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A Derivative-Free Algorithm for Constrained Global Optimization Based on Exact Penalty Functions

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Abstract Constrained global optimization problems can be tackled by using exact penalty approaches. In a preceding paper, we proposed an exact penalty algorithm for constrained problems which combines an unconstrained global minimization technique for minimizing a non-differentiable exact penalty function for given values of the penalty parameter, and an automatic updating of the penalty parameter that occurs only a finite number of times. However, in the updating of the penalty parameter, the method requires the evaluation of the derivatives of the problem functions. In this work, we show that an efficient updating can be implemented also without using the problem derivatives, in this way making the approach suitable for globally solving constrained problems where the derivatives are not available. In the algorithm, any efficient derivative-free unconstrained global minimization technique can be used. In particular, we adopt an improved version of the DIRECT algorithm. In addition, to improve the performances, the approach is enriched by resorting to derivative-free local searches, in a multistart framework. In this context, we prove that, under suitable assumptions, for every global minimum point there exists a neighborhood of attraction for the local search. An extensive numerical experience is reported.

Keywords Nonlinear optimization · Global optimization · Exact penalty functions · Derivative-free minimization · DIRECT algorithm

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1 Introduction

The search for global rather than local solutions of optimization problems has been receiving growing attention in many fields like engineering, economics and applied sciences. Many real-world problems in these fields are of the constrained type, while most of the algorithms in global optimization are for unconstrained or simply bound-constrained problems. For the unconstrained case, many algorithmic approaches, either deterministic or probabilistic, have been developed (see [1–7] and references therein). The more difficult case of nonlinearly-constrained global optimization problems has been investigated more recently and various approaches have been described in, e.g., [1, 7, 8]. In this context, a particular emphasis is given to the use of some Augmented Lagrangian function to deal with the general constraints (see [9–11]). However, the Augmented Lagrangian approach is sequential in nature and requires, in principle, an infinite number of global minimizations. Therefore, we adopt an exact penalty approach so that the number of unconstrained global minimizations needed to globally solving the original constrained problem would be finite. Moreover, by adopting an exact penalty approach we may take advantage of efficient derivative-free unconstrained algorithms. In this way, we can deal with constrained global optimization problems, where the derivatives of the functions are not available, as happens, for instance, in black-box simulation-based optimization, a topic that is receiving growing interest in the literature (see [12] and references therein). In particular, based on the approach described in [13], we develop a new strategy having the two following distinguishing features:

- differently from [13], derivatives of the objective and constraints functions are not required for the penalty parameter updating;
- a local search phase is used for speeding up the convergence to a global solution of the constrained problem: indeed, we prove that, under suitable assumptions, for every global minimum point there exists a neighborhood of attraction for the local search.

The paper is organized as follows. In Sect. 2, we give some preliminary results that will be used throughout the paper. In Sect. 3, we describe the Global Optimization Framework and we give some convergence results. In Sect. 4, we discuss the local search properties. In Sect. 5, we report an extended numerical experience using the proposed algorithm either alone, or combined with a derivative-free local minimization technique, and we discuss on the efficiency of our approach. Finally, we draw some conclusions in Sect. 6.

2 Preliminaries

In this paper, we are interested in the *global solution* of the general nonlinear programming problem:

$$\min f(x) \quad \text{s.t.} \quad g(x) \le 0, \quad l \le x \le u, \tag{1}$$

where $f : \Re^n \to \Re$, $g : \Re^n \to \Re^p$, $l, u \in \Re^n$ both finite, and we assume that f and g are continuously differentiable functions. We assume that no global information (convexity, Lipschitz constants, etc.) on the problem is available.

To simplify notations, we denote the bound constraints as

$$s(x) := \begin{bmatrix} l - x \\ x - u \end{bmatrix} \le 0.$$

We denote by $\mathcal{F}$ the feasible set of problem (1):

$$\mathcal{F} := \{x \in \Re^n : g(x) \le 0, s(x) \le 0\}.$$

In order to solve problem (1), we adopt an exact penalty function approach like that described in [13]. In particular, we use a non-differentiable penalty function having the following form:

$$P_q(x; \varepsilon) = f(x) + \frac{1}{\varepsilon} \left\| \left[\max\{0, g(x)\}', \max\{0, \tilde{s}(x)\}'\right]' \right\|_q, \tag{2}$$

where $1 < q < \infty$, and

$$\tilde{s}_j(x) = \frac{s_j(x)}{\alpha_j - s_j(x)},$$

with $\alpha_j > 0$, $j = 1, \dots, 2n$.

As pointed out in [13], even if, in general, the penalty function $P_q(x; \varepsilon)$ is non-differentiable, it turns out to be continuously differentiable at infeasible points.

We denote by $\mathcal{D}$ the set

$$\mathcal{D} := \{x \in \Re^n : s_j(x) \le \alpha_j, j = 1, \dots, 2n\}.$$

Then, we consider the problem

$$\min_{x \in \mathcal{D}} P_q(x; \varepsilon). \tag{3}$$

We make use of the following assumptions:

Assumption 2.1 $\mathcal{D}$ is a compact set.

Assumption 2.2 The Mangasarian–Fromovitz Constraint Qualification (MFCQ) holds at every global solution $x^\star$ of Problem (1).

We note that Assumption 2.1 is common in global optimization and is always satisfied in real-world problems and Assumption 2.2 is only needed in global minimizers; therefore they are very weak assumptions.

In the following theorem (see [13]), we recall the main exactness property of the penalty function P_q, which is at the basis of the proposed approach:

Theorem 2.1 *Under Assumption 2.2, there exists a threshold value $\bar{\varepsilon}$ such that, for any $\varepsilon \in]0, \bar{\varepsilon}]$, every global solution of Problem (1) is a global solution of Problem (3), and conversely.*

Remark 2.1 We note that by construction,

$$\lim_{x\to\partial\mathcal{D}} P_q(x;\varepsilon) = +\infty,$$

where $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$. Therefore Problem (3) has global solutions contained only in the interior of $\mathcal{D}$, so that determining a global solution of Problem (3) is essentially an unconstrained problem.

Remark 2.2 Theorem 2.1 implies that, for a sufficiently small value ε, Problem (3) has only unconstrained global minimizers.

Remark 2.3 The global solution of Problem (3) can be obtained by using any method for unconstrained global optimization.

On these bases, in [13] we were able to describe an algorithm for the global solution of the constrained problem (1), by combining any algorithm that globally solves the unconstrained global optimization problem (3) with an automatic updating rule for the penalty parameter which makes use of first-order derivatives of the problem functions.

In the paper, given a vector $v \in \Re^n$, we denote by v' its transpose, by $\|v\|_q$ its q-norm, and by $v^+ = \max\{0, v\}$ the n-vector $(\max\{0, v_1\}, \dots, \max\{0, v_n\})$.

3 Derivative-Free Exact Penalty Global Optimization Algorithm

In this section, we introduce the DF-EPGO (Derivative-Free Exact Penalty Global Optimization) algorithm model for finding a global solution of Problem (1) using the exact penalty function (2), and we analyze its convergence properties.

We assume that $F^k \in \Re^n$ approximates the gradient of the objective function f calculated in x^k, in the sense that there exists a value $\tau^k \geq 0$ such that:

$$\left\| F^k - \nabla f\left(x^k\right)\right\|_q \leq M_1 \tau^k, \tag{4}$$

with $M_1 > 0$. In the same way, we assume that V^k approximates the gradient of the function

$$\left\|\left[g^+(x)', \tilde{s}^+(x)'\right]'\right\|_q \tag{5}$$

calculated in $x^k \notin \mathcal{F}$, in the sense that there exists a value $\tau^k \geq 0$ such that

$$\left\| V^k - \nabla \left\|\left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)'\right]'\right\|_q \right\|_q \leq M_2 \tau^k, \tag{6}$$

with $M_2 > 0$.

In Fig. 1 we report the scheme of the DF-EPGO Algorithm, where we use the following condition to check if an updating of the penalty parameter is timely:

$$\varepsilon^k\left(\|F^k\|_q + \left\|\left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)'\right]'\right\|_q\right) > \rho \|V^k\|_q, \tag{7}$$

with $\rho \in]0, 1[$.

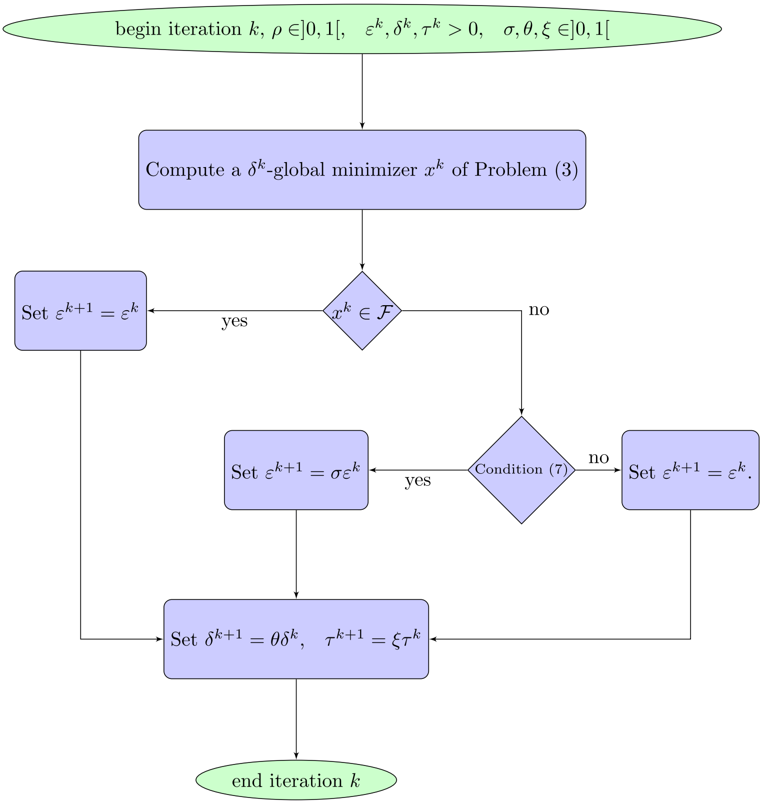


Fig. 1 Scheme of DF-EPGO algorithm (iteration k)

At iteration k of the algorithm, we first calculate x^k as a δ^k-global minimizer of Problem (3), defined as a point x^k such that

$$P_q\left(x^k, \varepsilon^k\right) \leq P_q\left(x, \varepsilon^k\right) + \delta^k, \quad \forall x \in \mathcal{D}.$$

In principle, any global unconstrained optimization method can be used to compute a δ^k-global minimizer x^k. Then, we check feasibility of x^k for Problem (1) and, if x^k is not feasible, we check if an updating of the penalty parameter is timely by means of condition (7). Finally, we reduce the value of δ^k in order to find a better approximation of the global solution of Problem (3) and the value τ^k to obtain a better approximation of the problem functions derivatives.

We point out that the DF-EPGO is a conceptual algorithm model; in practice, as usual in global optimization, it will be stopped when a δ^k-global minimizer x^k which is reputed a good approximation of a global minimizer $x^\star$ is found.

In order to state the convergence properties of the algorithm, we preliminary prove the following lemma.

Lemma 3.1 *Every accumulation point $\bar{x}$ of a sequence $\{x^k\}$ produced by DF-EPGO Algorithm belongs to the set $\mathcal{F}$.*

Proof We consider two different cases:

Case 1. An index $\bar{k}$ exists such that, for any $k \geq \bar{k}$, $\varepsilon^k = \bar{\varepsilon}$.

We assume, by contradiction, that there exists an accumulation point $\bar{x} \notin \mathcal{F}$. We have, for k sufficiently large, that the following inequality holds:

$$\begin{aligned}&\left\| \left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)' \right]' \right\|_q \\ &\quad \leq \frac{\rho}{\bar{\varepsilon}} \left\| \nabla \left\| \left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)' \right]' \right\|_q \right\|_q - \left\| \nabla f\left(x^k\right) \right\|_q + (M_1 + M_2)\tau^k.\end{aligned}$$

Let $\bar{x}$ be the limit of the subsequence $\{x^k\}_K$. Taking the limit for $k \to \infty$ on both sides, as $\tau^k \to 0$, we get

$$\left\| \left[g^+(\bar{x})', \tilde{s}^+(\bar{x})' \right]' \right\|_q \leq \frac{\rho}{\bar{\varepsilon}} \left\| \nabla \left\| \left[g^+(\bar{x})', \tilde{s}^+(\bar{x})' \right]' \right\|_q \right\|_q - \left\| \nabla f(\bar{x}) \right\|_q. \tag{8}$$

From this point, the proof follows the same steps of the proof of Lemma 1 in [13].

Case 2. $\{\varepsilon^k\} \to 0$.

The proof is a verbatim repetition of the proof of Case 2 of Lemma 1 in [13]. □

Then we can state the first convergence result.

Theorem 3.1 *Every accumulation point $\bar{x}$ of a sequence $\{x^k\}$ produced by DF-EPGO Algorithm is a global minimizer of Problem* (1).

Proof See proof of Theorem 2 in [13]. □

In next theorem, we prove that, if Assumption 2.2 holds, the penalty parameter ε is updated a finite number of times.

Theorem 3.2 *Let us assume that Assumption 2.2 holds. Let $\{x^k\}$ and $\{\varepsilon^k\}$ be the sequences produced by DF-EPGO Algorithm. Then an index $\bar{k}$ and a value $\bar{\varepsilon} > 0$ exist such that, for any $k \geq \bar{k}$, $\varepsilon^k = \bar{\varepsilon}$.*

Proof By contradiction, let us assume $\{\varepsilon_k\} \to 0$. Then, from the scheme of the DF-EPGO Algorithm, there exists a subsequence $\{x^k\}_K$ such that, for all $k \in K$, $x^k \notin \mathcal{F}$ and the test

$$\varepsilon^k \left(\|F^k\|_q + \left\| \left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)' \right]' \right\|_q \right) > \rho \|V^k\|_q \tag{9}$$

is satisfied. By rewriting (9) as

$$\begin{aligned}&\varepsilon^k \left(\left\| \nabla f\left(x^k\right) \right\|_q + M_1 \tau^k + \left\| \left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)' \right]' \right\|_q \right) \\ &\quad > \rho \left\| \nabla \left\| \left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)' \right]' \right\|_q \right\|_q - \rho M_2 \tau^k,\end{aligned}$$

and considering the fact that $\tau^k \to 0$, we obtain:

$$\lim_{k \in K, k \to \infty} \nabla \left\| \left[g^+\left(x^k\right)', \tilde{s}^+\left(x^k\right)' \right]' \right\|_q = 0. \tag{10}$$

From this point, the proof follows the same steps of the proof of Theorem 3 in [13]. □

4 Enriching the Global Search by a Derivative-Free Local Approach

In practice, we can enrich the global search by means of a derivative-free local minimization algorithm using as starting point the δ^k-global minimizer x^k obtained at iteration k by the DF-EPGO algorithm. In this section, we motivate analytically this enrichment by proving that, under reasonable assumptions, the sequence generated by the derivative-free local algorithm is attracted by a global solution of Problem (3).

To simplify notation, we will develop the analysis making reference to a locally Lipschitz function $\phi : \Re^n \to \Re$ and to the problem

$$\min_{x \in \Re^n} \phi(x). \tag{11}$$

Let us denote by $d^\circ \phi(x, d)$ the Clarke-directional derivative of ϕ at $x \in \Re^n$ in the direction $d \in \Re^n$:

$$d^\circ \phi(x, d) := \limsup_{y \to x t \downarrow 0^+} \frac{\phi(y + td) - \phi(y)}{t};$$

then we can recall the definition of stationary point for the non-smooth problem (11):

Definition 4.1 An x is a Clarke stationary point for ϕ, iff the following condition is satisfied:

$$d^\circ \phi(x, d) \geq 0, \quad \forall d \in \Re^n. \tag{12}$$

An alternative definition of Clarke stationary point for Problem (11) can by given by using the Clarke subdifferential $\partial \phi(x)$ of ϕ at x, given as:

$$\partial \phi(x) = \operatorname{conv}\{v \in \Re^n \mid \exists \{x^k\} : x^k \in \mathcal{T}, x^k \to x, \nabla \phi(x^k) \to v\},$$

$\mathcal{T}$ being the subset of $\Re^n$ where ϕ is differentiable. Even if we will not use the alternative definition, we will make use of the Clarke subdifferential in order to exploit some second-order properties of function ϕ. To this aim we need some additional notations and definitions.

Definition 4.2 The sequence $\{x^k\}$ converges to x in the direction d, denoted by $x^k \to_d x$, iff $x^k \neq x$ and the sequence $\{\frac{(x^k - x)}{\|x^k - x\|}\}$ converges to $\frac{d}{\|d\|}$.

Then, we can denote by $\partial_d \phi(x)$ the subset of $\partial \phi(x)$, defined as follows:

$$\partial_d \phi(x) := \{g : \exists x^k \text{ and } g^k \in \partial \phi(x^k) \text{ such that } x^k \to_d x \text{ and } g^k \to g\}.$$

Definition 4.3 Let $g \in \partial_d \phi(x)$. Then $\phi''_-(x, g, d)$ is defined as the infimum of all numbers

$$\liminf \frac{1}{(t^k)^2} \{\phi(x^k) - \phi(x) - g'(x^k - x)\} \tag{13}$$

taken over all triples of sequences $\{x^k\}$, $\{g^k\}$ and $\{t^k\}$, for which

- $t^k > 0$ for each k and $\{t^k\}$ converges to 0;
- $\{x^k\}$ converges to x and $(x^k - x)/t^k$ converges to d;
- $\{g^k\}$ converges to g with g^k in $\partial\phi(x^k)$ for each k.

Moreover, let us denote by $d_+\phi(x, d)$ the lower Dini-directional derivative at $x \in \Re^n$ in the direction $d \in \Re^n$:

$$d_+\phi(x, d) := \lim_{\bar{d}\to d} \inf_{t\downarrow 0} \frac{\phi(x + t\bar{d}) - \phi(x)}{t}.$$

We can now introduce the definition of second-order Dini stationary point for Problem (11).

Definition 4.4 An x is a second-order Dini stationary point for ϕ, iff the following conditions are satisfied:

(i) $d_+\phi(x, d) \geq 0$ for all unit vectors $d \in \Re^n$;
(ii) $\phi''_-(x, 0, d) \geq \beta > 0$ for all unit vectors $d \in \Re^n$ for which $d_+\phi(x, d) = 0$.

We state a preliminary result [14] that will be used to prove the main result:

Lemma 4.1 *Suppose that* $\phi(x) \geq \phi(\bar{x})$ *for all* $x \in B(\bar{x}, \delta)$. *Let* $0 \neq d \in \Re^n$, $t > 0$, $\alpha > 1$ *and* $0 < \eta < (\alpha\|d\|)^2$ *such that*

$$\phi(\bar{x} + td) - \phi(\bar{x}) \leq t\eta \quad \textit{and} \quad t\big(\|d\| + \eta^{1/2}\big) < \delta.$$

Then there exist $z \neq \bar{x}$ *in* $\Re^n$ *and* $g \in \partial\phi(z)$ *such that*

(i) $\|z - \bar{x} - td\| \leq t\eta^{1/2}\alpha^{-1} (< t\eta^{1/2})$;
(ii) $\phi(z) \leq \phi(\bar{x} + td)$; *and*
(iii) $\|g\| \leq \alpha\eta^{1/2}$.

Now we can prove the following result:

Proposition 4.1 *Let* $\bar{x} \in \Re^n$ *be a local minimum point of* ϕ *which satisfies the conditions for being a second-order Dini stationary point. Then there exists a neighborhood* $B(\bar{x}, \eta)$ *and a value* $\gamma > 0$ *such that*

$$\|x - \bar{x}\|^2 \leq \frac{1}{\gamma}\big(\phi(x) - \phi(\bar{x})\big), \quad \forall x \in B(\bar{x}, \eta).$$

Proof Suppose, by contradiction, that $\forall\eta^k > 0$ and $\forall\gamma^k > 0$ there exists a point $x^k \in B(\bar{x}, \eta^k)$ such that

$$\|x^k - \bar{x}\|^2 > \frac{1}{\gamma^k}\big(\phi\big(x^k\big) - \phi(\bar{x})\big). \tag{14}$$

Let $\frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} = d^k$; we consider the sequences $\{\gamma^k\}$, $\{\eta^k\}$, $\{d^k\}$ such that

$$\gamma^k \to 0, \quad \eta^k \to 0, \quad d^k \to d.$$

Now, by (14) and (i) of Definition 4.4, when k is large enough, we can write the following:

$$\|x^k - \bar{x}\| > \frac{1}{\gamma^k} \frac{(\phi(x^k) - \phi(\bar{x}))}{\|x^k - \bar{x}\|} \geq \frac{(\phi(x^k) - \phi(\bar{x}))}{\|x^k - \bar{x}\|} \geq d_+\phi(\bar{x}, d) \geq 0. \tag{15}$$

Taking the limit for $k \to \infty$, we have

$$d_+\phi(\bar{x}, d) = 0. \tag{16}$$

Now, we can denote

$$d^k = \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} \quad \text{and} \quad t^k = \|x^k - \bar{x}\|, \tag{17}$$

and, by (14), we can write

$$\phi\left(\bar{x} + t^k d^k\right) < \phi(\bar{x}) + \gamma^k\left(t^k\right)^2. \tag{18}$$

Using Lemma 4.1 with $\alpha = 2$ and $\eta = \gamma^k t^k$, there exists a sequence $\{z^k\}$ such that

(i) $\|z^k - \bar{x} - t^k d^k\| \leq \frac{t^k(\gamma^k t^k)^{1/2}}{2}$;
(ii) $\phi(z^k) \leq \phi(\bar{x} + t^k d^k)$; and
(iii) $\|g^k\| \leq 2(\gamma^k t^k)^{1/2}$.

By using (i) and (iii), we have that

$$\lim_{k\to\infty} z^k = \bar{x}, \tag{19}$$

$$\lim_{k\to\infty} \frac{z^k - \bar{x}}{t^k} = \lim_{k\to\infty} d^k = d, \tag{20}$$

$$\lim_{k\to\infty} \frac{\|z^k - \bar{x}\|}{t^k} = 1, \tag{21}$$

$$\lim_{k\to\infty} g^k = g = 0 \in \partial_d \phi(\bar{x}). \tag{22}$$

By using (ii) and (14), we have

$$\phi\left(z^k\right) \leq \phi\left(x^k\right) < \phi(\bar{x}) + \gamma^k\left(t^k\right)^2. \tag{23}$$

Then, we can write

$$\frac{\phi(z^k) - \phi(\bar{x})}{\|z^k - \bar{x}\|^2} \frac{\|z^k - \bar{x}\|^2}{(t^k)^2} \leq \gamma^k.$$

By using (21) and the fact that $\gamma^k \to 0$, we have

$$\lim_{k\to\infty} \frac{\phi(z^k) - \phi(\bar{x})}{\|z^k - \bar{x}\|^2} \leq 0. \tag{24}$$

Finally, taking into account (ii) of Definition 4.4, (13) and (16), we get the following contradiction:

$$0 \geq \lim_{k\to\infty} \frac{\phi(z^k) - \phi(\bar{x}) - g'(z^k - \bar{x})}{\|z^k - \bar{x}\|^2} \geq f''_-(\bar{x}, 0, d) \geq \beta > 0. \qquad \square$$

Now, we extend to the case of locally Lipschitz functions a result established in [15].

Theorem 4.1 *Let ϕ be a locally Lipschitz function and $\{x^k\}$ be a sequence of points generated by an iterative method*

$$x^{k+1} = x^k + \beta^k d^k$$

satisfying the following condition:

$$\phi\left(x^{k+1}\right) \leq \phi\left(x^k\right) - \nu\left(\beta^k\|d^k\|\right)^2,$$

for all k, where $\nu > 0$, and such that every accumulation point is a Clarke stationary point. Then, for every global minimum point $x^\star$ of $\phi(x)$, which is an isolated Clarke stationary point and a second-order Dini stationary point, there exists an open set $\mathcal{L}$ containing $x^\star$ such that, if $x^{\bar{k}} \in \mathcal{L}$ for some $\bar{k} \geq 0$, then we have:

- *$x^k \in \mathcal{L}$ for all $k \geq \bar{k}$*;
- *$\{x^k\} \to x^\star$*.

Proof Let $B(x^\star, \eta)$ be a neighborhood introduced in Proposition 4.1, where we have

$$\phi\left(x^{\bar{k}}\right) - \phi\left(x^\star\right) \geq \gamma\|x^{\bar{k}} - x^\star\|^2, \quad \forall x \in B\left(x^\star, \eta\right). \tag{25}$$

We consider now the following open set:

$$\mathcal{L} = \left\{x \in B\left(x^\star, \eta\right), \phi(x) < \phi\left(x^\star\right) + \frac{\eta^2\gamma\nu}{2(2\nu+\gamma)}\right\}. \tag{26}$$

Now we prove that, if $x^{\bar{k}} \in \mathcal{L}$ for some $\bar{k} \geq 0$, then $x^k \in \mathcal{L}$ for all $k \geq \bar{k}$ and $\{x^k\} \to x^\star$.

Recalling the hypothesis $\phi(x^{k+1}) \leq \phi(x^k) - \nu(\beta^k)^2\|d^k\|^2$, we have

$$\begin{aligned}\left(\beta^{\bar{k}}\right)^2\|d^{\bar{k}}\|^2 &\leq \frac{1}{\nu}\left(\phi\left(x^{\bar{k}}\right) - \phi\left(x^{\bar{k}+1}\right)\right) \\ &\leq \frac{1}{\nu}\left(\phi\left(x^{\bar{k}}\right) - \phi\left(x^\star\right)\right),\end{aligned} \tag{27}$$

where the second inequality follows from the hypothesis that $x^\star$ is a global minimum. By using (25) and (27), we have

$$\begin{aligned}\|x^{\bar{k}+1} - x^\star\|^2 &= \|x^{\bar{k}} - x^\star + \beta^{\bar{k}}d^{\bar{k}}\|^2 \\ &\leq 2\|x^{\bar{k}} - x^\star\|^2 + 2\left(\beta^{\bar{k}}\right)^2\|d^{\bar{k}}\|^2\end{aligned} \tag{28}$$

$$\leq \frac{4}{\gamma}\big(\phi\big(x^{\bar{k}}\big) - \phi\big(x^\star\big)\big) + \frac{2}{\nu}\big(\phi\big(x^{\bar{k}}\big) - \phi\big(x^\star\big)\big)$$
$$= \frac{2(2\nu + \gamma)}{\gamma\nu}\big(\phi\big(x^{\bar{k}}\big) - \phi\big(x^\star\big)\big).$$

Recalling that $x^{\bar{k}} \in \mathcal{L}$, we have

$$\phi\big(x^{\bar{k}}\big) - \phi\big(x^\star\big) < \frac{\eta^2\gamma\nu}{2(2\nu + \gamma)}.$$

Combining the above inequality with (28), we obtain

$$\|x^{\bar{k}+1} - x^\star\| < \eta. \tag{29}$$

Furthermore, by using again the hypothesis $\phi(x^{k+1}) \leq \phi(x^k) - \nu(\beta^k)^2\|d^k\|^2$, we have that $\phi(x^{k+1}) \leq \phi(x^k)$ for all k and hence

$$\phi\big(x^{\bar{k}+1}\big) \leq \phi\big(x^{\bar{k}}\big) < \phi\big(x^\star\big) + \frac{\eta^2\gamma\nu}{2(2\nu + \gamma)}. \tag{30}$$

Using the above two inequalities, it follows that $x^{\bar{k}+1} \in \mathcal{L}$ and similarly $x^k \in \mathcal{L}$ for all $k \geq \bar{k}$. In particular, using the fact that $\mathcal{L} \subset B(x^\star, \eta)$, then $x^k \in B(x^\star, \eta)$ for all $k \geq \bar{k}$. Then, the sequence $\{x^k\}$ is bounded and will have at least one limit point which must be a stationary point of Problem (11). Since within $B(x^\star, \eta)$ the only Clarke stationary point is $x^\star$, then $x^k \to x^\star$. □

As in the derivative-free local search used to enrich the global algorithm we minimize the exact penalty function P_q, which is a locally Lipschitz function, we can apply to it all the results given for ϕ in this section. Then, recalling Remark 2.1, we have that, under the assumptions of Theorem 4.1, there exists a neighborhood $\mathcal{L}$ of a global solution of Problem (3), where the sequence produced by the derivative-free local algorithm applied to P_q remains and is attracted by the global solution.

5 Numerical Results

In the numerical experimentation of the derivative-free exact penalty approach, we used:

- the set $\mathcal{A}$ of 20 small dimensional test problems reported in [9], where a global optimization algorithm for general constrained problems, which uses first- and second-order derivatives of the problem functions, is described;
- the set $\mathcal{B}$ of 97 problems from the GLOBALLib collection of COCONUT [16] with dimension $n \leq 10$.

These choices are motivated by the fact that derivative-free approaches cannot tackle large dimensional problems.

All test problems are of the kind

$$\min f(x) \quad \text{s.t.} \quad g(x) \le 0, \quad h(x) = 0, \quad l \le x \le u \tag{31}$$

with $h(x) : \Re^n \to \Re^t$, with $t \le n$. Of course, the DF-EPGO Algorithm can be easily adapted in order to take into account also the equality constraints $h(x) = 0$.

In the practical application of the proposed approach, we must choose:

- A value q for the vector norm. We chose $q = 2$.
- A method for approximating the derivatives. We chose the simple forward difference:

$$\frac{\partial P_q(x^k; \varepsilon)}{\partial x_i} \simeq \frac{P_q(x^k + \gamma e_i; \varepsilon) - P_q(x^k; \varepsilon)}{\gamma^k}, \tag{32}$$

  with $\gamma^{k+1} = \xi \gamma^k$, $\gamma^0 = 10^{-6}$ and $\xi = 0.9$. Of course, by this choice the conditions (4) and (6) are satisfied.
- A method for globally solving the problem with bounded feasible set. In principle, any method could be used. In our numerical experience we used:
  (1) the DIRECT method as improved in [17];
  (2) the DIRECT method as improved in [17], enriched by a derivative-free local search phase (DFN method) based on the same exact penalty function (2) used in the global search [18].
- Values for the data of the exact penalty approach. We set $\varepsilon^0 = 0.1$, $\sigma = 0.9$ and $\rho = 0.5$.

Our choice in case (1) is motivated by the fact that DIRECT is reputed to be one of the most efficient methods for problems with simple bounds. DIRECT belongs to the class of *derivative-free deterministic* methods, which perform the sampling of the objective function in points generated by the previous iterates of the method. The derivative-free algorithm used in case (2) seems to be very effective when first-order information is not available.

The practical computation of a δ^k-global minimizer x^k of Problem (3) has been performed in case (1) using as stopping rule the standard stopping rule of the DIRECT Algorithm [19]. Namely, the DIRECT algorithm stops when the diameter of the hyperrectangle containing the best found value of the objective function is less than a given threshold ζ^k. We set $\zeta^0 = 10^{-10}$ and $\zeta^{k+1} = \theta \zeta^k$ with $\theta = 0.1$. In case (2) the local search phase starts from the point obtained by the DIRECT algorithm as described before and stops as soon as the stepsize α in [18] is lower than 10^{-3}.

As concerns the checking of feasibility for the new point x^k, we consider x^k feasible if the constraints violation

$$cv = \max\{\|g^+(x)\|_\infty, \|h(x)\|_\infty\},$$

is lower than or equal to 10^{-3}, and infeasible otherwise.

The numerical experimentation was carried out on an Intel Core 2 Duo 3.16-GHz processor with 3.25-GB RAM. DF-EPGO and DFN algorithms were implemented in Fortran 90 (double precision).

Table 1 Performance of the DF-EPGO algorithm on the test set A

Problem	n	me	mi	CPU time	$f(x^\star)$	cv	$f^\star$
problem01	5	3	0	0.375	0.06256	2.35168E-07	0.02930
problem02a	5	0	10	3.328	−134.09839	2.16604E-04	−400.00000
problem02b	5	0	10	6.016	−705.13184	1.29510E-03	−600.00000
problem02c	5	0	10	2.109	−82.95282	2.16604E-04	−750.00000
problem02d	5	0	12	4.828	−399.76355	0.00000E+00	−400.00000
problem03a	6	4	1	2.000	−0.38612	3.14639E-06	−0.38880
problem03b	2	0	1	0.047	−0.38881	0.00000E+00	−0.38880
problem04	2	0	1	0.047	−6.66665	0.00000E+00	−6.66660
problem05	3	3	0	0.094	201.15915	2.48965E-04	201.16000
problem06	2	0	1	0.047	376.29266	0.00000E+00	376.29190
problem07	2	0	4	0.125	−2.80585	0.00000E+00	−2.82840
problem08	2	0	2	0.047	−118.70476	0.00000E+00	−118.70000
problem09	6	3	4	3.344	−13.40125	2.16372E-05	−13.40200
problem10	2	0	2	0.109	0.74178	0.00000E+00	0.74170
problem11	2	0	1	0.031	−0.50000	0.00000E+00	−0.50000
problem12	2	1	0	0.031	−16.73887	9.66333E-06	−16.73900
problem13	3	2	0	0.063	195.94547	3.28121E-04	189.35000
problem14	4	1	2	0.266	−4.35233	3.55751E-06	−4.51420
problem15	3	3	0	0.078	0.00000	4.94518E-05	0.00000
problem16	5	3	0	0.391	0.71809	1.11013E-04	0.70500

The results for the problem set $\mathcal{A}$ are collected in Tables 1 and 2, which refer respectively to case (1) and (2). In both tables we report the name of the problem, the number n of variables, the number *me* and *mi* of equality and inequality constraints (excluding simple bounds), the *CPU time* required to attain the stop condition, the optimal function value $f(x^\star)$, the constraints violation *cv*, and the reference value $f^\star$ reported in [9].

In Table 1, we note that the values $f(x^\star)$ and $f^\star$ are practically the same for all problems 3–12, 15 and 16, $f(x^\star)$ is lower than $f^\star$ for problem 2b, while $f^\star$ is lower than $f(x^\star)$ for problems 1, 2a, 2c, 2d, 13 and 14. These results can be considered satisfactory considering that the results reported in [9] are obtained using first- and second-order derivatives. Furthermore, we point out that the results reported in Table 1 are comparable with the ones reported in [13] where first-order derivatives of the problem functions are used. In particular, the values of $f(x^\star)$ and *cv* are practically the same, while there is a slight increase of the *CPU time*, as expected.

In Table 2, we note that the values $f(x^\star)$ are practically the same as the ones reported in Table 1, but we have better results in terms of constraints violation.

The results for the problem set $\mathcal{B}$ are collected in Tables 3–4. Again, in Tables 3–4 we report the name of the problem, the number n of variables, the number *me* and *mi* of equality and inequality constraints (excluding simple bounds), the *CPU time* required to attain the stop condition, the optimal function value $f(x^\star)$, the constraints violation *cv*. In order to synthesize the results, we make reference to Figs. 2 and 3. The

Table 2 Performance of the DF-EPGO+DFN algorithm on test set A

Problem	n	me	mi	CPU time	$f(x^\star)$	cv	$f^\star$
problem01	5	3	0	0.594	0.06256	8.53713E-08	0.02930
problem02a	5	0	10	3.641	−134.09631	4.23516E-22	−400.00000
problem02b	5	0	10	6.516	−705.13984	6.77626E-21	−600.00000
problem02c	5	0	10	2.266	−82.95170	0.00000E+00	−750.00000
problem02d	5	0	12	4.969	−399.81891	0.00000E+00	−400.00000
problem03a	6	4	1	2.313	−0.38612	1.87907E-08	−0.38880
problem03b	2	0	1	0.125	−0.38881	0.00000E+00	−0.38880
problem04	2	0	1	0.063	−6.66667	0.00000E+00	−6.66660
problem05	3	3	0	0.250	201.15933	2.43315E-07	201.16000
problem06	2	0	1	0.516	376.29195	0.00000E+00	376.29190
problem07	2	0	4	0.234	−2.80588	0.00000E+00	−2.82840
problem08	2	0	2	0.188	−118.70486	0.00000E+00	−118.70000
problem09	6	3	4	3.781	−13.40116	6.14165E-07	−13.40200
problem10	2	0	2	0.125	0.74178	0.00000E+00	0.74170
problem11	2	0	1	0.047	−0.50000	0.00000E+00	−0.50000
problem12	2	1	0	0.172	−16.73885	4.13864E-08	−16.73900
problem13	3	2	0	0.188	195.94551	3.90716E-05	189.35000
problem14	4	1	2	0.281	−4.35235	1.65255E-08	−4.51420
problem15	3	3	0	0.188	0.00000	4.94518E-05	0.00000
problem16	5	3	0	0.406	0.71809	1.68887E-09	0.70500

plot in Fig. 2 gives on the y-axis the number of obtained solutions whose constraints violation is smaller or equal than the value given on the x-axis. The plot in Fig. 3 gives on the y-axis the number of obtained solutions whose objective function value has a relative error, given by

$$\frac{|f(x^\star) - f^\star|}{\max\{1, |f^\star|\}},$$

where $f^\star$ is the best known value reported in [16], smaller or equal than 0.01 and whose constraints violation is smaller or equal than the value given on the x-axis. From both figures we note that the DF-EPGO algorithm guarantees quite good performances in terms of feasibility and optimality and that these performances are much improved by combining the DF-EPGO algorithm with a local search phase based on the DFN algorithm.

6 Conclusions

In this paper we have shown that the approach for constrained global optimization developed in [13], and based on the use of a non-differentiable exact penalty function, can be fruitfully modified in such a way that the derivatives of the problem functions are not required, thus making the approach derivative-free. Moreover, the approach

Table 3 Performance of the DF-EPGO algorithm on the test set B

Problem	n	me	mi	time	val funct	viol	$f^\star$
ex542	8	0	6	10.41	7478.98948	4.51760E+00	7512.22590
ex6212	4	2	0	0.68	0.28916	5.07053E-05	0.28920
chance	4	1	2	1.67	29.98343	1.01631E-04	29.89440
circle	3	0	10	0.80	4.57425	4.60491E-04	4.57420
dispatch	4	1	1	1.20	3155.95496	9.52624E-04	3155.28790
ex1411	3	0	4	0.43	0.00000	7.11417E-07	0.00000
ex1412	6	1	8	12.22	0.00000	7.93071E-01	0.00000
ex1413	3	0	4	1.17	0.00000	7.18566E-01	0.00000
ex1414	3	0	4	0.53	0.00000	2.38548E-08	0.00000
ex1415	6	4	2	6.54	0.00000	1.89759E-03	0.00000
ex1416	9	1	14	30.15	0.00000	2.36529E-04	0.00000
ex1417	10	1	16	76.31	9.99983	8.66320E-01	0.00000
ex1418	3	0	4	0.81	0.00000	2.13646E-02	0.00000
ex1419	2	0	2	0.16	0.00000	1.02138E-10	0.00000
ex1421	5	1	6	9.49	0.60995	8.32512E-07	0.00000
ex1422	4	1	4	2.39	0.47597	5.16869E-05	0.00000
ex1423	6	1	8	13.97	0.62500	5.48775E-03	0.00000
ex1424	5	1	6	12.60	0.61728	2.05562E-03	0.00000
ex1425	4	1	4	1.62	0.38095	2.46906E-04	0.00000
ex1426	5	1	6	14.01	0.62500	1.46327E-02	0.00000
ex1427	6	1	8	14.99	0.61728	2.98000E-06	0.00000
ex1428	4	1	4	3.68	0.27438	1.64625E-02	0.00000
ex1429	4	1	4	2.88	0.41155	2.54450E-07	0.00000
ex211	5	0	1	0.49	−16.99902	0.00000E+00	−17.00000
ex212	6	0	2	1.10	−24.46025	0.00000E+00	−213.00000
ex214	6	0	4	1.89	−9.76893	1.94638E-04	−11.00000
ex215	10	0	11	23.33	−266.23034	0.00000E+00	−268.01460
ex216	10	0	5	9.17	−38.99588	0.00000E+00	−39.00000
ex219	10	1	0	1.69	−0.24327	1.86710E-07	−0.37500
ex311	8	0	6	12.72	4738.66027	3.19336E-01	7049.20830
ex312	5	0	6	7.46	−30664.31710	9 0.00000E+00	−30665.54000
ex313	6	0	6	10.74	−9350.74943	0.00000E+00	−310.00000
ex314	3	0	3	0.52	−3.82507	0.00000E+00	−4.00000
ex411	1	0	0	0.00	−7.48729	0.00000E+00	−7.48730
ex412	1	0	0	0.00	−663.49989	0.00000E+00	−663.50010
ex413	1	0	0	0.00	−443.67169	0.00000E+00	−443.67170
ex414	1	0	0	0.00	0.00000	0.00000E+00	0.00000
ex415	2	0	0	0.00	0.35731	0.00000E+00	0.00000
ex416	1	0	0	0.00	7.00734	0.00000E+00	7.00000
ex417	1	0	0	0.01	−7.50000	0.00000E+00	−7.50000
ex418	2	1	0	0.30	−16.73887	9.65481E-06	−16.73890
ex419	2	0	2	0.31	−5.50801	3.85276E-05	−5.50800

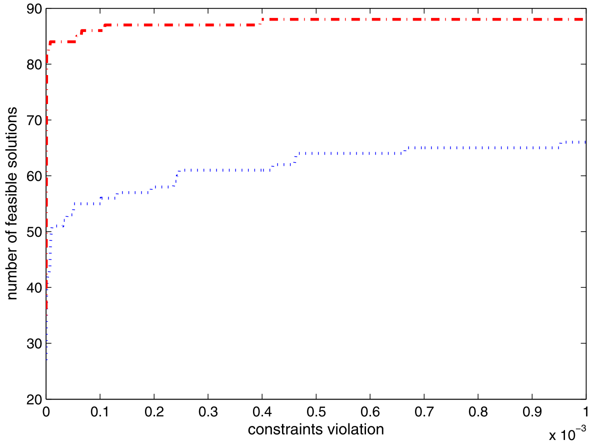
Table 3 (*Continued*)

Problem	n	me	mi	time	val funct	viol	$f^\star$
ex522case1	9	4	2	11.82	−1566.38978	3.49591E+02	−400.00000
ex522case2	9	4	2	13.93	−1606.91460	3.48753E+02	−600.00000
ex522case3	9	4	2	11.88	−1566.40248	3.49591E+02	−750.00000
ex524	7	1	5	15.95	−375.84342	8.47754E-06	−450.00000
ex611	8	6	0	8.07	−0.00666	4.95270E-01	−0.02020
ex612	4	3	0	1.67	−0.35714	8.27209E-01	−0.03250
ex614	6	4	0	3.89	−0.19229	8.68996E-01	−0.29450
ex6210	6	3	0	2.44	−3.05156	1.00000E-07	−3.05200
ex6211	3	1	0	0.10	0.00000	3.95557E-09	0.00000
ex6213	6	3	0	1.95	−0.21621	1.00000E-07	−0.21620
ex6214	4	2	0	0.76	−0.69527	1.00000E-07	−0.69540
ex625	9	3	0	5.65	−70.27806	4.62036E-04	−70.75210
ex626	3	1	0	0.08	0.00000	3.78794E-09	0.00000
ex627	9	3	0	3.54	−0.13872	4.19333E-04	−0.16080
ex628	3	1	0	0.12	−0.02701	1.04575E-05	−0.02700
ex629	4	2	0	0.43	−0.03407	1.00000E-07	−0.03410
ex721	7	0	14	60.39	1233.78232	5.79486E-06	1227.18960
ex722	6	4	1	4.06	−0.38747	7.72148E-06	−0.38880
ex723	8	0	6	15.75	6250.82305	5.26349E-02	7049.21810
ex724	8	0	4	5.20	4.15120	2.42912E-04	3.91800
ex725	5	0	6	9.05	10122.11468	1.31508E-04	10122.48280
ex726	3	0	1	0.49	−83.24917	0.00000E+00	−83.24990
ex727	4	0	2	0.95	−5.73910	0.00000E+00	−5.73990
ex728	8	0	4	5.69	−5.92094	0.00000E+00	−6.08200
ex729	10	0	6	7.98	1.58356	0.00000E+00	1.14360
ex731	4	0	7	1.43	1.69153	7.61315E-06	0.34170
ex732	4	0	7	2.26	1.20427	0.00000E+00	1.08990
ex733	5	2	6	2.86	18.59974	5.55021E-03	0.81750
ex736	1	2	0	0.07	0.00000	7.39103E-06	0.00000
ex811	2	0	0	0.00	−2.00171	0.00000E+00	−2.02180
ex812	1	0	0	0.00	−1.07086	0.00000E+00	−1.07090
ex813	2	0	0	0.01	6.30961	0.00000E+00	3.00000
ex814	2	0	0	0.01	0.00000	0.00000E+00	0.00000
ex815	2	0	0	0.01	0.00000	0.00000E+00	−1.03160
ex816	2	0	0	0.00	−0.49350	0.00000E+00	−10.08600
ex817	5	1	4	3.81	0.05927	2.30923E-07	0.02930
ex818	6	4	1	4.04	−0.38747	7.72148E-06	−0.38880
ex912	10	9	0	18.97	−14.78998	1.86894E-01	−16.00000
ex914	10	9	0	14.20	−48.28091	1.03696E+00	−37.00000
ex921	10	9	0	55.23	381.75430	1.08010E+01	17.00000
ex922	10	8	1	21.26	218.61468	4.26866E+00	99.99950

Table 3 (*Continued*)

Problem	n	me	mi	time	val funct	viol	$f^\star$
ex924	8	7	0	5.35	0.49028	8.85329E-02	0.50000
ex925	8	7	0	10.30	2.35042	2.39896E+00	5.00000
ex927	10	9	0	22.79	38.40080	6.06722E-01	17.00000
ex928	3	2	0	0.62	1.54572	1.90069E-06	1.50000
himmel11	9	3	0	7.10	−1882.80508	2.20301E+01	−30665.54000
house	8	4	4	11.99	−840.97921	1.35572E+01	−4500.00000
least	3	0	0	0.02	752888.00000	0 0.00000E+00	14085.13980
like	9	0	0	12.59	1131.41721	6.20713E-02	32341.50191
meanvar	7	2	0	2.29	7.30674	8.52345E-06	5.24340
mhw4d	5	3	0	3.05	64.36495	3.11330E-05	0.02930
process	8	6	0	20.80	−1928.29596	2.01812E+01	−1161.33660
rbrock	2	0	0	0.02	0.24211	0.00000E+00	0.00000
sample	4	0	2	1.74	700.74063	6.64664E-04	726.63670
wall	6	6	0	3.67	0.00000	1.00000E+00	−1.00000

Fig. 2 Performance comparison of the DF-EPGO algorithm (*dotted line*) and the DF-EPGO+DFN algorithm (*dash-dot line*) on the test set B: number of feasible solutions



has been enriched by a local search phase that uses the same penalty function used in the global search phase. This is motivated by the fact that the local searches are performed in such a way that the sequence produced by the algorithm, under suitable assumptions, is attracted by any global minimum point.

Avoiding the use of derivatives makes the approach suitable to deal with real-world problems where the analytical expressions of the problem functions are not available, as happens for instance in black-box simulation-based optimization.

The extensive numerical experimentation, which has been carried out on a well-known set of problems, shows that the new method is efficient and the results are comparable with those obtained by other methods which instead rely on the use of derivatives.

Table 4 Performance of the DF-EPGO+DFN algorithm on the test set B

Problem	n	me	mi	time	val funct	viol	$f^\star$
ex542	8	0	6	18.87	7526.54873	0.00000E+00	7512.22590
ex6212	4	2	0	1.02	0.29726	6.31016E-09	0.28920
chance	4	1	2	2.16	29.92691	6.64052E-10	29.89440
circle	3	0	10	1.01	4.57425	0.00000E+00	4.57420
dispatch	4	1	1	2.97	3155.46513	8.35561E-11	3155.28790
ex1411	3	0	4	0.48	0.00000	0.00000E+00	0.00000
ex1412	6	1	8	13.18	0.66512	2.62645E-11	0.00000
ex1413	3	0	4	4.67	0.00101	0.00000E+00	0.00000
ex1414	3	0	4	0.83	0.00000	0.00000E+00	0.00000
ex1415	6	4	2	8.16	0.00814	6.41406E-08	0.00000
ex1416	9	1	14	27.89	0.00019	3.45363E-12	0.00000
ex1417	10	1	16	85.00	2.65397	1.33627E-09	0.00000
ex1418	3	0	4	3.77	0.04142	0.00000E+00	0.00000
ex1419	2	0	2	0.22	0.00000	0.00000E+00	0.00000
ex1421	5	1	6	10.20	0.10675	1.36917E-10	0.00000
ex1422	4	1	4	3.19	0.15338	4.88037E-10	0.00000
ex1423	6	1	8	17.06	0.09784	7.89311E-11	0.00000
ex1424	5	1	6	14.56	0.09901	3.74105E-10	0.00000
ex1425	4	1	4	2.17	0.03701	7.61262E-10	0.00000
ex1426	5	1	6	15.83	0.35247	1.05200E-09	0.00000
ex1427	6	1	8	18.65	0.08666	1.63368E-11	0.00000
ex1428	4	1	4	5.02	0.01474	7.64843E-10	0.00000
ex1429	4	1	4	4.66	0.00341	1.03518E-08	0.00000
ex211	5	0	1	0.59	−17.00000	0.00000E+00	−17.00000
ex212	6	0	2	1.22	−24.46176	0.00000E+00	−213.00000
ex214	6	0	4	2.00	−10.73712	0.00000E+00	−11.00000
ex215	10	0	11	23.97	−267.38473	0.00000E+00	−268.01460
ex216	10	0	5	9.68	−39.00000	0.00000E+00	−39.00000
ex219	10	1	0	2.02	−0.29040	3.56220E-08	−0.37500
ex311	8	0	6	23.75	2115.03536	1.14983E+00	7049.20830
ex312	5	0	6	8.44	−30665.53920	4.94156E-07	−30665.54000
ex313	6	0	6	11.35	−249704.15463	1.14443E+02	−310.00000
ex314	3	0	3	0.66	−4.00000	0.00000E+00	−4.00000
ex411	1	0	0	0.00	−7.48729	0.00000E+00	−7.48730
ex412	1	0	0	0.00	−663.49989	0.00000E+00	−663.50010
ex413	1	0	0	0.00	−443.67169	0.00000E+00	−443.67170
ex414	1	0	0	0.00	0.00000	0.00000E+00	0.00000
ex415	2	0	0	0.00	0.35731	0.00000E+00	0.00000
ex416	1	0	0	0.00	7.00734	0.00000E+00	7.00000
ex417	1	0	0	0.01	−7.50000	0.00000E+00	−7.50000
ex418	2	1	0	0.60	−16.73887	1.36143E-09	−16.73890
ex419	2	0	2	0.53	−5.50801	0.00000E+00	−5.50800

Table 4 (*Continued*)

Problem	n	me	mi	time	val funct	viol	$f^\star$
ex522case1	9	4	2	13.89	−81.49865	6.51494E-05	−400.00000
ex522case2	9	4	2	19.68	−407.13450	5.66611E-05	−600.00000
ex522case3	9	4	2	13.65	−524.99480	1.08808E-04	−750.00000
ex524	7	1	5	21.52	−3900.02500	1.00000E+00	−450.00000
ex611	8	6	0	8.79	−0.01758	1.65232E-09	−0.02020
ex612	4	3	0	2.22	−0.02876	1.32790E-08	−0.03250
ex614	6	4	0	4.70	−0.29197	1.14183E-07	−0.29450
ex6210	6	3	0	2.86	−3.05156	1.20919E-09	−3.05200
ex6211	3	1	0	0.12	0.00002	1.09423E-09	0.00000
ex6213	6	3	0	2.58	−0.21621	2.83959E-08	−0.21620
ex6214	4	2	0	1.15	−0.69525	6.51810E-09	−0.69540
ex625	9	3	0	8.32	−70.54537	4.46143E-08	−70.75210
ex626	3	1	0	0.14	0.00000	1.11584E-10	0.00000
ex627	9	3	0	4.86	−0.13775	3.49582E-09	−0.16080
ex628	3	1	0	0.25	−0.02670	1.43204E-10	−0.02700
ex629	4	2	0	0.67	−0.03407	6.50114E-09	−0.03410
ex721	7	0	14	65.92	−832.58006	6.46226E-01	1227.18960
ex722	6	4	1	4.77	−0.38747	7.05484E-09	−0.38880
ex723	8	0	6	24.34	2100.00000	4.76207E-01	7049.21810
ex724	8	0	4	25.23	3.92148	0.00000E+00	3.91800
ex725	5	0	6	9.59	8570.47800	6.73740E-01	10122.48280
ex726	3	0	1	1.72	−83.24973	0.00000E+00	−83.24990
ex727	4	0	2	2.86	−5.73977	0.00000E+00	−5.73990
ex728	8	0	4	28.83	−6.08108	0.00000E+00	−6.08200
ex729	10	0	6	9.40	1.55547	0.00000E+00	1.14360
ex731	4	0	7	2.28	1.65056	0.00000E+00	0.34170
ex732	4	0	7	2.98	1.08986	0.00000E+00	1.08990
ex733	5	2	6	3.84	1.45836	7.56554E-08	0.81750
ex736	1	2	0	0.07	0.00000	7.39108E-06	0.00000
ex811	2	0	0	0.00	−2.00171	0.00000E+00	−2.02180
ex812	1	0	0	0.00	−1.07086	0.00000E+00	−1.07090
ex813	2	0	0	0.01	6.30961	0.00000E+00	3.00000
ex814	2	0	0	0.01	0.00000	0.00000E+00	0.00000
ex815	2	0	0	0.01	0.00000	0.00000E+00	−1.03160
ex816	2	0	0	0.00	−0.49350	0.00000E+00	−10.08600
ex817	5	1	4	4.95	0.06485	5.87662E-09	0.02930
ex818	6	4	1	4.64	−0.38747	7.05484E-09	−0.38880
ex912	10	9	0	19.41	−16.00000	6.70613E-09	−16.00000
ex914	10	9	0	28.27	−35.04563	4.46549E-06	−37.00000
ex921	10	9	0	56.11	17.00000	8.90821E-07	17.00000
ex922	10	8	1	24.28	101.38838	9.96012E-07	99.99950
ex924	8	7	0	14.59	0.55671	9.22622E-08	0.50000

Table 4 (*Continued*)

Problem	n	me	mi	time	val funct	viol	$f^\star$
ex925	8	7	0	11.31	5.00009	3.43983E-09	5.00000
ex927	10	9	0	24.51	20.64595	5.57030E-09	17.00000
ex928	3	2	0	1.01	1.50406	2.10443E-07	1.50000
himmel11	9	3	0	8.66	−30665.51717	1.92846E-09	−30665.54000
house	8	4	4	26.11	−3765.03212	5.05016E+00	−4500.00000
least	3	0	0	0.35	15067.13960	0.00000E+00	14085.13980
like	9	0	0	27.97	32341.50191	3.83495E-11	32341.50191
meanvar	7	2	0	8.24	5.47804	2.18619E-08	5.24340
mhw4d	5	3	0	2.78	0.03909	1.54289E-09	0.02930
process	8	6	0	20.80	−1928.29596	2.01812E+01	−1161.33660
rbrock	2	0	0	0.02	0.24211	0.00000E+00	0.00000
sample	4	0	2	2.80	400.00000	3.49000E-02	726.63670
wall	6	6	0	18.06	−0.99981	3.96594E-04	−1.00000

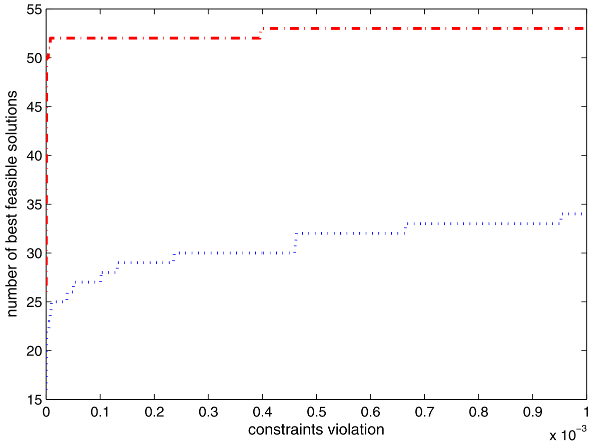


Fig. 3 Performance comparison of the DF-EPGO algorithm (*dotted line*) and the DF-EPGO+DFN Algorithm (*dash-dot line*) on the test set B: number of best feasible solutions

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