Portfolio optimization in a defaultable market under incomplete information

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Abstract: We consider the problem of maximization of expected utility from terminal wealth in a market model that is driven by a possibly not fully observable factor process and that takes explicitly into account the possibility of default for the individual assets as well as contagion (direct and information induced) among them. It is a multinomial model in discrete time that allows for an explicit solution. We discuss the solution within our defaultable and partial information setup, in particular we study its robustness. Numerical results are derived in the case of a log-utility function and they can be analogously obtained for a power utility function.

Keywords: Portfolio optimization, partial information, credit risk, dynamic programming, robust solutions.

JEL Classification: G11, C61, C11.

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1 Introduction

Our study concerns the classical portfolio optimization problem of maximization of expected utility from terminal wealth when the assets, in which one invests, may default. We put ourselves in a context where the dynamics of the asset prices are affected by exogenous factor processes, some of which may have an economic interpretation, some may not but, most importantly, NOT all of them may be directly observable. In credit risk models factors are often used to describe contagion: "physical" and "information induced". Information induced contagion arises due to the fact that the successive updating of the distributions of the latent (not observable) factors in reaction to incoming default observations leads to jumps in the default intensity of the surviving firms (this is sometimes referred to as "frailty approach", see e.g. Schönbucher 2003). As shown in Duffie et al. (2009), unobservable factor processes are needed on top of observable covariates in order to explain clustering of defaults in historical credit risk data. In general, the formulation of a model under incomplete information on the factors allows for greater model flexibility, avoids a possibly inadequate specification of the model itself, and the successive updating of the distribution of the unobserved factors (for constant factors one considers them from the Bayesian point of view as random variables) allows the model to "track the market" thus avoiding classical model calibration.

To keep the presentation at a possibly simple level, we shall consider only a single factor process that is supposed to be non directly observable and the observation history is given, in addition to the defaults, by the observed asset prices. Furthermore, we shall consider discrete time dynamics. With respect to continuous time models, this can be justified since trading takes place in discrete time anyway. Moreover, a solution is easier to compute in discrete time and, while it is more difficult to obtain qualitative results than in continuous time, once an explicit numerical solution is obtained, one can evaluate its performance also with respect to alternative criteria via simulation.

The outline of the paper is as follows. In Section 2 we describe our model and objective. The filter process, which allows for the transition from the partial information problem to a corresponding one under complete information, is studied in Section 3. Section 4 contains the main result on using Dynamic Programming to obtain the optimal investment strategy; we consider explicitly the log-utility case, but analogous results can be obtained for other utility functions, in particular power utility. The last section discusses numerical results from simulations, that were performed in order to investigate the effect of shorting as well as the robustness of the optimal strategy obtained for the partial information problem.

2 The model

Here we describe the model dynamics and the objective for our portfolio optimization problem. With a slight abuse of notation, in what follows we shall use the subscript n to indicate the instant t_n. All vectors will be row vectors and $'$ will indicate transposition.

2.1 Model dynamics

Given a discrete time set $t_0 = 0 < t_1 < \cdots < t_N = T$, let us introduce a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ ($\mathbb{G}$ stands for "global filtration"), where $\mathbb{G} = (\mathcal{G}_n)_n$ and, in addition to a nonrisky asset with price S_n^0, $S_0^0 = 1$ (S_n^0 is the price at time t_n), consider a set of M risky assets with prices S_n^m, $m = 1, \ldots, M$ that are subject to default, except for the first one S^1. Both for the applications (generally one invests in a pool of assets containing at least one non defaultable asset), as well as for formal reasons (see Remark 5) it is convenient to consider investment in at least one default-free risky asset. Let τ^m be the default time of the $m-$th asset and consider the default indicator process

$$H_n := (H_n^1, \ldots, H_n^M), \quad n = 0, \ldots, N, \tag{1}$$

where

$$H_n^m := \mathbb{1}_{\{t_n \geq \tau^m\}}$$

is the default indicator for the $m-$th firm. The possible values of H_n are the $M-$tuples $h^p = (h^{p,1}, \ldots, h^{p,M})$ for $p = 1, \ldots, 2^{M-1}$ with $h^{p,m} \in \{0, 1\}$. Since S^1 is assumed to be default free, we have

$$H^1 \equiv 0.$$

Furthermore, we arrange the values h^p according to a listing

$$h^1, h^2, \ldots, h^{2^{M-1}},$$

whereby, typically, $h^1 = (0, 0, \ldots, 0)$ and $h^{2^{M-1}} = (0, 1, \ldots, 1)$.

We let now the dynamics of the asset prices be given by

$$\begin{cases} S_{n+1}^0 = S_n^0(1 + r_n) \quad (\text{typically } r_n \equiv r) \\ \\ S_{n+1}^m = S_n^m \ \gamma^m(\xi_{n+1}) \ (1 - H_{n+1}^m), \quad S_0^m = s_0^m, \qquad m = 1, \ldots, M, \end{cases} \tag{2}$$

where ξ_n is a sequence of multinomial random variables with values in $\{\xi^1, \ldots, \xi^L\}$ and γ^m are positive measurable functions. Typically, $\gamma^m(\xi_{n+1}) \in (0, 1)$ when there is a downward movement in the dynamics of asset S^m during the period n, while $\gamma^m(\xi_{n+1}) > 1$ if the movement is upward. We want to point out that, while in our model the amplitude of the up- and downward movements may vary from asset to asset, in accordance with a common practice in trinomial and multinomial price evolution models it is intended that, if $\gamma^m(\xi_{n+1}) > 1$ for one asset m, the same holds for all the other assets (analogously when $\gamma^m(\xi_{n+1}) < 1$). In vector form, we may then write

$$S_{n+1} = diag\left(S_n \gamma(\xi_{n+1})\right) (\underline{1} - H_{n+1})' =: I(S_n, \xi_{n+1}, H_{n+1}), \tag{3}$$

where $\mathrm{diag}\left(S_n \gamma(\xi_{n+1})\right)$ is the $M \times M$ diagonal matrix, with elements $S_n^m \gamma^m(\xi_{n+1})$, $m = 1, \ldots, M$. The price evolution is thus driven by (ξ_n, H_n), defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ as follows. Given a $\mathbb{G}-$adapted finite state Markov chain $(Z_n)_n$ with values $Z_n \in \{z^1, \ldots, z^J\}$, with initial law μ and transition probability matrix

$$P^{ij} := \mathbb{P}(Z_n = z^j | Z_{n-1} = z^i), \quad \forall\, i, j \in \{1, \ldots, J\}, \ \forall n \ (\text{time homogeneous}), \tag{4}$$

conditionally on this process Z, the distribution of the driving processes (ξ, H) is supposed to be given, for every $z \in \{z^1, \dots, z^J\}$, by

$$\begin{aligned} \rho^{\ell;p,q}(z) \quad &:= \quad \mathbb{P}(\xi_n = \xi^\ell, H_n = h^q | H_{n-1} = h^p, Z_{n-1} = z) \\ &\qquad\qquad \forall\, \ell \in \{1, \dots, L\}, \forall\, p, q \in \{1, \dots, 2^{M-1}\} \end{aligned} \tag{5}$$

where $n = 1, \dots, N$. Notice that the dependence of $\rho^{\ell;p,q}$ on Z_{n-1}, in particular for what concerns H_n, allows to model contagion: only "physical" if Z_{n-1} is observed, and "information-induced" if Z_{n-1} is unobservable and its distribution is updated on the basis of the observed default state and of the defaultable asset prices. Furthermore, in the case when $q < p$, we assume that $\sum_{\ell=1}^{L} \rho^{\ell;p,q}(z) = 0$ for any $z \in \{z^1, \dots, z^J\}$.

2.2 Portfolios

To perform portfolio optimization, we evidently need to invest in the market and, for this purpose, we consider an investment strategy that may be defined either by specifying the number of units invested in the individual assets, namely $a_n = (a_n^0, a_n^1, \dots, a_n^M)$ (a_n^m is the number of units of asset m held in the portfolio in period t_n), or, restricting the attention to positive portfolio values, by equivalently specifying the ratios invested in the individual assets. More precisely, we shall consider the following relationships, that slightly differ from the standard ones for reasons that we shall explain below (see Remark 1), i.e.,

$$\phi_n^0 = \frac{a_{n+1}^0 S_n^0}{V_n^\phi}, \qquad \phi_n^m (1 - H_n^m) = \frac{a_{n+1}^m S_n^m}{V_n^\phi},\ m = 1, \dots, M, \tag{6}$$

where

$$V_n^\phi = V_n^a := \sum_{m=0}^{M} a_n^m S_n^m = \sum_{m=0}^{M} a_{n+1}^m S_n^m$$

is the (self-financing) portfolio value in period t_n. Notice that

$$\phi_n^0 = 1 - \sum_{m=1}^{M} \phi_n^m (1 - H_n^m)$$

so that, to define a self financing investment strategy $\bar{\phi}_n := (\phi_n^0, \phi_n^1, \dots, \phi_n^M)$, it suffices to define $\phi_n := (\phi_n^1, \dots, \phi_n^M)$.

It will be convenient to write the portfolio value at time t_{n+1} in terms of its value at time t_n and of the gain during the period n, namely

$$\begin{aligned}
V^{\phi}_{n+1} &= V^a_{n+1} = V^a_n + a^0_{n+1}(S^0_{n+1} - S^0_n) + \sum_{m=1}^{M} a^m_{n+1}(S^m_{n+1} - S^m_n) \\
&= V^a_n + a^0_{n+1}S^0_n r_n + \sum_{m=1}^{M} a^m_{n+1}S^m_n \left[\gamma^m(\xi_{n+1})(1 - H^m_{n+1}) - 1\right] \\
&= V^{\phi}_n + \phi^0_n V^{\phi}_n r_n + \sum_{m=1}^{M} \phi^m_n V^{\phi}_n (1 - H^m_n) \left[\gamma^m(\xi_{n+1})(1 - H^m_{n+1}) - 1\right] \\
&= V^{\phi}_n \left\{(1 + r_n) + \sum_{m=1}^{M} \phi^m_n (1 - H^m_n) \left[\gamma^m(\xi_{n+1})(1 - H^m_{n+1}) - (1 + r_n)\right]\right\}.
\end{aligned} \tag{7}$$

Remark 1 *With the given definitions (in particular, the presence of the factor $(1 - H^m_n)$ in the definition of ϕ^m_n in (6)) one has that investment in an asset automatically ceases as soon as it defaults. This implies the equivalence of the expressions for V^a_n and V^{ϕ}_n (namely, the next-to-last equality in (7) indeed holds true).*

Assuming first that the factor process Z is observed by the investor, the definitions above also imply that we consider $(a_n)_{n\geq 0}$ to be a predictable process (a_0 is $\mathcal{G}_0$-measurable and a_n is $\mathcal{G}_{n-1}-$measurable, $n \geq 1$, meaning that investment decisions a_{n+1}, taken at time t_n, are made on the basis of the information available at time t_n and kept until time t_{n+1}, when new quotations are available), while $(\phi_n)_{n\geq 0}$ is adapted.

2.3 The partial information problem

In view of formulating our partial information problem, let the default history be given by the filtration $\mathcal{H}_n := \sigma\{H_\nu,\ \nu \leq n\}$. With this filtration, we can reexpress the global filtration as

$$\mathcal{G}_n = \mathcal{F}^{\xi}_n \vee \mathcal{H}_n \vee \mathcal{F}^Z_n, \quad n = 0, \dots, N,$$

where $(\mathcal{F}^Z_n)_n$ and $(\mathcal{F}^{\xi}_n)_n$ denote, respectively, the natural filtration associated with Z and ξ, while, with $(\mathcal{F}^S_n)_n$ denoting the filtration given by the price observation history, the observation filtration (representing the information of an investor) is given by

$$\mathcal{F}_n = \mathcal{F}^S_n \vee \mathcal{H}_n \quad \subset \quad \mathcal{G}_n, \qquad n = 0, \dots, N.$$

Having specified a utility function $u : \mathbb{R}_+ \to \mathbb{R}$, of class $\mathcal{C}^1$, increasing and strictly concave, that satisfies the usual Inada's conditions:

$$\lim_{x\to 0^+} u'(x) = +\infty \quad \text{and} \quad \lim_{x\to +\infty} u'(x) = 0,$$

we can now give the following

Definition 2 *A self financing investment strategy $\phi_n = (\phi^1_n, \dots, \phi^M_n), n = 0, \dots, N$, is called* admissible *in our partial information problem, and we write $\phi \in \mathcal{A}$, if, besides implicit technical conditions, it is $\mathcal{F}_n-$adapted and such that V^{ϕ}_n belongs to the domain $\mathbb{R}_+$ of $u(\cdot)$.*

Notice that, in general, the set of admissible strategies is non empty (e.g., in the log and power utility cases it contains the strategy of not investing in the risky assets) and it is a convex set that may be unbounded; by possibly bounding it (e.g., imposing that, at any time t_n, $\phi_n^m \geq -C, m = 1, \ldots, M$) it can be transformed into a set with compact closure (for details in the log-utility case see Lemma 7 below). We come now to define our

Problem: Given an initial wealth v_0, determine an admissible ϕ^* such that

$$E\left[u\left(V_N^{\phi^*}\right)\right] \geq E\left[u\left(V_N^{\phi}\right)\right], \quad \forall\ \phi \in \mathcal{A}.$$

Our problem is a partial information problem in that the factor process Z cannot be observed; on the other hand, the investment strategy can depend only on observable quantities. The usual approach in this situation (see e.g., Bensoussan 1992 and Bertsekas 1976, see also Corsi et al. 2008 for a problem related to the one of the present paper) consists in transforming the partial observation problem into one under full information, by replacing the unobservable quantities Z_n by their conditional distributions, given the current observation history. These conditional distributions are the so-called *filter distributions* or just *filters* and they can be computed recursively, as we are going to show in the next section.

We conclude this section by recalling a fundamental result on the absence of arbitrage opportunities (AOA, see e.g., Prop. 2.7.1 in Dana and Jeanblanc 1998).

Lemma 3 *If the above Problem has a solution, then there are no arbitrage opportunities. The converse also holds true, i.e., there is equivalence between the existence of an optimal solution and the AOA, in the case when the utility function u is strictly concave,* strictly *increasing and of class $\mathcal{C}^1$.*

3 The filter

Since the investment strategy ϕ is by definition $\mathbb{F}-$adapted, the information coming from observing (S, V, H) (namely the asset prices, the portfolio value, and the default state) is equivalent to that of observing just (S, H).

Defining $(S^n, H^n) := ((S_1, H_1), \ldots, (S_n, H_n))$, the filter distribution for Z at time t_n is the random vector $\Pi_n = (\Pi_n^1, \ldots, \Pi_n^J)$ with components

$$\Pi_n^j := \mathbb{P}\left(Z_n = z^j | \mathcal{F}_n\right) = \mathbb{P}\left(Z_n = z^j | (S^n, H^n)\right)\ ,\ \ j = 1, \ldots, J\ ,$$

taking values in the J-simplex $\mathcal{K}_J \subset \mathbb{R}^J$ (here $|\cdot|_1$ denotes the l^1-norm)

$$\mathcal{K}_J = \left\{ x = (x^j) \in \mathbb{R}^J : x^j \geq 0, j = 1, \ldots, J \ \text{ and } |x|_1 = \sum_{j=1}^{J} x^j = 1 \right\}.$$

By applying the *recursive Bayes' formula*, one obtains, for $j = 1, \ldots, J$,

$$\begin{aligned} \Pi_n^j &= \mathbb{P}\left(Z_n = z^j | S_n = s_n, H_n = h_n, (S^{n-1}, H^{n-1})\right) \propto \\ &\propto \textstyle\sum_{i=1}^J \mathbb{P}\left(Z_n = z^j, Z_{n-1} = z^i | S_n = s_n, H_n = h_n, (S^{n-1}, H^{n-1})\right) \propto \qquad (8) \\ &\propto \textstyle\sum_{i=1}^J P^{ij}\ \mathbb{P}\left(S_n = s_n, H_n = h_n | Z_{n-1} = z^i, S_{n-1}, H_{n-1}\right)\ \Pi_{n-1}^i, \end{aligned}$$

where P^{ij} was defined in (4) and with the observation distribution (likelihood function) given by

$$\begin{aligned}&\mathbb{P}\left(S_n = s_n, H_n = h^q | Z_{n-1} = z^i, S_{n-1} = s_{n-1}, H_{n-1} = h^p\right) = \\ &\sum_{\ell=1}^{L} \rho^{\ell;p,q}(z^i) \mathbb{1}_{\{s_n = I(s_{n-1}, \xi^\ell, h^q)\}} =: F(z^i; s_n, s_{n-1}, h^q, h^p),\end{aligned} \tag{9}$$

where $I(s, \xi, h)$ was defined in (3) and the proportionality coefficient in (8) is obtained through the normalization condition $\sum_{j=1}^{J} \Pi_n^j = 1$.

Remark 4 *Since the model may not correspond exactly to reality, there may be no $\xi^\ell \in \{\xi^1, \ldots, \xi^L\}$ so that, for the actually observed values of s_{n-1} and s_n, one has $s_n = I(s_{n-1}, \xi^\ell, h^q)$. Following standard usage, we shall then consider the value of ℓ for which $I(s_{n-1}, \xi^\ell, h^q)$ comes closest to the actually observed value of s_n ("nearest neighbor").*

Given the current observations (s_n, h_n) and the previous ones (s_{n-1}, h_{n-1}), setting

$$F(s_n, s_{n-1}, h_n, h_{n-1}) := diag\left(F(z; s_n, s_{n-1}, h_n, h_{n-1})\right), \tag{10}$$

which is a $J \times J$ diagonal matrix with elements $F(z^i; s_n, s_{n-1}, h_n, h_{n-1}), i = 1, \ldots, J$, the recursions (8) can be expressed in vector form as

$$\begin{cases} \Pi_0' = \mu \qquad \text{and, for } n \geq 1, \\ \Pi_n' = \dfrac{P' F(s_n, s_{n-1}, h_n, h_{n-1}) \Pi_{n-1}'}{\left|P' F(s_n, s_{n-1}, h_n, h_{n-1}) \Pi_{n-1}'\right|_1} =: \bar{F}\left(\Pi_{n-1}, s_n, s_{n-1}, h_n, h_{n-1}\right). \end{cases} \tag{11}$$

Remark 5 *By having assumed that at least one asset in the market is default free, the filter is well defined at every time step. Indeed, if we had considered only defaultable assets, in the case of default of all assets by time t_n we would have found $S_n = (0, \ldots, 0)$ and we would have lost all the information on ξ_n necessary to update the filter.*

4 Dynamic Programming for the "equivalent full information problem"

Under full information corresponding to $\mathbb{G}$ the tuple (S, V, H, Z) is Markov. In the full information setting equivalent to the partial information problem, the process Z has to be replaced by the filter process Π. Indeed, from (8) it is easily seen (for details we refer, e.g., to Corsi et al. 2008) that, in the partial information filtration $\mathcal{F}_n$, it is the tuple (S, V, H, Π) that is Markov.

Denoting by $U_n(s, v, h, \pi)$ the optimal value at time t_n for $S_n = s, V_n^\phi = v, H_n = h, \Pi_n = \pi$, i.e.,

$$U_n(s, v, h, \pi) = \sup_{\phi \in \mathcal{A}} \mathbb{E}\left\{u(V_N^\phi) \big| (S, V^\phi, H, \Pi)_n = (s, v, h, \pi)\right\}$$

(recall that $\mathcal{A}$ denotes the set of admissible strategies over the entire investment interval), an application of the Dynamic Programming Principle (see, e.g., Chapter

4 in Bertsekas 1976, that concerns the Dynamic Programming Approach to stochastic control problems with imperfect state information; in particular, see Equations (16) and (17) in Section 4.2 therein) leads to the backward recursions

$$\begin{cases} U_N(s,v,h,\pi) = u(v) \quad \text{and, for } n \in \{1,\dots,N\}, \\ \\ U_{n-1}(s,v,h,\pi) = \\ \max_{\phi_{n-1}} \mathbb{E}\left\{ U_n(S_n, V_n^\phi, H_n, \Pi_n) \middle| (S, V^\phi, H, \Pi)_{n-1} = (s,v,h,\pi) \right\}. \end{cases} \tag{12}$$

4.1 Explicit solution in the log-utility case

In the log-utility case (and analogously in the power utility case), assuming for simplicity that $r_n \equiv r$, we have the following result.

Theorem 6 *For $n = 0,\dots,N$ and supposing that $H_n = h^p$ for some $p \in \{1,\dots,2^{M-1}\}$ we have*

$$U_n(s,v,h^p,\pi) = \log v + K_n(s,h^p,\pi), \tag{13}$$

with

$$\begin{cases} K_N(s,h^p,\pi) &= 0 \qquad \text{for every } s \in \mathbb{R}_+^M, p \in \{1,\dots,2^{M-1}\}, \pi \in \mathcal{K}_J, \\ \\ K_n(s,h^p,\pi) &= k(h^p,\pi) + \sum_{i=1}^J \pi^i \sum_{\ell=1}^L \sum_{q=1}^{2^{M-1}} \rho^{\ell;p,q}(z^i) \cdot \\ & \quad \cdot K_{n+1}\left(I(s,\xi^\ell,h^q), h^q, \bar{F}\left(\pi, I(s,\xi^\ell,h^q), s, h^q, h^p\right)\right), \end{cases}$$

where $\bar{F}(\cdot)$ was defined in (11) and where

$$\begin{aligned} k(h^p,\pi) = \max_{\phi=(\phi^1,\dots,\phi^M)} \Big\{ & \sum_{i=1}^J \pi^i \sum_{\ell=1}^L \sum_{q=1}^{2^{M-1}} \rho^{\ell;p,q}(z^i) \cdot \\ & \cdot \log\left[(1+r) + \sum_{m=1}^M \phi^m (1-h^{p,m}) \left[\gamma^m(\xi^\ell)(1-h^{q,m}) - (1+r)\right]\right] \Big\}. \end{aligned} \tag{14}$$

Notice that, in each period t_n, the additive term $K_n(\cdot)$ results from the sum of a current additive term $k(h^p,\pi)$ and the conditional expectation of the previously obtained $K_{n+1}(\cdot)$. For clarity, we split the proof into two parts: in the first one, that immediately follows, we straightforwardly obtain the result by backward induction on n and in the second part, in Subsection 4.1.1, we show that, indeed, $k(h^p,\pi)$ defined in Equation (14) exists and it is unique. To this (second) end, an intermediate technical result will be required and it will be given in Lemma 7.

We first notice that the result holds true for $n = N$. We now suppose that Equation (13) is verified at time t_{n+1} and we show that it remains valid at time t_n. We have, given Equation (12) and recalling Equation (7), where the portfolio value at time t_{n+1} is written as a function of its value at time t_n (we omit the subscript n

in the investment strategy $\phi_n = (\phi_n^1, \ldots, \phi_n^M)$)

$$\begin{aligned}
U_n(s,v,h^p,\pi) &= \max_\phi \mathbb{E}\left\{ U_{n+1}(S_{n+1}, V_{n+1}^\phi, H_{n+1}, \Pi_{n+1}) \big| {\scriptstyle (S,V^\phi,H,\Pi)_n=(s,v,h^p,\pi)} \right\} \\
&= \max_\phi \mathbb{E}\left\{ \log V_{n+1}^\phi + K_{n+1}(S_{n+1}, H_{n+1}, \Pi_{n+1}) \big| {\scriptstyle (S,V^\phi,H,\Pi)_n=(s,v,h^p,\pi)} \right\} \\
&= \log v + \max_\phi \mathbb{E}\Big\{ \log \Big[{\scriptstyle (1+r)+\sum_{m=1}^M \phi^m (1-h^{p,m}) \left(\gamma^m(\xi_{n+1})(1-H_{n+1}^m)-(1+r)\right)} \Big] \\
&\quad + K_{n+1}\Big({\scriptstyle I(S_n,\xi_{n+1},H_{n+1}),\ H_{n+1},\ \bar{F}\left(\Pi_n, I(S_n,\xi_{n+1},H_{n+1}), S_n, H_{n+1}, H_n\right)} \Big) \\
&\quad \big| (S,H,\Pi)_n = (s,h^p,\pi) \Big\},
\end{aligned}$$

where I and $\bar{F}$ were introduced, respectively, in Equations (3) and (11).

We now use iterated conditional expectations and we introduce a conditional expectation with respect to a larger filtration containing Z_n. This allows us to explicitly compute this conditional expectation, that will be a function of Z_n, and we find

$$\begin{aligned}
U_n(s,v,h^p,\pi) &= \log v + \max_\phi \mathbb{E}\Big\{ \sum_{\ell=1}^{L} \sum_{q=1}^{2^{M-1}} \rho^{\ell;p,q}(Z_n) \Big[\log \Big((1+r) \\
&\quad + \sum_{m=1}^{M} \phi^m (1-h^{p,m}) \big(\gamma^m(\xi^\ell)(1-h^{q,m}) - (1+r) \big) \Big) \\
&\quad + K_{n+1} \Big(I(s,\xi^\ell,h^q), h^q, \bar{F}\big(\pi, I(s,\xi^\ell,h^q), s, h^q, h^p\big) \Big) \Big] \\
&\quad \big| (S,H,\Pi)_n = (s,h^p,\pi) \Big\}
\end{aligned}$$

It suffices now to recall that the conditional distribution of Z_n given the investor's information at time t_n is, by definition, the filter at time t_n, so that we finally have

$$\begin{aligned}
U_n(s,v,h^p,\pi) &= \log v + \max_\phi \Big\{ \sum_{i=1}^{J} \pi^i \sum_{\ell=1}^{L} \sum_{q=1}^{2^{M-1}} \rho^{\ell;p,q}(z^i) \log \Big[(1+r) \\
&\quad + \sum_{m=1}^{M} \phi^m (1-h^{p,m}) \Big(\gamma^m(\xi^\ell)(1-h^{q,m}) - (1+r) \Big) \Big] \Big\} \\
&\quad + \sum_{i=1}^{J} \pi^i \sum_{\ell=1}^{L} \sum_{q=1}^{2^{M-1}} \rho^{\ell;p,q}(z^i) K_{n+1} \Big({\scriptstyle I(s,\xi^\ell,h^q), h^q, \bar{F}\left(\pi, I(s,\xi^\ell,h^q), s, h^q, h^p\right)} \Big) \\
&= \log v + K_n(s,h^p,\pi). \qquad (15)
\end{aligned}$$

The theorem is proved once we show that $k(h^p,\pi)$, defined in (14), exists and this is the subject of the next subsection.

4.1.1 Existence of $k(h^p,\pi)$

Recalling that $k(h^p,\pi)$ is defined in (14) as

$$k(h^p,\pi) = \max_{\phi=(\phi^1,\ldots,\phi^M)} \Big\{ \sum_{i=1}^{J} \pi^i \sum_{\ell=1}^{L} \sum_{q=1}^{2^{M-1}} \rho^{\ell;p,q}(z^i) \cdot$$

$$\cdot \log \Big[(1+r) + \sum_{m=1}^{M} \phi^m (1-h^{p,m}) \left[\gamma^m(\xi^\ell)(1-h^{q,m}) - (1+r) \right] \Big] \Big\},$$

in this second part of the proof to Theorem 6 we show that the maximum exists and it is unique. Restricting our attention, as it is reasonable from an economic point of view, to a truncated domain for $\phi^1, \dots, \phi^M$, denoted by $\mathcal{D}_C$, we show, in Lemma 7, that $\mathcal{D}_C$ has a compact closure. The existence (and the uniqueness) of $k(h^p, \pi)$, for any $p \in \{1, \dots, 2^{M-1}\}$, then follows.

At every time step, the maximization problem is defined for $\phi = (\phi^1, \dots, \phi^M) \in \mathcal{D}$, where $\mathcal{D}$ is such that the above logarithms are well defined (recall that in the Definition 2 of admissibility it was required that V_n^ϕ belongs to the domain $\mathbb{R}_+$ of $u(\cdot)$). In particular, $\mathcal{D}$ is non empty (it contains at least the point $(0, \dots, 0)$) and it is delimited by the intersection of a maximum of $2^{M-1} \times L \times 2^{M-1}$ half-planes of the form

$$1 + r + \sum_{m=1}^{M} \phi^m (1 - h^{p,m}) \big(\gamma^m(\xi^\ell)(1 - h^{q,m}) - 1 - r\big) > 0, \tag{16}$$

where p and q vary in $\{1, \dots, 2^{M-1}\}$ and ℓ is in $\{1, \dots, L\}$. Since it is reasonable from an economic point of view to impose on an investor not to take short positions in the risky assets for more than a proportion C of his current wealth, we truncate $\mathcal{D}$ from below, by imposing the condition

$$\phi^m > -C, \quad m = 1, \dots, M,$$

for a suitable $C > 0$, thus restricting our attention to a domain $\mathcal{D}_C$ that is a subset of $\mathcal{D}$, $\mathcal{D}_C \subseteq \mathcal{D}$. In the following Lemma 7 we show that the closure of $\mathcal{D}_C$, namely $\bar{\mathcal{D}}_C$, is compact.

Once we have restricted our attention to a domain with a compact closure, the maximizing ϕ^* exists and it is unique. Noticing that the boundary of $\mathcal{D}_C$ partly coincides with the boundary of $\mathcal{D}$, the common boundary will be called "natural boundary" of $\mathcal{D}_C$, while the boundary resulting from the truncation of $\mathcal{D}$ will be the "artificial boundary" of $\mathcal{D}_C$. We then have:

- if $\mathcal{D}$ is bounded by itself, we have to maximize over $\mathcal{D}$ a strictly concave and continuous function, namely (see (15)) the sum over q and ℓ of logarithms of the left hand side of (16), that goes to $-\infty$ on $\partial\mathcal{D}$;

- otherwise, if the domain has been artificially bounded, then we have to maximize over $\mathcal{D}_C$ a strictly concave and continuous function that goes to $-\infty$ on the "natural boundary" of $\mathcal{D}_C$ and that is well defined on the "artificial boundary" of $\mathcal{D}_C$.

The maximum point then exists (possibly only in the truncated domain $\mathcal{D}_C$), it is automatically admissible and it is unique. Notice that it can be on the "artificial boundary". We only state here that ϕ^* can be numerically obtained (this will be clarified in Section 5, which is devoted to numerical examples).

Lemma 7 *The closure $\bar{\mathcal{D}}_C$ of the admissibility domain $\mathcal{D}_C$ is compact.*

Being, by definition, $\mathcal{D}_C$ obtained as a truncation from below of the domain $\mathcal{D}$, by imposing the condition $\phi^m > -C$, $m = 1, \dots, M$, for a suitable $C > 0$, we have to show that $\mathcal{D}_C$ is bounded also from above in each variable. Let us recall that $\mathcal{D}$ is delimited by the intersection of a maximum of $2^{M-1} \times L \times 2^{M-1}$ half-planes of the form as given in (16), where p and q vary in $\{1,\dots,2^{M-1}\}$ and ℓ is in $\{1,\dots,L\}$. Let us then set, without loss of generality, $r = 0$ and consider the half plane in equation (16) identified by $p = 1$ and $q = 2^{M-1}$ (i.e., for $h^p = (0,0,\dots,0)$ and $h^q = (0,1,\dots,1)$), namely

$$1 + \phi^1(\gamma^1(\xi^\ell) - 1) - \phi^2 - \cdots - \phi^M > 0 \tag{17}$$

(the other cases, namely when $h^{p,m} = 1$ for some $m \in \{2,\dots,M\}$, are simpler to treat).

Next denote by $\underline{\ell}$ and $\bar{\ell} \in \{1,\dots,L\}, \bar{\ell} \neq \underline{\ell}$, the indexes such that $\gamma^m(\xi^{\underline{\ell}}) \in (0,1)$ and $\gamma^m(\xi^{\bar{\ell}}) > 1$, respectively, for every $m \in \{1,\dots,M\}$ (recall the description after (2)). Since by definition of $\mathcal{D}_C$ we have $-\phi^m < C$ for all m, focusing on ϕ^1 we find that a necessary condition for $\phi^1 \in \mathcal{D}_C$ is that

$$\phi^1(1 - \gamma^1(\xi^\ell)) < 1 - \phi^2 - \cdots - \phi^M < 1 + C(M-1) \quad \forall\, \ell \in \{1,\dots,L\}.$$

By taking $\ell = \underline{\ell}$, so that $1 - \gamma^1(\xi^{\underline{\ell}}) > 0$, we find that the boundedness from below of $\mathcal{D}_C$ ensures its boundedness from above with respect to ϕ^1. For what concerns ϕ^2 (the reasoning is the same for $\phi^3,\dots,\phi^M$), taking $\ell = \underline{\ell}$ in equation (17) we have that a necessary condition for $\phi^2 \in \mathcal{D}_C$ is that

$$\phi^2 < 1 - \phi^1(1 - \gamma^1(\xi^{\underline{\ell}})) - \phi^3 - \cdots - \phi^M < 1 + C(1 - \gamma^1(\xi^{\underline{\ell}})) + C(M-2).$$

This allows us to conclude that, given the boundedness from below, the domain $\mathcal{D}_C$ is also bounded from above in each variable and its closure is compact (for details in the simpler binomial model see Callegaro 2010).

4.2 Particular cases

We now consider three particular cases, namely the full information case, the case when $Z_n \equiv Z$ with Z unobservable and when it is observable. As previously done, we suppose, for simplicity, that $r_n \equiv r$.

4.2.1 Full information about Z_n

In this case the Markovian tuple is (S, V, H, Z), so that we replace Π by Z and the optimal wealth at time t_n is

$$U_n(s,v,h,z) = \sup_{\phi\,\in\mathcal{A}} \mathbb{E}\left\{ u(V_N^\phi) \middle| S_n = s, V_n^\phi = v, H_n = h, Z_n = z \right\}.$$

In the log-utility case we find the following corollary of Theorem 6. Having fixed $Z_n = z^i$, we just substitute π by z^i in $K(\cdot)$ and in $k(\cdot)$ and we drop the $\sum_{i=1}^J \pi^i$ everywhere. Moreover, since $K_n(s,h^p,z^i)$ is the conditional expectation of $K_{n+1}(S_{n+1},H_{n+1},Z_{n+1})$, in the definition of $K_n(s,h^p,z^i)$ we will also find the sum $\sum_{j=1}^J P^{ij} K_{n+1}(\cdot,\cdot,z^j)$. We obtain

Corollary 8 *For $n = 0,\dots,N$, we have*

$$U_n(s,v,h^p,z^i) = \log v + K_n(s,h^p,z^i), \tag{18}$$

with $K_N(s,h^p,z^i)=0$ *for every* $s\in\mathbb{R}^M_+, p\in\{1,\dots,2^{M-1}\}, i\in\{1,\dots,J\}$ *and*

$$K_n(s,h^p,z^i)=k(h^p,z^i)+\sum_{\ell=1}^{L}\sum_{q=1}^{2^{M-1}}\rho^{\ell;p,q}(z^i)\sum_{j=1}^{J}P^{ij}K_{n+1}\left(I(s,\xi^\ell,h^q),h^q,z^j\right),$$

where

$$k(h^p,z^i)=\max_{\phi}\sum_{\ell=1}^{L}\sum_{q=1}^{2^{M-1}}\rho^{\ell;p,q}(z^i)\log\left[{\scriptstyle 1+r+\sum_{m=1}^{M}\phi^m(1-h^{p,m})\left[\gamma^m(\xi^\ell)(1-h^{q,m})-1-r\right]}\right].$$

4.2.2 $Z_n\equiv Z$ unobserved

In the case when $Z_n\equiv Z$, the factor process reduces to an unobserved parameter that, in accordance with the Bayesian point of view, is considered as a random variable Z, with given *a priori* law μ. Even if Z is modeled as not time varying, the successive updating of its conditional distribution, i.e.,

$$\Pi^j_n:=\mathbb{P}(Z=z^j\mid(S^n,H^n)),\quad j=1,\dots,J,\ \ n\le N$$

makes the context dynamic. The solution is obtained as in the general case and here it simplifies considerably. In fact, the recursive Bayes' formula (8) reduces to the ordinary one, that here becomes

$$\begin{aligned}\Pi^j_n &= \mathbb{P}\left(Z=z^j|S_n=s_n,H_n=h_n,(S^{n-1},H^{n-1})\right)\\ &\propto \mathbb{P}\left(S_n=s_n,H_n=h_n|Z=z^j,S_{n-1}=s_{n-1},H_{n-1}=h_{n-1}\right)\cdot\Pi^j_{n-1}.\end{aligned}$$

Having fixed the previous observations (s_{n-1},h_{n-1}) and recalling the definition (10) of the diagonal matrix F, Equation (11) then becomes

$$\begin{cases}\Pi'_0=\mu \qquad \text{and, for } n\ge 1,\\[2ex] \Pi'_n=\dfrac{F(s_n,s_{n-1},h_n,h_{n-1})\,\Pi'_{n-1}}{\left|F(s_n,s_{n-1},h_n,h_{n-1})\,\Pi'_{n-1}\right|_1}:=\bar F\left(\Pi_{n-1},s_n,s_{n-1},h_n,h_{n-1}\right).\end{cases}\tag{19}$$

With these changes the statement of Theorem 6 remains valid in the same form also for the present case.

4.2.3 $Z_n\equiv Z$ fully observed

In this case the factor Z has no relevance anymore, the model is fully defined. Defining, in perfect analogy with Equation (5),

$$\rho^{\ell;p,q}:=\mathbb{P}(\xi_n=\xi^\ell,H_n=h^q|H_{n-1}=h^p),$$

for $\ell=1,\dots,L$ and for $p,q\in\{1,\dots,2^{M-1}\}$, one immediately finds

Corollary 9 *For* $n=0,\dots,N$, *we have*

$$U_n(s,v,h^p)=\log v+K_n(s,h^p),\tag{20}$$

with $K_N(s,h^p)=0$ *for every* $s\in\mathbb{R}_+^M, p\in\{1,\dots,2^{M-1}\}$ *and*

$$K_n(s,h^p)=k(h^p)+\sum_{\ell=1}^{L}\sum_{q=1}^{2^{M-1}}\rho^{\ell;p,q}\;K_{n+1}\Big(I(s,\xi^\ell,h^q),h^q\Big),$$

where

$$k(h^p)=\max_\phi\sum_{\ell=1}^{L}\sum_{q=1}^{2^{M-1}}\rho^{\ell;p,q}\;\log\left[1+r+\sum_{m=1}^{M}\phi^m(1-h^{p,m})\Big[\gamma^m(\xi^\ell)(1-h^{q,m})-1-r\Big]\right].\tag{21}$$

Remark 10 *Due to the (assumed) time homogeneity of* ρ*, i.e., of the processes* ξ *and* H*, the maximizing investment strategy* ϕ^* *does not depend on time. It does not depend on the current values* s *and* v *of the prices and the wealth either, it depends however on the current default state* h*.*

5 Numerical results and the issue of robustness

Numerical results from simulations are presented in the case when

- $M=3$, i.e., there are one non-defaultable and two defaultable risky assets on the market (it is the smallest value of M allowing for contagion);
- $L=2$, i.e., $\xi_n\in\{\xi^1,\xi^2\}$ (binomial model). Here ξ^1 corresponds to an "up" movement in asset prices and ξ^2 to a "down" movement;
- $J=2$, i.e., $Z_n\in\{0,1\}$, $\forall\ n$, with the following economic interpretation

$$\begin{cases} Z_n=0: & \text{good state} \quad \text{(bull market)},\\ Z_n=1: & \text{bad state} \quad \text{(bear market)};\end{cases}$$

- $r_n\equiv r=0$;
- $u(x)=\log(x), x>0$.

The initial law μ of the Markov chain Z is fixed by assigning

$$\mathbb{P}(Z_0=0)=0.5,\quad \mathbb{P}(Z_0=1)=0.5$$

and its transition probability matrix is supposed to be

$$P=\begin{pmatrix}P^{11} & P^{12}\\ P^{21} & P^{22}\end{pmatrix}=\begin{pmatrix}0.6 & 0.4\\ 0.4 & 0.6\end{pmatrix}.$$

For simplicity of presentation, in this section we assume that, given Z, ξ and H are independent. The conditional distribution of ξ given Z is then assigned, for every $n=1,\dots,N$, as

$$p(0):=p^1(0)=\mathbb{P}(\xi_n=\xi^1|Z_{n-1}=0)=0.6,\quad p(1):=p^1(1)=0.4,$$

meaning that, when the economy is in good state, the probability of having an "up" movement in asset prices is equal to 0.6, while when the economic situation is bad, this probability decreases to 0.4. It is also useful to introduce the following notation

$$\gamma^m(\xi^1) = u^m \quad \text{and} \quad \gamma^m(\xi^2) = d^m, \quad m = 1, 2, 3,$$

where $u^{\cdot}$ stands for "up" and $d^{\cdot}$ for "down" and, typically, $0 < d^m < 1 < u^m, m = 1, 2, 3$. We fix the following listing of the possible default states $h^p, p = 1, \ldots, 4$:

$$h^1 = (0,0,0), \quad h^2 = (0,1,0), \quad h^3 = (0,0,1), \quad h^4 = (0,1,1)$$

and we assign, in the next two matrices, the values of $\rho^{p,q}(z) := \mathbb{P}(H_n = h^q | H_{n-1} = h^p, Z_{n-1} = z)$, for $n = 1, \ldots, N$ and $p, q \in \{1, \ldots, 4\}$, according to the value of z,

$$\{z=0\} : \begin{pmatrix} 0.91 & 0.03 & 0.03 & 0.03 \\ 0 & 0.80 & 0 & 0.20 \\ 0 & 0 & 0.80 & 0.20 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \{z=1\} : \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0.50 & 0 & 0.50 \\ 0 & 0 & 0.50 & 0.50 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In the simulations we consider three cases:

GOOD : full information, where the true model is known and corresponds to the case $\{Z_n \equiv Z = 0\}$ (see Section 4.2.3);

BAD : full information, where the true model is known and corresponds to the case $\{Z_n \equiv Z = 1\}$;

PARTIAL : partial information, where there is uncertainty about the true model (Z_n is unobserved and evolves according to the Markov chain specified by the initial law μ and the transition probability matrix P).

We have two goals in mind:

i) investigating for each one of the three cases the effect of allowing for shorting in the risky assets;

ii) investigating the "robustness" of the optimal solution obtained in the partial information case (case "PARTIAL").

5.1 Shorting vs. no shorting

We analyze and compare two possible situations: the first one corresponds to the case when no shorting is possible and the investment strategy is constrained from above, namely (recall that ϕ_n^m is the proportion of wealth invested in S^m at time t_n)

$$\phi_n^m \in [0, 2], \quad m = 1, 2, 3, \ \forall\, n,$$

while in the second one shorting is allowed and the strategy is constrained from above and below, i.e.,

$$\phi_n^m \in [-2, 2], \quad m = 1, 2, 3, \ \forall\, n.$$

It can furthermore be easily seen that, in order for V_n^ϕ to be in the domain of $u(x) = \log(x)$, in the case when no-shorting is allowed (i.e., $C = 0$), we even have

$$\phi_n^2, \phi_n^3 < 1, \qquad \forall\ n$$

(it suffices to look at the function to be maximized in Equation (14) and to consider the case when $r = 0, H_{n+1} = h^4, H_n \neq h^4$, as was done in Lemma 7 to show that the domain $\mathcal{D}_C$ is bounded).

Remark 11 *In the context just described we thus consider investment strategies in a truncated domain, $\phi \in \mathcal{D}_C$ (this notation was introduced in the proof of Theorem 6), with $C = 0$ in the case of no shorting and $C = 2$ when shorting is possible. According to Theorem 6 the optimal strategy exists and it is unique. In our calculations it was obtained by means of a "search procedure", performed on the basis of a numerical code written in* C *on a grid of points constructed on the admissibility domain (for further details see Callegaro 2010). The precision of the grid is fixed to* 0.01.

Numerical values of the optimal investment strategy and of the corresponding expected terminal utility have been computed in cases "GOOD" and "BAD" in the simulations reported in Callegaro (2010), for various values of the parameters $u^m, d^m, m = 1, 2, 3$. As pointed out in Remark 10, these values do not depend on s and v, but only on the current default configuration h. Numerical results are reported in Callegaro (2010) also for case "PARTIAL" and here the optimal strategy depends on the current default configuration as well as on the prices of the assets.

From the numerical simulations in Callegaro (2010) it results that:

1. *When no shorting is possible:* in state "BAD" the optimal solution consists in not investing at all in risky assets and in placing all the money in the bank account. On the contrary, in state "GOOD", it is optimal to invest as much as one can in the default-free risky asset, regardless of the default state. In case "PARTIAL", it is never optimal to invest in the defaultable assets.

2. *When shorting is allowed:* for "reasonable" returns on the assets (up to a certain level of rewarding) both in the states "GOOD" and "BAD" it is optimal to invest all the wealth in S^1, but if the defaultable assets have a very high yield, then it becomes interesting to invest also in them; this latter fact happens also in case "PARTIAL".

To better illustrate this analysis, in Figure 1 we show in three diagrams the optimal expected utility of terminal wealth in the log-utility case, when

$$v_0 = 1, \quad H_0 = h^1 \quad \text{and} \quad N = 1, 2, \ldots, 5.$$

The first diagram corresponds to the situation when no shorting is possible, while in the second and third diagrams shorting is allowed, whereby in the second diagram one has "reasonable" asset returns and in the third one the defaultable assets have a high yield. Notice that, due to the fact that in the case of no shorting the optimal strategy in state "BAD" consists in not investing in the risky assets, the corresponding optimal portfolio value (in red in Figure 1 (a)) remains constant over time and is always lower than in state "GOOD". Notice also that in case "PARTIAL" the optimal value is lower than in state "BAD".

When shorting is allowed, up to a certain level of "return" on the risky assets (Figure 1 (b)), the optimal value in state "BAD" is superior to that in state "GOOD", which is due to the returns on the defaultable assets as well as on the fact that they are subject to default risk: these facts make it convenient to go short in them. Beyond that level, when it becomes convenient to invest in S^2 and S^3 (Figure 1 (c)), the optimal value in state "GOOD" is superior than in state "BAD", as one would expect.

5.2 Robustness

For what concerns *robustness*, it is here intended in the sense of obtaining a solution that works well for a variety of possible models. This is an important issue because the "exact model" is practically never known and, on the other hand, the solution may be rather sensitive to the model.

From the numerical calculations, which we show in the next set of diagrams in Figures 2 and 3 (notice that in both cases the graph on the right-hand side is a zoom of the one on the left-hand side) for the case when shorting is not allowed, respectively in case "GOOD" and "BAD", it turns out that the solution obtained for the model under incomplete information possesses this property of robustness in the sense that (as can be seen from Figure 2 and Figure 3)

- while it underperforms the solution under a hypothetical full information about the model,
- it performs much better with respect to using the wrong solution for the wrong model.

Remark 12 *From Theorem 6 it follows that the optimal strategy in case "PARTIAL" depends, in addition to the current default configuration, also on the current filter values, while under full information (cases "GOOD" and "BAD") it depends only on the default configuration. The strategy in case "PARTIAL" is therefore more refined and as such can be applied also in the cases "GOOD" and "BAD".*

In the four diagrams of Figure 2 and Figure 3 we now show the "robustness" of the partial information optimal strategy (case "PARTIAL") with respect to using the wrong full information optimal strategy in the wrong state, in the case of no shorting in the risky assets. Figure 2 concerns state "GOOD", Figure 3 the case "BAD" and we plot the optimal expected utility from terminal wealth as a function of t_N when

$$v_0 = 1, \quad H_0 = h^1 \quad \text{and} \quad N = 1, 2, 3.$$

In particular, in Figure 2 (recall that the graph on the right-hand side is a zoom of the one on the left-hand side) we plot the optimal expected utility of terminal wealth in the situation when the true state is "GOOD" (analogously, in Figure 3 when the true state is "BAD") in the following three cases:

- using the optimal solution for case "GOOD", dark blue line (upper benchmark case);
- using the optimal solution for case "PARTIAL", orange line;

- using the optimal solution for case "BAD"(lower benchmark case), light blue line.

In addition, for comparison purposes, we also plot the optimal expected utility of terminal wealth for case "PARTIAL" using the corresponding optimal strategy (fuchsia line).

It is evident from the figures that the optimal investment solution obtained in the partial information case is robust, in the sense specified at the outset.

The two additional diagrams in Figure 4 illustrate the robustness of the optimal strategy for case "PARTIAL" in the situation when the true state is "GOOD" and "BAD", respectively, when shorting is allowed and with the defaultable assets having a considerably high return (analogous to case (c) in Figure 1).

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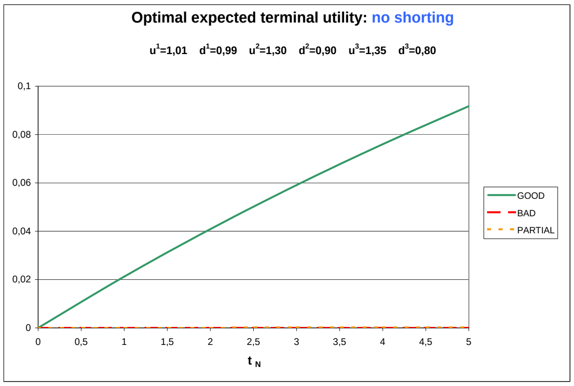
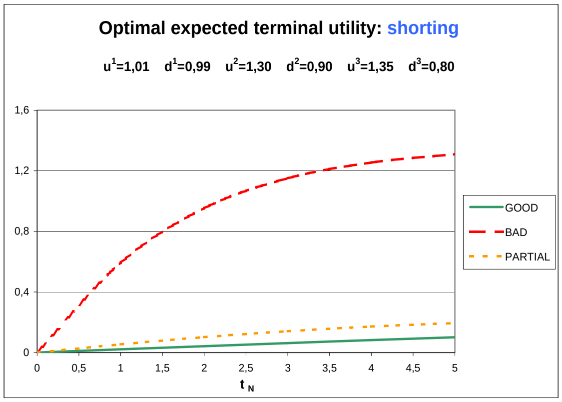
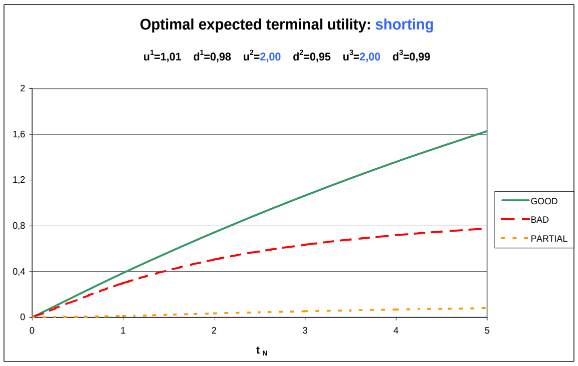


(a) Shorting not allowed.



(b) Shorting allowed, reasonable assets' returns.



(c) Shorting allowed, high defaultable assets' returns.

Figure 1: Optimal expected utility from terminal wealth, when $V_0 = 1, h_0 = h^1$.

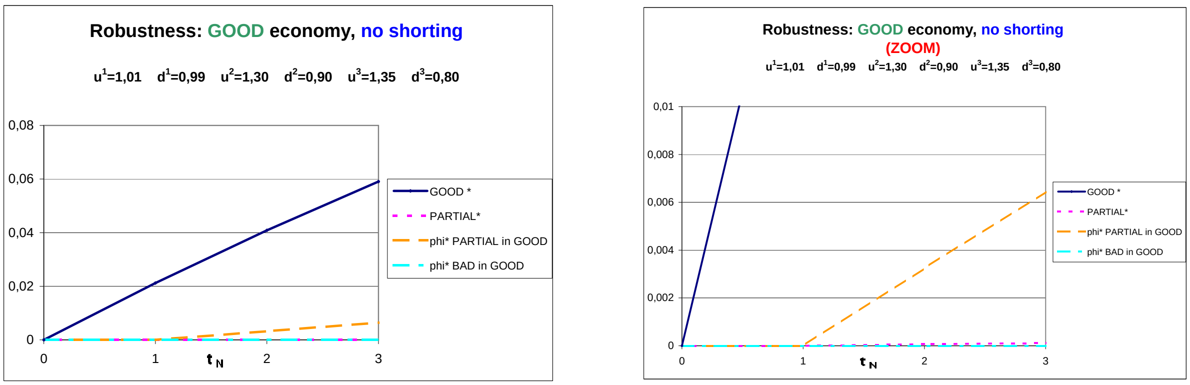


Figure 2: Robustness: shorting not allowed, GOOD state.

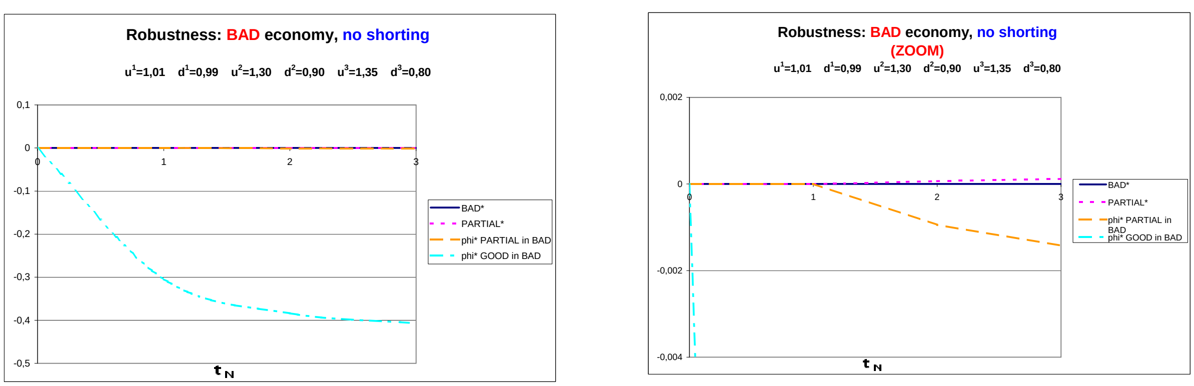
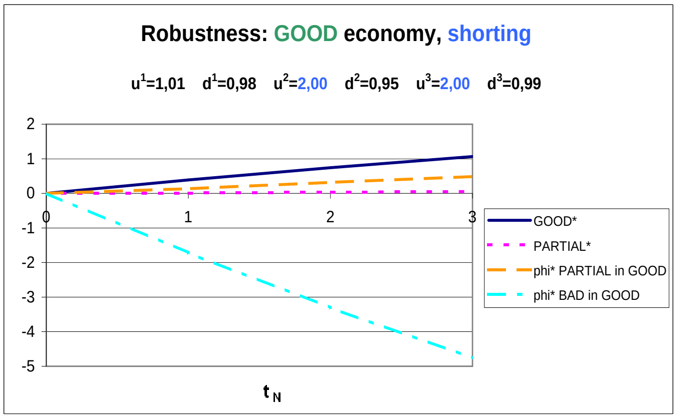
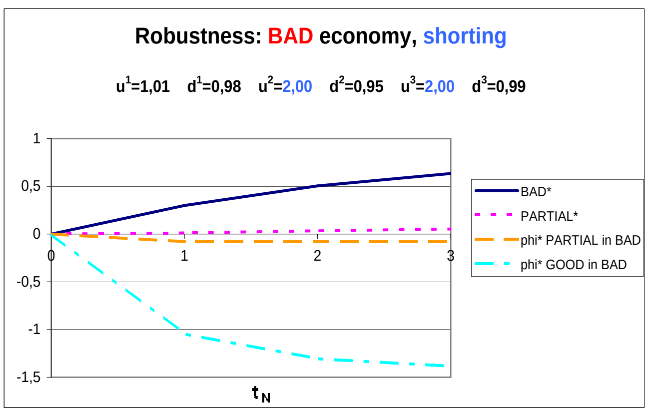


Figure 3: Robustness: shorting not allowed, BAD state.



(a) Robustness: shorting allowed, high assets' returns, GOOD state.



(b) Robustness: shorting allowed, high assets' returns, BAD state.

Figure 4: Robustness: shorting allowed, high assets' returns.