Optim Lett
DOI 10.1007/s11590-011-0417-9

ORIGINAL PAPER

An exact penalty global optimization approach for mixed-integer programming problems

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Received: 19 November 2010 / Accepted: 15 October 2011
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Abstract In this work, we propose a global optimization approach for mixed-integer programming problems. To this aim, we preliminarily define an exact penalty algorithm model for globally solving general problems and we show its convergence properties. Then, we describe a particular version of the algorithm that solves mixed-integer problems and we report computational results on some MINLP problems.

Keywords Mixed-integer programming · Global optimization · Exact penalty functions

1 Introduction

Many real-world problems in Engineering, Economics and Applied Sciences can be formulated as a nonlinear minimization problem where some of the variables only assume integer values. A reasonable approach can be that of transforming the original problem into an equivalent continuous problem. A number of different transformations have been proposed in the literature (see, e.g. [1,4,8,15–17]). A particular continuous reformulation, which comes out by relaxing the integer constraints on the variables and by adding a penalty term to the objective function, was first described by Ragavachari in [18] to solve zero-one linear programming problems. There are many other papers closely related to the one by Ragavachari (see, e.g. [5,7,9,10,14,19]). In [7], the exact penalty approach has been extended to general nonlinear integer programming

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Published online: 28 October 2011

problems. In [19], various penalty terms have been proposed for solving zero-one concave programming problems. In [14], the results described in [7] have been generalized. Furthermore, it has been shown that a general class of penalty functions, including the ones proposed in [19], can be used for solving general nonlinear integer problems.

In this work, we propose an exact penalty method for globally solving mixed-integer programming problems. We consider a continuous reformulation of the original problem using a penalty term like that proposed in [14]. It is possible to show that, under weak assumptions, there exists a threshold value $\bar{\varepsilon} > 0$ of the penalty parameter ε such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, any solution of the continuous problem is also a solution of the related integer problem (see [14] for further details). On these bases, we describe an algorithm that combines a global optimization technique for solving the continuous reformulation for a given value of the penalty parameter ε and an automatic updating of ε occurring a finite number of times. The main feature of the algorithm is that the sequence of points $\{x^k\}$ generated is convergent to a solution of the original mixed-integer programming problem.

The paper is organized as follows. In Sect. 2 we recall a general result concerning the equivalence between an unspecified optimization problem and a parameterized family of problems. In Sect. 3, we describe an exact penalty global optimization algorithm model for solving general problems based on the equivalence result reported in Sect. 2 and we show its convergence properties. In Sect. 4, we describe an exact penalty algorithm for globally solving mixed integer problems based on the model described in Sect. 3. Finally, in Sect. 5 we report preliminary computational results on some MINLP problems.

2 A general equivalence result

We start from the general nonlinear constrained problem:

$$\min_{x \in W} f(x) \tag{1}$$

where $W \subset \mathbb{R}^n$ and $f(x) : \mathbb{R}^n \to \mathbb{R}$.

For any $\varepsilon \in \mathbb{R}_+$, we consider the following problem:

$$\min_{x \in X} f(x) + \varphi(x, \varepsilon). \tag{2}$$

where $W \subseteq X \subset \mathbb{R}^n$, and $\varphi(\cdot, \varepsilon) : \mathbb{R}^n \to \mathbb{R}$. In (1), (2) and in the sequel, "min" denotes global minimum.

Throughout the paper, we make the following two assumptions:

Assumption 1 f is bounded on X, and there exists an open set $A \supset W$ and real numbers $\alpha, L > 0$, such that, $\forall\, x, y \in A$, f satisfies the following condition:

$$|f(x) - f(y)| \leq L\|x - y\|^{\alpha}. \tag{3}$$

Assumption 2 the function φ satisfies the following conditions:

(i) $\forall x, y \in W$, and $\forall\, \varepsilon \in \mathbb{R}_+$

$$\varphi(x, \varepsilon) = \varphi(y, \varepsilon).$$

(ii) There exists a value $\hat{\varepsilon}$ and, $\forall z \in W$, there exists a neighborhood $S(z)$ such that $\forall\, x \in S(z) \cap (X \setminus W)$, and $\varepsilon \in]0, \hat{\varepsilon}]$, we have:

$$\varphi(x, \varepsilon) - \varphi(z, \varepsilon) \geq \hat{L}\|x - z\|^{\alpha} \tag{4}$$

where $\hat{L} > L$ and α is chosen as in (3). Furthermore, let $S = \bigcup_{z \in W} S(z)$, $\exists\, \hat{x} \notin S$ such that:

$$\lim_{\varepsilon \to 0} [\varphi(\hat{x}, \varepsilon) - \varphi(z, \varepsilon)] = +\infty, \quad \forall\, z \in W, \tag{5}$$

$$\varphi(x, \varepsilon) \geq \varphi(\hat{x}, \varepsilon), \quad \forall\, x \in X \setminus S,\ \forall\, \varepsilon > 0. \tag{6}$$

The following Theorem shows that, when assumptions on f and φ hold, Problems (1) and (2) are equivalent.

Theorem 1 *Let W and X be compact sets. Let $\|\cdot\|$ be a suitably chosen norm. Then, $\exists\, \tilde{\varepsilon} \in \mathbb{R}_+$ such that, $\forall \varepsilon \in]0, \tilde{\varepsilon}]$, Problems (2) and (1) have the same minimum points.*

Proof See [14]. □

3 An exact penalty algorithm model

In this section, we introduce the EXP (EXact Penalty) algorithm model for finding a solution of Problem (1) and we analyze its convergence properties.

EXP Algorithm

Data. $k = 0$, $\varepsilon^0 > 0$, $\delta^0 > 0$, $\alpha > 0$, $\beta > 0$, $\sigma \in (0, 1)$.

Step 1. Compute $x^k \in X$ such that

$$f(x^k) + \varphi(x^k, \varepsilon^k) \leq f(x) + \varphi(x, \varepsilon^k) + \delta^k \tag{7}$$

$\forall\, x \in X$.

Step 2. If $x^k \notin W$ and

$$[f(x^k) + \varphi(x^k, \varepsilon^k)] - [f(z^k) + \varphi(z^k, \varepsilon^k)] \leq \varepsilon^k \beta \|x^k - z^k\|^{\alpha} \tag{8}$$

where $z^k \in W$ minimizes the distance between x^k and $S(z^k)$,

then $\varepsilon^{k+1} = \sigma \varepsilon^k$, $\delta^{k+1} = \delta^k$.

Else $\varepsilon^{k+1} = \varepsilon^k$, $\delta^{k+1} = \sigma \delta_k$.

Step 3. Set $k = k + 1$ and go to Step 1.

In the algorithm, at Step 1 the point x^k is a δ^k-global minimizer of Problem (2). At Step 2, we check feasibility of the current solution x^k, and, in case x^k is feasible, we reduce the value of δ^k for finding a better approximation of the optimal solution of Problem (2). When x^k is not feasible, we use test (8) to verify if an updating of the penalty parameter is timely. The sets $S(z^k)$ are those ones used in Assumption 2.

Throughout the paper we assume, without any loss of generality, that the scaling parameter $\beta = 1$. We preliminarily prove the following Lemma, that will be used to state the convergence properties of the EXP Algorithm. In the Lemma, we assume that the sequence $\{x^k\}$ is well defined. It means that the δ^k-global minimizer of the penalty function can always be found. The compactness of X is sufficient to ensure that this assumption holds.

Lemma 1 *Let $\{x^k\}$ be the sequence produced by the EXP Algorithm. One of the following possibilities hold:*

(1) *an index $\bar{k}$ exists such that for any $k \geq \bar{k}$, $\varepsilon^k = \bar{\varepsilon}$ and every accumulation point of the sequence belongs to the set W;*
(2) $\{\varepsilon^k\} \to 0$*, and every accumulation point of a subsequence $\{x^k\}_K$, with $k \in K$ the set of indices such that test (8) is satisfied, belongs to the set W;*

Proof We consider two different cases:

Case (1) an index $\bar{k}$ exists such that for any $k \geq \bar{k}$, $\varepsilon^k = \bar{\varepsilon}$: By contradiction, let us assume that there exists a subsequence $\{x^k\}_K \to \bar{x}$ such that $\bar{x} \notin W$. Since for any $k \geq \bar{k}$, we have that $\varepsilon^k = \bar{\varepsilon}$, then the test (8) is not satisfied:

$$f(z^k) - f(x^k) < \varphi(x^k, \bar{\varepsilon}) - \varphi(z^k, \bar{\varepsilon}) - \bar{\varepsilon}\|x^k - z^k\|^\alpha \tag{9}$$

from which we have

$$f(z^k) + \varphi(z^k, \bar{\varepsilon}) < f(x^k) + \varphi(x^k, \bar{\varepsilon}) - \bar{\varepsilon}\|x^k - z^k\|^\alpha \tag{10}$$

and, by using (7), we get the following contradiction

$$\begin{aligned} f(z^k) + \varphi(z^k, \bar{\varepsilon}) < f(x^k) + \varphi(x^k, \bar{\varepsilon}) - \bar{\varepsilon}\|x^k - z^k\|^\alpha \leq f(z^k) + \varphi(z^k, \bar{\varepsilon}) \\ + \delta^k - \bar{\varepsilon}\|x^k - z^k\|^\alpha \end{aligned} \tag{11}$$

where $\delta^k \to 0$ and $\bar{\varepsilon}\|x^k - z^k\|^\alpha \to \bar{\rho} > 0$.
Case (2) $\{\varepsilon^k\} \to 0$:
Once again, by contradiction, we assume that there exists a limit point $\bar{x}$ of the subsequence $\{x^k\}_K$ such that $\bar{x} \notin W$. We define a subsequence $\{x^k\}_{\hat{K}} \to \bar{x}$, with $\hat{K} \subset K$.
If a subsequence $\{x^k\}_{\tilde{K}}$, with $\tilde{K} \subset \hat{K}$, exists such that $x^k \in S(z^k)$, by taking into account (4) into assumption (ii) of Theorem 1 and the test (8), we obtain

$$\hat{L}\|x^k - z^k\|^\alpha \leq \varphi(x^k, \varepsilon^k) - \varphi(z^k, \varepsilon^k) \leq f(z^k) - f(x^k) + \varepsilon^k\|x^k - z^k\|^\alpha. \tag{12}$$

Then by assumption 1 we can write

$$\hat{L}\|x^k - z^k\|^\alpha \le \varphi(x^k, \varepsilon^k) - \varphi(z^k, \varepsilon^k) \le f(z^k) - f(x^k) + \varepsilon^k \|x^k - vz^k\|^\alpha \\ \le (L + \varepsilon^k)\|x^k - z^k\|^\alpha, \tag{13}$$

and by taking into account the fact that $\{\varepsilon^k\} \to 0$, we obtain the contradiction $\hat{L} \le L$.

On the other hand, if the subsequence $\{x^k\}_{\hat{K}}$ is such that $x^k \notin S(z^k)$, the choice of z^k guarantees that $x^k \in X \backslash S$ for every $k \in \hat{K}$. By taking into account (6) into assumption (ii) of Theorem 1 and the test (8), we obtain

$$\varphi(\hat{x}, \varepsilon^k) - \varphi(\hat{z}, \varepsilon^k) \le \varphi(x^k, \varepsilon^k) - \varphi(z^k, \varepsilon^k) \le f(z^k) - f(x^k) + \varepsilon^k \|x^k - z^k\|^\alpha, \tag{14}$$

where $\hat{z}$ is any point belonging to the set W. Now, due to the fact that x^k and z^k belong to a compact set, we have that

$$\varphi(\hat{x}, \varepsilon^k) - \varphi(\hat{z}, \varepsilon^k) \le M \tag{15}$$

for every $k \in \hat{K}$, which contradicts (5) into assumption (ii) of Theorem 1.

□

In the next proposition, we show that in the EXP Algorithm the penalty parameter ε is updated only a finite number of times.

Lemma 2 *Let $\{x^k\}$ and $\{\varepsilon^k\}$ be the sequences produced by the EXP Algorithm. Then an index $\bar{k}$ exists such that for any $k \ge \bar{k}$, $\varepsilon^k = \bar{\varepsilon}$.*

Proof By contradiction, let us assume $\{\varepsilon_k\} \to 0$, and there exists a subsequence $\{x^k\}_K$ converging to $\bar{x}$ such that the test (8) is satisfied for all $k \in K$. By Lemma 1, we have that $\bar{x} \in W$, then there exists an index $\tilde{k}$ such that for any $k \ge \tilde{k}$ and $k \in K$ we obtain $x^k \in S(z^k) = S(\bar{x})$. By using (4) of Assumption 2 we have

$$\varphi(x^k, \varepsilon^k) - \varphi(\bar{x}, \varepsilon^k) \ge \hat{L}\|x^k - \bar{x}\|^\alpha \ge L\|x^k - \bar{x}\|^\alpha + (\hat{L} - L)\|x^k - \bar{x}\|^\alpha \tag{16}$$

where L is the constant used in Assumption 1. Now, by means of Assumption 1 we obtain the following inequality:

$$\varphi(x^k, \varepsilon^k) - \varphi(\bar{x}, \varepsilon^k) \ge |f(x^k) - f(\bar{x})| + (\hat{L} - L)\|x^k - \bar{x}\|^\alpha, \tag{17}$$

and when k is sufficiently large, we have

$$\varphi(x^k, \varepsilon^k) - \varphi(\bar{x}, \varepsilon^k) \ge f(\bar{x}) - f(x^k) + (\hat{L} - L)\|x^k - \bar{x}\|^\alpha > f(\bar{x}) - f(x^k) \\ + \varepsilon^k \|x^k - \bar{x}\|^\alpha \tag{18}$$

with $k \in K$, which contradicts the fact that test (8) is satisfied for all $k \in K$. □

Then, we can state the main convergence result.

Theorem 2 *Every accumulation point $\bar{x}$ of a sequence $\{x^k\}$ produced by the EXP Algorithm is a global minimizer of the Problem* (1).

Proof By using Lemma 2, and the fact that $\delta^k \to 0$, we can write

$$f(\bar{x}) + \varphi(\bar{x}, \bar{\varepsilon}) \leq f(x) + \varphi(x, \bar{\varepsilon}) \tag{19}$$

for all $x \in X$. Then

$$f(\bar{x}) + \varphi(\bar{x}, \bar{\varepsilon}) \leq f(z) + \varphi(z, \bar{\varepsilon}) \tag{20}$$

for all $z \in W$. By Lemma 1 we have that $\bar{x} \in W$, and by (i) of Assumption 2, we obtain

$$f(\bar{x}) \leq f(z) \tag{21}$$

for all $z \in W$. □

4 An exact penalty algorithm for solving mixed integer problems

Let us consider now the problem

$$\begin{aligned} &\min f(x) \\ &x \in C \\ &x_i \in Z \qquad i \in I_z \end{aligned} \tag{22}$$

with $f : R^n \to R, C \subset R^n$ a compact convex set, and $I_z \subseteq \{1, \ldots, n\}$. We notice that, due to the compactness of C, there always exist finite values l_i and u_i such that $l_i \leq x_i \leq u_i, i = 1, \ldots, n$. Using Theorem 1, we can show that the mixed-integer Problem (22) is equivalent to the following continuous formulation:

$$\min f(x) + \varphi(x, \varepsilon), \quad x \in C, \tag{23}$$

where $\varepsilon \in (0, \bar{\varepsilon}]$, and $\varphi(x, \varepsilon)$ is a suitably chosen penalty term.

In [14], the equivalence between (22) and (23) has been proved for a class of penalty terms including the following two:

$$\varphi(x, \varepsilon) = \sum_{i \in I_z} \min_{\substack{l_i \leq d_j \leq u_i, \\ d_j \in Z}} \{\log[|x_i - d_j| + \varepsilon]\} \tag{24}$$

$$\varphi(x, \varepsilon) = \frac{1}{\varepsilon} \sum_{i \in I_z} \min_{\substack{l_i \leq d_j \leq u_i, \\ d_j \in Z}} \{[|x_i - d_j| + \varepsilon]^p\} \tag{25}$$

with $\varepsilon > 0$ and $0 < p < 1$.

The following proposition shows the equivalence between Problems (22) and (23).

Proposition 1 *Let us consider the penalty terms* (24) *and* (25)*. We have that:*

(i) *when* $S(z) = \{x \in \mathbb{R}^n : \|x - z\|_\infty < \rho\}$ *and* ρ *is a sufficiently small positive value, the two terms satisfy Assumption* 2*;*
(ii) *there exists a value* $\bar{\varepsilon} > 0$ *such that, for any* $\varepsilon \in (0, \bar{\varepsilon}]$*, Problem* (22) *and Problem* (23) *have the same minimum points.*

Proof See [14]. □

Remark 1 When dealing with problems with boolean variables, we can define specific penalty terms:

$$\varphi(x, \varepsilon) = \sum_{i \in I_z} \min\{\log(x_i + \varepsilon), \log[(1 - x_i) + \varepsilon]\} \tag{26}$$

$$\varphi(x, \varepsilon) = \frac{1}{\varepsilon} \sum_{i \in I_z} \min\{(x_i + \varepsilon)^p, [(1 - x_i) + \varepsilon]^p\} \tag{27}$$

$$\varphi(x, \varepsilon) = \sum_{i \in I_z} \{\log(x_i + \varepsilon) + \log[(1 - x_i) + \varepsilon]\} \tag{28}$$

$$\varphi(x, \varepsilon) = \frac{1}{\varepsilon} \sum_{i \in I_z} \left\{(x_i + \varepsilon)^p + [(1 - x_i) + \varepsilon]^p\right\} \tag{29}$$

where $\varepsilon > 0$ and $0 < p < 1$.

We state a result that will be useful to describe a specific version of the EXP Algorithm for Mixed-Integer Bound Constrained Problems. The symbol $[\cdot]_R$ indicates the scalar rounding to the nearest integer.

Proposition 2 *Let* $S(z)$ *be a set defined as follows*

$$S(z) = \{x \in \mathbb{R}^n : \|x - z\|_\infty < \rho\}.$$

Given a point $x \in R^n$ *and a sufficiently small positive value* ρ*, the point* $z \in Z$ *that minimizes the distance between* x *and* $S(z)$ *is*

$$z = [x]_R.$$

Proof Let $z^* \in Z$ be the point such that $z^* = [x]_R$. If $x \in S(z^*)$, then the distance between x and $S(z^*)$ is equal to 0 and the proposition is trivially proved.

Now, let us assume $x \notin S(z^*)$ and, by contradiction, there exists a point $\tilde{z} \in Z$ such that the distance between x and $S(\tilde{z})$ is lower than the distance between x and $S(z^*)$, that is

$$\inf_{p \in S(\tilde{z})} \|x - p\|_\infty < \inf_{p \in S(z^*)} \|x - p\|_\infty. \tag{30}$$

Hence, we can find two points $\tilde{p}$ and p^* such that:

$$\tilde{p} = \arg \inf_{p \in S(\tilde{z})} \|x - p\|_\infty \tag{31}$$

$$p^* = \arg \inf_{p \in S(z^*)} \|x - p\|_\infty. \tag{32}$$

Then we have

$$\|\tilde{p} - \tilde{z}\|_\infty = \|p^* - z^*\|_\infty = \rho. \tag{33}$$

Furthermore, from (31), (32) and (33), we can write

$$\begin{aligned} \|x - \tilde{z}\|_\infty &= \|x - \tilde{p}\|_\infty + \|\tilde{p} - \tilde{z}\|_\infty \\ \|x - z^*\|_\infty &= \|x - p^*\|_\infty + \|p^* - z^*\|_\infty. \end{aligned} \tag{34}$$

Using the fact that $z^* = [x]_R \neq \tilde{z}$, we have

$$\|x - z^*\|_\infty \leq 0.5 \ \|x - \tilde{z}\|_\infty > 0.5. \tag{35}$$

Then, by (30), (33) and (34),

$$\|x-\tilde{z}\|_\infty - \|x-z^*\|_\infty = \|x-\tilde{p}\|_\infty + \|\tilde{p}-\tilde{z}\|_\infty - \|x-p^*\|_\infty - \|p^*-z^*\|_\infty < 0,$$

but this contradicts (35). □

Now, we can describe a version of the EXP Algorithm for solving Problem (22). We set

$$W = \left\{ x \in R^n : x \in C, \ x_i \in Z, \ i \in I_z \right\}, \quad X = C.$$

We remark that the set X is obtained by relaxing the integrality constraints in the set W. When we have

$$W = \left\{ x \in R^n : x \in C, \ x_i \in \{0, 1\}, \ i \in I_z \right\}$$

the relaxed set we obtain is

$$X = \left\{ x \in R^n : x \in C, \ 0 \leq x_i \leq 1, \ i \in I_z \right\}.$$

EXP Algorithm for mixed-integer problems (EXP-MIP)

Data. $k = 0, \varepsilon^0 > 0, \delta^0 > 0, \alpha > 0, \beta > 0, \sigma \in (0, 1)$.

Step 1. Compute $x^k \in X$ such that

$$f(x^k) + \varphi(x^k, \varepsilon^k) \leq f(x) + \varphi(x, \varepsilon^k) + \delta^k \tag{36}$$

$\forall \, x \in X$.

Step 2. If $x^k \notin W$ and

$$[f(x^k) + \varphi(x^k, \varepsilon^k)] - [f(z^k) + \varphi(z^k, \varepsilon^k)] \leq \varepsilon^k \beta \|x^k - z^k\|^\alpha \quad (37)$$

where $z^k = [x^k]_R$,

then $\varepsilon^{k+1} = \sigma \varepsilon^k$, $\delta^{k+1} = \delta^k$.

Else $\varepsilon^{k+1} = \varepsilon^k$, $\delta^{k+1} = \sigma \delta^k$.

Step 3. Set $k = k + 1$ and go to Step 1.

At Step 1, we can obtain a δ^k-global minimizer by using a specific global optimization method like, e.g. BARON [20,21], α-BB [2,3] or DIRECT [11–13]. What we want to highlight here is that when we have a good global optimization algorithm for continuous programming problems, if we include this algorithm in our exact penalty framework, we can also handle problems with integer variables. Anyway, there already exist various global optimization algorithms able to solve MINLP problems efficiently (see e.g. BARON). In this case, we can see our approach as a different way to handle the integrality constraints. As we have already said, Step 2 is basically a way to test if the updating of the penalty parameter is timely. In principle, we could get rid of this updating operation by setting the penalty parameter to a very small value thus guaranteeing the equivalence between the original integer problem and the continuous reformulation. However, such a choice of the parameter can make the continuous reformulation very hard to be solved (see Sect. 5 for further details).

By taking into account Theorem 2, we can state the result related to the convergence of the EXP-MIP Algorithm.

Corollary 1 *Every accumulation point $\bar{x}$ of a sequence $\{x^k\}$ produced by the EXP-MIP Algorithm is a global minimizer of the Problem* (22).

5 Preliminary computational results

In this section we report the results of a preliminary numerical experience. We implemented the EXP-MIP Algorithm in MATLAB using BARON/GAMS to solve the continuous global problem at Step 1. We compared our EXP-MIP Algorithm with BARON on a set of well-known MINLP problems (see [6] for further details). All the codes have been run on an Intel Pentium 4 CPU 3.2 GHz with 1 GB main memory. The EXP-MIP Algorithm has been run with the following starting parameters:

- penalty parameter $\varepsilon = 10^{-1}$;
- relative termination tolerance $EpsR = 10^{-2}$.

In the following table, we report, besides the problem name, the total number of variables n, the number of integer variables n_i, the number of constraints m, the total number of branch and reduce iterations (nodes), the CPU time in seconds (time) and the objective function value of the best solution found f^*. As we can easily notice by taking a look at Table 1, the EXP-MIP Algorithm shows a good behavior.

Table 1 Comparison between EXP-MIP and BARON

Problem	n	n_i	m	EXP-MIP			BARON		
				Nodes	Time	f^*	Nodes	Time	f^*
P 12.2.1	5	3	5	2	0.078	7.667	1	0.093	7.667
P 12.2.2	3	1	3	0	0.093	1.076	0	0.078	1.076
P 12.2.3	7	4	9	2	0.078	4.579	5	0.078	4.579
P 12.2.4	11	8	7	1	0.093	−0.912	0	0.093	−0.904
P 12.2.5	2	2	4	1	0.109	31	1	0.109	31
P 12.2.6	2	1	3	1	0.093	−17	1	0.062	−17

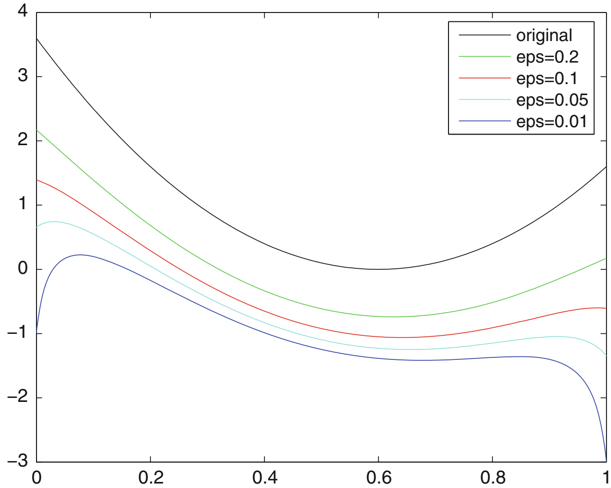

Fig. 1 Graph of the function (38) (in *black*) and of the penalty function (39) for various values of the parameter ε (eps)

In order to better assess the importance of the Step 2 in our method, we used BARON to solve the continuous penalty reformulation related to Problem 3 where we fixed the penalty parameter $\varepsilon = 10^{-6}$. For this particular choice, BARON needs 0.25 s of CPU time and the total number of nodes is 18. Then the adoption of an updating rule like the one adopted at Step 2 can help improve the performance. In Fig. 1 we show the graph of the function

$$f(x) = 10(x - 0.6)^2 \tag{38}$$

and of the related penalty function

$$f(x) + \varphi(x, \varepsilon) = 10(x - 0.6)^2 + \log(x + \varepsilon) + \log[(1 - x) + \varepsilon] \tag{39}$$

for various choices of ε. As we can easily see, when the penalty parameter is very small, the slope of the graph gets very large close to 0 or 1, thus making the problem, in some cases, hard to be solved.

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