J Glob Optim
DOI 10.1007/s10898-010-9582-0

An approach to constrained global optimization based on exact penalty functions

G. Di Pillo · S. Lucidi · F. Rinaldi

Received: 22 June 2010 / Accepted: 29 June 2010

Abstract In the field of global optimization many efforts have been devoted to solve unconstrained global optimization problems. The aim of this paper is to show that unconstrained global optimization methods can be used also for solving constrained optimization problems, by resorting to an exact penalty approach. In particular, we make use of a nondifferentiable exact penalty function $P_q(x; \varepsilon)$. We show that, under weak assumptions, there exists a threshold value $\bar{\varepsilon} > 0$ of the penalty parameter ε such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, any global minimizer of P_q is a global solution of the related constrained problem and conversely. On these bases, we describe an algorithm that, by combining an unconstrained global minimization technique for minimizing P_q for given values of the penalty parameter ε and an automatic updating of ε that occurs only a finite number of times, produces a sequence $\{x^k\}$ such that any limit point of the sequence is a global solution of the related constrained problem. In the algorithm any efficient unconstrained global minimization technique can be used. In particular, we adopt an improved version of the DIRECT algorithm. Some numerical experimentation confirms the effectiveness of the approach.

Keywords Nonlinear programming · Global optimization · Exact penalty functions · DIRECT algorithm

This work was supported by MIUR PRIN National Research Program 20079PLLN7 "Nonlinear Optimization, Variational Inequalities and Equilibrium Problems".

G. Di Pillo (✉) · S. Lucidi · F. Rinaldi
Dipartimento di Informatica e Sistemistica, Università di Roma "La Sapienza", via Ariosto 25, 00185 Rome, Italy
e-mail: dipillo@dis.uniroma1.it

S. Lucidi
e-mail: lucidi@dis.uniroma1.it

F. Rinaldi
e-mail: rinaldi@dis.uniroma1.it

Published online: 15 July 2010

1 Introduction

Many real-world problems in Engineering, Economics and Applied Sciences can be modelled as nonlinear global minimization problems (see [5,6,8,18,20]). This motivates a growing attention in the search for *global* rather than *local* solutions of nonlinear optimization problems. In the last decades, most research efforts have been devoted to globally solving either unconstrained problems or problems with simple constraints. For these cases, many algorithmic approaches, either deterministic or probabilistic, have been developed (see [5,9–13,20,21] and references therein). Recently, the more difficult case of global optimization problems with general constraints has been also investigated, and various approaches are described (see [5,19,21] and references therein); in this context, a particular emphasis is given to the use of some Augmented Lagrangian function to deal with the general constraints (see [1,17,22]).

In this paper we are concerned with determining a *global solution* of the following general nonlinear programming problem:

$$\begin{aligned} \min\ & f(x) \\ & g(x) \le 0 \\ & s(x) \le 0 \end{aligned} \tag{1}$$

where $f : R^n \to R$, $g : R^n \to R^p$, $s : R^n \to R^m$, and we assume that f, g and s are continuously differentiable functions. We assume that $g(x)$ is a set of general nonlinear constraints without any particular structure, and that $s(x)$ denotes a set of simple convex constraints, for instance a set of bounds,

$$s(x) = \begin{bmatrix} l - x \\ x - u \end{bmatrix} \le 0,$$

with $l,\ u \in R^n$, or a spherical constraint,

$$s(x) = \|x\|^2 - r^2 \le 0,$$

with $r \in R$.
We assume that no global information (convexity, Lipschitz constants, ...) on the problem is available.

We denote by $\mathcal{F}$ the feasible set of problem (1):

$$\mathcal{F} = \{x \in R^n :\ g(x) \le 0,\ s(x) \le 0\}.$$

In order to deal with problem (1), we adopt an exact penalty function approach, based on some theoretical results developed in [2–4]. In particular, we consider the following class of non-differentiable penalty functions:

$$P_q(x; \varepsilon) = f(x) + \frac{1}{\varepsilon}\|[\max\{0, g(x)\}', \max\{0, \tilde{s}(x)\}']'\|_q$$

where $1 < q < \infty$, and

$$\tilde{s}_j(x) = \frac{s_j(x)}{\alpha_j - s_j(x)},$$

with $\alpha_j > 0$, $j = 1, \ldots, m$.

We point out that in general the penalty function $P_q(x; \varepsilon)$ is non-differentiable, however it turns out to be continuously differentiable at infeasible points (see, for instance, Proposition 4 of [2]).

We denote by $\mathcal{D}$ the set

$$\mathcal{D} = \{x \in R^n : s_j(x) \leq \alpha_j, \quad j = 1, \ldots, m\}.$$

Then, we consider the problem

$$\min_{x \in \mathcal{D}} P_q(x; \varepsilon). \tag{2}$$

We show that, under weak assumptions, there exists a threshold value $\bar{\varepsilon} > 0$ of the penalty parameter ε such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, any global solution of Problem (1) is a global solution of Problem (2) and conversely. Therefore, while the Augmented Lagrangian approach, which is sequential in nature, requires in principle an infinite number of global minimizations, the number of global minimizations required by the exact penalty approach turns out to be finite. On these bases, we describe an algorithm that combines a global minimization technique for solving Problem (2) for given values of the penalty parameter ε and an automatic updating of ε that occurs only a finite number of times. The resulting algorithm produces a sequence $\{x^k\}$ such that any limit point of the sequence is a global solution of Problem (1). In the algorithm any efficient global minimization technique for solving Problem (2), which is a problem with simple constraints, could be employed. In particular, in the more frequent case that simple constraints are given by bound constraints, we adopt an efficient implementation of the DIRECT algorithm (see [16]), which is well suited for our purpose. Some numerical experimentation confirms the effectiveness of the approach.

We wonder that an exact penalty approach does not seem to have been widely experimented before, so that our work appears to be the first practical attempt. A possible explanation may be the fact that, in exact penalty approaches, determining a small enough penalty parameter is considered a hard task (see [18]). In this paper, the value of the penalty parameter is determined by an automatic updating rule, and we prove that the updating occurs only a finite number of times, thus making viable the approach based on the exact penalty function.

The paper is organized as follows. In Sect. 2, we introduce the main exactness properties of the penalty function $P_q(x; \varepsilon)$. In Sect. 3, we describe the Exact Penalty Global Optimization (EPGO) algorithm and we state its convergence results. In Sect. 4, we report some numerical results that are promising of further improvements. Finally, we draw some conclusions in Sect. 5.

In the paper, given a vector $v \in R^n$ we denote by v' its transpose, by $\|v\|_q$ its q-norm, and by $v^+ = \max\{0, v\}$ the n-vector $(\max\{0, v_1\}, \ldots, \max\{0, v_n\})$.

2 Non-differentiable exact penalty function for problem (1)

In this section we introduce the needed assumptions and the main exactness properties of the penalty function.

Assumption 1 *$\mathcal{D}$ is a compact set.*

This assumption is common in global optimization and is always satisfied in real-world problems. In the following, we assume that Assumption 1 holds everywhere.

Since, by definition, $\mathcal{F} \subset \mathcal{D}$, by Assumption 1 also $\mathcal{F}$ is a compact set, and Problem (1) admits a global solution contained in the interior of $\mathcal{D}$.

In association with Problem (1), we consider the following class of non-differentiable penalty functions:

$$P_q(x; \varepsilon) = f(x) + \frac{1}{\varepsilon} \|[g^+(x)', \tilde{s}^+(x)']'\|_q \tag{3}$$

that is

$$P_q(x;\varepsilon) = f(x) + \frac{1}{\varepsilon}\left[\sum_{i=1}^{p}(g_i^+(x))^q + \sum_{j=1}^{m}(\tilde{s}_j^+(x))^q\right]^{1/q},$$

with $1 < q < \infty$, and the problem

$$\min_{x\in\mathcal{D}} \; P_q(x;\varepsilon). \tag{4}$$

Remark 1 We note that, by construction,

$$\lim_{x\to\partial\mathcal{D}} P_q(x;\varepsilon) = +\infty,$$

where $\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$. Therefore, Problem (4) has global solutions contained only in the interior of $\mathcal{D}$, so that determining a global solution of Problem (4) is essentially an unconstrained problem.

We remind that the *Mangasarian-Fromovitz constraint qualification* (MFCQ) holds at a point $x \in \mathcal{F}$, if there exists a $z \in R^n$ such that

$$\nabla g_i(x)'z < 0, \quad i \in I_0(x)$$

$$\nabla s_j(x)'z < 0, \quad j \in J_0(x),$$

where $I_0(x) = \{i \in \{1,\dots,p\} : \; g_i(x) = 0\}$ and $J_0(x) = \{j \in \{1,\dots,m\} : \; s_j(x) = 0\}$. By using the theorem of the alternative, the MFCQ can be restated as follows. The MFCQ holds at a point $x \in \mathcal{F}$ if there exists no $u_i, \; i \in I_0(x)$, and $v_j, \; j \in J_0(x)$, such that

$$\sum_{i\in I_0(x)} u_i\nabla g_i(x) + \sum_{j\in J_0(x)} v_j\nabla s_j(x) = 0,$$

$$u_i \geq 0, \quad i \in I_0(x),$$

$$v_j \geq 0, \quad j \in J_0(x),$$

$$(u_i, \; i \in I_0(x), \; v_j, \; j \in J_0(x)) \neq 0.$$

Then we can state the following assumption:

Assumption 2 *The MFCQ holds at every global solution x^* of Problem* (1).

Then we can state the main exactness property of function P_q of concern here.

Theorem 1 *Under Assumption 2, there exists a threshold value $\bar{\varepsilon}$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, every global solution of Problem (1) is a global solution of Problem (4), and conversely.*

Proof The proof of this result can be deduced from the proof of Theorem 5 in [4]. □

Remark 2 Theorem 1 implies that for a sufficiently small value ε, Problem (4) has only unconstrained global minimizers.

On the basis of Remark 2, the global solution of Problem (4) can be obtained by using any method for unconstrained global optimization. We point out that Assumption 2 is only needed in global minimizers, therefore it is a very weak assumption.

3 Exact penalty global optimization algorithm

In this section, we introduce the EPGO (Exact Penalty Global Optimization) algorithm model for finding a global solution of Problem (1) using the exact penalty function (3), and we analyze its convergence properties.

EPGO Algorithm Model

Data. $k = 0, \varepsilon^0 > 0, \sigma \in (0, 1), \rho \in (0, 1), \delta^0 > 0, \theta \in (0, 1)$.

Step 1. Compute $x^k \in \mathcal{D}$ such that

$$P_q(x^k, \varepsilon^k) \leq P_q(x, \varepsilon^k) + \delta^k, \quad \forall x \in \mathcal{D}.$$

Step 2. If $\|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q = 0$, set $\varepsilon^{k+1} = \varepsilon^k$ and go to Step 4.

Step 3. If $\varepsilon^k(\|\nabla f(x^k)\|_q + \|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q) > \rho \|\nabla \|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q\|_q$,

set $\varepsilon^{k+1} = \sigma\varepsilon^k, \quad \delta^{k+1} = \delta^k, \quad k = k + 1$, and go to Step 1.

Else set $\varepsilon^{k+1} = \varepsilon^k$.

Step 4. Set $\delta^{k+1} = \theta\delta^k, \quad k = k + 1$, and go to Step 1.

In the algorithm, at Step 1 x^k is a δ^k-global minimizer of Problem (2). The δ^k-global minimizer x^k can be obtained by using any global unconstrained optimization method. At Step 2 we check feasibility of x^k and, if x^k is feasible, we reduce the value of δ^k in order to find a better approximation of the global solution of Problem (2). Step 3 is intended to determine if an updating of the penalty parameter is timely. We point out that at Step 3 the constraint violation gradient can be evaluated because, as already said, at infeasible points the penalty function is continuously differentiable. As described, EPGO is a conceptual algorithm model; in practice, as usual in global optimization, it will be stopped when a sufficiently small value of δ^k is reached and the δ^k-global minimizer x^k can be considered a good approximation of a global minimizer $x^\star$.

In order to state the convergence properties of the algorithm, we preliminary prove the following Lemma.

Lemma 1 *Every accumulation point $\bar{x}$ of a sequence $\{x^k\}$ produced by EPGO Algorithm belongs to the set $\mathcal{F}$.*

Proof We consider two different cases:

Case (1) an index $\bar{k}$ exists such that for any $k \geq \bar{k}$, $\varepsilon^k = \bar{\varepsilon}$: By contradiction, let us assume that there exists an accumulation point $\bar{x} \notin \mathcal{F}$. For k sufficiently large, we have

$$\|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q \leq \frac{\rho}{\bar{\varepsilon}} \|\nabla \|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q\|_q - \|\nabla f(x^k)\|_q.$$

Let $\bar{x}$ be the limit of the subsequence $\{x^k\}_K$. Taking the limit for $k \to \infty$ on both sides, we get

$$\|[g^+(\bar{x})', \tilde{s}^+(\bar{x})']'\|_q \leq \frac{\rho}{\bar{\varepsilon}} \|\nabla \|[g^+(\bar{x})', \tilde{s}^+(\bar{x})']'\|_q\|_q - \|\nabla f(\bar{x})\|_q. \tag{5}$$

Since by Step 4 $\delta^k \to 0$, we have by Step 1 that $\bar{x}$ is a global minimizer of $P_q(x, \bar{\varepsilon})$ on $\mathcal{D}$ and, from Remark 1, satisfies the following

$$\frac{1}{\bar{\varepsilon}} \|\nabla \|[g^+(\bar{x})', \tilde{s}^+(\bar{x})']'\|_q\|_q = \|\nabla f(\bar{x})\|_q$$

Hence, from (5) we get

$$\|[g^+(\bar{x})', \tilde{s}^+(\bar{x})']'\|_q + \frac{1-\rho}{\bar{\varepsilon}}\|\nabla\|[g^+(\bar{x})', \tilde{s}^+(\bar{x})']'\|_q\|_q \leq 0,$$

and

$$\|[g^+(\bar{x})', \tilde{s}^+(\bar{x})']'\|_q = 0,$$

but this contradicts the fact that $\bar{x} \notin \mathcal{F}$.

Case (2) $\{\varepsilon^k\} \to 0$:
From the definition of x^k, for any $z \in \mathcal{F}$ we have:

$$f(x^k) + \frac{1}{\varepsilon^k}\|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q \leq f(z) + \delta^k.$$

Taking the limit for $k \to \infty$ on both sides , we get

$$\lim_{k\to\infty} \|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q \leq \lim_{k\to\infty} \varepsilon_k[f(z) - f(x^k) + \delta^k],$$

from which we obtain

$$\lim_{k\to\infty} \|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q = 0,$$

and every accumulation point belongs to $\mathcal{F}$.

□

Then we can state the first convergence result.

Theorem 2 *Every accumulation point $\bar{x}$ of a sequence $\{x^k\}$ produced by EPGO Algorithm is a global minimizer of Problem* (1).

Proof From the definition of x^k, for any z global minimizer of Problem (1), we have:

$$f(x^k) \leq f(x^k) + \frac{1}{\varepsilon^k}\|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q \leq f(z) + \delta^k.$$

Let $\bar{x}$ be the limit of the subsequence $\{x^k\}_K$. Taking the limit for $k \to \infty$ on both sides, we get

$$f(\bar{x}) \leq f(z).$$

From Lemma 1, we have that $\bar{x}$ is feasible, then $\bar{x}$ is a global minimizer of Problem (1). □

In the next theorem, we state the convergence result of main interest, that is that if Assumption 2 holds, the penalty parameter ε is updated a finite number of times.

Theorem 3 *Let us assume Assumption 2 holds. Let $\{x^k\}$ and $\{\varepsilon^k\}$ be the sequences produced by EPGO Algorithm. Then an index $\bar{k}$ and a value $\bar{\varepsilon} > 0$ exist such that for any $k \geq \bar{k}$, $\varepsilon^k = \bar{\varepsilon}$.*

Proof By contradiction, let us assume $\{\varepsilon_k\} \to 0$. Then, from Step 2 and 3 of EPGO Algorithm, there exists a subsequence $\{x^k\}_K$ such that for all $k \in K$, $x^k \notin \mathcal{F}$ and the test

$$\varepsilon^k(\|\nabla f(x^k)\|_q + \|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q) > \rho\|\nabla\|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q\|_q$$

is satisfied. Then, we obtain:

$$\lim_{k\in K,\, k\to\infty} \nabla\|[g^+(x^k)', \tilde{s}^+(x^k)']'\|_q = 0. \tag{6}$$

By construction, we have from (6):

$$\lim_{k\in K,\ k\to\infty} \sum_{i=1}^{p} u_i^k \nabla g_i(x^k) + \sum_{j=1}^{m} v_j^k \nabla \tilde{s}_j(x^k) = 0, \tag{7}$$

with

$$u_i^k = \frac{[g_i(x^k)^+]^{q-1}}{[\|g_i(x^k)^+, \tilde{s}_j(x^k)\|_q]^{q-1}} \quad v_j^k = \frac{[\tilde{s}_j(x^k)^+]^{q-1}}{[\|g_i(x^k)^+, \tilde{s}_j(x^k)^+\|_q]^{q-1}}. \tag{8}$$

By (8), we can write

$$\left\| \begin{matrix} u^k \\ v^k \end{matrix} \right\|_q = 1. \tag{9}$$

By (9), there exists a further subset $\tilde{K} \subset K$, such that

$$\{x^k\}_{\tilde{K}} \to \bar{x}, \quad \{u^k\}_{\tilde{K}} \to \bar{u} \geq 0, \quad \{v^k\}_{\tilde{K}} \to \bar{v} \geq 0. \tag{10}$$

By Lemma 1, we have $\bar{x} \in \mathcal{F}$. By continuity, we have that, for k sufficiently large,

$$\{i : g_i(x^k) < 0\} \supseteq \{i : g_i(\bar{x}) < 0\},\ \{j : \tilde{s}_j(x^k) < 0\} \supseteq \{j : \tilde{s}_j(\bar{x}) < 0\},$$

from which, we have that

$$\bar{u}_i = 0, \quad \forall\, i \in \{i : g_i(\bar{x}) < 0\} \tag{11}$$

$$\bar{v}_j = 0, \quad \forall\, j \in \{j : \tilde{s}_j(\bar{x}) < 0\}. \tag{12}$$

Then, from (7), we have

$$\lim_{k\in \tilde{K},\ k\to\infty} \sum_{i=1}^{p} u_i^k \nabla g_i(x^k) + \sum_{j=1}^{m} v_j^k \nabla \tilde{s}_j(x^k) = \sum_{i\in I_0(\bar{x})} \bar{u}_i \nabla g_i(\bar{x}) + \sum_{j\in J_0(\bar{x})} \bar{v}_j \nabla \tilde{s}_j(\bar{x}) = 0,$$

where, by (9), (10), (11) and (12), we have

$$(\bar{u}_i,\ i \in I_0(\bar{x}),\ \bar{v}_j,\ j \in J_0(\bar{x})) \neq 0,$$

with

$$\bar{u}_i \geq 0, \quad i \in I_0(\bar{x}), \quad \bar{v}_j \geq 0, \quad j \in J_0(\bar{x}).$$

Noting that

$$\nabla \tilde{s}(\bar{x}) = c \nabla s(\bar{x})$$

with $c > 0$, we can write

$$\sum_{i\in I_0(\bar{x})} \bar{u}_i \nabla g_i(\bar{x}) + \sum_{j\in J_0(\bar{x})} \bar{v}_j\, c\, \nabla s_j(\bar{x}) = 0, \tag{13}$$

Then, by Theorem 2, we have that $\bar{x}$ is a global minimizer of Problem (1), and by (13) we get a contradiction with Assumption 2. □

4 Numerical results

In order to evaluate the performance of the exact penalty approach we used a set of 20 small dimensional test problems reported in the recent paper [1]. All test problems are of the kind

$$\begin{aligned} &\min\ f(x)\\ &g(x) \le 0\\ &h(x) = 0\\ &l \le x \le u \end{aligned} \tag{14}$$

with $h(x) : R^n \to R^t$. Of course, the EPGO Algorithm can be easily adapted in order to take into account also the equality constraints $h(x) = 0$. In practical applications of the exact penalty approach, one must choose a method for globally solving the problem with simple constraints at Step 1 of EPGO Algorithm. In principle, any method could be used. In our numerical experience we used the DIRECT method [16], tailored for problems with simple bounds. Our use of DIRECT is motivated by the following reasons:

– DIRECT is reputed to be one of the most efficient methods for problems with simple bounds;
– all test problems are of the kind of problem (1), with the constraint $s(x) \le 0$ given by simple bounds.

DIRECT belongs to the class of *deterministic* methods, which perform the sampling of the objective function in points generated by the previous iterates of the method.

The numerical experimentation was carried out on an Intel Core 2 Duo 3.16 GHz processor with 3.25 GB RAM. EPGO Algorithm was implemented in Fortran 90 (double precision). The results are collected in Table 1. In Table 1 we report the name of the problems, the number **n** of variables, the number **me** and **mi** of equality and inequality constraints (excluding simple bounds), the number **nf** of penalty function evaluations, the CPU time, the optimal function value $\mathbf{f}(\mathbf{x}^\star)$, the constraints violation **cv** measured as

$$cv = \max\left\{ \|g^+(x)\|_\infty,\ \|h(x)\|_\infty \right\},$$

and the reference value $\mathbf{f}^\star$ reported in [1]. In the algorithm we set $\sigma = 0.9$, $\rho = 0.5$, $\varepsilon^0 = 0.1$ and $q = 2$. The practical computation of the point x^k at Step 1 has been performed using as stopping rule the standard stopping rule of the DIRECT Algorithm [7]. Namely, the DIRECT algorithm terminates the optimization when the diameter of the hyperrectangle containing the best found value of the objective function is less than a given threshold d^k. We set $d^0 = 10^{-10}$ and $d^{k+1} = \theta d^k$ with $\theta = 0.1$. As concerns Step 2, the penalty parameter ε is updated when the infeasibilty, measured by **cv**, is larger than 10^{-3}.

The results reported in Table 1 , even if obtained by a preliminary code, not yet optimized, are comparable with the ones reported in [1].

5 Conclusions

In this paper we have developed a global optimization algorithm which is based on the use of a non-differentiable exact penalty function, in order to reduce a general nonlinear programming problem to a problem with simple constraints. The main feature of the algorithm is that it incorporates an automatic updating rule for the penalty parameter and that, under weak assumptions, the penalty parameter is updated only a finite number of times. Preliminary

Table 1 Performace of the EPGO algorithm

PROBLEM	n	me	mi	nf	CPU time	$f(x^*)$	cv	f^*
problem01	5	3	0	39,575	0.328	0.0625	2.3516E−007	0.0293
problem02a	5	0	10	115,107	2.078	−134.1127	8.4319E−004	−400.0000
problem02b	5	0	10	120,057	3.828	−768.4569	5.2957E−004	−600.0000
problem02c	5	0	10	102,015	0.953	−82.9774	8.4319E−004	−750.0000
problem02d	5	0	12	229,773	2.328	−385.1704	0.0000E+000	−400.0000
problem03a	6	4	1	48,647	1.234	−0.3861	1.0215E−006	−0.3888
problem03b	2	0	1	3,449	0.031	−0.3888	0.0000E+000	−0.3888
problem04	2	0	1	3,547	0.031	−6.6666	0.0000E+000	−6.6666
problem05	3	3	0	14,087	0.078	201.1593	1.6622E−004	201.1600
problem06	2	0	1	1,523	0.000	0.4701	2.0495E−005	376.2919
problem07	2	0	4	13,187	0.125	−2.8058	0.0000E+000	−2.8284
problem08	2	0	2	7,621	0.046	−118.7044	0.0000E+000	−118.7000
problem09	6	3	4	68,177	2.171	−13.4026	1.3544E−004	−13.4020
problem10	2	0	2	6,739	0.078	0.7420	0.0000E+000	0.7417
problem11	2	0	1	3,579	0.031	−0.5000	0.0000E+000	−0.5000
problem12	2	1	0	3,499	0.015	−16.7389	5.3584E−006	−16.7390
problem13	3	2	0	8,085	0.078	195.9553	9.2057E−004	189.350
problem14	4	1	2	19,685	0.250	−4.3460	9.2160E−005	−4.5142
problem15	3	3	0	1,645	0.000	0.0000	4.9407E−005	0.0000
problem16	5	3	0	22,593	0.312	0.7181	2.0022E−004	0.7050

numerical results are comparable with those obtained by using other approaches, so that this work could be a base for promising future developments. In particular, we note that the overall performance of the exact penalty approach relies significantly on the performance of the algorithm for simple constraints employed at Step 1 of EPGO Algorithm. Therefore, we are currently investigating the possibility of improving the performance of EPGO Algorithm along two different lines:

- by developing a more efficient version of the DIRECT algorithm based on local search [14], or based on the use of estimates of the global information of the problem [15];
- by using a different global optimization algorithm such as [5,21], able also to exploit some particular structure of the problem, for solving the problem at Step 1.

These improvements, which will be fruitful also in order to tackle larger dimensional problems, and a more extensive numerical experimentation will be matter of future work.

As a final remark, we point out that, with some additional technical efforts, the approach described here could be extended to treat also problems where the functions f and g are nonsmooth. Indeed, the present work is largely based on the results developed in [2], where the more general case of an exact penalty approach for nonsmooth problems is considered.

References

1. Birgin, E.G., Floudas, C.A., Martinez, J.M.: Global minimization using an augmented Lagrangian method with variable lower-level constraints. Math. Program. Ser. A. (2009). doi:10.1007/s10107-009-264-y

2. Di Pillo, G., Facchinei, F.: Exact barrier function methods for Lipschitz programs. Appl. Math. Optim. **32**, 1–31 (1995)
3. Di Pillo, G., Grippo, L.: On the exactness of a class of nondifferentiable penalty functions. J. Optim. Theory Appl. **57**(3), 399–410 (1988)
4. Di Pillo, G., Grippo, L.: Exact penalty functions in constrained optimization. Siam J. Control Optim. **27**, 1333–1360 (1989)
5. Floudas, C.A.: Deterministic Global Optimization: Theory, Methods and Application. Kluwer Academic Publishers, Dordrecht (1999)
6. Floudas, C.A., Akrotirianakis, I.G., Caratsoulas, S., Meyer, C.A., Kallrath, J.: Global optimization in the 21st century: advances and challenges. Comput. Chem. Eng. **29**, 1185–1202 (2005)
7. Gablonsky, J.M.: DIRECT version 2.0, User Guide (2001)
8. Gao, D.Y.: Duality Principles in Nonconvex Systems: Theory, Methods and Applications. Kluwer Academic Publishers, Boston (2000)
9. Gao D.Y., Sherali H.D.: Complementarity and duality in global optimization, Special issue of Journal of Global Optimization (2004)
10. Horst, R., Pardalos, P.M., Thoai, N.V.: Introduction to Global Optimization. Kluwer Academic Publishers, Dordrecht (2000)
11. Horst, R., Tuy, H.: Global Optimization: Deterministic Approaches. Springer, Berlin (1990)
12. Jones, D.R., Perttunen, C.D., Stuckman, B.E.: Lipschitzian optimization without the lipschitz constant. J. Optim. Theory Appl. **79**(1), 157–181 (1993)
13. Jones, D.R.: The DIRECT global optimization algorithm. In: Floudas, C., Pardalos, P. (eds.) Encyclopedia of Optimization, pp. 431–440. Kluwer Academic Publishers, Dordrecht (2001)
14. Liuzzi, G., Lucidi, S., Piccialli, V.: A DIRECT-based approach exploiting local minimizations for the solution of large-scale global optimization problems. Accepted for publication on Computational Optimization and Applications (2008). doi:10.1007/s10589-008-9217-2
15. Liuzzi, G., Lucidi, S., Piccialli, V.: A partition-based global optimization algorithm. Accepted for publication on Journal of Global Optimization (2009). doi:10.1007/s10898-009-9515-y
16. Liuzzi, G., Lucidi, S., Piccialli, V.: A DIRECT-type algorithm for constrained global optimization. CNR-IASI Technical Report, to appear
17. Luo, H.Z., Sun, X.L., Li, D.: On the convergence of augmented Lagrangian methods for constrained global optimization. SIAM J. Optim. **18**, 1209–1230 (2007)
18. Neumaier, A.: Complete search in continuous global optimization and constraints satisfaction. Acta Numerica **13**, 271–369 (2004)
19. Neumaier, A., Shcherbina, O., Huyer, W., Vinkó, T.: A comparison of complete global optimization solvers. Math. Program. **103**, 335–356 (2005)
20. Pinter, J.D.: Global Optimization in Action. Kluwer Academic Publishers, Dordrecht (1996)
21. Tawarmalani, M., Sahinidis, N.V.: Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming. Kluwer Academic Publishers, Dordrecht (2002)
22. Wang, C.Y., Li, D.: Unified theory of augmented Lagrangian methods for constrained global optimization. J. Glob. Optim. **44**, 433–458 (2009)