E_{a_0}$ of stitch$_P(hp)$ is satisfied in the store $\rho$ of the state $s = (\rho, C_0)$, that is, $\alpha_{\text{store}}(\{\rho\}) \leq a_0$.

If the unfolding for the trace $(\rho, C_0)\sigma$ is actually started by applying $\text{tr}^\text{out}_P((\rho, C_0)\sigma)$ then:

(iii) the first state $(\rho, C_0)$ is unfolded into the following sequence of two states of $P_{hp}$:

$$\langle \rho, L_0 : \text{guard } E_{a_0} \rightarrow t_0 \rangle \langle \rho, t_0 : \text{act}(C_0) \rightarrow I_1 \rangle;$$

(iv) in turn, the unfolding of the residual trace $\sigma$ is carried on by applying $\text{tr}^\text{in}_P(\sigma)$.

Let us now focus on the function $\text{tr}^\text{in}_P$. A function application $\text{tr}^\text{in}_P(s\sigma)$—here the superscript $\text{in}$ suggests that we are currently inside the hot path $hp$ so that $\text{tr}^\text{in}_P(s\sigma)$ could either exit from the unfolding of $hp$ or advance with the unfolding of $hp$—carries on the unfolding of $hp$ as a trace in $P_{hp}$ when the current state $s$ is such that:

(i) $s = (\rho, C_i)$, where $i \in [1, n-1]$, meaning that the command $C_i$ is strictly inside $hp$, i.e., $C_i$ is different from the first command $C_0$ and the last command $C_n$ of $hp$;
(ii) the guarded conditional guard $E_{a_i}$ is satisfied in the store $\rho$ of the state $s = (\rho, C_i)$, that is, $\alpha_{\text{store}}(\{\rho\}) \leq a_i$.

If one of these two conditions does not hold then the trace transformation $\text{tr}^\text{in}_P((\rho, C_i)\sigma)$, after a suitable unfolding step for $(\rho, C_i)$, jumps back to the “outside of $hp$” modality by progressing with $\text{tr}^\text{out}_P(\sigma)$.

**Example 6.3.** Consider the transform $P_{hp}$ of Example 5.3 for the program $P$ in Example 2.1 w.r.t. the hot path $hp = \langle T, C_1, T, C_2, T, C_3 \rangle$. In particular, we refer to the notation $H_1, H_2$ used to denote the commands in the stitch of $hp$ into $P$. Consider the following trace fragment $\tau$ in Trace$_P$:

$$\tau = ([x/3], C_0)([x/0], C_1)([x/0], C_2)([x/1], C_3)([x/1], C_1)([x/2], C_5)([x/2], C_1)([x/2], C_2)([x/3], C_3)([x/3], C_4)$$

Then, we have that the dynamic transformation $\text{tr}^\text{out}_P(\tau)$ acts as follows:

$$\text{tr}^\text{out}_P(\tau) = ([x/3], C_0)\text{tr}^\text{out}_P(\tau_\rightarrow) = ([x/3], C_0)([x/0], H_0)([x/0], H_1)\text{tr}^\text{in}_P(\tau_\rightarrow)$$

$$\text{tr}^\text{in}_P(\tau_\rightarrow) = ([x/0], H_2)([x/0], H_3)\text{tr}^\text{in}_P(\tau_\rightarrow)$$

$$\text{tr}^\text{in}_P(\tau_\rightarrow) = ([x/1], H_4)([x/1], H_5)\text{tr}^\text{in}_P(\tau_\rightarrow)$$

$$\ldots$$

$$\text{tr}^\text{in}_P(\tau_\rightarrow) = \text{tr}^\text{in}_P(([x/3], C_3)([x/3], C_4)) = ([x/3], H_4)([x/3], H_5)\text{tr}^\text{out}_P(([x/3], C_4))$$

$$= ([x/3], H_4)([x/3], H_5)\text{tr}^\text{out}_P(\epsilon)$$

$$= ([x/3], H_4)([x/3], H_5)([x/3], C_4)$$

Summing up, using the colors in the flow graph of $P_{hp}$ in Fig. 3 and representing traces as sequences of commands only, we have that:

$$\tau \equiv C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_4$$

$$\text{tr}^\text{out}_P(\tau) \equiv C_0 \rightarrow H_0 \rightarrow H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow H_4 \rightarrow H_5 \rightarrow H_0 \rightarrow H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow H_4 \rightarrow H_5 \rightarrow C_4$$
\[ hp = (a_0, C_0, ..., a_n, C_n) \] is a given hot path

\[
\begin{align*}
\text{tr}_h^{\text{out}}(\epsilon) & \triangleq \epsilon \\
\text{tr}_h^{\text{out}}(s \sigma) & \triangleq \\
& \begin{cases} 
\langle \rho, L_0 : \text{guard } E_{a_0} \rightarrow \ell_0 \rangle \langle \rho, \ell_0 : \text{act}(C_0) \rightarrow 1_i \rangle \text{tr}_h^{\text{out}}(\sigma) \\
& \text{if } s = \langle \rho, C_0 \rangle, \alpha_{\text{store}}(\{\rho\}) \leq a_0 \\
\langle \rho, L_0 : -\text{guard } E_{a_0} \rightarrow \ell_0 \rangle \langle \rho, \ell_0 : \text{act}(C_0) \rightarrow L_i \rangle \text{tr}_h^{\text{out}}(\sigma) \\
& \text{if } s = \langle \rho, C_0 \rangle, \alpha_{\text{store}}(\{\rho\}) \leq a_0 \\
\langle \rho, L_0 : \text{guard } E_{a_0} \rightarrow \ell_0 \rangle \langle \rho, \ell_0 : -\text{act}(C_0) \rightarrow L_i \rangle \text{tr}_h^{\text{out}}(\sigma) \\
& \text{if } s = \langle \rho, \text{cmpl}(C_0) \rangle, \alpha_{\text{store}}(\{\rho\}) \leq a_0 \\
s \cdot \text{tr}_h^{\text{out}}(\sigma) & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{tr}_h^{\text{in}}(\epsilon) & \triangleq \epsilon \\
\text{tr}_h^{\text{in}}(s \sigma) & \triangleq \\
& \begin{cases} 
\langle \rho, 1_i : \text{guard } E_{a_i} \rightarrow \ell_i \rangle \langle \rho, \ell_i : \text{act}(C_i) \rightarrow 1_{i+1} \rangle \text{tr}_h^{\text{in}}(\sigma) \\
& \text{if } s = \langle \rho, C_i \rangle, i \in [1, n-1], \alpha_{\text{store}}(\{\rho\}) \leq a_i \\
\langle \rho, 1_n : \text{guard } E_{a_n} \rightarrow \ell_n \rangle \langle \rho, \ell_n : \text{act}(C_n) \rightarrow L_0 \rangle \text{tr}_h^{\text{out}}(\sigma) \\
& \text{if } s = \langle \rho, C_n \rangle, \alpha_{\text{store}}(\{\rho\}) \leq a_n \\
\langle \rho, 1_i : -\text{guard } E_{a_i} \rightarrow L_i \rangle \langle \rho, C_i \rangle \text{tr}_h^{\text{out}}(\sigma) \\
& \text{if } s = \langle \rho, C_i \rangle, i \in [1, n], \alpha_{\text{store}}(\{\rho\}) \leq a_i \\
\langle \rho, 1_i : \text{guard } E_{a_i} \rightarrow \ell_i \rangle \langle \rho, \ell_i : -\text{act}(C_i) \rightarrow L_{n_{\text{next}(i)}}^{\text{ext}} \rangle \text{tr}_h^{\text{out}}(\sigma) \\
& \text{if } s = \langle \rho, \text{cmpl}(C_i) \rangle, i \in [1, n], \alpha_{\text{store}}(\{\rho\}) \leq a_i \\
\langle \rho, 1_i : -\text{guard } E_{a_i} \rightarrow L_i \rangle \langle \rho, \text{cmpl}(C_i) \rangle \text{tr}_h^{\text{out}}(\sigma) \\
& \text{if } s = \langle \rho, \text{cmpl}(C_i) \rangle, i \in [1, n], \alpha_{\text{store}}(\{\rho\}) \leq a_i \\
s \cdot \text{tr}_h^{\text{out}}(\sigma) & \text{otherwise}
\end{cases}
\end{align*}
\]

Fig. 4. Definitions of \( \text{tr}_h^{\text{out}} \) and \( \text{tr}_h^{\text{in}} \).

where red boxes denote commands of \( \tau \) and \( \text{tr}_{hp}^{\text{out}}(\tau) \) outside of the hot path \( hp \), black boxes with red commands denote commands of \( \tau \) inside \( hp \), while black boxes with blue commands denote commands of \( \text{tr}_{hp}^{\text{out}}(\tau) \) in \( \text{stitch}_P(hp) \). Hence, \( \text{tr}_{hp}^{\text{out}}(\tau) \) carries out the unfolding of the hot path \( hp \) for the execution trace \( \tau \) of \( P \), and therefore provides an execution trace of the transformed program \( P_{hp} \). \( \square \)

It turns out that \( \text{tr}_{hp}^{\text{out}} \) maps traces of \( P \) into traces of \( P_{hp} \) and does not alter store change sequences.

**Lemma 6.4.** \( \text{tr}_{hp}^{\text{out}} \) is well-defined and for any \( \sigma \in \text{Trace}_P \), \( \text{sc}(\text{tr}_{hp}^{\text{out}}(\sigma)) = \text{sc}(\sigma) \).

**Proof.** We first show that: (1) \( \text{tr}_{hp}^{\text{out}} \) is well-defined, i.e., for any \( \sigma \in \text{Trace}_P \), \( \text{tr}_{hp}^{\text{out}}(\sigma) \in \text{Trace}_{P_{hp}} \), and (2) for any \( \sigma \in \text{Trace}_P \), if \( \text{cmd}(\sigma_0) \notin \{C_0, \text{cmpl}(C_0)\} \) then...
\[ hp = (a_0, C_0, ..., a_n, C_n) \] is a given hot path

\[
\begin{align*}
\text{rtr}_{hp}(\epsilon) & \equiv \epsilon \\
\text{rtr}_{hp}(\sigma) & \equiv \begin{cases} 
\langle \text{store}(s), C_i \rangle & \text{if } \sigma = \epsilon, \text{act}(s) \in \{\text{guard } E_a, \neg\text{guard } E_a\}, i \in [1, n] \\
\text{rtr}_{hp}(\sigma') & \text{if } \sigma \neq \epsilon, \text{act}(s) \in \{\text{guard } E_a, \neg\text{guard } E_a\}, i \in [1, n] \\
\langle \rho, C_0 \rangle \text{rtr}_{hp}(\sigma) & \text{if } s = \langle \rho, L_0 : \text{act}(C_0) \rightarrow L_1 \rangle \\
\langle \rho, C_i \rangle \text{rtr}_{hp}(\sigma) & \text{if } s = \langle \rho, L_0 : \neg\text{act}(C_0) \rightarrow L_1 \rangle \\
\langle \rho, C_i \rangle \text{rtr}_{hp}(\sigma) & \text{if } s = \langle \rho, \ell_i : \text{act}(C_i) \rightarrow L_{i+1} \rangle, i \in [1, n-1] \\
\langle \rho, C_n \rangle \text{rtr}_{hp}(\sigma) & \text{if } s = \langle \rho, \ell_i : \neg\text{act}(C_i) \rightarrow L_{\text{next}(i)} \rangle, i \in [1, n] \\
s \cdot \text{rtr}_{hp}(\sigma) & \text{otherwise}
\end{cases}
\]

(iii) for the clauses 1, 2 and 4 that define \( \text{tr}^{\text{in}}_{hp}(\sigma) \) in Fig. 4 we have that \( \text{tr}^{\text{out}}_{hp}(\sigma) = s's''\text{tr}^{\text{in}}_{hp}(\sigma) \) or \( \text{tr}^{\text{in}}_{hp}(\sigma) = s's''\text{tr}^{\text{out}}_{hp}(\sigma) \), where \( s' \) is a guard command of \( P_{hp} \) and \( s'' \) is in turn a legal sub-execution trace of \( P_{hp} \);

(ii) for the last clause that defines \( \text{tr}^{\text{out}}_{hp}(\sigma) \) in Fig. 4 we have that \( \text{cmd}(s) \notin \{C_0, \text{cmpl}(C_0)\} \), hence \( s \) is a legal state in \( P_{hp} \) and, in turn, \( \text{tr}^{\text{out}}_{hp}(\sigma) = s \cdot \text{tr}^{\text{out}}_{hp}(\sigma) \) is a trace of \( P_{hp} \);

(iii) for the clauses 1, 2 and 4 that define \( \text{tr}^{\text{in}}_{hp}(\sigma) \) in Fig. 4 we have that \( \text{tr}^{\text{in}}_{hp}(\sigma) = s's''\text{tr}^{\text{in}}_{hp}(\sigma) \) or \( \text{tr}^{\text{in}}_{hp}(\sigma) = s's''\text{tr}^{\text{out}}_{hp}(\sigma) \), where \( s' \) is a guard command and \( s'' \) is an action command such that \( s'' \) is a legal sub-execution trace of \( P_{hp} \);

(iv) for the clauses 3 and 5 that define \( \text{tr}^{\text{in}}_{hp}(\sigma) \) in Fig. 4 we have that \( \text{tr}^{\text{in}}_{hp}(\sigma) = s's''\text{tr}^{\text{in}}_{hp}(\sigma) \) where \( s' \) is a guard command and \( s'' \) turns out to be a legal sub-execution trace of \( P_{hp} \);

(v) for the last clause that defines \( \text{tr}^{\text{in}}_{hp}(\sigma) \) in Fig. 4 we have that \( \text{cmd}(s) \notin \{C_i, \text{cmpl}(C_i) \mid i \in [1, n]\} \); by hypothesis, \( \text{cmd}(s) \notin \{C_0, \text{cmpl}(C_0)\} \), so that \( \text{cmd}(s) \notin \{C_i, \text{cmpl}(C_i) \mid i \in [0, n]\} \), hence \( s \) is a legal state in \( P_{hp} \) and in turn \( \text{tr}^{\text{in}}_{hp}(\sigma) = s \cdot \text{tr}^{\text{out}}_{hp}(\sigma) \) is a trace of \( P_{hp} \);

(vi) \( \text{tr}^{\text{in}}_{hp}(\sigma) \) is never recursively called by a function application \( \text{tr}^{\text{out}}_{hp}(s_0s\sigma) \) when \( \text{cmd}(s) \in \{C_0, \text{cmpl}(C_0)\} \).

Then, it is immediate to check from the definitions in Fig. 4 that if \( \text{tr}^{\text{out}}_{hp}(\sigma) = s's''\tau \) then \( \text{store}(s) = \text{store}(s') = \text{store}(s'') \). Therefore, for any \( \sigma \in \text{Trace}_P \), we obtain that \( \text{sc}(\text{tr}^{\text{out}}_{hp}(\sigma)) = \text{sc}(\sigma) \). \( \square \)

Vice versa, it is a simpler task to define a reverse transformation function \( \text{rtr}_{hp} \) that “decompiles” an execution trace \( \sigma \) of \( P_{hp} \) into an execution trace of \( P \) by removing guarded commands in \( \sigma \), as generated by the hot path \( hp \), and by mapping the relabeled

---

Fig. 5. Definition of \( \text{rtr}_{hp} \).
commands of \(hp\) in \(\sigma\) back to their corresponding source commands of \(hp\). This function \(\text{rtr}_{hp} : \text{Trace}_{hp} \rightarrow \text{Trace}_P\) is correctly defined by the clauses in Fig. 5 and it preserves store change sequences.

**Lemma 6.5.** \(\text{rtr}_{hp}\) is well-defined and for any \(\sigma \in \text{Trace}_{hp}\), \(sc(\text{rtr}_{hp}(\sigma)) = \text{rtr}_{hp}(\sigma)\).

**Proof.** We show that \(\text{rtr}_{hp}\) is well-defined, i.e., for any \(\sigma \in \text{Trace}_{hp}\), \(\text{rtr}_{hp}(\sigma) \in \text{Trace}_P\). This follows by an easy induction on the length of the execution trace \(\sigma\) by observing that:

(i) the first clause that defines \(\text{rtr}_{hp}(\sigma)\) in Fig. 5 is an extremal base case where \(so = s\) and the command action of \(s\) is a guard command guard \(E_a\), (or its complement); in this case, we simply retain the store of \(s\) and pick the command \(C_i\) of \(P\).

(ii) the clause 2 of \(\text{rtr}_{hp}(\sigma)\) in Fig. 5 simply removes the states whose commands are some guard \(E_a\); since guard \(E_a\) does not alter stores, this removal preserves the sequence of store changes.

(iii) the clauses 3-7 of \(\text{rtr}_{hp}(\sigma)\) in Fig. 5 map a state \(s\) of \(P_{hp}\) whose command \(H_i\) is a relabeled action \(\text{act}(C_i)\) or \(\neg\text{act}(C_i)\) of the hot path \(hp\) to a corresponding state of \(P\) that has the same store \(s\) and whose command is: \(C_i\) for \(\text{act}(C_i)\) and \(C_i^c\) for \(\neg\text{act}(C_i)\); here, we observe that since guards in \(\sigma\) are removed, by induction, these definitions allow us to obtain that \(so\) is mapped to a legal trace of \(P\) that does not alter the sequence of store changes.

(iv) the clause 8 of \(\text{rtr}_{hp}(\sigma)\) in Fig. 5 states that if \(s\) is already a state of \(P\) then it is left unchanged.

Hence, the above points also show that the sequence of store changes is not affected by \(\text{rtr}_{hp}\), i.e., for any \(\sigma \in \text{Trace}_{hp}\), \(sc(\text{rtr}_{hp}(\sigma)) = sc(\sigma)\). \(\square\)

**Example 6.6.** We carry on Example 6.3 by considering the following trace fragment \(\sigma \in \text{Trace}_{hp}\), where the transformed program \(P_{hp}\) is in Example 5.3.

\[
\sigma = \langle [x/2], H_4 \rangle \langle [x/2], H_5 \rangle \langle [x/2], H_0 \rangle \langle [x/2], H_1 \rangle \langle [x/2], H_2 \rangle \langle [x/2], H_3 \rangle \langle [x/3], H_4 \rangle \langle [x/3], H_5^R \rangle \langle [x/3], C_4 \rangle \langle [x/6], C_1 \rangle
\]

Here, the decompilation of \(\sigma\) back into an execution trace of \(P\) through \(\text{rtr}_{hp}\) yields:

\[
\text{rtr}_{hp}(\sigma) = \text{rtr}_{hp}(\sigma_1^-) = \langle [x/2], C_5^p \rangle \text{rtr}_{hp}(\sigma_2^-) = \langle [x/2], C_5^p \rangle \text{rtr}_{hp}(\sigma_3^-)
\]

\[
= \langle [x/2], C_5^p \rangle \langle [x/2], C_1 \rangle \langle [x/2], C_2 \rangle \text{rtr}_{hp}(\sigma_4^-)
\]

\[
= \langle [x/2], C_5^p \rangle \langle [x/2], C_4 \rangle
\]

Indeed, \(\langle [x/2], C_5^p \rangle \langle [x/2], C_1 \rangle \langle [x/2], C_2 \rangle \langle [x/3], C_3 \rangle \langle [x/3], C_4 \rangle \langle [x/6], C_1 \rangle\) is a well-defined execution trace of \(P\). \(\square\)

We are now in the position to prove Theorem 6.2.

**Proof of Theorem 6.2** With an abuse of notation for \(\text{rtr}_{hp}\), let us define two functions \(\text{tr}_{hp} : \wp(\text{Trace}_{hp}) \rightarrow \wp(\text{Trace}_P)\) and \(\text{rtr}_{hp} : \wp(\text{Trace}_{hp}) \rightarrow \wp(\text{Trace}_P)\) which are the collecting versions of \(\text{tr}_{hp}^{out}\) and \(\text{rtr}_{hp}\), that is, \(\text{tr}_{hp}(T) \triangleq \{\text{tr}_{hp}^{out}(\sigma) \mid \sigma \in T\}\) and
rtr\textsubscript{hp}(T) \triangleq \{ rtr\textsubscript{hp}(\sigma) \mid \sigma \in T \}. As consequences of the above lemmata, we have the following properties.

(A) $\alpha\text{sc} \circ \text{tr}_{hp} = \alpha\text{sc}$: by Lemma\ref{lem:sc-trhp}.
(B) $\text{tr}_{hp}(T[P]) \subseteq T[P_\text{hp}]$: because, by Lemma\ref{lem:sc-trhp}, $\text{tr}_{hp}$ is well-defined.
(C) $\alpha\text{sc} \circ \text{rtr}_{hp} = \alpha\text{sc}$: by Lemma\ref{lem:sc-rtrhp}.
(D) $\text{rtr}_{hp}(T[P_\text{hp}]) \subseteq T[P]$: because, by Lemma\ref{lem:sc-rtrhp}, $\text{rtr}_{hp}$ is well-defined.

We therefore obtain:

\[
\begin{align*}
\alpha\text{sc}(T[P]) &= \text{[By point (A)]} \\
\alpha\text{sc}(\text{tr}_{hp}(T[P])) &\subseteq \text{[By point (B)]} \\
\alpha\text{sc}(T[P_\text{hp}]) &= \text{[By point (C)]} \\
\alpha\text{sc}(\text{rtr}_{hp}(T[P_\text{hp}]))) &\subseteq \text{[By point (D)]} \\
\alpha\text{sc}(T[P]) &= \text{[By point (D)]}
\end{align*}
\]

and this closes the proof. \hfill \Box

### 6.2. Correctness of Hot Path Optimizations

Guarded hot paths are a key feature of our tracing compilation model and are meant to be dynamically recorded by a hot path monitor. An abstract guard for a command $C$ of some stitched hot path $\text{stitch}_P(hp)$ encodes a property of program stores which is represented as an element of an abstract domain $\text{Store}^\alpha$ and is guaranteed to hold at the entry of $C$. This information on program stores, as encapsulated by the abstract guards in $\text{stitch}_P(hp)$, can then be used in hot path optimizations, namely, to optimize the commands in $hp$.

We follow a modular approach for proving the correctness of hot path optimizations. A hot path optimization $O$ should optimize $P$ along some hot path $hp$ of $P$, by relying on the abstract store information recorded in $hp$, while leaving unchanged the commands outside of $hp$. Hence, in our framework, fixed $P \in \text{Program}$, an optimization $O$ is defined to be a program transform of the commands in $\text{stitch}_P(hp)$, that is,

\[ O : \{ \text{stitch}_P(hp) \mid hp \in \alpha^N_{\text{hot}}(\text{Trace}_P) \} \rightarrow \text{Program} \]

where Program may allow new optimized expressions and/or actions introduced by $O$, as it will be the case of type-specific additions $+_{\text{Type}}$ in the type specialization optimization described in Section\ref{sec:typespec}. Let $P_{\neg hp} \triangleq \text{extr}_{hp}(P) \setminus \text{stitch}_P(hp)$ denote the commands outside of the stitched hot path. Then, the corresponding full optimization $O_{\text{full}}$ of the whole program $P$ w.r.t. the hot path $hp$ should extract and simultaneously optimize $hp$, namely, this is defined by

\[ O_{\text{full}}(P, hp) \triangleq P_{\neg hp} \cup O(\text{stitch}_P(hp)) \]

where $O_{\text{full}}(P, hp)$ is required to be a well-formed program, i.e., $O_{\text{full}}(P, hp) \in \text{Program}$. This full optimization $O_{\text{full}}(P, hp)$ has to be proved correct w.r.t. some observational abstraction $\alpha_o : \varphi(\text{Trace}_P) \rightarrow A$ of program traces, which is assumed to be more abstract than the store changes abstraction $\alpha_{\text{sc}}$ (cf. Section\ref{sec:observ}). Then, this full optimization is correct for $\alpha_o$ when:

\[ \alpha_o(T[O_{\text{full}}(P, hp)]) = \alpha_o(T[P]) \]

Since Theorem\ref{th:tr-extr} ensures that the unoptimized trace extraction transform is already correct for the store changes abstraction $\alpha_{\text{sc}}$, which is more precise than $\alpha_o$, the intuition is that in order to prove the correctness of $O_{\text{full}}$ w.r.t. $\alpha_o$, it is enough to focus
on the correctness of the optimization $O$ along the stitched hot path $\text{stitch}_P(h_p)$. This therefore leads to the following definition of correctness for a hot path optimization.

**Definition 6.7 (Correctness of hot path optimization).** $O$ is correct for the observational abstraction $\alpha_o$ if for any $P \in \text{Program}$ and for any $h_p \in \alpha_{\text{hot}}(\text{Trace}_P)$, $\alpha_o(T[O(\text{stitch}_P(h_p))]) = \alpha_o(T[\text{stitch}_P(h_p)])$. \[\square\]

In order to prove that this correctness of a hot path optimization implies the correctness of the corresponding full optimization, we define two functions

$$t\text{odo} : \text{Trace}_{\text{stitch}_P(h_p)} \rightarrow \text{Trace}_{O(\text{stitch}_P(h_p))}$$

which must be well-defined, i.e. they have to map well-formed traces into well-formed traces, and, intuitively, encode the effect of optimizing (function $t\text{odo}$) and de-optimizing (function $t\text{do}$) execution traces along a stitched hot path. Since $\text{Trace}_{\text{stitch}_P(h_p)} \subseteq \text{Trace}_{\text{extr}_P(P)}$ and $\text{Trace}_{O(\text{stitch}_P(h_p))} \subseteq \text{Trace}_{\text{full}_P(P,h_p)}$, we then extend $t\text{odo}$ and $t\text{do}$ to two functions

$$t\text{odo}_{\text{full}} : \text{Trace}_{\text{extr}_P(P)} \rightarrow \text{Trace}_{\text{full}_P(P,h_p)} \quad t\text{do}_{\text{full}} : \text{Trace}_{\text{full}_P(P,h_p)} \rightarrow \text{Trace}_{\text{extr}_P(P)}$$

which simply apply $t\text{odo}$ and $t\text{do}$ to maximal subtraces, respectively, in $\text{Trace}_{\text{stitch}_P(h_p)}$ and $\text{Trace}_{O(\text{stitch}_P(h_p))}$, while leaving unchanged the remaining states. Let us formalize this idea. If $\sigma \in \text{Trace}_{\text{extr}_P(P)}$ is nonempty and, for some $k \in [0,|\sigma|]$, $\text{cmd}(\sigma_k) \in \text{stitch}_P(h_p)$ then $\sigma_{[k,n_{st}]}$ denotes the maximal subtrace of $\sigma$ beginning at $\sigma_k$ which belongs to $\text{Trace}_{\text{stitch}_P(h_p)}$, that is, the index $n_{st} \geq k$ is such that: (1) $\text{cmd}(\sigma_{n_{st}}) \in \text{stitch}_P(h_p)$, (2) if $n_{st} < |\sigma| - 1$ then $\text{cmd}(\sigma_{n_{st}+1}) \notin \text{stitch}_P(h_p)$, (3) for any $j \in [k,n_{st}]$, $\text{cmd}(\sigma_j) \in \text{stitch}_P(h_p)$. Analogously, if $\tau \in \text{Trace}_{O(\text{full}_P(P,h_p))}$ is nonempty and $\text{cmd}(\tau_k) \in O(\text{stitch}_P(h_p))$ then $\tau_{[k,n_{st}]}$ denotes the maximal subtrace of $\tau$ beginning at $\tau_k$ which belongs to $\text{Trace}_{O(\text{stitch}_P(h_p))}$. Then, the formal definition of $t\text{odo}_{\text{full}}$ goes as follows:

$$t\text{odo}_{\text{full}}(\sigma) = \begin{cases} 
\epsilon & \text{if } \sigma = \epsilon \\
\sigma_0 \text{t\text{odo}_{\text{full}}}(\sigma_{[0,n_{st}]}^{-}) & \text{if } \sigma \neq \epsilon, \text{cmd}(\sigma_0) \notin \text{stitch}_P(h_p) \\
\text{to}_{\text{full}}(\sigma_{[n_{st}+1]^{-}}) & \text{if } \sigma \neq \epsilon, \text{cmd}(\sigma_0) \in \text{stitch}_P(h_p)
\end{cases}$$

and analogously for $t\text{do}_{\text{full}}$. Since $t\text{odo}$ and $t\text{do}$ are supposed to be well-defined, it turns out that $t\text{odo}_{\text{full}}$ and $t\text{do}_{\text{full}}$ are well-defined once we make the weak and reasonable assumption that $t\text{odo}$ and $t\text{do}$ do not modify the entry (which is always $L_0$) and exit labels of the stitched hot path. This assumption, e.g., for $t\text{odo}$ can be formalized as follows: if $\sigma \in \text{Trace}_{\text{stitch}_P(h_p)}$ and $t\text{odo}(\sigma) = \tau$ then (i) if $\text{lbl}(\sigma_0) = L_0$ then $\text{lbl}(\tau_0) = L_0$; (ii) if $\text{suc}(\sigma_{|\sigma|-1}) = L' \notin \text{labels}(P)$ then $\text{suc}(\tau_{|\tau|-1}) = L'$. In the following, $t\text{odo}_{\text{full}}$ and $t\text{do}_{\text{full}}$ are also used to denote their corresponding collecting functions defined on sets of traces.

**Lemma 6.8.** Assume that $\alpha_o \circ t\text{odo}_{\text{full}} = \alpha_o = \alpha_o \circ t\text{do}_{\text{full}}$. If $O$ is correct for $\alpha_o$ then $O_{\text{full}}$ is correct for $\alpha_o$.

**Proof.** We have that:

$$\alpha_o(T[O_{\text{full}}(P,h_p)]) = [\text{By } \alpha_o \circ t\text{do}_{\text{full}} = \alpha_o]$$

$$\alpha_o(t\text{do}_{\text{full}}(T[O_{\text{full}}(P,h_p)])) \subseteq [\text{Since } t\text{do}_{\text{full}} \text{ is well-defined}]$$

$$\alpha_o(T[\text{extr}_P(P)]) = [\text{By } \alpha_o \circ t\text{odo}_{\text{full}} = \alpha_o]$$

$$\alpha_o(t\text{odo}_{\text{full}}(T[\text{extr}_P(P)])) \subseteq [\text{Since } t\text{odo}_{\text{full}} \text{ is well-defined}]$$

$$\alpha_o(T[O_{\text{full}}(P,h_p)])$$

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Thus, $\alpha_o(T[O_{full}(P,hp)]) = \alpha_o(T[extr_{hp}(P)])$. By Theorem 6.2 $\alpha_{sc}(T[extr_{hp}(P)]) = \alpha_{sc}(T[P])$, so that, since $\alpha_{sc}$ is more precise than $\alpha_o$, $\alpha_o(T[extr_{hp}(P)]) = \alpha_o(T[P])$, and, in turn, $\alpha_o(T[O_{full}(P,hp)]) = \alpha_o(T[P])$. [\[
\]
]

We will see in Sections 7 and 8 two significant examples of hot path optimizations, namely, type specialization and constant folding.

7. TYPE SPECIALIZATION

One key optimization for dynamic languages like JavaScript and PHP is type specialization, that is, the use of type-specific primitives in place of generic untyped operations whose runtime execution can be costly. As a paradigmatic example, a generic addition operation could be defined on more than one type, so that the runtime environment must check the type of its operands and execute a different operation depending on these types: this is the case of the addition operation in JavaScript (see its runtime semantics in the ECMA-262 standard [Ecma International 2015, Section 12.7.3.1]) and of the semantics of $+$ in our language as given in Section 2.3. Of course, type specialization avoids the overhead of dynamic type checking and dispatch of generic untyped operations. When a type is associated to each variable before the execution of a command in some hot path, this type environment can be used to replace generic operations with type-specific primitives. In this section, we show that type specialization can be viewed as a particular optimization which can be proved correct according to our definition in Section 5.2.

7.1. Type Abstraction

Let us recall that the set of type names is $\text{Types} = \{\top, \text{Int}, \text{String}, \text{Undef}, \bot\}$, which can be viewed as the following finite lattice $\langle \text{Types}, \leq \rangle$:

\[
\begin{array}{c}
\top \\
\text{Int} \quad \text{String} \quad \text{Undef} \\
\bot
\end{array}
\]

The abstraction $\alpha_{type} : \wp(\text{Value}_u) \rightarrow \text{Types}$ and concretization $\gamma_{type} : \text{Types} \rightarrow \wp(\text{Value}_u)$ functions are defined as follows:

\[
\alpha_{type}(S) \triangleq \begin{cases} 
\bot & \text{if } S = \emptyset \\
\text{Int} & \text{if } \emptyset \neq S \subseteq \mathbb{Z} \\
\text{String} & \text{if } \emptyset \neq S \subseteq \text{Char}^* \\
\text{Undef} & \text{if } \emptyset \neq S = \{\text{undefined}\} \\
\top & \text{otherwise}
\end{cases}
\]

\[
\gamma_{type}(T) \triangleq \begin{cases} 
\emptyset & \text{if } T = \bot \\
\mathbb{Z} & \text{if } T = \text{Int} \\
\text{Char}^* & \text{if } T = \text{String} \\
\{\text{undefined}\} & \text{if } T = \text{Undef} \\
\text{Value}_u & \text{if } T = \top
\end{cases}
\]

Thus, $\alpha_{type}(S)$ provides the smallest type in $\langle \text{Types}, \leq \rangle$ for a set $S$ of values. In particular, given $v \in \text{Value}_u$, $\alpha_{type}(\{v\})$ coincides with $\text{type}(v)$. Following the approach described in Section 3.2.1 we then consider a simple nonrelational store abstraction for types

\[
\text{Store}^t \triangleq \langle \text{Var} \rightarrow \text{Types}, \leq_1 \rangle
\]

where $\leq_1$ is the standard pointwise lifting of $\leq_1$, so that $\lambda x.\bot$ and $\lambda x.\top$ are, respectively, the bottom and top abstract stores in $\text{Store}^t$. The abstraction and concretization maps $\alpha_{store} : \wp(\text{Store}) \rightarrow \text{Store}^t$ and $\gamma_{store} : \text{Store}^t \rightarrow \wp(\text{Store})$ are defined as a straight instantiation of the definitions in Section 3.2.1.
For instance, we have that:

\[
\begin{align*}
\text{An Abstract Interpretation-based Model of Tracing Just-In-Time Compilation} \\
\text{of expressions is defined as the best correct approximation of the concrete collecting semantics}
\end{align*}
\]

whose corresponding action semantics is automatically induced, as defined in Section 5, by the Galois connection \((\cdot \gamma)\) denoted by \(\cdot\). Hence, this definition leads to the following equalities:

\[
\begin{align*}
\mathcal{E}[^{\text{E}}][\mathcal{E}]\rho^t & \triangleq \alpha_{\text{type}}(\mathcal{E}[^{\text{E}}]_{\gamma_{\text{store}}} \rho^t).
\end{align*}
\]

For instance, we have that:

\[
\begin{align*}
\mathcal{E}[^{\text{E}}][v]\rho^t &= \text{type}(v) \\
\mathcal{E}[^{\text{E}}][x]\rho^t &= \rho^t(x) \\
\mathcal{E}[^{\text{E}}][E_1 + E_2]\rho^t &= \begin{cases} \\
\perp_T & \text{if } \exists i. \mathcal{E}[^{\text{E}}][E_i]\rho^t = \perp_T \\
\mathcal{E}[^{\text{E}}][E_1]\rho^t & \text{else if } \mathcal{E}[^{\text{E}}][E_1]\rho^t = \mathcal{E}[^{\text{E}}][E_2]\rho^t \in \{\text{Int}, \text{String}\} \\
\text{Undef} & \text{else if } \forall i. \mathcal{E}[^{\text{E}}][E_i]\rho^t < T_T \\
\perp_T & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}[^{\text{E}}][x + y]\rho^t &= \text{type}(x + y)\rho^t, \\
\mathcal{E}[^{\text{E}}][x]\rho^t &= \rho^t(x), \\
\mathcal{E}[^{\text{E}}][E_1 + E_2]\rho^t &= \begin{cases} \\
\perp_T & \text{if } \exists i. \mathcal{E}[^{\text{E}}][E_i]\rho^t = \perp_T \\
\mathcal{E}[^{\text{E}}][E_1]\rho^t & \text{else if } \mathcal{E}[^{\text{E}}][E_1]\rho^t = \mathcal{E}[^{\text{E}}][E_2]\rho^t \in \{\text{Int}, \text{String}\} \\
\text{Undef} & \text{else if } \forall i. \mathcal{E}[^{\text{E}}][E_i]\rho^t < T_T \\
\perp_T & \text{otherwise}
\end{cases}
\end{align*}
\]

Being defined as best correct approximation, it turns out that the abstract type semantics \(\mathcal{E}[^{\text{E}}}\) of expressions is correct by definition.

**COROLLARY 7.1.** If \(\rho \in \gamma_{\text{store}}(\rho^t)\) then \(\mathcal{E}[^{\text{E}}][\mathcal{E}]\rho \in \mathcal{E}[^{\text{E}}][\mathcal{E}]\rho^t\).

According to Section 5 for any abstract type store (that we also call type environment) \(\{x_i/T_i \mid x_i \in \text{Var}\} \in \text{Store}^t\) we consider a corresponding Boolean action guard denoted by

\[
\text{guard } x_0 : T_0, \ldots, x_n : T_n \in \text{BExp}
\]

whose corresponding action semantics is automatically induced, as defined in Section 5 by the Galois connection \((\cdot \gamma)\): for any \(\rho \in \text{Store}\),

\[
\begin{align*}
\mathcal{A}[\text{guard } x_0 : T_0, \ldots, x_n : T_n] \rho &= \begin{cases} \\
\rho & \text{if } \rho \in \gamma_{\text{store}}([x_i/T_i \mid x_i \in \text{Var}]) \\
\perp & \text{otherwise}
\end{cases}
\end{align*}
\]

For example, we have that:

\[
\begin{align*}
\mathcal{A}[\text{guard } x : \text{String}, y : \text{String}] [x/\text{foo}, y/\text{bar}] &= [x/\text{foo}, y/\text{bar}], \\
\mathcal{A}[\text{guard } x : \text{String}, y : \perp_T] [x/\text{foo}, y/\text{bar}] &= [x/\text{foo}, y/\text{bar}], \\
\mathcal{A}[\text{guard } x : \text{String}, y : \perp_T] [x/1, y/3] &= \perp, \\
\mathcal{A}[\text{guard } x : \text{String}, y : \text{Undef}] [x/\text{foo}] &= [x/\text{foo}].
\end{align*}
\]
7.2. Type Specialization of Hot Paths

Let us consider some hot path \( hp = \langle \rho^1_0, C_0, \ldots, \rho^n_n, C_n \rangle \in \alpha_{hot}^N(\text{Trace}_P) \) on the type abstraction \((\text{Store}^\rho_{\leq i}), \) where each \( \rho^i_0 \) is therefore a type environment for \( P. \) Thus, in the transformed program \( \text{extr}_{hp}(P) \), the stitched hot path \( \text{stitch}_P(hp) \) contains \( n + 1 \) typed guards, that, for any \( i \in [0, n], \) we simply denote as \( \text{guard} \, \rho^i_0. \) Typed guards allow us to perform type specialization of commands in the stitched hot path. In order to keep the notation simple, we only focus on type specialization of addition operations occurring in assignments, while one could also consider an analogous type specialization of Boolean comparisons in conditional commands. This is defined as a program transform that instantiates most type-specific addition operations in place of generic untyped additions by exploiting the type information dynamically recorded by typed guards in \( \text{stitch}_P(hp). \) Note that if \( C \in \text{stitch}_P(hp) \) and \( \text{act}(C) \equiv x := E_1 + E_2 \) then \( C \equiv \ell_i : x := E_1 + E_2 \rightarrow L', \) for some \( i \in [0, n], \) where \( L' \in \{1_{i+1}, L_0\}. \) Let \( C' \) denote the extended set of commands which includes type specific additions \(+_{\text{Int}} \) and \(+_{\text{String}} \) and, in turn, let Program\(^t\) denote the possibly type-specialized programs with commands ranging in \( C'. \) The semantic function \( E \) for expressions is then updated to type specific additions as follows:

\[
E[E_1 +_{\text{Int}} E_2]_\rho = \begin{cases} 
E[E_1]_\rho +_{\text{Int}} E[E_2]_\rho & \text{if type} (E[E_1]_\rho) = \text{Int} \\
\text{undef} & \text{otherwise}
\end{cases}
\]

\[
E[E_1 +_{\text{String}} E_2]_\rho = \begin{cases} 
E[E_1]_\rho \cdot E[E_2]_\rho & \text{if type} (E[E_1]_\rho) = \text{String} \\
\text{undef} & \text{otherwise}
\end{cases}
\]

Given a hot path \( hp = \langle \rho^0_0, C_0, \ldots, \rho^n_n, C_n \rangle, \) the type specialization function \( \text{ts}_{hp} : \text{stitch}_P(hp) \rightarrow C' \) is defined as follows:

\[
\text{ts}_{hp}(\ell_i : x := E_1 +_{\text{Int}} E_2 \rightarrow L') \triangleq \begin{cases} 
\ell_i : x := E_1 +_{\text{Int}} E_2 \rightarrow L' & \text{if } E'[E_1 +_{\text{Int}} E_2]_\rho^i = \text{Int} \\
\ell_i : x := E_1 +_{\text{String}} E_2 \rightarrow L' & \text{if } E'[E_1 +_{\text{String}} E_2]_\rho^i = \text{String} \\
\ell_i : x := E_1 + E_2 \rightarrow L' & \text{otherwise}
\end{cases}
\]

\[
\text{ts}_{hp}(C) \triangleq C \quad \text{if } C \neq \ell_i : x := E_1 + E_2 \rightarrow L'
\]

Hence, if a typed guard \( \text{guard} \, \rho^i_0 \) preceding a command \( \ell_i : x := E_1 + E_2 \rightarrow L' \) allows us to derive abstractly on \( \text{Store}^\rho \) that \( E_1 \) and \( E_2 \) have the same type \((\text{Int} \text{ or } \text{String})\) then the addition \( E_1 + E_2 \) is accordingly type specialized. This function allows us to define the hot path type specialization optimization

\[
O^{ts} : \{\text{stitch}_P(hp) \mid hp \in \alpha_{hot}^N(\text{Trace}_P)\} \rightarrow \text{Program}^t
\]

simply by

\[
O^{ts}(\text{stitch}_P(hp)) \triangleq \{\text{ts}_{hp}(C) \mid C \in \text{stitch}_P(hp)\}.
\]

In turn, as described in Section 6.2 this induces the full type specialization optimization

\[
O^{ts}_{full}(P, hp) \triangleq \text{extr}_{hp}(P) \setminus \text{stitch}_P(hp) \cup O^{ts}(\text{stitch}_P(hp)).
\]

\( O^{ts}_{full}(P, hp) \) is also called typed trace extraction since it extracts and simultaneously type specializes a typed hot path \( hp \) in a program \( P. \) The correctness of this program optimization can be proved for the store changes observational abstraction by relying on Lemma 6.8.

**Theorem 7.2 (Correctness of Typed Trace Extraction).** For any typed hot path \( hp \in \alpha_{hot}^N(\text{Trace}_P), \) we have that \( \alpha_{sc}(T[O^{ts}_{full}(P, hp)]) = \alpha_{sc}(T[P]). \)
PROOF. Let $td : \text{Trace}_{OP}(\text{stitch}_{hp}) \rightarrow \text{Trace}_{stitch_{hp}}$ be the following type despecialization function, where Type is either int or string:

$$td(\epsilon) \equiv \epsilon$$

$$td(s_{\sigma}) \equiv \begin{cases} \langle \rho, \ell_i : x := E_1 + E_2 \rightarrow L' \rangle & \text{if } s = \langle \rho, \ell_i : x := E_1 + \text{Type } E_2 \rightarrow L' \rangle, \\
\text{type}(E[1 + E_2]\rho) \neq \text{Type} & \\
\langle \rho, \ell_i : x := E_1 + E_2 \rightarrow L' \rangle \cdot td(\sigma) & \text{if } s = \langle \rho, \ell_i : x := E_1 + \text{Type } E_2 \rightarrow L' \rangle, \\
\text{type}(E[1 + E_2]\rho) = \text{Type} & \\
s \cdot td(\sigma) & \text{otherwise}
\end{cases}$$

Let us explain the first defining clause of $td(s_{\sigma})$, i.e., $s = \langle \rho, \ell_i : x := E_1 + \text{Type } E_2 \rightarrow L' \rangle$ and $\text{type}(E[1 + E_2]\rho) \neq \text{Type}$. These conditions can never hold in an inductive call of the function $td$; in fact, when $td(s_{\sigma})$ is recursively called by $td(s'_{\sigma})$, we necessarily have that $s' = \langle \rho, \ell_i : \text{guard } \rho^i \rightarrow \ell_i \rangle$, so that $\rho \in \gamma_{\text{store}}(\rho^i)$, and, in turn, by Corollary 7.1 $E[1 + E_2]\rho \in E[1 + E_2]\rho^i$, which implies $\text{type}(E[1 + E_2]\rho) = \text{Type}$, which is a contradiction. Thus, the first defining clause of $td(s_{\sigma})$ only applies to type specialized traces in $\text{Trace}_{OP}(\text{stitch}_{hp})$ whose first state is $s = \langle \rho, \ell_i : x := E_1 + \text{Type } E_2 \rightarrow L' \rangle$: in this case, we necessarily have that $s = \epsilon$, because $A[E[1 + \text{Type } E_2]\rho = \text{undef}$ so that $S_s = \varnothing$. This clarifies the definition of $td$ in this particular case. Also, observe that in this case, $sc(td(s)) = sc(s)$ trivially holds. In all the remaining cases, it is clear that $td$ maps type specialized traces into legal unspecialized traces of $\text{stitch}_{hp}$ since labels are left unchanged. Moreover, $sc \circ td = sc$ holds, in particular because in the second defining clause of $td(s_{\sigma})$, the condition $\text{type}(E[1 + E_2]\rho) = \text{Type}$ guarantees that $E[1 + E_2]\rho = E[1 + \text{Type } E_2]\rho$.

On the other hand, we define a trace specialization function $sp : \text{Trace}_{stitch_{hp}} \rightarrow \text{Trace}_{OP}(\text{stitch}_{hp})$ as follows:

$$sp(\epsilon) \equiv \epsilon$$

$$sp(\langle \mu_0, H_0 \rangle \cdots \langle \mu_k, H_k \rangle) \equiv \begin{cases} \langle \mu_0, \text{ts}_{hp}(H_0) \rangle & \text{if } \text{ts}_{hp}(H_0) \equiv \ell_i : x := E_1 + \text{Type } E_2 \rightarrow L', \\
\mu_0 \notin \gamma_{\text{store}}(\rho^i) & \\
\langle \mu_0, \text{ts}_{hp}(H_0) \rangle \cdots \langle \mu_k, \text{ts}_{hp}(H_k) \rangle & \text{otherwise}
\end{cases}$$

Let us comment on this definition. If $\sigma \in \text{Trace}_{stitch_{hp}}$ and $\sigma \neq \epsilon$ then it may happen that the first state $\langle \mu_0, H_0 \rangle$ of $\sigma$ is such that the command $H_0$ is $\ell_i : x := E_1 + E_2 \rightarrow L'$ and, since $E^i[1 + E_2]\rho^i = \text{Type}$ (int or string), $H_0$ is type specialized to $\text{ts}_{hp}(H_0) \equiv \ell_i : x := E_1 + \text{Type } E_2 \rightarrow L'$, while the store $\mu_0$ is not approximated by the abstract store $\rho^i$, i.e., $\mu_0 \notin \gamma_{\text{store}}(\rho^i)$. Thus, in this case, the trace in $O^i(\text{stitch}_{hp})$ beginning at $\langle \mu_0, \text{ts}_{hp}(H_0) \rangle$ is stuck, because the concrete semantics of addition is $E[E_1 + \text{Type } E_2]\mu_0 = \text{undef}$, and in turn $A[x := E_1 + \text{Type } E_2]\mu_0 = \perp$, so that we necessarily have to define $sp(\sigma) = \langle \mu_0, \text{ts}_{hp}(H_0) \rangle$. Otherwise, $sp(\sigma)$ simply type specializes through $\text{ts}_{hp}$ all the commands (actually, addition expressions) occurring in $\sigma$. Here, it turns out that $sp$ is well-defined, i.e. $sp(\sigma)$ is a legal trace of $O^i(\text{stitch}_{hp})$, because any state $\langle \rho, \ell_i : x := E_1 + E_2 \rightarrow L' \rangle$ of $\sigma$ is always preceded by the state $\langle \rho, \ell_i : \text{guard } \rho^i \rightarrow \ell_i \rangle$ and $\rho \in \gamma_{\text{store}}(\rho^i)$ must hold. Thus, by Corollary 7.1 $E[E_1 + E_2]\rho \in E[E_1 + E_2]\rho^i = \text{Type}$, so that $A[x := E_1 + \text{Type } E_2]\rho = A[x := E_1 + E_2]\rho$ holds. Consequently, the trace fragment

$$sp(\langle \rho, \ell_i : \text{guard } \rho^i \rightarrow \ell_i \rangle \langle \rho, \ell_i : x := E_1 + E_2 \rightarrow L' \rangle) = \langle \rho, \ell_i : \text{guard } \rho^i \rightarrow \ell_i \rangle \langle \rho, \ell_i : x := E_1 + \text{Type } E_2 \rightarrow L' \rangle$$
is legal in $O^{ls}(\text{stitch}_P(hp))$. Furthermore, let us also observe that $sc \circ ts = sc$ trivially holds.

Thus, following the scheme in Section 6.2 these two functions $td$ and $ts$ allow us to define $td_{full} : \text{Trace}_{exr_a}(P, hp) \rightarrow \text{Trace}_{exr_a}(P)$ and $ts_{full} : \text{Trace}_{exr_a}(P) \rightarrow \text{Trace}_{exr_a}(P, hp)$ such that $\alpha_{sc} \circ td_{full} = \alpha_{sc} \circ ts_{full}$, so that the thesis follows by Lemma 6.8. □

Example 7.3. Let us consider the following sieve of Eratosthenes in a Javascript-like language—this is taken from the running example in [Gal et al. 2009]—where $\text{primes}$ is an array initialized with 100 true values:

$$\begin{align*}
\text{for (var } i = 2; \ i < 100; \ i = i + 1) \text{ do} \\
&\text{ if (} \neg \text{primes}[i] \text{) then continue} \\
&\text{ for (var } k = i + i; \ k < 100; \ k = k + i) \text{ do } \text{primes}[k] = \neg \text{false}
\end{align*}$$

With a slight abuse, we assume that our language is extended with arrays and Boolean values ranging in the type $\text{Bool}$. The semantics of read and store for arrays is standard:

first, the index expression is checked to be in bounds, then the value is read or stored into the array. If the index is out of bounds then the corresponding action command gives $\bot$, that is, we assume that the program generates an error (e.g., it is aborted).

The above program is encoded in our language as follows:

$$P = \{ C_0 \equiv L_0 : i := 2 \rightarrow L_1, C_1 \equiv L_1 : i < 100 \rightarrow L_2, C_1' \equiv L_1 : \neg(i < 100) \rightarrow L_8, C_2 \equiv L_2 : \text{primes}[i] = \text{tt} \rightarrow L_3, C_2' \equiv L_2 : \neg(\text{primes}[i] = \text{ff}) \rightarrow L_7, C_3 \equiv L_3 : k := i + i \rightarrow L_4, C_4 \equiv L_4 : k < 100 \rightarrow L_5, C_5 \equiv L_5 : \neg(k < 100) \rightarrow L_6, C_6 \equiv L_6 : \text{false} \rightarrow L_4, C_7 \equiv L_7 : i := i + 1 \rightarrow L_1, C_8 \equiv L_8 : \text{skip} \rightarrow L_1 \}.$$ 

Let us consider the following type environment

$$\rho' \triangleq \{ \text{primes}[n]/\text{Bool}, i/\text{Int}, k/\text{Int} \} \in \text{Store}^f$$

where $\text{primes}[n]/\text{Bool}$ is a shorthand for $\text{primes}[0]/\text{Bool}, \ldots, \text{primes}[99]/\text{Bool}$. Then the first traced 2-hot path on the type abstraction $\text{Store}^f$ is $hp_1 \triangleq (\rho', C_4, \rho', C_5, \rho', C_6)$. As a consequence, the typed trace extraction of $hp_1$ yields:

$$P_1 \triangleq O^{ls}(P, hp_1) = P \setminus \{ C_4, C_4' \} \cup \{ L_4 : k < 100 \rightarrow L_5, L_4 : \neg(k < 100) \rightarrow L_7 \} \cup O^{ls}(\text{stitch}_P(hp_1))$$

where:

$$O^{ls}(\text{stitch}_P(hp_1)) = \{ H_0 \equiv L_4 : \text{guard}(\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \rightarrow \ell_0, H_0' \equiv L_4 : \neg\text{guard}(\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \rightarrow \text{true}, H_1 \equiv L_5 : k < 100 \rightarrow L_1, H_1' \equiv L_5 : \neg(k < 100) \rightarrow L_7, H_2 \equiv L_1 : \text{guard}(\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \rightarrow \ell_1, H_2' \equiv L_1 : \neg\text{guard}(\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \rightarrow L_5, H_3 \equiv L_4 : \text{primes}[k] = \text{false} \rightarrow L_2, H_3' \equiv L_4 : \neg\text{primes}[k] : \text{Bool}, i : \text{Int}, k : \text{Int} \rightarrow \ell_2, H_5 \equiv L_2 : \text{guard}(\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \rightarrow \ell_2, H_5' \equiv L_2 : \neg\text{guard}(\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \rightarrow L_6, H_6 \equiv L_5 : k := k + \text{Int} \rightarrow L_4 \}.$$ □
8. CONSTANT VARIABLE FOLDING

Constant variable folding, a.k.a. constant propagation [Wegman and Zadeck 1991], is a standard and well-known program optimization, whose goal is to detect which program variables at some program point are constant on all possible executions and then to propagate these constant values as far forward through the program as possible. Guo and Palsberg [2011] show how to define this optimization along hot paths and then to propagate these constant values as far forward through the program as possible. As a significant example, we show here how to specify and prove its correctness. As a significant example, we show here how to specify and prove the correctness w.r.t. the store changes abstraction \( \alpha_{sc} \) of this simple hot path optimization according to the approach defined in Section 6.2.

The constant propagation store abstraction \( CP_{st} \) and its corresponding GI \((\alpha_{CP}, \varphi(\text{Store}), CP_{st}, \gamma_{CP})\) have been defined in Example 3.1. Following Section 5 any abstract store \([x_i/a_i \mid x_i \in \text{Var}] \in CP_{st}\), where, as usual, the bindings \(x_i/\text{undefined} \) are omitted, defines a corresponding guard \( x_0 : a_0, \ldots, x_n : a_n \in \text{BExp} \) whose semantics is induced by the GI \((\alpha_{CP}, \varphi(\text{Store}), CP_{st}, \gamma_{CP})\), as defined in Section 5 for any \( \rho \in \text{Store}, \)

\[
A[\text{guard } x_0 : a_0, \ldots, x_n : a_n] \rho = \begin{cases} 
\rho & \text{if } \rho \in \gamma_{CP}([x_i/a_i \mid x_i \in \text{Var}]) \\
\bot & \text{otherwise}
\end{cases}
\]

Therefore, we have that:

\[
A[\text{guard } x : 2, y : \text{foo}] [x/2, y/3] = \bot,
A[\text{guard } x : 2, y : \text{foo}] [x/2, y/\text{foo}, z/4] = \bot,
A[\text{guard } x : 2, y : \text{foo}] [x/2] = \bot,
A[\text{guard } x : 2, y : \text{foo}] [x/2, y/\text{foo}] = [x/2, y/\text{foo}],
A[\text{guard } x : 2, y : \top] [x/2, y/\text{foo}] = [x/2, y/\text{foo}],
A[\text{guard } x : 2, y : \top] [x/2] = [x/2].
\]

Let us consider some hot path \( hp = \langle \rho_0, C_0, \ldots, \rho_n, C_n \rangle \in \alpha_{hot}^{\text{Trace}}(P) \) on the constant propagation abstraction \( CP_{st} \), where each \( \rho_i \) is therefore an abstract store in \( CP_{st} \), whose corresponding guard in \( \text{stitch}_P(hp) \) will be denoted by \( \text{guard } \rho_i \). The constant value information encoded in these guards is used to define the variable folding in the stitched hot path. Following [Guo and Palsberg [2011] Section 2.4], let \( \text{FV} : \varphi(C) \rightarrow \varphi(\text{Var}) \) denote the function that returns the “free” variables occurring in some set of commands (in particular, a well-defined program), i.e., \( \text{FV}(P) \) is the set of variables occurring in \( P \) which are never-assigned-to in some command of \( P \). As in [Guo and Palsberg [2011]], constant variable folding is restricted to expressions \( E \) of some assignment \( x := E \) and is defined as a program transform which exploits the constant information recorded by abstract guards in \( \text{stitch}_P(hp) \). The constant folding function \( \text{cf}_{hp} : \text{stitch}_P(hp) \rightarrow C \) is defined as follows:

\[
\text{cf}_{hp}(\ell_i : x := E \rightarrow L') \triangleq 
\begin{cases} 
\ell_i : x := E[y_1/v_{y_1}, \ldots, y_k/v_{y_k}] \rightarrow L' \text{ if } \{y_1, \ldots, y_k\} = \{y \in \text{vars}(E) \cap \text{FV}(\text{stitch}_P(hp)) \mid \rho_i(y) = v_y \in \text{Value} \} \neq \emptyset \\
\ell_i : x := E \rightarrow L' \quad \text{otherwise}
\end{cases}
\]

\[
\text{cf}_{hp}(C) \triangleq C \quad \text{if } C \neq \ell_i : x := E \rightarrow L'
\]

where \( E[y_1/v_{y_1}, \ldots, y_k/v_{y_k}] \) denotes the standard syntactic substitution of variables \( y_j \in \text{vars}(E) \) with constant values \( \rho_i(y_j) = v_{y_j} \in \text{Value} \). Hence, when the abstract
guard $\mathit{guard}\, \rho_c^f$ which precedes an assignment $\ell_i : x := E \to L'$ tells us that a free variable $y$ occurring in the expression $E$ is definitely a constant value $v_y \in \mathit{Value}$ then $\mathit{cf}_{hp}$ performs the corresponding variable folding in $E$. Thus, the hot path constant folding optimization is defined by

$$O_{\mathit{cf}}(\mathit{stitch}_P(hp)) \triangleq \{ \mathit{cf}_{hp}(C) \mid C \in \mathit{stitch}_P(hp) \}$$

and, in turn, this induces the full constant folding optimization $O_{\mathit{full}}^f(P, hp)$. The correctness of this constant folding optimization can be proved for the store changes observational abstraction

**Theorem 8.1 (Correctness of Constant Folding Optimization).** For any hot path $hp \in \alpha_{\mathit{hot}}^N(\mathit{Trace}_P)$ w.r.t. the constant propagation store abstraction $\mathit{CP}_{\mathit{st}}$, $\alpha_{\mathit{sc}}(T[O_{\mathit{full}}^f(P, hp)]) = \alpha_{\mathit{sc}}(T[P])$.

This proof is omitted, since it follows the same pattern of Theorem 7.2 for the correctness of typed trace extraction, in particular it relies on Lemma 6.3.

**Example 8.2.** Let us consider the following program written in a while-language:

$x := 0; a := 2$;
\[\begin{array}{l}
\textbf{while } (x \leq 15) \textbf{ do} \\
\quad \textbf{if } (x \leq 5) \textbf{ then } x := x + a \\
\quad \textbf{else } \{ a := a + 1; x := x + a; \}
\end{array}\]

whose translation as $P \in \mathit{Program}$ goes as follows:

$$P = \{ C_0 \equiv \ell_0 : x := 0 \to L_1, C_1 \equiv \ell_1 : a := 2 \to L_2 \}
\quad C_2 \equiv \ell_2 : x \leq 15 \to L_3, C_3 \equiv \ell_3 : \neg(x \leq 15) \to L_7, \\
\quad C_4 \equiv \ell_4 : x := a \to L_5, C_5 \equiv \ell_5 : a := a + 1 \to L_6
\quad C_6 \equiv \ell_6 : x := x + a \to L_2, C_7 \equiv \ell_7 : \mathit{skip} \to L_1 \}
$$

The first traced 2-hot path for the abstraction $\mathit{CP}_{\mathit{st}}$ is:

$$hp = \langle [x/\top, a/2], C_2, [x/\top, a/2], C_3, [x/\top, a/2], C_4 \rangle.$$

In fact, the initial prefix of the complete trace of $P$ which corresponds to the terminating run of $P$ is as follows:

$$\langle [], C_0 \rangle \langle [x/0], C_1 \rangle \langle [x/0, a/2], C_2 \rangle \langle [x/0, a/2], C_3 \rangle \langle [x/0, a/2], C_4 \rangle \langle [x/2, a/2], C_2 \rangle \langle [x/4, a/2], C_2 \rangle \langle [x/4, a/2], C_3 \rangle \langle [x/4, a/2], C_4 \rangle$$

so that $hp \in \alpha_{\mathit{hot}}^2(\mathit{Trace}_P)$. Hence, the constant folding optimization $O_{\mathit{cf}}$ along $hp$ provides:

$$O_{\mathit{full}}^f(P, hp) = P \setminus \{ C_2, C_3 \} \cup \{ \ell_2 : x \leq 15 \to L_3, \ell_2 : \neg(x \leq 15) \to L_7 \} \cup O_{\mathit{cf}}(\mathit{stitch}_P(hp))$$

where:

$$O_{\mathit{cf}}(\mathit{stitch}_P(hp)) = \{ H_0 \equiv \ell_0 : \mathit{guard} [x/\top, a/2] \to \ell_0, H_0^c \equiv \ell_2 : \neg\mathit{guard} [x/\top, a/2] \to \ell_2, \\
H_1 \equiv \ell_1 : x \leq 15 \to \ell_1, H_1^c \equiv \ell_0 : \neg(x \leq 15) \to L_7, \\
H_2 \equiv \ell_2 : \mathit{guard} [x/\top, a/2] \to \ell_1, H_2^c \equiv \ell_1 : \neg\mathit{guard} [x/\top, a/2] \to L_3, \\
H_3 \equiv \ell_3 : x \leq 5 \to \ell_2, H_3^c \equiv \ell_1 : \neg(x \leq 5) \to L_5, \\
H_4 \equiv \ell_4 : \mathit{guard} [x/\top, a/2] \to \ell_3, H_4^c \equiv \ell_1 : \neg\mathit{guard} [x/\top, a/2] \to L_4, \\
H_5 \equiv \ell_5 : x := x + 2 \to L_2 \}.$$
Therefore, this hot path optimization allows us to fold the constant value 2 for the variable \( a \), in the hot path command \( H_5 \equiv \ell_2 : x := x + 2 \to L_2 \).

9. NESTED HOT PATHS

Once a first hot path \( hp_1 \) has been extracted by transforming \( P \) to \( P_1 \equiv extr_{hp_1}(P) \), it may well happen that a new hot path \( hp_2 \) in \( P_1 \) contains \( hp_1 \) as a nested sub-path. Following TraceMonkey’s trace recording strategy \([Gal et al. 2009]\), we attempt to nest an inner hot path inside the current trace: during trace recording, an inner hot path is called as a kind of “subroutine”, this executes a loop to a successful completion and then returns to the trace recorder that may therefore register the inner hot path as part of a new hot path.

In order to handle nested hot paths, we need a more general definition of hot path which takes into account previously extracted hot paths and a corresponding program transform for extracting nested hot paths. Let \( P \) be the original program and let \( P' \) be a hot path transform of \( P \) so that \( P' \setminus P \) contains all the commands (guards included) in the hot path. We define a function \( hotcut : Trace_P \to (State_{P'})^* \) that cuts from an execution trace \( \sigma \) of \( P' \) all the states whose commands appear in some previous hot path \( hp \) except for the entry and exit states of \( hp \):

\[
hotcut(\sigma) \triangleq \begin{cases} \epsilon & \text{if } \sigma = \epsilon \\ hotcut((\rho_1, C_1)(\rho_2, C_2)(\rho_3, C_3)\sigma') & \text{if } \sigma = (\rho_1, C_1)(\rho_2, C_2)(\rho_3, C_3)\sigma' \land C_1, C_2, C_3 \notin P \\ \sigma_0 hotcut(\sigma_{\text{\text{-}1}}) & \text{otherwise} \end{cases}
\]

In turn, we define \( outerhot^N : Trace_{P'} \to \varphi((State_{P'})^*) \) as follows:

\[
outerhot^N(\sigma) \triangleq \{ (a_i, C_i), \ldots (a_j, C_j) \in (State_{P'})^* \mid \exists \rho_1, C_1 \ldots \rho_j, C_j \in \text{loop}(hotcut(\sigma)) \text{ such that } i \leq j, \alpha_{\text{store}}((\rho_i, C_i) \cdots (\rho_j, C_j)) = (a_i, C_i) \cdots (a_j, C_j), \\
\alpha_{\text{count}}(\alpha_{\text{store}}(hotcut(\sigma))), (a_i, C_i) \cdots (a_j, C_j) \gneq N \}. 
\]

Clearly, when \( P' = P \) it turns out that \( hotcut = \lambda \sigma. \sigma \) so that \( outerhot^N = hot^N \). We define the usual collecting version of \( outerhot^N \) on \( \varphi(Trace_{P'}) \) as the abstraction map \( \alpha_{outerhot}^N : \lambda T. \cup_{\sigma \in T} outerhot^N(\sigma) \). Then, \( \alpha_{outerhot}^N(T[P']) \) provides the set of \( N \)-hot paths in \( P' \).

Example 9.1. Let us consider again Example 5.3 where Store^ε is the trivial one-point store abstraction \( \{ \} \). In Example 5.3, we first extracted \( hp_1 \equiv (\top, C_1, \top, C_2, \top, C_3) \) by transforming \( P \) to \( P_1 \equiv extr_{hp_1}(P) \). We then consider the following trace in \( T[P_1] \):

\[
\sigma = ([], C_0)([x/0], H_0)([x/0], H_1)([x/0], H_2)([x/0], H_3)([x/1], H_4)([x/1], H_5)([x/2], H_6)([x/3], H_7)([x/3], C_4)([x/4], H_8)([x/4], H_9)([x/5], H_{10})([x/5], C_4)([x/6], H_0) \cdots (N_{next(i)}).
\]

Thus, we have that

\[
hotcut(\sigma) = ([], C_0)([x/0], H_0)([x/0], H_5)([x/3], C_4)([x/6], H_0)([x/9], H_{10})([x/9], C_4) \cdots
\]

so that \( hp_2 = (\top, H_0, \top, H_5, \top, C_4) \in \alpha_{outerhot}^N(T[P_1]) \). Hence, \( hp_2 \) contains a nested hot path, which is called at the beginning of \( hp_2 \) and whose entry and exit commands are, respectively, \( H_0 \) and \( H_5 \).

Let \( hp = (a_0, C_0, \ldots, a_n, C_n) \in \alpha_{outerhot}^N(T[P']) \) be a \( N \)-hot path in \( P' \), where, for all \( i \in [0, n] \), we assume that \( C_i \equiv L_i : A_i \to L_{next(i)} \). Let us note that:

- If for all \( i \in [0, n] \), \( C_i \in P \) then \( hp \) actually is a hot path in \( P \), i.e., \( hp \in \alpha_{hot}^N(T[P]) \).
– Otherwise, there exists some \( C_k \notin P \). If \( C_i \in P \) and \( C_{i+1} \notin P \) then \( C_{i+1} \) is the entry command of some inner hot path; on the other hand, if \( C_i \notin P \) and \( C_{i+1} \in P \) then \( C_i \) is the exit command of some inner hot path.

The transform of \( P' \) for extracting \( hp \) is then given as the following generalization of Definition 5.1.

**Definition 9.2.** *(Nested trace extraction transform).* The nested trace extraction transform of \( P' \) for the hot path \( hp = \langle a_0, C_0, \ldots, a_n, C_n \rangle \) is:

\[
eqtr_{hp}(P') \triangleq P
\]

\[
\begin{align*}
(1) & \quad \{ \{ C_0 | C_0 \in P \} \cup \{ \text{cmpl}(C_0) | \text{cmpl}(C_0) \in P \} \\
(2) & \quad \{ \overline{P_0} : \text{act}(C_0) \rightarrow L_1 | C_0 \in P \} \cup \{ \overline{P_0} : \neg \text{act}(C_0) \rightarrow L_1 | \text{cmpl}(C_0) \in P \} \\
(3) & \quad \{ L_0 : \text{guard } E_{a_0} \rightarrow h_0, L_0 : \neg \text{guard } E_{a} \rightarrow \overline{L_0} | C_0 \in P \} \\
(4) & \quad \{ h_i : \text{act}(C_i) \rightarrow h_{i+1} | i \in [0, n - 1], C_i, C_{i+1} \in P \} \cup \{ h_n : \text{act}(C_n) \rightarrow L_0 | C_n \in P \} \\
(5) & \quad \{ h_i : \neg \text{act}(C_i) \rightarrow L_{\text{next}(i)} | i \in [0, n], C_i, \text{cmpl}(C_i) \in P \} \\
(6) & \quad \{ h_i : \text{guard } E_{a} \rightarrow h_i, h_i : \neg \text{guard } E_{a} \rightarrow L_i | i \in [1, n], C_i \in P \} \\
(7) & \quad \{ h_i : \text{act}(C_i) \rightarrow L_{i+1} | i \in [0, n - 1], C_i \in P, C_{i+1} \notin P \} \\
(8) & \quad \{ \{ C_i | i \in [0, n - 1], C_i \notin P, C_{i+1} \in P \} \\
(9) & \quad \{ L_i : \text{act}(C_i) \rightarrow h_{i+1} | i \in [0, n - 1], C_i \notin P, C_{i+1} \in P \}
\end{align*}
\]

where we define \( \text{stitch}_{hp}(hp) \triangleq (3) \cup (4) \cup (5) \cup (6) \cup (7) \cup (9) \). \( \square \)

Let us observe that:

– Clauses (1)–(6) are the same clauses of the trace extraction transform of Definition 5.1 with the additional constraint that all the commands \( C_i \) of \( hp \) are required to belong to the original program \( P \). This is equivalent to ask that any \( C_i \) is not the entry or exit command of a nested hot path inside \( hp \), i.e., \( C_i \notin P' \setminus P \). In Definition 5.1 where no previous hot path extraction is assumed, any command \( C_i \) of \( hp \) belongs to \( P \), so that this constraint is trivially satisfied.

– Clause (7) where \( C_i \in P \) and \( C_{i+1} \notin P \), namely \( \text{next}(C_i) \) is the call program point of a nested hot path \( nhp \) and \( C_{i+1} \) is the entry command of \( nhp \), performs a relabeling that allows to neatly nest \( nhp \) in \( hp \).

– Clauses (8)–(9) where \( C_i \notin P \) and \( C_{i+1} \in P \), i.e., \( C_i \) is the exit command of a nested hot path \( nhp \) that returns to the program point \( \text{lbl}(C_{i+1}) \), performs the relabeling of \( \text{succ}(C_i) \) in \( C_i \) in order to return from \( nhp \) to \( hp \);

\( \overline{P_0}, h_i \) and \( h_i \) are meant to be fresh labels, i.e., they have not been already used in \( P' \).

**Example 9.3.** Let us go on with Example 9.1 The second traced hot path in \( \alpha^2_{\text{outerhot}}(T[P_1]) \) is:

\[
hp_2 = \langle \top, H_0 \equiv L_1 : \text{guard } E \top \rightarrow \ell_0, \top, H_5 \equiv \ell_2 : (x \% 3 = 0) \rightarrow L_4, \top, C_4 \equiv L_4 : x := x + 3 \rightarrow L_1 \rangle.
\]
According to Definition 9.2, trace extraction of $hp_2$ in $P_1$ yields the following transform:

$$ \text{extr}_{hp_2}(P_1) = \begin{cases} 
\text{by clause (8)} & P_1 \setminus \{H^c_5\} \\
\text{by clause (9)} & \cup \{\ell_2 : (x \% 3 = 0) \to h_2\} \\
\text{by clause (6)} & \cup \{h_2 : \text{guard } E_T \to h_2, h_2 : \neg \text{guard } E_T \to L_4\} \\
\text{by clause (4)} & \cup \{h_2 : x := x + 3 \to L_1\}
\end{cases} $$

where we used the additional fresh labels $b_2$ and $h_2$. \qed

**Example 9.4.** Let us consider again Example 7.3. After the trace extraction of $hp_1$ that transforms $P$ to $P_1$, a second traced 2-hot path is the following:

$$ hp_2 \triangleq \langle \rho', C_1, \rho, C_2, \rho', C_3, \rho', H_0, \rho, H^c_1, \rho, C_7 \rangle $$

where $\rho' = \{\text{primes}[n]/\text{Bool}, i/\text{Int}, k/\text{Int}\} \in \text{Store}'$. Thus, $hp_2$ contains a nested hot path which is called at $\text{suc}(C_3) = L_4$ and whose entry and exit commands are, respectively, $H_0$ and $H^c_5$. Here, typed trace extraction according to Definition 9.2 provides the following transform of $P_1$:

$$ P_2 \triangleq \text{O}^\text{te}_{\text{full}}(P_1, hp_2) = P_1 \setminus \{C_1, C^c_5\} \cup \left\{ \begin{array}{l}
\emptyset_0 : i < 100 \to L_2, \emptyset_0 : -(i < 100) \to L_8, \\
H_6 \equiv L_1 : \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to h_0, \\
H^c_6 \equiv L_1 : \neg \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to \emptyset_0, \\
H_7 \equiv h_0 : i < 100 \to h_1, H^c_7 \equiv h_0 : -(i < 100) \to L_8, \\
H_8 \equiv h_1 : \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to h_1, \\
H^c_8 \equiv h_1 : \neg \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to L_2, \\
h_9 \equiv h_1 : \text{primes}[i] = tt \to h_1, H^c_9 \equiv h_1 : \neg (\text{primes}[i] = tt) \to L_7, \\
h_{10} \equiv h_2 : \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to h_2, \\
H^c_{10} \equiv h_2 : \neg \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to L_3, \\
H_{11} \equiv h_2 : k := i + _\text{Int} i \to L_4 \\
\end{array} \right\} \cup \left\{ \begin{array}{l}
\{H^c_5\} \cup \{(H^c_7)' \equiv \ell_0 : -(k < 100) \to h_3, \\
H_{12} \equiv h_3 : \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to h_3, \\
H^c_{12} \equiv h_3 : \neg \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to L_7, \\
H_{13} \equiv h_3 : i := i + _\text{Int} 1 \to L_1 \}
\right\}. $$

Finally, a third traced 2-hot path in $P_2$ is $hp_3 \triangleq \langle \rho', H_6, \rho', H^c_5, \rho', C_7 \rangle$ which contains a nested hot path which is called at the beginning of $hp_3$ and whose entry and exit commands are, respectively, $H_6$ and $H^c_5$. Here, typed trace extraction of $hp_3$ yields:

$$ P_3 \triangleq \text{O}^\text{te}_{\text{full}}(P_2, hp_3) = P_2 \setminus \{H^c_5\} \cup \left\{ \begin{array}{l}
\{H^c_5\}' \equiv h_1 : -(\text{primes}[i] = tt) \to j_2, \\
j_2 : \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to j_2, \\
j_2 : \neg \text{guard } (\text{primes}[n]) : \text{Bool}, i : \text{Int}, k : \text{Int} \to L_7, \\
j_2 : i := i + _\text{Int} 1 \to L_1 \}
\right\}. $$

We have thus obtained the same three trace extraction steps described by Gal et al. [2009] Section 2. In particular, in $P_1$ we specialized the typed addition operation $k + _\text{Int} i$, in $P_2$ we specialized $i + _\text{Int} i$ and $i + _\text{Int} 1$, while in $P_3$ we specialized once
again \(i +_{\text{int}} 1\) in a different hot path. Thus, in \(P_3\) all the addition operations occurring in assignments have been type specialized.

\[ \square \]

10. COMPARISON WITH GUO AND PALSBERG’S FRAMEWORK

A formal model for tracing JIT compilation has been put forward at POPL 2011 symposium by [Guo and Palsberg 2011]. Its main distinctive feature is the use of a bisimulation relation [Milner 1995] to model the operational equivalence between source and optimized programs. In this section, we show how this model can be expressed within our framework.

10.1. Language and Semantics

[Guo and Palsberg 2011] rely on a simple imperative language (without jumps and with while loops and a so-called bail construct. Its syntax is as follows:

\[
\begin{align*}
E &::= v | x | E_1 + E_2 \\
B &::= \text{tt} | \text{ff} | E_1 \leq E_2 | \neg B | B_1 \land B_2 \\
\text{Cmd} \ni c &::= \text{skip} | x := E | \text{if } B \text{ then } S | \text{while } B \text{ do } S | \text{bail } B \text{ to } S
\end{align*}
\]

\(\text{Stm} \ni S ::= \epsilon | cS\)

where \(\epsilon\) stands for the empty string. Thus, any statement \(S \in \text{Stm}\) is a (possibly empty) sequence of commands \(c^n\), with \(n \geq 0\). We follow [Guo and Palsberg 2011] in making an abuse in program syntax by assuming that if \(S_1, S_2 \in \text{Stm}\) then \(S_1 S_2 \in \text{Stm}\), where \(S_1 S_2\) denotes a simple string concatenation of \(S_1\) and \(S_2\). We denote by \(\text{State}_{\text{GP}} \triangleq \text{Store} \times \text{Stm}\) the set of states for this language. The baseline small-step operational semantics \(\rightarrow_B\) \(\subseteq \text{State}_{\text{GP}} \times \text{State}_{\text{GP}}\) is standard and is given in continuation-style (where \(K \in \text{Stm}\)):

\[
\begin{align*}
\langle \rho, \epsilon \rangle & \not\rightarrow_B \\
\langle \rho, \text{skip}; K \rangle & \rightarrow_B \langle \rho, K \rangle \\
\langle \rho, x := E; K \rangle & \rightarrow_B \langle \rho[x/E][E][\rho], K \rangle \\
\langle \rho, (\text{if } B \text{ then } S)K \rangle & \rightarrow_B \begin{cases} 
\langle \rho, K \rangle & \text{if } B[\rho] = \text{false} \\
\langle \rho, SK \rangle & \text{if } B[\rho] = \text{true}
\end{cases} \\
\langle \rho, (\text{while } B \text{ do } S)K \rangle & \rightarrow_B \langle \rho, (\text{if } B \text{ then } (S \text{ while } B \text{ do } S))K \rangle \\
\langle \rho, (\text{bail } B \text{ to } S)K \rangle & \rightarrow_B \begin{cases} 
\langle \rho, K \rangle & \text{if } B[\rho] = \text{false} \\
\langle \rho, S \rangle & \text{if } B[\rho] = \text{true}
\end{cases}
\end{align*}
\]

The relation \(\rightarrow_B\) is clearly deterministic and we denote by

\[
\text{Trace}_{\text{GP}} \triangleq \{ \sigma \in \text{State}_{\text{GP}}^+ \mid \forall i \in [0, |\sigma| - 1). \sigma_i \rightarrow_B \sigma_{i+1} \}
\]

the set of generic program traces for Guo and Palsberg’s language. Then, given a program \(S \in \text{Stm}\), so that \(\text{Store}_{S} \triangleq \text{vars}(S) \rightarrow \text{Value}_{\text{a}}\) denotes the set of stores for \(S\), its partial trace semantics is

\[
\text{T}_{\text{GP}}[S] = \text{Trace}_{S}^{\text{GP}} \triangleq \{ \sigma \in \text{Trace}_{\text{GP}} \mid \sigma_0 = \langle \rho, S \rangle, \rho \in \text{Store}_{S} \}.
\]

Notice that, differently from our trace semantics, a partial trace of the program \(S\) always starts from an initial state, i.e., \(\langle \rho, S \rangle\).

10.2. Language Compilation

Programs in \(\text{Stm}\) can be compiled into \(\text{Program}\) by resorting to an injective labeling function \(l : \text{Stm} \rightarrow \text{L}\) that assigns different labels to different statements.
Definition 10.1 (Language compilation). The “first command” compilation function $C : \text{Stm} \rightarrow \mathcal{F}(\mathcal{C})$ is defined as follows:

- $C(\epsilon) \triangleq \{l(\epsilon) : \text{skip} \rightarrow L\}$
- $C(S' \equiv (\text{skip}, K)) \triangleq \{l(S') : \text{skip} \rightarrow l(K)\}$
- $C(S' \equiv (x := E; K)) \triangleq \{l(S') : x := E \rightarrow l(K)\}$
- $C(S' \equiv ((\text{if } B \text{ then } S)K)) \triangleq \{l(S') : B \rightarrow l(SK), l(S') : \neg B \rightarrow l(K)\}$
- $C(S' \equiv ((\text{while } B \text{ do } S)K)) \triangleq \{l(S') : \text{skip} \rightarrow l((\text{if } B \text{ then } (S \text{ while } B \text{ do } S))K)\}$
- $C(S' \equiv ((\text{bail } B \text{ to } S)K)) \triangleq \{l(S') : B \rightarrow l(S), l(S') : \neg B \rightarrow l(K)\}$

Then, the full compilation function $C : \text{Stm} \rightarrow \mathcal{F}(\mathcal{C})$ is recursively defined by the following clauses:

- $C(\epsilon) \triangleq C(\epsilon)$
- $C(\text{skip}, K) \triangleq C(\text{skip}, K) \cup C(K)$
- $C(x := E; K) \triangleq C(x := E; K) \cup C(K)$
- $C((\text{if } B \text{ then } S)K) \triangleq C((\text{if } B \text{ then } S)K) \cup C(SK) \cup C(K)$
- $C((\text{while } B \text{ do } S)K) \triangleq C((\text{while } B \text{ do } S)K) \cup C((\text{if } B \text{ then } (S \text{ while } B \text{ do } S))K)$
- $C((\text{bail } B \text{ to } S)K) \triangleq C((\text{bail } B \text{ to } S)K) \cup C(S) \cup C(K)$

Given $S \in \text{Stm}$, $l(S)$ is the initial label of $C(S)$, while $L$ is, as usual, the undefined label where the execution becomes stuck.

It turns out that the recursive function $C$ is well-defined—the easy proof is standard and is omitted, let us just observe that $C((\text{while } B \text{ do } S)K)$ is a base case—so that, for any $S \in \text{Stm}$, $C(S)$ is a finite set of commands. Let us observe that, by Definition 10.1, if $(\rho, S) \rightarrow_B (\rho', S')$ then $C(S') \subseteq C(S)$ (this can be proved through an easy structural induction on $S$). Consequently, if $(\rho, S) \rightarrow^*_B (\rho', S')$ then $C(S') \subseteq C(S)$.

Example 10.2. Consider the following program $S \in \text{Stm}$ in Guo and Palsberg’s syntax:

$x := 0;$
while $B_1 \text{ do } x := 1;$
x := 2;
bail $B_2 \text{ to } x := 3;$
x := 4;

$S$ is then compiled in our language by $C$ in Definition 10.1 as follows:

$C(S) = \{l(S) : x := 0 \rightarrow l_{\text{while}}, l_{\text{while}} : B_1 \rightarrow l_3, l_{\text{while}} : \neg B_1 \rightarrow l_2, l_1 : x := 1 \rightarrow l_{\text{while}}, l_2 : x := 2 \rightarrow l_{\text{bail}}, l_{\text{bail}} : B_2 \rightarrow l_3, l_{\text{bail}} : \neg B_2 \rightarrow l_4, l_3 : x := 3 \rightarrow l_e, l_4 : x := 4 \rightarrow l_e, l_e : \text{skip} \rightarrow L\}.$

Notice that in the command $l_{\text{bail}} : B_2 \rightarrow l_3$, the label $l_3$ stands for $l(x := 3 ;)$ so that $C(x := 3 ;) \equiv l_3 : x := 3 \rightarrow l_e$, i.e., after the execution of $x := 3$ the program terminates.

Correctness for the above compilation function $C$ means that for any $S \in \text{Stm}$: (i) $C(S) \in \text{Program}$ and (ii) program traces of $S$ and $C(S)$ have the same store sequences.
\[ C^*(\langle \rho, \epsilon \rangle) \triangleq \langle \rho, I(\epsilon) : \text{skip} \rightarrow L \rangle \]
\[ C^*(\langle \rho, S \equiv (\text{skip}; K) \rangle) \triangleq \langle \rho, I(S) : \text{skip} \rightarrow I(K) \rangle \]
\[ C^*(\langle \rho, S \equiv (x := E; K) \rangle) \triangleq \langle \rho, I(S) : x := E \rightarrow I(K) \rangle \]
\[ C^*(\langle \rho, S \equiv ((\text{if } B \text{ then } S')K) \rangle) \triangleq \begin{cases} \langle \rho, I(S) : B \rightarrow I(S'K) \rangle & \text{if } B[\rho] = \text{true} \\ \langle \rho, I(S) : \neg B \rightarrow I(K) \rangle & \text{if } B[\rho] = \text{false} \end{cases} \]
\[ C^*(\langle \rho, S \equiv ((\text{while } B \text{ do } S')K) \rangle) \triangleq \langle \rho, I(S) : \text{skip} \rightarrow I((\text{if } B \text{ then } (S' \text{ while } B \text{ do } S'))K) \rangle \]
\[ C^*(\langle \rho, S \equiv ((\text{bail } B \text{ to } S')K) \rangle) \triangleq \begin{cases} \langle \rho, I(S) : B \rightarrow I(S') \rangle & \text{if } B[\rho] = \text{true} \\ \langle \rho, I(S) : \neg B \rightarrow I(K) \rangle & \text{if } B[\rho] = \text{false} \end{cases} \]

Fig. 6. Definition of the state compile function \( C^* : \text{State}_{GP} \rightarrow \text{State} \).

In the proof we will make use of a “state compile” function \( C^* : \text{State}_{GP} \rightarrow \text{State} \) as defined in Figure 6. In turn, \( C^* \) allows us to define a “trace compile” function \( C^t : T_{GP}[S] \rightarrow T^t[\mathcal{C}(S)] \) which applies state-by-state the function \( C^* \) to traces as follows:

\[ C^t(\epsilon) \triangleq \epsilon ; \ C^t(s)T \triangleq C^*(s)C^t(T). \]

**Lemma 10.3.**

1. \( \langle \rho, S \rangle \rightarrow_B \langle \rho', S' \rangle \Leftrightarrow C^*(\langle \rho', S' \rangle) \in S(C^*(\langle \rho, S \rangle)) \)
2. \( C^t \) is well-defined.

**Proof.** We show the equivalence (1) by structural induction on \( S \in \text{Stm} \).

- \( [S \equiv \epsilon] \): Trivially true, since \( \langle \rho, S \rangle \not\in B \) and \( S(\rho, I(\epsilon) : \text{skip} \rightarrow L) = \emptyset \).
- \( [S \equiv x := E; K] \) (\( \Rightarrow \)): If \( \langle \rho, x := E; K \rangle \rightarrow_B \langle \rho[x/E][E][\rho], K \rangle, C^*(\langle \rho, x := E; K \rangle) = \langle \rho, I(S) : x := E \rightarrow I(K) \rangle \) and \( C^*(\rho[x/E][E][\rho], K) = \langle \rho[x/E][E][\rho], I(K) : A \rightarrow I(S') \rangle \) for some action \( A \) and statement \( S' \), then, by definition of the transition semantics \( S \), \( \langle \rho[x/E][E][\rho], I(K) : A \rightarrow I(S') \rangle \in S(\rho, I(S) : x := E \rightarrow I(K)) \).
- \( (\Leftarrow) \): If \( \langle \rho', C \rangle = C^*(\langle \rho', S' \rangle) \in S(\rho, I(S) : x := E \rightarrow I(K)) \) then: (1) \( E[\rho] \neq \text{undef} \), (2) \( \rho'' = \rho[x/E][E][\rho] \), and therefore \( \rho' = \rho[x/E][E][\rho] \); (3) \( \text{lbl}(C) = I(K) \), and therefore \( S' = K \). Hence, \( \langle \rho, x := E; K \rangle \rightarrow_B \langle \rho[x/E][E][\rho], K \rangle = \langle \rho', S' \rangle \).

- \( [S \equiv \text{skip}; K] \) Analogous to \( S \equiv x := E; K \).
- \( [S \equiv (\text{if } B \text{ then } T)K] \) (\( \Rightarrow \)): Assume that \( B[\rho] = \text{false} \), so that \( \langle \rho, (\text{if } B \text{ then } T)K \rangle \rightarrow_B \langle \rho, K \rangle, C^*(\langle \rho, (\text{if } B \text{ then } T)K \rangle) = \langle \rho, I(S) : \neg B \rightarrow I(K) \rangle \) and \( C^*(\rho, K) = \langle \rho, I(K) : A \rightarrow I(T') \rangle \) for some \( A \) and \( T' \in \text{Stm} \). Hence, by definition of \( S \), \( \langle \rho, I(K) : A \rightarrow I(T') \rangle \in S(\rho, I(S) : \neg B \rightarrow I(K)) \). On the other hand, if \( B[\rho] = \text{true} \) then \( \langle \rho, (\text{if } B \text{ then } T)K \rangle \rightarrow_B \langle \rho, I(T)K \rangle, C^*(\rho, (\text{if } B \text{ then } T)K) = \langle \rho, I(S) : B \rightarrow I(T)K \rangle \) and \( C^*(\rho, T)K = \langle \rho, I(T) : A \rightarrow I(T') \rangle \) for some \( A \) and \( T' \). Hence, \( \langle \rho, I(T)K : A \rightarrow I(T') \rangle \in S(\rho, I(S) : B \rightarrow I(T)K) \).
- \( (\Leftarrow) \): Assume that \( B[\rho] = \text{false} \), so that \( C^*(\langle \rho, (\text{if } B \text{ then } T)K \rangle) = \langle \rho, I(S) : \neg B \rightarrow I(K) \rangle \), and \( \rho'' \), \( C = C^*(\rho'', S') \in S(\rho, I(S) : \neg B \rightarrow I(K)) \). Hence: (1) \( \rho'' = \rho \) and therefore \( \rho' = \rho' \); (2) \( \text{lbl}(C) = I(K) \), and therefore \( S' = K \). Hence, \( \langle \rho, (\text{if } B \text{ then } T)K \rangle \rightarrow_B \langle \rho, I(T)K \rangle \) and \( \langle \rho', C \rangle = C^*(\rho'', S') \in S(\rho, I(S) : B \rightarrow I(T)K) \). We thus have that: (1) \( \rho'' = \rho \) and therefore \( \rho' = \rho' \); (2) \( \text{lbl}(C) = I(T)K \), and therefore \( S' = TK \). Hence, \( \langle \rho, (\text{if } B \text{ then } T)K \rangle \rightarrow_B \langle \rho, TK \rangle = \langle \rho', S' \rangle \).

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\[
[S \equiv (\text{while } B \text{ do } T)K] \implies \text{We have that } \langle \rho, (\text{while } B \text{ do } T)K \rangle \rightarrow_B
\langle \rho, (\text{if } B \text{ then } (T \text{ while } B \text{ do } T))K \rangle \quad \text{and} \quad C^*(\langle \rho, (\text{while } B \text{ do } T)K \rangle) = \langle \rho, I(S) : \text{skip} \rightarrow I((\text{if } B \text{ then } (T \text{ while } B \text{ do } T))K) \rangle.
\]

Also, given a set \(C\), we define a "trace de-compile" function \(\rho'' = \rho\), and therefore \(\rho' = \rho\); (2) \(\text{lb}(C) = I((\text{if } B \text{ then } (T \text{ while } B \text{ do } T))K)\), and therefore \(S' = (\text{if } B \text{ then } (T \text{ while } B \text{ do } T))K\). Hence, \(\langle \rho, (\text{while } B \text{ do } T)K \rangle \rightarrow_B
\langle \rho, (\text{if } B \text{ then } (T \text{ while } B \text{ do } T))K \rangle\).

\[
[S \equiv (\text{bail } B \text{ to } T)K] \quad \text{Analogous to } S \equiv (\text{if } B \text{ then } T)K.
\]

Let us now turn to point (2). By the \(\Rightarrow\) implication of the equivalence (1), we have that if \(\tau \in T_{GP}[S]\) then \(C^\tau(\tau) \in T[\text{C}(S)]\): this can be shown by an easy induction on the length of \(\tau\) and by using the fact that if \(C^\tau(\tau) = \langle \rho_0, C_0 \rangle \langle \rho_1, C_1 \rangle \cdots \langle \rho_n, C_n \rangle\) then, for any \(i, C_i \in \text{C}(S)\). Moreover, since \(I(S)\) is the initial label of the compiled program \(\text{C}(S)\) and \(\text{lb}(C_0) = I(S)\), we also notice that \(C^\tau(\tau) \in T'[\text{C}(S)]\). Therefore, \(C^\tau\) is a well-defined function.

Let \(st : \text{Trace}^{GP} \cup \text{Trace} \rightarrow \text{Store}^\ast\) be the function that returns the store sequence of any trace, that is:

\[
st(\epsilon) \triangleq \epsilon \quad \text{and} \quad st((\rho, S)\sigma) \triangleq \rho \cdot st(\sigma).
\]

Also, given a set \(X\) of traces, let \(\alpha_{st}(X) \triangleq \{st(\sigma) \mid \sigma \in X\}\). Then, correctness of the compilation function \(C\) goes as follows:

**Theorem 10.4 (Correctness of Language Compilation).** If \(S \in \text{Stm}\) then \(\text{C}(S) \in \text{Program}\) and \(\alpha_{st}(T_{GP}[S]) = \alpha_{st}(T'[\text{C}(S)]).

**Proof.** We define a "trace de-compile" function \(D^\tau : T'[\text{C}(S)] \rightarrow T_{GP}[S]\) as follows. Consider a trace \(\sigma = \langle \rho_0, C_0 \rangle \cdots \langle \rho_n, C_n \rangle \in T'[\text{C}(S)]\), so that \(\text{lb}(C_0) = I(S)\), for any \(i \in [0, n]\), \(C_i \in \text{C}(S)\) and for any \(i \in [0, n]\), \(\langle \rho_i, C_i+1 \rangle \in \text{S}[\text{C}(S)](\rho_i, C_i).\) Since \(\text{lb}(C_0) = I(S)\), by definition of \(C^\ast\), we have that \(\langle \rho_0, C_0 \rangle = C^\ast(\langle \rho_0, S \rangle)\). Then, since \(\langle \rho_1, C_1 \rangle \in \text{S}[\text{C}(S)](C^\ast(\langle \rho_0, S \rangle))\), there exists \(S_1 \in \text{Stm}\) such that \(\text{lb}(C_1) = I(S_1)\), so that, \(\langle \rho_1, C_1 \rangle = C^\ast(\langle \rho_1, S_1 \rangle)\). Hence, from \(\langle \rho_1, S_1 \rangle \in \text{S}[\text{C}(S)](C^\ast(\langle \rho_0, S \rangle))\), by the implication \(\Rightarrow\) of Lemma 10.3(1), we obtain that \(\langle \rho_0, S \rangle \rightarrow_B \langle \rho_1, S_1 \rangle \rightarrow_B \cdots \rightarrow_B \langle \rho_n, S_n \rangle\).

and \(C^\ast(\langle \rho_1, S_1 \rangle) = \langle \rho_1, C_1 \rangle\). We therefore define \(D^\tau(\sigma) \triangleq \langle \rho_0, S \rangle \langle \rho_1, S_1 \rangle \cdots \langle \rho_n, S_n \rangle \in T_{GP}[S]\). Moreover, we notice that \(st(D^\tau(\sigma)) = st(\sigma)\). Let us also observe that \(st \circ C^\tau = st\), since \(C^\tau\) does not affect stores.
Summing up, we obtain:
\[
\alpha_{st}(T_{GP}[S]) = \begin{cases} 
\text{since } st \circ C' = st & \text{[since } st \circ C' = st] \\
\alpha_{st}(C'(T_{GP}[S])) \subseteq & \text{[by Lemma 10.3(2), } C' \text{ is well-defined]} \\
\alpha_{st}(T'[C(S)]) = & \text{[since } st \circ D' = st] \\
\alpha_{st}(D'(T'[C(S)])) \subseteq & \text{[since } D' \text{ is well-defined]} \\
\end{cases}
\]

and this closes the proof. □

10.3. Bisimulation

Correctness of trace extraction in [Guo and Palsberg 2011] relies on a notion of bisimulation relation, parameterized by program stores. Let us recall this definition. If \(\langle \rho, S \rangle \rightarrow_B \langle \rho', S' \rangle\) then this “silent” transition that does not change the store is also denoted by \(\langle \rho, S \rangle \overset{\text{st}}{\rightarrow}_B \langle \rho', S' \rangle\). Moreover, for the assignment transition \(\langle \rho, x := E; K \rangle \rightarrow_B \langle \rho[x/E][E]_\rho, K \rangle\), if \(\delta = [x/E][E]_\rho\) denotes the corresponding store update of \(\rho\) then this transition is also denoted by \(\langle \rho, x := E; K \rangle \overset{\delta}{\rightarrow}_B \langle \rho[x/E][E]_\rho, K \rangle\). Let \(Act \overset{\text{st}}{=} \{\delta \mid \delta \text{ is a store update}\} \cup \{\tau\}\). Then, for a nonempty sequence of actions \(s = a_1 \cdots a_n \in Act^+\), we define:

\[
\langle \rho, S \rangle \overset{s}{\rightarrow}_B \langle \rho', S' \rangle \quad \text{iff} \quad \langle \rho, S \rangle \overset{\tau}{\rightarrow}_B \circ a_1 \overset{\tau}{\rightarrow}_B \circ \cdots \circ a_n \overset{\tau}{\rightarrow}_B \overset{\tau}{\rightarrow}_B \langle \rho', S' \rangle,
\]

namely, there may be any number of silent transitions either in front of or following any \(a_i\)-transition \(\overset{a_i}{\rightarrow}_B\). Moreover, if \(s \in Act^+\) is a nonempty sequence of actions then \(\overset{s}{\rightarrow}_B\) denotes the possibly empty sequence of actions where all the occurrences of \(\tau\) are removed.

Definition 10.5 (Guo and Palsberg 2011). A relation \(R \subseteq \text{Store} \times \text{Stm} \times \text{Stm}\) is a bisimulation when \(R(\rho, S_1, S_2)\) implies:

1. if \(\langle \rho, S_1 \rangle \overset{s}{\rightarrow}_B \langle \rho', S'_1 \rangle\) then \(\langle \rho, S_2 \rangle \overset{s}{\rightarrow}_B \langle \rho', S'_2 \rangle\), for some \(\langle \rho', S'_2 \rangle\) such that \(R(\rho', S'_1, S'_2)\);
2. if \(\langle \rho, S_2 \rangle \overset{s}{\rightarrow}_B \langle \rho', S'_2 \rangle\) then \(\langle \rho, S_1 \rangle \overset{s}{\rightarrow}_B \langle \rho', S'_1 \rangle\), for some \(\langle \rho', S'_1 \rangle\) such that \(R(\rho', S'_1, S'_2)\).

\(S_1\) is bisimilar to \(S_2\) for a given \(\rho \in \text{Store}\), denoted by \(S_1 \approx_\rho S_2\), if \(R(\rho, S_1, S_2)\) for some bisimulation \(R\). □

Let us remark that if \(\langle \rho, S_1 \rangle \overset{\tau}{\rightarrow}_B \langle \rho', S'_1 \rangle\) then \(\tau = \epsilon\), so that \(\langle \rho, S_2 \rangle \overset{\tau}{\rightarrow} \langle \rho, S_2 \rangle\) is allowed to be the matching (empty) transition sequence.

It turns out that bisimilarity can be characterized through an abstraction of traces that observes store changes. By a negligible abuse of notation, the store changes \(\text{Stm} \subseteq \text{Trace} \cup \text{Trace}^{GP} \rightarrow \text{Store}^+\) in turn, given \(\rho \in \text{Store}\), the function \(\alpha_{sc}^\rho : \varphi(\text{Trace}^{GP}) \rightarrow \varphi(\text{Store}^+)\) is then defined as follows:

\[
\alpha_{sc}^\rho(X) \overset{\text{st}}{=} \{sc(\tau) \in \text{Store}^+ \mid \tau \in X, \exists S, \tau', \tau = (\rho, S)\tau'\}.
\]

It is worth remarking that \(\alpha_{sc}^\rho\) is a weaker abstraction than \(\alpha_{sc}\) defined in Section 6 that is, for any \(X, Y \in \varphi(\text{Trace}^{GP})\), \(\alpha_{sc}(X) = \alpha_{sc}(Y) \Rightarrow \alpha_{sc}^\rho(X) = \alpha_{sc}^\rho(Y)\) (while the converse does not hold in general).

Theorem 10.6. For any \(S_1, S_2 \in \text{Stm}\), \(\rho \in \text{Store}\), we have that \(S_1 \approx_\rho S_2\) iff \(\alpha_{sc}^\rho(T_{GP}[S_1]) = \alpha_{sc}^\rho(T_{GP}[S_2])\).
PROOF. (⇒): We prove that if \( R(\rho, S_1, S_2) \) holds for some bismutation \( R \) then \( \alpha^\rho_a(T_{GP}[S_1]) \subseteq \alpha^\rho_a(T_{GP}[S_2]) \) (the reverse containment is symmetric), that is, if \( sc(\tau) \in Store^\rho \) for some \( \tau \in T_{GP}[S_1] \) such that \( \tau = \langle \rho, S_1 \rangle \tau' \) then there exists some \( \psi \in T_{GP}[S_2] \) such that \( \psi = \langle \rho, S_2 \rangle \psi' \) and \( sc(\tau) = sc(\psi) \). Let us then consider \( \tau \in T_{GP}[S_1] \) such that \( \Gamma = \langle \rho, S_1 \rangle \tau' \). If \( \tau' = \epsilon \) then we pick \( \langle \rho, S_2 \rangle \in T_{GP}[S_2] \) so that \( sc(\langle \rho, S_1 \rangle) = \rho = sc(\langle \rho, S_2 \rangle) \). Otherwise, \( \tau = \langle \rho, S_1 \rangle \tau' \in T_{GP}[S_1] \), with \( \epsilon \neq \tau' = \tau''(\mu, S) \). We prove by induction on \( |\tau'| \geq 1 \) that there exists \( \psi = \langle \rho, S_2 \rangle \psi''(\mu, T) \in T_{GP}[S_2] \) such that \( sc(\tau) = sc(\psi) \) and \( R(\mu, S, T) \).

\( |\tau'| = 1 \): In this case, \( \tau = \langle \rho, S_1 \rangle \psi \in T_{GP}[S_1] \), so that \( \langle \rho, S_1 \rangle \overset{\Delta}{\Rightarrow}_B \langle \mu, S \rangle \). Since, by hypothesis, \( R(\rho, S_1, S_2) \) holds, we have that \( \langle \rho, S_2 \rangle \overset{\Delta}{\Rightarrow}_B \langle \mu, T \rangle \), for some \( T \), and \( R(\mu, S, T) \). Let \( \psi \in T_{GP}[S_2] \) be the trace corresponding to the sequence of transitions \( \langle \rho, S_2 \rangle \overset{\Delta}{\Rightarrow}_B \langle \mu, T \rangle \). Then, by definition of \( \overset{\Delta}{\Rightarrow}_B \), we have that \( sc(\tau) = sc(\psi) \), and, by definition of bismimulation, \( R(\mu, S, T) \) holds.

\( |\tau'| > 1 \): Here, \( \tau' = \tau''(\mu, S) \) and \( \tau = \langle \rho, S_1 \rangle \tau' \in T_{GP}[S_1] \), with \( |\tau'| = |\tau''| - 1 \geq 1 \). Hence, \( \tau'' = \tau''(\eta, U) \). By inductive hypothesis, there exists \( \psi = \langle \rho, S_2 \rangle \psi''(\eta, V) \in T_{GP}[S_2] \) such that \( sc(\langle \rho, S_1 \rangle \tau''(\eta, U)) = sc(\langle \rho, S_2 \rangle \psi''(\eta, V)) \) and \( \eta, U, V \) holds. Since \( \langle \eta, U \rangle \overset{\Delta}{\Rightarrow}_B \langle \mu, S \rangle \) and \( \eta, U, V \) holds, we obtain that \( \langle \eta, V \rangle \overset{\Delta}{\Rightarrow}_B \langle \mu, T \rangle \) for some \( T \), and \( R(\mu, S, T) \) holds. Let \( \langle \eta, V \rangle \cdots \langle \mu, T \rangle \) be the sequence of states corresponding to the sequence of transitions \( \langle \eta, V \rangle \cdots \langle \mu, T \rangle \) so that we pick \( \langle \rho, S_2 \rangle \psi''(\eta, V) \cdots \langle \mu, T \rangle \in T_{GP}[S_2] \). The condition \( R(\mu, S, T) \) already holds. Moreover, by definition of \( \overset{\Delta}{\Rightarrow}_B \), we have that \( \langle \eta, U \rangle \overset{\Delta}{\Rightarrow}_B \langle \mu, S \rangle \) and \( \eta, U, V \), hence \( sc(\langle \eta, U \rangle \langle \mu, S \rangle) = sc(\langle \eta, V \rangle \cdots \langle \mu, T \rangle) \), and therefore we obtain \( sc(\tau) = sc(\langle \rho, S_1 \rangle \tau''(\eta, U) \langle \mu, S \rangle) = sc(\langle \rho, S_2 \rangle \psi''(\eta, V) \cdots \langle \mu, T \rangle) \).

\( (\Leftarrow) \): We first observe the following property (\( \ast \)), which is a straightforward consequence of the fact that \( \rightarrow_B \) is a deterministic relation: If \( S \subseteq Stn \) and \( \tau, \sigma \in T[S] \) are such that \( \sigma_0 = \mu, S \) = \( \tau_0 \) and \( |\tau| \leq |\sigma| \) then there exists some \( \psi \) such that \( \tau = \tau \psi \). Given \( \rho \in Store \), we assume that \( \alpha^\rho_a(T_{GP}[S_1]) = \alpha^\rho_a(T_{GP}[S_2]) \) and we then define the following relation \( R \):

\[
R \triangleq \{(\rho, S_1, S_2) \} \cup \{(\mu, T_1, T_2) \mid (\rho, S_1, \cdots, \mu, T_1) \in T[S_1], (\rho, S_2, \cdots, \mu, T_1) \in T[S_2],
sc(\langle \rho, S_1, \cdots, \mu, T_1 \rangle) = sc(\langle \rho, S_2, \cdots, \mu, T_1 \rangle)\}.
\]

We show that \( R \) is a bismilation, so that \( R(\rho, S_1, S_2) \) follows.

(case A) Assume that \( (\rho, S_1) \overset{a}{\rightarrow}_B (\rho', S_1') \). Then, since \( (\rho, S_1, \rho', S_1') \in T[S_1] \) and \( \alpha^\rho_a(T_{GP}[S_1]) = \alpha^\rho_a(T_{GP}[S_2]) \), we have that there exists \( \tau = \langle \rho, S_2, \cdots \rangle \in T[S_2] \) such that \( sc(\langle \rho, S_1, \rho', S_1' \rangle) = sc(\tau) \). Hence, \( \tau \) necessarily has the following shape:

\[
\tau = \langle \rho, S_2, \cdots \rangle \langle \mu, U_1, \cdots, \mu, U_n, \rho', V_1, \cdots, \rho', V_m \rangle
\]

where \( n \geq 0 (n = 0 \text{ means that } (\mu, U_1, \cdots, \mu, U_n) \text{ is the empty sequence}) \) and \( m \geq 1 \). This therefore means that \( (\rho, S_2, \cdots) \overset{a}{\Rightarrow}_B (\rho', V_m) \), so that, by definition of \( R \), \( R(\rho', S_1', V_m) \) holds.

(case B) Assume now that \( (\mu, T_1, T_2) \) holds because \( \delta = \langle \rho, S_1, \cdots \rangle \in T[S_1] \), \( \sigma = \langle \mu, S_2, \cdots \rangle \in T[S_2] \) and \( sc(\delta) = sc(\sigma) \). Hence, let us suppose that \( (\mu, T_1) \overset{\Delta}{\Rightarrow}_B (\mu', T_1') \). Then, since \( \delta(\mu', T_1') \in T[S_1] \) and \( \alpha^\rho_a(T_{GP}[S_1]) = \alpha^\rho_a(T_{GP}[S_2]) \), we have that there exists \( \tau = \langle \rho, S_2, \cdots \rangle \in T[S_2] \) such that \( sc(\delta(\mu', T_1')) = sc(\tau) \).

(case B1). If \( |\tau| \leq |\sigma| \) then, by the property (\( \ast \)) above, \( \sigma = \tau \psi \), for some \( \psi \). Hence, \( sc(\tau) = sc(\delta(\mu', T_1')) \) is a prefix of \( sc(\sigma) = sc(\delta) \). Consequently, \( sc(\delta(\mu', T_1')) \) can be a prefix of \( sc(\delta) \) only if \( sc(\delta(\mu', T_1')) = sc(\delta) \), so that the action \( a \) is \( \tau \) and \( \mu' = \mu \), that is, \( (\mu, T_1) \overset{a}{\rightarrow}_B (\mu, T_1') \). We thus consider the empty transition sequence.
(T1) \(\langle \rho, (\textbf{if} \ B \ \textbf{then} \ (S \ \textbf{while} \ B \ \textbf{do} \ S)) \rangle \rightarrow_T \langle \rho, (\textbf{while} \ B \ \textbf{do} \ S) \rangle \) if \(B[\rho] = \text{true}\)

(T2) \(\langle \rho, K_w, t, \textbf{skip} \rangle \rightarrow_T \langle \rho, K_w, t(\textbf{skip}) \rangle \)

(T3) \(\langle \rho, K_w, t, x := E; K \rangle \rightarrow_T \langle \rho[x/E][E[\rho]], K_w, t(x := E_i); K \rangle \)

(T4) \(\langle \rho, K_w, t, (\textbf{if} \ B \ \textbf{then} \ S) \rangle \rightarrow_T \begin{cases} \langle \rho, K_w, t(\textbf{bail} \ B (SK)), K \rangle \quad & \text{if } B[\rho] = \text{false} \\ \langle \rho, K_w, t(\textbf{bail} \ K (SK)), K \rangle \quad & \text{if } B[\rho] = \text{true} \end{cases} \)

(T5) \(\langle \rho, K_w, t, (\textbf{while} \ B \ \textbf{do} \ S) \rangle \rightarrow_T \begin{cases} \langle \rho, K_w, t(\textbf{skip}), (\textbf{if} \ B \ \textbf{then} \ (S \ \textbf{while} \ B \ \textbf{do} \ S)) \rangle \\ (\rho, O(\textbf{while} \ B \ \textbf{do} t), \rho) \end{cases} \)

(T6) \(\langle \rho, K_w, t, S \rangle \rightarrow_T \langle \rho', S' \rangle \) if \(K_w \neq S\) and \(\langle \rho, S \rangle \rightarrow_B \langle \rho', S' \rangle \)

Fig. 7. Definition of the tracing relation \(\rightarrow_T\).

\(\langle \mu, T_2 \rangle \xrightarrow{R} \langle \mu, T_2 \rangle\), so that from \(sc(\delta(\mu, T_1)) = sc(\sigma)\), by definition of \(R\) we obtain that \(R(\mu_1, T_1, T_2)\) holds.

(10.4. Hot Paths)

Let us recall the set of rules that define the tracing transitions in Guo and Palsberg [2011] model. Let \(\text{tState}_{\text{GP}} \triangleq \text{Store} \times \text{Stm} \times \text{Stm} \times \text{Stm}\) denote the set of states in trace recording mode, whose components are, respectively, the current store, the entry point of the recorded trace (this is always a while statement), the current trace (i.e., a sequence of commands) and the current program to be evaluated. In turn, \(\text{State}_{\text{GP}} \triangleq \text{State}_{\text{GP}} \cup \text{tState}_{\text{GP}}\) denotes the corresponding extended notion of state, which encompasses the trace recording mode. Then, the relation \(\rightarrow_T \subseteq \text{State}_{\text{GP}} \times \text{State}_{\text{GP}}\) is defined by the clauses in Figure 7 where \(O : \text{Stm} \times \text{Store} \to \text{Stm}\) is a “sound” optimization function that depends on a given store. Correspondingly, the trace semantics \(T_{\text{GP}}[S] \subseteq (\text{State}_{\text{GP}})^* \) of a program \(S \in \text{Stm}\) is naturally extended to the relation \(\rightarrow_{B,T} \triangleq \rightarrow_B \cup \rightarrow_T \subseteq \text{State}_{\text{GP}} \times \text{State}_{\text{GP}}\).

Let us notice that in Guo and Palsberg’s model of hot paths:

(i) By clause \((T_1)\), trace recording is always triggered by an unfolded while loop, and the loop itself is not included in the hot path.

(ii) By clause \((T_4)\), when we bail out of a hot path \(t\) through a \textbf{bail} command, we cannot anymore re-enter into \(t\).

(iii) By clause \((T_5)\)—the second condition of this clause is called stitch rule in [Guo and Palsberg 2011]—the store used to optimize a hot path \(t\) is recorded at the end of the first loop iteration. This is a concrete store which is used by \(O\) to optimize the stitched hot path \textbf{while} \(B \ \textbf{do} \ t\).

(iv) Hot paths actually are 1-hot paths according to our definition, since, by clause \((T_1)\), once the first iteration of the traced while loop is terminated, trace recording necessarily discontinues.

(v) There are no clauses for trace recording \textbf{bail} commands. Hence, when trying to trace a loop that already contains a nested hot path, by clause \((T_6)\), trace recording...
is aborted when a bail command is encountered. In other terms, in contrast to our approach described in Section 9 nested hot paths are not allowed.

(vi) Observe that when tracing a loop while B do S whose body S does not contain branching commands, i.e. if or while statements, it turns out that the hot path t coincides with the body S, so that while B do t ≡ while B do S, namely, in this case the hot path transform does not change the subject while loop.

In the following, we show how this hot path extraction model can be formalized within our trace-based approach. To this aim, we do not consider optimizations of hot paths, which is an orthogonal issue here, so that we assume that O performs no optimization, that is, O(while B do t, ρ) = while B do t.

A sequence of commands t ∈ Stm is defined to be a GP hot path for a program Q ∈ Stm when we have the following transition sequence:

(ρ, Q) →∗ B,T ⟨ρ′, (while B do S)K⟩ →∗ B,T ⟨ρ″, (while B do S)K⟩, (while B do S)K).

Since the operational semantics →∗ B,T is given in continuation-style, without loss of generality, we assume that the program Q begins with a while statement, that is Q ≡ (while B do S)K. Guo and Palsberg’s hot loops can be modeled in our framework by exploiting a revised loop selection map loop_GP : Trace → ϕ(C+), defined as follows:

loop_GP((ρ0, C0) · · · (ρn, Cn)) ≜ { CiCi+1 · · · Cj | 0 ≤ i ≤ j < n, Ci < Cj,

suc(Ci) = lbl(Ci), ∀k ∈ (i, j), Ck /∈ {Ci, cmpl(Ci)} }.

Thus, loop_GP(τ) contains sequences of commands without store. The map α_hot : ϕ(Trace) → ϕ(C+) then lifts loop_GP to sets of traces as usual: α_hot(T) ≜ ∪τ∈T loop_GP(τ).

Then, let us consider a GP hot path t as recorded by a transition sequence τ:

τ ≜ ⟨ρ, S0 ≡ (while B do S)K⟩ → B
⟨ρ, S1 ≡ (if B then (S while B do S)K)⟩ → T
⟨ρ, (while B do S)K, t, S2 ≡ S(while B do S)K⟩ → T

. . . → T
⟨ρ′, (while B do S)K, t, S′n⟩ → T
⟨ρ″, (while B do S)K, t, S″n+1 ≡ (while B do S)K⟩

where B[B]ρ = true. Hence, the Si’s occurring in τ are the current statements to be evaluated. With a negligible abuse of notation, we assume that τ ∈ T_GP[(while B do S)K], that is, the arrow symbols → B and → T are taken out of the sequence τ. By Lemma [10,3](2), we therefore consider the corresponding execution trace C′(τ) of the compiled program C((while B do S)K), where the state compile function C* in Figure 6 when applied to states in trace recording mode, is assumed to act on the current state and the program to be evaluated, that is, C*(⟨ρ, K_w, t, S⟩) = C*(⟨ρ, S⟩). We thus obtain:

C′(τ) ≜ ⟨ρ, C0 ≡ I((while B do S)K) : skip → I((if B then (S while B do S)K))
⟨ρ, C1 ≡ I((if B then (S while B do S))K) : B → I(S(while B do S)K)
⟨ρ, C2 ≡ I(S(while B do S)K) : A2 → I(T)

. . .
⟨ρ′, Cn ≡ I(Sn) : A_n → I((while B do S)K)
⟨ρ″, Cn+1 ≡ I((while B do S)K) : skip → I((if B then (S while B do S))K))

We therefore obtain a hot path hp_t = C0C1 · · · Cn ∈ loop_GP(C′(τ)), i.e. hp_t ∈ α_hot(T | C((while B do S)K)), where lbl(C0) = I((while B do S)K) = suc(Cn). This is a consequence of the fact that for all k ∈ (0, n], Ck cannot be the entry command C0.
or its complement command, because, by the stitch rule of clause \((T_5)\), \(S_{n+1}\) is necessarily the first occurrence of \((\text{while } B \text{ do } S)\) as current program to be evaluated in the trace \(\tau\), so that, for any \(k \in (0, n]\), \(\text{lbl}(C_k) \neq I(\text{while } B \text{ do } S)\). We have thus shown that any GP hot path arising from a trace \(\tau\) generates a corresponding hot path extracted by our selection map \(\text{loop}_{GP}\) on the compiled trace \(C^1(\tau)\):

**Lemma 10.7.** Let \(Q_w \equiv (\text{while } B \text{ do } S) K\). If \(t\) is a GP hot path for \(Q_w\) where \(\tau = (\rho, Q_w) \rightarrow^{*}_{B,T}(\rho', Q_w, t, Q_w)\) is the transition sequence \(\frac{1}{2}\) that records \(t\), then there exists a hot path \(\text{hp}_t = C_0 C_1 \cdots C_n \in \alpha^G_{\text{hot}}(T'[\lceil C(Q_w) \rceil])\) such that, for any \(i \in [0, n]\), \(\text{lb}(C_i) = I(S_i)\), and, in particular, \(\text{lb}(C_0) = I(Q_w) = \text{succ}(C_0)\).

**Example 10.8.** Let us consider the while statement \(Q_w\) of the program in Example 10.1:

\[
Q_w \equiv \text{while } (x \leq 20) \text{ do } (x := x + 1; \text{ if } (x \% 3 = 0) \text{ then } x := x + 3; )
\]

This program is already written in Guo and Palsberg language, so that \(Q_w\) is a well formed statement in \(\text{Stm}\). The tracing rules in Figure 7 yield the following trace \(t\) for \(Q_w\):

\[
t \equiv x := x + 1; \text{ \{bail } (x \% 3 = 0) \text{ to } (x := x + 3; Q_w)\).
\]

On the other hand, the compiled program \(C(Q_w) \in \wp(C)\) is as follows:

\[
C(Q_w) = \{D_0 \equiv L_{\text{while : skip } \rightarrow L_{\text{ifwhile}}},
D_1 \equiv L_{\text{ifwhile : } (x \leq 20) \rightarrow L_1,}
D_1^* \equiv L_{\text{ifwhile : } \lnot (x \leq 20) \rightarrow L_1},
D_2 \equiv L_1 : x := x + 1 \rightarrow L_{\text{if}},
D_3 \equiv L_{\text{if : } (x \% 3 = 0) \rightarrow L_2,}
D_5 \equiv L_{\text{if : } \lnot (x \% 3 = 0) \rightarrow L_{\text{while}},}
D_4 \equiv L_2 : x := x + 3 \rightarrow L_{\text{while}},
D_5 \equiv L_e : \text{skip } \rightarrow L\},
\]

where labels have the following meaning:

\[
\begin{align*}
L_{\text{while}} & \equiv L(Q_w) \\
L_{\text{ifwhile}} & \equiv L(\text{if } (x \leq 20) \text{ then } (x := x + 1; \text{ if } (x \% 3 = 0) \text{ then } x := x + 3; ) Q_w)) \\
L_1 & \equiv L(x := x + 1; \text{ if } (x \% 3 = 0) \text{ then } x := x + 3; ) Q_w)) \\
L_{\text{if}} & \equiv L(\text{if } (x \% 3 = 0) \text{ then } x := x + 3; Q_w)) \\
L_2 & \equiv L(x := x + 3; Q_w).
\end{align*}
\]

Hence, in correspondence with the trace \(t\), we obtain the hot path \(\text{hp}_t = D_0 D_1 D_2 D_5 \in \alpha^G_{\text{hot}}(T'[\lceil C(Q_w) \rceil])\). In turn, this hot path \(\text{hp}_t\) corresponds to the 2-hot path \(\text{hp}_1\) consisting of the analogous sequence of commands, which has been selected in Example 10.1.

**10.5. GP Trace Extraction**

In the following, we conform to the notation used in Section 5 for our trace extraction transform. Let us consider a while program \(Q_w \equiv (\text{while } B \text{ do } S) K \in \text{Stm}\) and its compilation \(P_w \equiv C(Q_w) \in \wp(C)\). Observe that, by Definition 10.1 of compilation \(C\), a hot path \(C_0 \cdots C_n \in \alpha^G_{\text{hot}}(T[P_w])\) for the compiled program \(P_w\) always arises in correspondence with some while loop \(\text{while } B' \text{ do } S'\) occurring in \(Q_w\) and therefore has

\[
\text{ACM Transactions on Programming Languages and Systems, Vol. V, No. N, Article A, Publication date: January YYYY.}
\]
necessarily the following shape:

\[ C_0 \equiv l((\text{while } B' \text{ do } S') : J) : \text{skip} \to l((\text{if } B' \text{ then } (S' \text{ while } B' \text{ do } S')) : J) \]

\[ C_1 \equiv l((\text{if } B' \text{ then } (S' \text{ while } B' \text{ do } S')) : J) : B' \to l(S' \text{ while } B' \text{ do } S') : J \]

\[ C_2 \equiv l(S' \text{ while } B' \text{ do } S') : A_2 \to l(T_3) \]

\[ \ldots \]

\[ C_n \equiv l(T_n) : A_n \to l((\text{while } B' \text{ do } S') : J) \]

The GP hot path extraction scheme for \( Q_w \) described by the rules in Figure 7 can be defined in our language by the following simple transform of \( P_w \).

**Definition 10.9** (GP trace extraction transform). The GP trace extraction transform \( \text{extr}^{GP}_h \) of \( P_w \) for the hot path \( h_p = C_0 C_1 \cdots C_n \in \alpha_{\text{hot}}(T[P_w]) \) is defined as follows:

1. If for any \( i \in [2, n], \text{cmpl}(C_i) \notin P_w \) then \( \text{extr}^{GP}_h(P_w) \equiv P_w \);
2. Otherwise:
\[
\text{extr}^{GP}_h(P_w) \equiv P_w \cup \{ \ell_i : \text{act}(C_i) \to \ell_{\text{next}(i)} \mid i \in [0, n] \} \\
\cup \{ \ell_i : \neg \text{act}(C_i) \to \ell'_{\text{next}(i)} \mid i \in [0, n], \text{cmpl}(C_i) \in P_w \}. \]

Clearly, \( \text{extr}^{GP}_h(P) \) remains a well-formed program. Also observe that the case (1) of Definition 10.9 means that the traced hot path \( h_p \) does not contain conditional commands (except from the entry conditional \( C_1 \)) and therefore corresponds to point (vi) in Section 10.4

**Example 10.10.** Let us consider the programs \( Q_w \) and \( C(Q_w) \) of Example 10.8 and the hot path \( h_{pt} = D_0 D_1 D_2 D_3 \in \alpha_{\text{hot}}(T[C(Q_w)]) \) which corresponds to the trace \( t = x := x + 1; \text{bail } (x \% 3 = 0) \to (x := x + 3; Q_w) \) of \( Q_w \). Here, the GP trace extraction of \( h_{pt} \), according to Definition 10.9 provides the following program transform:

\[
\text{extr}^{GP}_{h_{pt}}(C(Q_w)) \equiv \{ D_0 \equiv l_{\text{while}} : \text{skip} \to l_{\text{ifwhile}}, D_1 \equiv l_{\text{ifwhile}} : (x \leq 20) \to l_1, \\
D_2 \equiv l_{\text{if}} : (x \% 3 = 0) \to l_2, D_3 \equiv l_{\text{if}} : \neg (x \% 3 = 0) \to l_{\text{while}}, \\
D_4 \equiv l_2 : x := x + 3 \to l_{\text{while}}, D_5 \equiv l_2 : \text{skip} \to l_3 \} \cup \\
\{ \ell_0 : \text{skip} \to \ell_1, \ell_1 : x \leq 20 \to \ell_2, \ell_2 : \neg (x \leq 20) \to \ell_1, \\
\ell_3 : \text{skip} \to \ell_4, \ell_4 : (x \% 3 = 0) \to \ell_0, \ell_5 : (x \% 3 = 0) \to l_2 \}.
\]

On the other hand, the stitch rule \( (T_5) \) transforms \( Q_w \) into the following program \( Q_t \):

\[
\text{while } (x \leq 20) \text{ do } x := x + 1; \text{bail } (x \% 3 = 0) \text{ to } (x := x + 3; Q_w)
\]

whose compilation yields the following program:

\[
C(Q_t) = \{ l_{\text{while}} : \text{skip} \to l_{\text{ifwhile}}, l_{\text{ifwhile}} : (x \leq 20) \to l_{1t}, l_{\text{ifwhile}} : \neg (x \leq 20) \to l_c, \\
l_{1t} : x := x + 1 \to l_{\text{bail}}, l_{\text{bail}} : (x \% 3 = 0) \to l_{\text{bail}}_\text{first}, l_{\text{bail}} : \neg (x \% 3 = 0) \to l_{\text{while}}, \\
l_{\text{while}} : \text{skip} \to l_{\text{ifwhile}}, l_{\text{ifwhile}} : (x \leq 20) \to l_1, l_{\text{ifwhile}} : \neg (x \leq 20) \to l_c, \\
l_{1} : x := x + 1 \to l_{\text{if}}, l_{\text{if}} : (x \% 3 = 0) \to l_2, l_{\text{if}} : \neg (x \% 3 = 0) \to l_{\text{while}}, \\
l_{\text{bail}}_\text{first} : x := x + 3 \to l_{\text{while}}, l_c : \text{skip} \to l_3 \}
\]
with the following new labels:

\[ l_{\text{while}} \triangleq l(\text{while } (x \leq 20) t) \]
\[ l_{\text{ifwhile}} \triangleq l(\text{if } (x \leq 20) \text{ then } (t (\text{while } (x \leq 20) t))) \]
\[ l_{1:} \triangleq l(t (\text{while } (x \leq 20) t)) \]
\[ l_{\text{bail}} \triangleq l(\text{bail } (x \% 3 = 0) \text{ to } (x := x + 3; Q_w)(\text{while } (x \leq 20) t)) \]

while observe that \( l_{\text{bail}} \triangleq l(x := x + 3; Q_w) = l_2 \). It is then immediate to check that the programs \( C(Q_t) \) and \( \text{extr}_{hp_1}^GP(C(Q_w)) \) are equal up to the following label renaming of \( \text{extr}_{hp_1}^GP(C(Q_w)) \):

\[ \{ \ell_0 \rightarrow l_{\text{while}}, \ell_1 \rightarrow l_{\text{ifwhile}}, \ell_2 \rightarrow l_{1:}, \ell_3 \rightarrow l_{\text{bail}} \} \]

The equivalence of this GP trace extraction with the stitch of hot paths by Guo and Palsberg [2011] goes as follows.

**Theorem 10.11 (Equivalence with GP Trace Extraction).** Let \( t \) be a GP trace such that \( \langle \rho, (\text{while } B \text{ do } S)K \rangle \rightarrow_B \langle \rho', (\text{while } B \text{ do } S)K, t, (\text{while } B \text{ do } S)K \rangle \) and let \( hp_1 \in \alpha^{GP}(T, \llbracket C((\text{while } B \text{ do } S)K) \rrbracket) \) be the corresponding GP hot path as determined by Lemma 10.7. Then, \( C((\text{while } B \text{ do } S)K) \cong \text{extr}_{hp_1}^GP(C((\text{while } B \text{ do } S)K)) \).

**Proof.** Let the GP hot path \( t \) be recorded by the following transition sequence for \( (\rho, (\text{while } B \text{ do } S)K) \):

\[ \langle \rho, S_{-2} \equiv (\text{while } B \text{ do } S)K \rangle \rightarrow_B \langle \rho, S_{-1} \equiv (\text{if } B \text{ then } (S \text{ while } B \text{ do } S)K) \rangle \rightarrow_T \text{[with } B[B^\rho = \text{true}] \]
\[ \langle \rho_0 \equiv \rho, (\text{while } B \text{ do } S)K, t_0 \equiv \epsilon, S_0 \equiv S(\text{while } B \text{ do } S)K \rangle \rightarrow_T \]
\[ \langle \rho_1, (\text{while } B \text{ do } S)K, t_1 \equiv c_1, S_1 \rangle \rightarrow_T \]
\[ \langle \rho_2, (\text{while } B \text{ do } S)K, t_2 \equiv t_{n-1}c_n, S_2 \rangle \rightarrow_T \]
\[ \langle \rho_n, (\text{while } B \text{ do } S)K, t_n \equiv t_{n-1}c_n, S_n \rangle \rightarrow_T \]
\[ \langle \rho_{n+1} \equiv \rho', (\text{while } B \text{ do } S)K, t \equiv t_{n}c_{n+1}, S_{n+1} \equiv (\text{while } B \text{ do } S)K \rangle \]

where \( n \geq 0 \), so that the body \( S \) is assumed to be nonempty, i.e., \( S \neq \epsilon \) (there is no loss of generality since for \( S = \epsilon \) the result trivially holds). Hence, \( t = c_1 \ldots c_n c_{n+1} \), for some commands \( c_i \in \text{Cmd} \), and the corresponding hot path \( hp_1 = H_{-2}H_{-1}H_{0} \ldots H_{n} \) as determined by Lemma 10.7 is as follows:

\[ H_{-2} \triangleq l(S_{-2}) : \text{skip} \rightarrow l(S_{-1}) \]
\[ H_{-1} \triangleq l(S_{-1}) : B \rightarrow l(S_0) \quad [\text{because } B[B^\rho = \text{true}] \]
\[ H_0 \triangleq l(S_0) : A_0 \rightarrow l(S_1) \]
\[ H_1 \triangleq l(S_1) : A_1 \rightarrow l(S_2) \]
\[ \ldots \]
\[ H_n \triangleq l(S_n) : A_n \rightarrow l(S_{-2}) \]

where the action \( A_i \), with \( i \in [0, n] \), and the command \( c_{i+1} \) depend on the first command of the statement \( S_i \) as follows (this range of cases will be later referred to as (*)):

1. \( S_i \equiv \text{skip} : J \quad \Rightarrow \quad A_i \equiv \text{skip} \quad \& \quad c_{i+1} \equiv \text{skip}; \quad \& \quad S_{i+1} \equiv J \)
2. \( S_i \equiv x := E : J \quad \Rightarrow \quad A_i \equiv x := E \quad \& \quad c_{i+1} \equiv x := E; \quad \& \quad S_{i+1} \equiv J \)
3. \( S_i \equiv (\text{if } B' \text{ then } S')J \quad \& \quad B[B^\rho_1 = \text{true}] \quad \Rightarrow \quad A_i \equiv B' \quad \& \quad c_{i+1} \equiv \text{bail} \rightarrow B' \text{ to } J \quad \& \quad S_{i+1} \equiv S'J \)
(4) \( S_i \equiv (\text{if } B' \text{ then } S')J \) & \( B[B'][\rho_i = \text{false}] \Rightarrow A_i \equiv \neg B' \& c_{i+1} \equiv \text{bail } B' \text{ to } (S'J) \) & \( S_{i+1} \equiv J \)

(5) \( S_i \equiv (\text{while } B' \text{ do } S')J \) & \( (\text{while } B' \text{ do } S')J \neq (\text{while } B \text{ do } S)K \) \( \Rightarrow A_i \equiv \text{skip } \& c_{i+1} \equiv \text{skip} \) & \( S_{i+1} \equiv (\text{if } B' \text{ then } (S'(\text{while } B' \text{ do } S')))J \)

If, for any \( i \in [0, n], H_i \) is not a conditional command then, by case (1) of Definition [10.9] we have that \( \text{extr}_{GP}(C((\text{while } B \text{ do } S)K)) = C((\text{while } B \text{ do } S)K) \). Also, for any \( i \in [0, n], A_i \) is either a skip or an assignment, so that \( c_{i+1} = A_i \), and, in turn, \( t = S \). Hence, \( (\text{while } B \text{ do } t)K \equiv (\text{while } B \text{ do } S)K \), so that the thesis follows trivially.

Thus, we assume that \( H_k \), with \( k \in [0, n] \), is the first conditional command occuring in the sequence \( H_0 \ldots H_n \). Case (2) of Definition [10.9] applies, so that:

\[
\text{extr}_{GP}(C((\text{while } B \text{ do } S)K)) = C((\text{while } B \text{ do } S)K) \cup \{ \ell_{-2} : \text{skip} \rightarrow \ell_{-1}, \ell_{-1} : B \rightarrow \ell_0, \ell_{-1} : \neg B \rightarrow I(K), \\
\ell_0 : A_0 \rightarrow \ell_1, \ldots, \ell_n : A_n \rightarrow \ell_{-2} \} \cup \\
\{ \ell_i : \neg A_i \rightarrow I(S_{\text{next}(i)})^c | i \in [0, n], A_i \in \text{BExp} \}.
\]

Moreover, we have that:

\[
C((\text{while } B \text{ do } S)K) = \\
\{ I((\text{while } B \text{ do } S)K) : \text{skip} \rightarrow I((\text{if } B \text{ then } (S \text{ while } B \text{ do } S)))K), \\
I((\text{if } B \text{ then } (S \text{ while } B \text{ do } S)))K) : B \rightarrow I(S \text{ while } B \text{ do } S)K), \\
I((\text{if } B \text{ then } (S \text{ while } B \text{ do } S)))K) : \neg B \rightarrow I(K) \} \\
\cup C(S \text{ while } B \text{ do } S)K) \cup C(K).
\]

\[
C((\text{while } B \text{ do } t)K) = \\
\{ I((\text{while } B \text{ do } t)K) : \text{skip} \rightarrow I((\text{if } B \text{ then } (t \text{ while } B \text{ do } t)))K), \\
I((\text{if } B \text{ then } (t \text{ while } B \text{ do } t)))K) : B \rightarrow I(t \text{ while } B \text{ do } t)K), \\
I((\text{if } B \text{ then } (t \text{ while } B \text{ do } t)))K) : \neg B \rightarrow I(K) \} \\
\cup C(t \text{ while } B \text{ do } t)K) \cup C(K).
\]

We first show that \( C((\text{while } B \text{ do } t)K) \subseteq_{\sim} \text{extr}_{GP}(C((\text{while } B \text{ do } S)K)) \). We consider the following label renaming:

\[
I((\text{while } B \text{ do } t)K) \mapsto \ell_{-2} \\
I((\text{if } B \text{ then } (t \text{ while } B \text{ do } t)))K) \mapsto \ell_{-1} \\
I(t \text{ while } B \text{ do } t)K) \mapsto \ell_0
\]

so that it remains to show that \( C(t \text{ while } B \text{ do } t)K) \subseteq_{\sim} \text{extr}_{GP}(C((\text{while } B \text{ do } S)K)) \). Since \( t = c_1 t' \), with \( t' = c_2 \ldots c_{n+1} \), let us analyze the five different cases for the first command \( c_1 \) of \( t \).

(i) \( c_1 \equiv x : E \). Thus, \( S_0 \equiv x := E; T(\text{while } B \text{ do } S)K, S_1 \equiv T(\text{while } B \text{ do } S)K, A_0 \equiv x := E \). In this case,

\[
C(t \text{ while } B \text{ do } t)K) = \{ I(t \text{ while } B \text{ do } t)K) : x := E \rightarrow I(t' \text{ while } B \text{ do } t)K) \\
\cup C(t' \text{ while } B \text{ do } t)K).
\]

Hence, it is enough to consider the relabeling \( I(t' \text{ while } B \text{ do } t)K) \mapsto \ell_1 \) and to show that \( C(t' \text{ while } B \text{ do } t)K) \subseteq_{\sim} \text{extr}_{GP}(C((\text{while } B \text{ do } S)K)) \).
Thus, in order to prove this containment, it remains to show that

\[ C(t(\text{while } B \text{ do } t)K) = \{ l(\text{while } B \text{ do } t)K : \text{skip} \rightarrow l'(\text{while } B \text{ do } t)K\} \]

\[ \cup C'(\text{while } B \text{ do } t)K). \]

Again, it is enough to consider the relabeling \( l'(\text{while } B \text{ do } t)K) \rightarrow \ell_1 \) and to show that \( C'(\text{while } B \text{ do } t)K) \subseteq \text{extr}_{hp}^{GP}(C((\text{while } B \text{ do } S)K)) \).

(iv) \( c_1 \equiv \text{bail} \ 
eg B' \to (T(\text{while } B \text{ do } S)K) \), with \( S_0 \equiv (\text{if } B' \text{ then } S')T(\text{while } B \text{ do } S)K \) and \( B'[\rho_0 = false] \). In this case:

\[ C(t(\text{while } B \text{ do } t)K) = \{ l(\text{while } B \text{ do } t)K : \neg B' \rightarrow l(T(\text{while } B \text{ do } S)K), \]

\[ l(\text{while } B \text{ do } t)K : B' \rightarrow l'(\text{while } B \text{ do } t)K)\} \]

\[ \cup C(T(\text{while } B \text{ do } S)K) \cup C'(\text{while } B \text{ do } t)K), \]

\[ C(S(\text{while } B \text{ do } S)K) = \{ l(S(\text{while } B \text{ do } S)K) : B' \rightarrow l(S'T(\text{while } B \text{ do } S)K), \]

\[ l(S(\text{while } B \text{ do } S)K) : \neg B' \rightarrow l(T(\text{while } B \text{ do } S)K)\} \]

\[ \cup C(S'T(\text{while } B \text{ do } S)K) \cup C(T(\text{while } B \text{ do } S)K). \]

Hence, since \( l(\text{while } B \text{ do } t)K) \rightarrow \ell_0 \) and \( A_0 \equiv B' \), it is enough to consider the relabeling \( l'(\text{while } B \text{ do } t)K) \rightarrow \ell_1 \) and to show that \( C'(\text{while } B \text{ do } t)K) \subseteq \text{extr}_{hp}^{GP}(C((\text{while } B \text{ do } S)K)) \).

(v) \( c_1 \equiv \text{bail} B' \to (S'T(\text{while } B \text{ do } S)K), \) with \( S_0 \equiv (\text{if } B' \text{ then } S')T(\text{while } B \text{ do } S)K \) and \( B'[\rho_0 = false] \). In this case:

\[ C(t(\text{while } B \text{ do } t)K) = \{ l(\text{while } B \text{ do } t)K : \neg B' \rightarrow l(T(\text{while } B \text{ do } S)K), \]

\[ l(\text{while } B \text{ do } t)K : B' \rightarrow l'(\text{while } B \text{ do } t)K)\} \]

\[ \cup C(T(\text{while } B \text{ do } S)K) \cup C'(\text{while } B \text{ do } t)K), \]

\[ C(S(\text{while } B \text{ do } S)K) = \{ l(S(\text{while } B \text{ do } S)K) : B' \rightarrow l(S'T(\text{while } B \text{ do } S)K), \]

\[ l(S(\text{while } B \text{ do } S)K) : \neg B' \rightarrow l(T(\text{while } B \text{ do } S)K)\} \]

\[ \cup C(S'T(\text{while } B \text{ do } S)K) \cup C(T(\text{while } B \text{ do } S)K). \]

Thus, in order to prove this containment, it remains to show that \( C'(\text{while } B \text{ do } t)K) \subseteq \text{extr}_{hp}^{GP}(C((\text{while } B \text{ do } S)K)). \) If \( t' = \epsilon \) then the containment boils down to \( C((\text{while } B \text{ do } t)K) \subseteq \text{extr}_{hp}^{GP}(C((\text{while } B \text{ do } S)K)) \) which is therefore proved. Otherwise, \( t' = c_1 t'' \), so that the containment can be inductively proved by using the same five cases (i)-(v) above.

Let us now show the reverse containment, that is, \( \text{extr}_{hp}^{GP}(C((\text{while } B \text{ do } S)K)) \subseteq \text{extr}_{hp}^{GP}(C((\text{while } B \text{ do } S)K)). \) For the trace \( t = c_1 c_2 \ldots c_{n+1} \), we know by (\( \ast \)) that each command of \( c_i \) either is in \{skip; \( x := E; \)\) or is one of the two following bail commands (cf. cases (3) and (4) in (\( \ast \))):

\[ \text{bail } \neg B' \to (T(\text{while } B \text{ do } S)K), \quad \text{bail } B' \to (S'T(\text{while } B \text{ do } S)K). \]

Furthermore, at least a bail command occurs in \( t \) because there exists at least a conditional command \( H_k \) in \( \mathcal{H}_t \). Let \( c_k \), with \( k \in [1, n+1] \), be the first bail command occurring in \( t \). Thus, since the sequence \( c_1 \ldots c_{k-1} \) consists of skip and assignment commands
only, we have that \( C(t(\text{while } B \text{ do } t)K) \supseteq C(\ell_0 : B \rightarrow t_0, \ell_1 : \neg B \rightarrow l(K)) \). Hence, either
\[ C(\ell_0 : B \rightarrow t_0, \ell_1 : \neg B \rightarrow l(K)) \] or \( C(\ell_0 : B \rightarrow t_0, \ell_1 : \neg B \rightarrow l(K)) \) is contained in \( C(t(\text{while } B \text{ do } t)K) \). In both cases, we obtain that \( C(\ell_0 : B \rightarrow t_0, \ell_1 : \neg B \rightarrow l(K)) \) is contained in \( C(t(\text{while } B \text{ do } t)K) \). Thus, it remains to show that
\[
\{ \ell_2 : \text{skip} \rightarrow \ell_1, \ell_1 : B \rightarrow t_0, \ell_0 : \neg B \rightarrow l(K) \} \cup \\
\{ \ell_i : A_i \rightarrow \ell_{next(i)} \mid i \in [0, n] \} \cup \{ \ell_i : \neg A_i \rightarrow l(S_{next(i)}^c) \mid i \in [0, n], A_i \in \text{BExp} \}
\] is contained in \( C(t(\text{while } B \text{ do } t)K) \). We consider the following label renaming:
\[ \begin{align*}
\ell_2 & \rightarrow l(\text{while } B \text{ do } t)K) \\
\ell_1 & \rightarrow l((\text{if } B \text{ then } t(\text{while } B \text{ do } t)))K) \\
\ell_0 & \rightarrow l((\text{while } B \text{ do } t)K)
\end{align*} \]
so that it remains to check that for any \( i \in [0, n] \), the commands \( \ell_i : A_i \rightarrow \ell_{next(i)} \) and \( \ell_i : \neg A_i \rightarrow l(S_{next(i)}^c) \), when \( A_i \in \text{BExp} \), are in \( C(t(\text{while } B \text{ do } t)K) \). We analyze the possible five cases listed in (i) for the action \( A_0 \):

(i) \( A_0 \equiv \text{skip} \) because \( S_0 \equiv \text{skip} ; T(\text{while } B \text{ do } S)K \). Here, \( t = \text{skip} ; t' \). Hence, \( l(\text{while } B \text{ do } t)K : \text{skip} \rightarrow l(t'(\text{while } B \text{ do } t)K) \in C(t(\text{while } B \text{ do } t)K) \) and it is enough to use the relabeling \( \ell_0 \rightarrow l(t'(\text{while } B \text{ do } t)K) \).

(ii) \( A_0 \equiv x := e \) because \( S_0 \equiv x := E; T(\text{while } B \text{ do } S)K \), so that \( t = e \). Hence, \( t \rightarrow s \).

(iii) \( A_0 = \text{skip} \), \( S_0 = (\text{while } B \text{ do } S')T(\text{while } B \text{ do } S)K \). Here, \( t = \text{skip} ; t' \). Hence, \( l(\text{while } B \text{ do } t)K : \text{skip} \rightarrow l(t'(\text{while } B \text{ do } t)K) \in C(t(\text{while } B \text{ do } t)K) \), so that it is enough to use the relabeling \( \ell_1 \rightarrow l(t'(\text{while } B \text{ do } t)K) \).

(iv) \( A_0 \equiv B' \) because \( S_0 \equiv (\text{if } B' \text{ then } S')T(\text{while } B \text{ do } S)K \) and \( B[B']_{\rho_0} = \text{true} \). Thus, \( t = (\text{bail } \neg B' \text{ to } (T(\text{while } B \text{ do } S))]t' \) and \( S_1 \equiv T(\text{while } B \text{ do } S)K \). Note that \( l(S_1^c) = l(T(\text{while } B \text{ do } S)K) \). Hence,
\[
\begin{align*}
l(t(\text{while } B \text{ do } t)K) & : \neg B' \rightarrow l(T(\text{while } B \text{ do } S)K) \in C(t(\text{while } B \text{ do } t)K), \\
l(t(\text{while } B \text{ do } t)K) & : B' \rightarrow l(T(\text{while } B \text{ do } S)K) \in C(t(\text{while } B \text{ do } t)K).
\end{align*}
\]

Once again, the relabeling \( \ell_0 \rightarrow l(t'(\text{while } B \text{ do } t)K) \) allows us to obtain that \( \ell_0 : B' \rightarrow \ell_0 \) and \( \ell_0 : \neg B' \rightarrow l(T(\text{while } B \text{ do } S)K) \). Hence,
\[
\begin{align*}
l(t(\text{while } B \text{ do } t)K) & : B' \rightarrow l(S'T(\text{while } B \text{ do } S)K) \in C(t(\text{while } B \text{ do } t)K), \\
l(t(\text{while } B \text{ do } t)K) & : \neg B' \rightarrow l(T(\text{while } B \text{ do } S)K) \in C(t(\text{while } B \text{ do } t)K).
\end{align*}
\]
Then, through the relabeling \( \ell_1 \rightarrow l(t'(\text{while } B \text{ do } t)K) \) we obtain that \( \ell_0 : B' \rightarrow \ell_0 \) and \( \ell_1 : \neg B' \rightarrow l(S'T(\text{while } B \text{ do } S)K) \) are in \( C(t(\text{while } B \text{ do } t)K) \).

This case analysis (i)-(v) for the action \( A_0 \) can be iterated for all the other actions \( A_i \), with \( i \in [1, n] \), and this allows us to close the proof. \( \square \)

Finally, we can also state the correctness of the GP trace extraction transform for the store changes abstraction as follows.

**Theorem 10.12 (Correctness of GP Trace Extraction).** For any \( P \in \text{Program} \), \( hp = C_0 \cdots C_n \in \alpha_{hot}^{GP}(T[P]) \), we have that \( \alpha_{sc}(T[\text{extr}^{GP}_{hp}(P)]) = \alpha_{sc}(T[P]) \).
The proof of Theorem 10.12 is omitted, since it is a conceptually straightforward adaptation of the proof technique for the analogous Theorem 6.2 on the correctness of trace extraction. Let us observe that since $\alpha_{sc}$ is a stronger abstraction than $\alpha_{\rho_{sc}}$ and, by Theorem 10.6, we know that $\alpha_{\rho_{sc}}$ characterizes bisimilarity, we obtain the so-called Stitch lemma in [Guo and Palsberg 2011, Lemma 3.6] as a straight consequence of Theorem 10.12:

$$\alpha_{\rho_{sc}}(T[\extr{GP}(P)]) = \alpha_{\rho_{sc}}(T[P]).$$

11. CONCLUSION AND FURTHER WORK

This article put forward a formal model of tracing JIT compilation which allows:

1. an easy definition of program hot paths—that is, most frequently executed program traces;
2. to prove the correctness of a hot path extraction transform of programs;
3. to prove the correctness of dynamic optimizations confined to hot paths, such as dynamic type specialization along a hot path.

Our approach is based on two main ideas: the use of a standard trace semantics for modeling the behavior of programs and the use of abstract interpretation for defining the notion of hot path as an abstraction of the trace semantics and for proving the correctness of hot path extraction and optimization. We have shown that this framework is more flexible than the Guo and Palsberg [2011] model of tracing JIT compilation, which relies on a notion of correctness based on operational program bisimulations, and allows to overcome some limitations of [Guo and Palsberg 2011] on selection and annotation of hot paths and on the correctness of optimizations such as dead store elimination. We expect that most optimizations employed by tracing JIT compilers can be formalized and proved correct using the proof methodology of our framework.

We see a number of interesting avenues for further work on this topic. As a significant example of optimization implemented by a practical tracing compiler, it would be worth to cast in our model the allocation removal optimization for Python described by Bolz et al. [2011] in order to formally prove its correctness. Then, we think that our framework could be adapted in order to provide a model of whole-method just-in-time compilation, as used, e.g., by IonMonkey [Mozilla Foundation 2013], the current JIT compilation scheme in the Firefox JavaScript engine. Finally, the main ideas of our model could be useful to study and relate the foundational differences between traditional static vs dynamic tracing compilation.

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