On Classical and Restricted Impulse Stochastic Control for the Exchange Rate

Giulio Bertola
Department of Mathematics
University of Padua, Italy
giulio@powerlinux.it*

Wolfgang J. Runggaldier
Department of Mathematics
University of Padua, Italy
runggal@math.unipd.it†

Kazuhiro Yasuda
Faculty of Science and Engineering
Hosei University, Japan
k_yasuda@hosei.ac.jp‡

Abstract

Our problem is motivated by an exchange rate control problem, where the control is composed of a direct impulsive intervention and an indirect, continuously acting intervention given by the control of the domestic interest rate. Similarly to [2] we formulate it as a mixed classical-impulse control problem. Analogously to [2], our approach builds on a quasi-variational inequality, which we consider here in a weakened version, and we too start by conjecturing the optimal solution to have a specific structure. While in [2] the horizon is infinite thus leading to a time-homogeneous solution and the value function is supposed to be of class $\mathcal{C}^1$ throughout, we have a finite horizon T and the value function is allowed not to be $\mathcal{C}^1$ at the boundaries of the continuation region. By suitably restricting the class of impulse controls, we obtain a fully analytical solution.

Mathematics Subject Classification: 93E20, 91G80, 49N25, 60H30, 49L20.

Keywords: Impulse control, quasi-variational inequalities, verification theorem, regularity of value function, exchange rate control.

1 Introduction

In [2] Cadenillas and Zapatero consider the problem of a Central Bank to optimally control the exchange rate by using two non-excluding tools: direct intervention in the foreign exchange market and indirect intervention through determination of the interest rate levels. Interest rates have in fact an effect on the exchange rate through the attraction or deflection of foreign capital. The control of the interest rate is of the type of a continuously acting control, while the direct intervention on the exchange rate is of the type of an impulsive control. The problem thus concerns a mixed classical-impulsive stochastic control problem and the authors in [2] aim at determining this control in order to balance the purpose of keeping the exchange rate as close as possible to a given target set and on the other hand to minimize the expected total cost of the intervention. The approach in [2] builds on the notion of quasi-variational inequality (QVI). The control horizon in [2] is infinite, which leads to a time-homogeneous solution and the authors search for such a solution within a specific class,

*Postal address: Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy.
†Postal address: Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy.
‡Postal address: Faculty of Science and Engineering, Hosei University, 3-7-2, Kajino-cho, Koganei-shi, Tokyo 184-8584, Japan.

for which the value function has to be $\mathcal{C}^1$ throughout and $\mathcal{C}^2$ except for two boundary points. They end up with six conditions on four parameters, but they nevertheless come up with a solution by using partly a numerical approach.

In this paper we consider basically the same problem as in [2] concentrating on the case of a finite-horizon that leads more generally to a time-non-homogeneous solution but our technique can also be applied to the infinite-horizon time-homogeneous case. In addition to the theoretical interest of studying the problem on a finite horizon, for which the solution is time-inhomogeneous and thus more complex, there is also a financial interest. Like in other economic problems, where one searches for a strategy that achieves a certain objective, one tries to reach this objective within a finite planning horizon. In fact, the context for a given problem might change within a foreseeable amount of time, requiring the problem to be reformulated anew at the end of the planning horizon. For our exchange rate problem notice also that, analogously to what was done in the relevant literature, we consider only two currencies: the given domestic one and one specific foreign currency. This specific foreign currency may in fact be of interest only over a given horizon, after which the interest may shift to another one thus changing the problem.

We aim at a fully analytical solution within a specific class, which contains however the optimal solution obtained in [2]. It will allow us to drop the requirement of having a $\mathcal{C}^1-$value function throughout thus ending up with four conditions on four parameters, which allows then for a fully analytical solution. For our specific class of strategies, which we shall denote by $\mathcal{A}$, we assume, as in [2] and in general in the literature, that it consists in an indirect intervention through a continuously acting control of the interest rate when the exchange rate is in a specific interval, namely the *continuation region* that in the finite-horizon case depends on the current time. On the other hand, the impulse control intervenes when the exchange rate reaches either of the two boundaries of that interval, namely when it enters the *intervention region*. At that point the Central Bank is supposed to intervene directly by pushing the exchange rate to yet another interval, the *preferred region*. We thus consider a restricted form of impulse control and the determination of the control action reduces thus to the determination of the boundaries of the continuation region and of the amount by which the exchange rate is pushed back when it reaches either one of the two boundaries. More precisely, the solution is supposed to be given by four continuous functions $a(\cdot) < \alpha(\cdot) < \beta(\cdot) < b(\cdot)$ with $\beta(\cdot) > 0$, where $a(\cdot)$ and $b(\cdot)$ represent the boundaries of the continuation region, while $\alpha(\cdot)$ and $\beta(\cdot)$ represent the boundaries of the preferred region, namely the values to which the exchange rate is shifted when it reaches $a(\cdot)$ or $b(\cdot)$ respectively. Finding a fully optimal solution within this class is still not possible in a purely analytical way and this is why in [2] the solution was obtained by a combination with a numerical approach. In our search for a completely analytical solution we therefore make a further assumption that the value function is given, within the continuation region, by the quadratic solution (20) of the HJB equation (18) below and is linear outside this region. This assumption is motivated by the following:

i) It is intuitive, given the quadratic costs.

ii) It is consistent with the solution obtained in [2] for the $\infty-$horizon problem.

iii) It allows one to drop the requirement of having a $\mathcal{C}^1$ value function throughout; in fact, the numerical results in section 6 show that the solution may not be $\mathcal{C}^1$ at the boundary points.

Restricting the solution of the HJB equation within the continuation region to a quadratic function implies that our optimal solution may actually be only an upper bound to the true optimal solution; it leads however to an explicitly computable strategy. The true optimal solution, even with controls in the class $\mathcal{A}$, may have a very complicated structure within the continuation region and we do not know of any result to this effect in the literature.

The control of the exchange rate has been studied previously. It was initiated as an application of stochastic impulse control in [4] and then further developed in [5]. Indirect intervention by using the control also of the interest rate appears in [7]. Building on [1], the authors in [2] then generalized the previous approaches. The solution in [2] can however not be obtained in a fully analytic way and the aim here is to fill this gap. As example within a more general context of impulse control, but without indirect intervention through the control of the interest rate, the problem has also been studied in [6] (see [3] as well), but over an infinite horizon and with a long-term average cost criterion. On the other hand, our controller may act in two complementary ways through a continuously acting control represented by the indirect intervention through the interest rate and an impulse control given by direct interventions on the exchange rate. Furthermore, we concentrate on a given finite horizon and this makes the problem time inhomogeneous.

In the next section 2 we present the model, thereby building mainly on [2]. In the following section 3 we state our problem and introduce various relevant concepts, among them a weaker notion of a QVI, define the class of solutions that we consider and derive some preliminary results. The solution of the resulting HJB equation and of the four equations for the four parameters is derived in section 4. In section 5 we prove a verification theorem showing that the solution derived in section 4 is indeed optimal within the class specified in section 3. Finally, some numerical illustrations are provided in section 6.

2 The model

Our model corresponds to that in [2] that we recall in this section. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and W be a one dimensional $\mathcal{F}_t$-Brownian motion. Consider a given foreign currency (e.g. the Dollar) and a domestic currency (e.g. the Euro) and define, for each $t \in \mathbb{R}_+$ the exchange rate at time t as

$$X_t \doteq \text{units of domestic currency for one unit of foreign currency.}$$

Suppose that X is an adapted stochastic process given by

$$X_t = x + \int_0^t \big(\mu\, X_s + K\, u_s\big)\, \mathrm{d}s + \sigma \int_0^t X_s\, \mathrm{d}W_s + \sum_{n=1}^{\infty} 1_{\{\tau_n \leq t\}} \xi_n, \tag{1}$$

where x is the initial value of the exchange rate and u is the process defined by

$$u_t \doteq \log \frac{r_t}{\bar{r}}.$$

Here, $\mu \in \mathbb{R}$ represents the exogenous economic pressure on the level of the exchange rate: $\mu > 0$ indicates a pressure towards a devaluation of the domestic currency, for example, as a consequence of political reasons, while $\mu < 0$ indicates the opposite situation. The constant $K \in (-\infty, 0)$ represents the influence of the interest rate on the level of the exchange rate and the constant $\sigma \in (0, \infty)$ is the exogenous volatility of the exchange rate. The stochastic process r is the domestic interest rate and the constant $\bar{r}$ is the target. Furthermore, τ_n is the time of the n-th intervention and ξ_n represents the amount of the n-th intervention. The stochastic process u measures the relationship between the interest rate level r set by the Central Bank and the target $\bar{r}$. Since we consider the logarithm of the ratio, if the interest rate level r is above the target $\bar{r}$, u will be positive and, therefore, the interest rate pushes X downwards; if r is below $\bar{r}$, the opposite happens. The justification is given by the fact that $\bar{r}$ is perceived as a natural equilibrium rate on the long run and deviations from this rate make the domestic assets more or less attractive.

Definition 1. A *mixed classical-impulsive stochastic control* is a triple

$$(u, \mathcal{T}, \xi) = (u;\, \tau_1,\, \tau_2,\, \tau_3, \ldots,\, \tau_n, \ldots;\, \xi_1,\, \xi_2,\, \xi_3, \ldots,\, \xi_n, \ldots).$$

Here u is a classical stochastic control, namely $u : \mathbb{R}_+ \times \Omega \longmapsto \mathbb{R}$ is an $\mathcal{F}_t$-adapted stochastic process. Furthermore, the pair $(\mathcal{T}, \xi)$ is an impulsive control, namely $0 \leq \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_n < \ldots$ is an increasing sequence of stopping times and $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of random variables such that $\xi_n : \Omega \longmapsto \mathbb{R}$ is $\mathcal{F}_{\tau_n}$-measurable. The Central Bank (the controller) decides to act at time τ_n adding the quantity ξ_n to the value of the exchange rate at that moment of time, namely $X_{\tau_n+} = X_{\tau_n} + \xi_n$. Since we want $X_{\tau_n+} > 0$, we consider only those mixed stochastic controls for which

$$\mathbb{P}\Big(X_t \in \mathbb{R}_+ \quad \forall\, t \in \mathbb{R}_+\Big) = 1. \tag{2}$$

In this paper we shall consider a finite horizon $T > 0$ and thus require that $\tau_i \le T$. We shall call *generically admissible* the mixed classical-impulsive stochastic control that possesses the above properties and denote their class by $\mathcal{A}^g$. Given the horizon $T > 0$, consider the following cost functional

$$\mathcal{J}(t,x;u,\mathcal{T},\xi) \doteq \mathbb{E}_{t,x}\left\{ \int_t^T e^{-\lambda(s-t)} f(X_s,u_s)\,\mathrm{d}s + e^{-\lambda(T-t)} h(X_T) + \sum_{n=1}^{\infty} e^{-\lambda(\tau_n - t)} g(\xi_n)\, 1_{\{t\le\tau_n\le T\}} \right\}.$$

where

$$f(x,u) \doteq (x-\rho)^2 + k\,u^2, \tag{3}$$

$$g(\xi) \doteq \begin{cases} C + c\,\xi & \text{if } \xi > 0 \\ \min(C,\,D) & \text{if } \xi = 0 \\ D - d\,\xi & \text{if } \xi < 0 \end{cases}, \tag{4}$$

$$h(x) \doteq \ell\,(x-\rho)^2, \quad \ell > 0$$

$$k, \ell, \lambda,\ \rho,\ C,\ c,\ D,\ d \quad \text{are positive constants.}$$

Here, f is the running cost caused by the deviation from the target that has been set, both for the exchange rate as well as for the interest rate: ρ is the target for the exchange rate, $\bar{r}$ is the target for the interest rate, k is a positive constant (by increasing k one penalizes the use of the continuous control), $h(X_T)$ represents the terminal cost. Furthermore, C and D represent the fixed intervention costs if the Central Bank pushes the exchange rate upward or downward respectively, c and d represent proportional costs for each intervention when the Central Bank pushes the exchange rate upward or downward respectively and λ is a discount rate. It is for the sake of generality that we consider here also a discount factor with intensity λ. However, as pointed out in [3], an exchange rate is not an asset and so the functions f, h, g do not represent a tangible cost leaving a discount factor without a clear economic interpretation. This would not change the situation for the finite horizon case that is our major setting here; for the infinite horizon case an average cost criterion could then be preferable though.

In this context we may now consider the following problem: the Central Bank aims at selecting an admissible triple $(u,\,\mathcal{T},\,\xi)$ in the class $\mathcal{A}^g$, which minimizes the functional $\mathcal{J}$ defined by

$$\mathcal{J}(0,x;u,\mathcal{T},\xi) = \mathbb{E}_{0,x}\left\{ \int_0^T e^{-\lambda s} f(X_s,u_s)\,\mathrm{d}s + e^{-\lambda T} h(X_T) + \sum_{n=1}^{\infty} e^{-\lambda\tau_n} g(\xi_n)\, 1_{\{\tau_n\le T\}} \right\}. \tag{5}$$

In this generality the problem is very difficult to solve and so we shall look for an optimal solution within a subclass of strategies and value functions that is however still rather general, contains the optimal solution in [2], and above all admits an analytical solution.

3 The specific problem

3.1 Definitions and preliminary notions

Let $\Sigma \doteq [0,T]\times\mathbb{R}$. For each $(t,x)\in\Sigma$ define the value function as

$$V(t,x) \doteq \inf\Big\{ \mathcal{J}(t,x;u,\mathcal{T},\xi) :\ (u,\mathcal{T},\xi)\in\mathcal{A} \Big\}, \tag{6}$$

where as $\mathcal{A}$ we shall take a subclass of $\mathcal{A}^g$ according to Definition 3 below. For each continuous function $\phi : \Sigma \to \mathbb{R}$ define now the *minimal cost operator* M as

$$M\phi(t,x) \doteq \inf \Big\{ \phi(t, x+\eta) + g(\eta) : \, \eta \in \mathbb{R}, \, x+\eta \in (0,\infty) \Big\}$$

and notice that, for $\phi \in \mathcal{C}^2$, the following differential operator is well defined

$$\mathcal{L}^u \phi(t,x) \doteq \frac{\sigma^2}{2} \, x^2 \phi_{xx}(t,x) + \big(\mu\, x + Ku\big)\phi_x(t,x) - \lambda\, \phi(t,x).$$

3.2 Quasi-variational inequalities (QVI)

As in [2], our solution methodology is based on the notion of a quasi-variational inequality (QVI) that we use here in a weaker form than what is commonly done. More precisely, we introduce the following

Definition 2 (QVI)**.** We say that a function $v : \Sigma \longrightarrow \mathbb{R}_+$ satisfies the (weak) quasi-variational inequality (QVI) for Problem 1, defined in the next subsection 3.3, if

$$\left(v_t + \mathcal{L}^u v\right)(t,x) + f(x,u) \geq 0, \quad \forall t \in [0,T] \text{ and a.a. } x \in \mathbb{R} \text{ with } v(T,x) = h(x) \tag{7}$$

and

$$Mv(t,x) - v(t,x) \geq 0 \quad \text{everywhere.} \tag{8}$$

Furthermore, noticing that from (7) we obtain

$$v_t(t,x) + \inf_{u \in \mathbb{R}} \Big\{ \mathcal{L}^u v(t,x) + f(x,u) \Big\} \geq 0, \quad \forall t \in [0,T] \text{ and a.a. } x \in \mathbb{R} \tag{9}$$

we require that at least one of the two inequalities, (8), (9), holds as an equality.

Observe that a solution v of the QVI separates Σ into two disjoint regions: the continuation region

$$\mathcal{C} \doteq \Big\{ (t,x) \in \Sigma : \, v(t,x) < Mv(t,x), \quad v_t(t,x) + \inf_{u\in\mathbb{R}} \Big\{ \mathcal{L}^u v(t,x) + f(x,u) \Big\} = 0 \Big\}$$

and the intervention region

$$\mathcal{I} \doteq \Big\{ (t,x) \in \Sigma : \, v(t,x) = Mv(t,x), \quad v_t(t,x) + \inf_{u\in\mathbb{R}} \Big\{ \mathcal{L}^u v(t,x) + f(x,u) \Big\} \geq 0, \ \forall t \in [0,T] \text{ and a.a. } x \in \mathbb{R} \Big\}.$$

3.3 Specific class of solutions

As suggested by the results in [2], we conjecture that the optimal solution of Problem 1, defined below in this subsection, satisfies QVI and has the following structure: for given continuous functions $a(\cdot) < \alpha(\cdot) < \beta(\cdot) < b(\cdot)$ with $\beta(\cdot) > 0$ let the value function $V(t,x)$ have the form

$$V(t,x) = \begin{cases} \Phi(t,\alpha(t)) + C + c(\alpha(t) - x) & \text{if } x \leq a(t) \\ \Phi(t,x) & \text{if } a(t) < x < b(t) \\ \Phi(t,\beta(t)) + D + d(x - \beta(t)) & \text{if } x \geq b(t) \end{cases} \tag{10}$$

where $\Phi(t,x)$ satisfies

$$\begin{aligned} &\big(\Phi_t + \mathcal{L}^{\hat{u}}\Phi\big)\big(t,\hat{X}_t\big) + f\big(\hat{X}_t, \hat{u}_t\big) = \Phi_t\big(t,\hat{X}_t\big) + \inf_{u\in\mathbb{R}} \Big\{ \mathcal{L}^u \Phi\big(t,\hat{X}_t\big) + f\big(\hat{X}_t, u\big) \Big\} = 0 \\ &\Phi(T,x) = h(x) = \ell\,(x-\rho)^2 \end{aligned} \tag{11}$$

with $\hat{u}$ a control achieving the inf on the right hand side in (11) and where $\hat{X}_t$ is the process X_t corresponding to this control $\hat{u}$. Such a minimizing control $\hat{u}$ exists indeed and is given, for each $t \in [0,T]$, by

$$\hat{u}_t = -\frac{K}{2k}\,\Phi_x\big(t, \hat{X}_t\big), \tag{12}$$

implying that the corresponding optimal interest rate is

$$\hat{r}_t = \bar{r}\,\exp\left(-\frac{K}{2k}\,\Phi_x\big(t, \hat{X}_t\big)\right). \tag{13}$$

Furthermore, given always the continuous functions $a(\cdot) < \alpha(\cdot) < \beta(\cdot) < b(\cdot)$ with $\beta(\cdot) > 0$, the mixed classical-impulsive stochastic control $(u, \mathcal{T}, \xi)$ is taken to be in the class where u is as in (12) and $(\mathcal{T}, \xi)$ are of the form

$$\tau_n = \inf\Big\{t > \tau_{n-1} : X_t \notin \big(a(t), b(t)\big)\Big\}, \quad (\tau_0 = 0) \tag{14}$$

$$X_{\tau_n+} = X_{\tau_n} + \xi_n = \beta(\tau_n)\, 1_{\{b(\tau_n)\}}(X_{\tau_n}) + \alpha(\tau_n)\, 1_{\{a(\tau_n)\}}(X_{\tau_n}) \tag{15}$$

Since, modulo a slight adjustment, the optimal solution obtained by [2] belongs to the class described above, we shall restrict the original class of generic admissible controls $\mathcal{A}^g$ to a subclass $\mathcal{A}$ of *admissible* controls according to the following definition and search for an optimal solution within this class.

Definition 3. We say that a mixed classical-impulsive stochastic control $(u, \mathcal{T}, \xi)$ is admissible if u is of the form as in (12) and there exists four continuous functions $a(\cdot) < \alpha(\cdot) < \beta(\cdot) < b(\cdot)$ with $\beta(\cdot) > 0$ such that (14) and (15) are satisfied. Furthermore, the functional $\mathcal{J}(0, x; u, \mathcal{T}, \xi)$ in (5) has to be finite. We shall denote by $\mathcal{A}$ this class of admissible mixed classical-impulsive stochastic controls for the process X_t.

Limiting ourselves to the class of solutions as specified above, the continuation region can then more specifically be expressed as

$$\mathcal{C} = \{(t,x) \in \Sigma \,:\, a(t) < x < b(t)\} \tag{16}$$

and, consequently, the intervention region is then $\mathcal{I} = \{(t,x) \in \Sigma \setminus \mathcal{C}\}$. Next we formalize the assumption of a quadratic solution as it was described in the Introduction, namely

Assumption 1. The value function $V(t,x)$ is supposed to be given, within the continuation region $\mathcal{C}$, by $V(t,x) = \Phi(t,x)$ where $\Phi(t,x)$ is quadratic in x (see (20)).

We can now formulate our specific problem as follows

Problem 1. Determine the value function $V(t,x)$, defined in (6), in the form as given in (10) where, for $a(t) < x < b(t)$, the function $\Phi(t,x)$ is given by the quadratic solution (34) of the HJB equation (32), (33) below. Furthermore, determine the optimal mixed classical-impulsive control $(u, \mathcal{T}, \xi) \in \mathcal{A}$ that achieves the *inf* in (6).

Remark 1. i) Note that a control in $\mathcal{A}$ depends on the choice of the four boundaries $a(\cdot), \alpha(\cdot), \beta(\cdot), b(\cdot)$ and, if X_t ($t \in [0,T]$) is controlled by a control in $\mathcal{A}$, then

$$\mathbb{P}\Big(X_t \in [a(t), b(t)], \quad \forall\, t > 0\Big) = 1. \tag{17}$$

Now, for admissibility in $\mathcal{A} \subset \mathcal{A}^g$ we had required (see (2)) that $X_t > 0$ a.s. A sufficient condition for this is that the constants $d,\ D,\ \ell,\ \rho$ are positive, as we had assumed. In fact, from Lemma 1 and Remark 3 in Section 4.1 below it then follows first that $p(t) > 0$, while $q(t) < 0$. This, as well as $b(\cdot) \geq \beta(\cdot)$ and the fact that $\beta(\cdot)$ will be determined by the second equation in (42), lead then to $b(t) \geq \beta(t) > 0$. By (15) this implies that an impulse shift at the upper boundary cannot bring X_t to assume negative values. Furthermore, the fact that $q(t) < 0$ as well as the structure of the continuous control in (12) with $\Phi(t,x)$ of the form of (34) below imply, via a comparison theorem applied to (1), that $X_t > 0$ a.s. Finally, even if we would have $a(t) < 0$, no impulsive control would be applied at the lower boundary and so we still would always have $X_t > 0$.

ii) Thanks to the particular structure of our class of admissible impulse controls, namely by defining them through the strategic boundaries $a(\cdot) < \alpha(\cdot) < \beta(\cdot) < b(\cdot)$ with $\beta(\cdot) > 0$, as well as to the structure of the value function as given in (10), in the proof of the verification theorem below (see Lemma 2) we do not need to use Itô's formula on the entire state space. We only need to apply it on the open interval $(a(\cdot), b(\cdot))$, which allows us to avoid the requirement of C^1-regularity throughout as it is required by the smooth fit condition in general stopping and impulse control problems (see e.g. [9]). Notice also that if we would apply the generalized Itô's formula with local times (see e.g. in [8]) as it is often done, then we would run into difficulties to manage the local time terms. With the controls restricted to our admissible class, the state space is restricted to the open interval $(a(\cdot), b(\cdot))$ which, on one hand, prevents the application of the generalized Itô's formula, on the other hand it allows us however to apply the standard Itô's formula on the process in the open interval $(a(\cdot), b(\cdot))$. Finally, we want to point out that a further consequence of our class of admissible impulse controls is that in equation (9) equality may hold also in the intervention region $\mathcal{I}$.

3.4 Preliminaries in view of the solution approach

Notice that in $\mathcal{C}$ the value function $V(t,x)$ has to satisfy (11) and so in the continuation region the optimal $\hat{u}$ and $\hat{r}$ are given by (12) and (13) respectively. Substituting (12) into (11) we obtain that, within our solution class, the optimal value function has to satisfy, in $\mathcal{C}$, the following HJB equation

$$V_t(t,x) + \frac{\sigma^2}{2} x^2 V_{xx}(t,x) - \frac{K^2}{4k} \Big(V_x(t,x)\Big)^2 + \mu\, x\, V_x(t,x) - \lambda\, V(t,x) + (x-\rho)^2 = 0, \tag{18}$$

with the terminal condition

$$V(T,x) = \ell\,(x-\rho)^2. \tag{19}$$

It is a Cauchy problem for a nonlinear partial differential equation for which, according to Assumption 1, we search for a solution of the form

$$V(t,x) = p(t)\,x^2 + q(t)\,x + R(t) \tag{20}$$

that will be explicitly determined in in Section 4.

On the other hand, to be a QVI solution, in the intervention region the value function has to satisfy $V(t,x) = M\,V(t,x)$ and so it has to have the form

$$\begin{aligned} V(t,x) &= V(t,\alpha(t)) + C + c(\alpha(t) - x), \quad \forall\, x \in (-\infty, a(t)] \\ V(t,x) &= V(t,\beta(t)) + D + d(x - \beta(t)), \quad \forall\, x \in [b(t), \infty) \end{aligned} \tag{21}$$

implying that, if $V(t,x)$ is differentiable with respect to x in $x = \alpha(t)$ and $x = \beta(t)$, then one has

$$V_x(t,\alpha(t)) = -c \qquad \text{and} \qquad V_x(t,\beta(t)) = d. \tag{22}$$

Remark 2. A solution $V(t,x)$, that in $\mathcal{C}$ satisfies (18) and is of the form $V(t,x) = p(t)\,x^2 + q(t)\,x + R(t)$, implies that $V_x(t,x) = 2p(t)x + q(t)$ which is then bounded in $\mathcal{C}$. The derivative $V_x(t,x)$ is bounded also in $(-\infty, a(t)]$ and $[b(t), \infty)$ since, by (21), we have $V_x(t,x) = -c$ in $(-\infty, a(t)]$ and $V_x(t,x) = d$ in $[b(t), \infty)$. Taking also into account that $\hat{u}$ is given by (12), for each admissible control we then have that the following properties are satisfied

$$\mathbb{E}\left\{\int_0^T e^{-\lambda s}\, X_s^2 \,\mathrm{d}s\right\} < \infty, \tag{23}$$

$$\mathbb{E}\left\{\int_0^T e^{-\lambda s}\, u_s^2 \,\mathrm{d}s\right\} < \infty, \tag{24}$$

$$\mathbb{E}\left\{\sum_{n=1}^{\infty} e^{-\lambda\tau_n}\, 1_{\{\tau_n \le T\}}\right\} < \infty, \tag{25}$$

$$\mathbb{E}\left\{\sum_{n=1}^{\infty} e^{-\lambda\tau_n} |\xi_n|\, 1_{\{\tau_n \le T\}}\right\} < \infty. \tag{26}$$

From here it follows that, for each admissible control, the functional $\mathcal{J}(0, x; u, \mathcal{T}, \xi)$ in (5) is automatically finite. Furthermore, inequality (25) implies that

$$\mathbb{P}\left(t \le \lim_{n\to\infty} \tau_n \le T\right) = 0. \tag{27}$$

It also follows that, for our finite horizon $T > 0$, the admissible controls here coincide with those in [2] (see Definition 2.2 in [2]).

Summarizing, we are searching for a solution of Problem 1 within the class as described by (10) and (11) and with controls having the structure (12) (which is equivalent to (13)) as well as (14), (15). This solution depends on the choice of the four continuous functions a, b, α, β that, by (21) and (22), have to satisfy the following system of equations

$$V(t, a(t)) = V(t, \alpha(t)) + C + c\big(\alpha(t) - a(t)\big), \tag{28}$$

$$V(t, b(t)) = V(t, \beta(t)) + D + d\big(b(t) - \beta(t)\big), \tag{29}$$

$$V_x(t, \alpha(t)) = -c, \tag{30}$$

$$V_x(t, \beta(t)) = d, \tag{31}$$

with V satisfying (18), (19) and given in the form of (20). The system (28) to (31) forms, for each $t \in [0, T]$, a system of four equations in the four unknowns $a(\cdot), b(\cdot), \alpha(\cdot), \beta(\cdot)$. In section 5 we shall provide a verification result showing that a solution of (28) to (31) with V satisfying (18)-(20) leads indeed to an optimal solution of our Problem 1.

In their infinite time horizon model, Cadenillas and Zapatero in [2] add two more conditions to assure the continuity of the first derivative of the value function also in the points a and b (in the infinite horizon setup the value function does not depend on time and so also the functions a and b are constants, i.e. $a(t) \equiv a$, $b(t) \equiv b$) and so to be able to apply the standard Ito formula in the verification theorem to prove the optimality of their solution. In this way however the authors in [2] obtain six conditions on only the four parameters a, b, α, β. This creates a problem that they try to overcome by using a numerical procedure. In our context we obtain a completely analytic solution that we then illustrate by numerical results.

4 The solution of the HJB equation

In this section we present the quadratic solution to the HJB equation (18), (19) as well as the solution of the equations (28)-(31).

4.1 Solution of the HJB equation

We want to find an explicit solution for the Cauchy problem given, for a generic function $\Phi(t,x)$, by the PDE

$$\Phi_t(t,x) + \frac{\sigma^2}{2}\, x^2\, \Phi_{xx}(t,x) - \frac{K^2}{4k}\Big(\Phi_x(t,x)\Big)^2 + \mu\, x\, \Phi_x(t,x) - \lambda\, \Phi(t,x) + (x-\rho)^2 = 0, \tag{32}$$

and the terminal condition

$$\Phi(T,x) = \ell\,(x-\rho)^2. \tag{33}$$

According to section 3, see in particular (20), we search for a solution of the form

$$\Phi(t,x) = p(t)\, x^2 + q(t)\, x + R(t), \tag{34}$$

where p, q and R are appropriate functions in $\mathcal{C}^1([0,T];\mathbb{R})$. The partial derivatives are

$$\begin{aligned} \Phi_t(t,x) &= p'(t)\, x^2 + q'(t)\, x + R'(t),\\ \Phi_x(t,x) &= 2p(t)\, x + q(t),\\ \Phi_{xx}(t,x) &= 2p(t). \end{aligned}$$

Observe that the terminal condition (33) is equivalent to the following terminal conditions on p, q and R:

$$p(T) = \ell, \qquad q(T) = -2\ell\rho, \qquad R(T) = \ell\rho^2.$$

Substituting (34) into (32) we obtain

$$\begin{aligned} &\left[p'(t) - \frac{K^2}{k}\big(p(t)\big)^2 + \big(-\lambda + 2\mu + \sigma^2\big)p(t) + 1\right] x^2 \\ &\qquad + \left[q'(t) + \left(-\frac{K^2}{k}p(t) + \mu - \lambda\right) q(t) - 2\rho\right] x \\ &\qquad\qquad + R'(t) - \lambda R(t) - \frac{K^2}{4k}\big(q(t)\big)^2 + \rho^2 = 0. \end{aligned}$$

It thus suffices to solve the following system of ordinary differential equations

$$p'(t) - \frac{K^2}{k}\big(p(t)\big)^2 + \big(-\lambda + 2\mu + \sigma^2\big)p(t) + 1 = 0, \tag{35}$$

$$q'(t) + \left(-\frac{K^2}{k}p(t) + \mu - \lambda\right) q(t) - 2\rho = 0, \tag{36}$$

$$R'(t) - \lambda R(t) - \frac{K^2}{4k}\big(q(t)\big)^2 + \rho^2 = 0, \tag{37}$$

with the corresponding terminal conditions

$$p(T) = \ell, \qquad q(T) = -2\ell\rho, \qquad R(T) = \ell\rho^2.$$

In the following we give the solutions of these ODEs, the detailed calculations that lead to these solutions can be found in the Appendix (section 7).

Putting

$$\Delta = \sqrt{\big(-\lambda + 2\mu + \sigma^2\big)^2 + 4\frac{K^2}{k}},$$

the solution of (35) is

$$p(t) = \frac{k}{K^2}\left(\frac{-\lambda + 2\mu + \sigma^2 - \Delta}{2} + \frac{\Delta}{C_1\, e^{\Delta t} + 1}\right),$$

where the constant C_1 is determined by the terminal condition $p(T) = \ell$, i.e.

$$C_1 = \left(\frac{2\Delta}{2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta} - 1\right) e^{-\Delta T}.$$

Lemma 1. For each $t \in [0, T]$, one has $p(t) > 0$.

Proof. The proof follows easily from the definition of C_1. If $C_1 \geq 0$ we have

$$\begin{aligned} C_1 e^{\Delta t} &= \left(\frac{2\Delta}{2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta} - 1 \right) e^{\Delta(t-T)} \\ &\leq \frac{2\Delta}{2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta} - 1, \end{aligned}$$

from which

$$\begin{aligned} p(t) &\geq \frac{k}{K^2}\left[\frac{-\lambda + 2\mu + \sigma^2 - \Delta}{2} + \frac{\Delta\left(2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta\right)}{2\Delta} \right] \\ &= \ell > 0. \end{aligned}$$

If $C_1 < 0$ it results

$$p(t) \geq \frac{k}{K^2}\left(\frac{-\lambda + 2\mu + \sigma^2 - \Delta}{2} + \Delta \right),$$

which is positive by the definition of Δ. □

Equation (36) has as solution

$$q(t) = \frac{C_1 + 1}{C_1 e^{\Delta t} + 1}\, e^{\frac{1}{2}(\lambda + \sigma^2 + \Delta)t} \left\{ \frac{4\rho}{C_1 + 1}\left[C_1 \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 + \Delta)t} - 1}{-\lambda - \sigma^2 + \Delta} + \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 - \Delta)t} - 1}{-\lambda - \sigma^2 - \Delta} \right] + C_2 \right\},$$

where the constant C_2 is determined by the terminal condition $q(T) = -2\ell\rho$, namely

$$C_2 = -\frac{2\rho}{C_1 + 1}\left\{ 2\left[C_1 \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 + \Delta)T} - 1}{-\lambda - \sigma^2 + \Delta} + \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 - \Delta)T} - 1}{-\lambda - \sigma^2 - \Delta} \right] + \ell\left(C_1 e^{\Delta T} + 1\right) e^{-\frac{1}{2}(\lambda + \sigma^2 + \Delta)T} \right\}.$$

Remark 3. Notice that, for all $t \in [0, T]$, the deterministic function $q(t)$ is negative. This is best seen from the fact that the ODE (36) can be written as $q'(t) = F(t)q(t) + 2\rho$ with $q(T) = -2\ell\rho$ where one sets $F(t) = \frac{K^2}{k}p(t) - \mu + \lambda$. Then the solution can be expressed as $q(t) = e^{\int_0^t F(u)du}(-2\rho \int_t^T e^{-\int_0^s F(u)du} ds - 2\ell\rho e^{-\int_0^T F(u)du})$, so that from the assumption $\rho,\ \ell > 0$, we have $q(t) < 0$.

Since the optimal continuous control $\hat{u}_t$ in (12), and thus also the solution of Problem 1, does not depend on $R(t)$, we present the solution of (37) in implicit form

$$R(t) = e^{\lambda t}\left\{ \int_0^t \left[\frac{K^2}{4k}\left(q(s)\right)^2 - \rho^2 \right] e^{-\lambda s}\, \mathrm{d}s + C_3 \right\},$$

with C_3 defined by $R(T) = \ell\rho^2$, namely

$$C_3 = \ell\rho^2 e^{-\lambda T} - \int_0^T \left[\frac{K^2}{4k}\left(q(s)\right)^2 - \rho^2 \right] e^{-\lambda s}\, \mathrm{d}s.$$

4.2 Determining a, b, α, β

We determine now the four continuous functions that characterize the solution (14), (15), (12) and (13). We derive these functions explicitly from the equations (see (28)-(31)), written for a generic function $\Phi(t, x)$,

$$\Phi(t, a(t)) = \Phi(t, \alpha(t)) + C + c\big(\alpha(t) - a(t)\big), \tag{38}$$

$$\Phi(t, b(t)) = \Phi(t, \beta(t)) + D + d\big(b(t) - \beta(t)\big), \tag{39}$$

$$\Phi_x(t, \alpha(t)) = -c, \tag{40}$$

$$\Phi_x(t, \beta(t)) = d, \tag{41}$$

for each $t \in [0, T]$, where Φ is defined in (34) (namely as is $V(t, x)$ in (20)). From (40) and (41) we obtain

$$\alpha(t) = \frac{-c - q(t)}{2p(t)}, \qquad \beta(t) = \frac{d - q(t)}{2p(t)} \quad \forall t \in [0, T]. \tag{42}$$

while the identities (38) and (39) lead to

$$a(t) = \frac{-c - q(t) - 2\sqrt{Cp(t)}}{2p(t)}, \qquad b(t) = \frac{d - q(t) + 2\sqrt{Dp(t)}}{2p(t)} \tag{43}$$

for each $t \in [0, T]$. Notice that the above functions are continuous as requested (see Definition 3).

Remark 4. Note that the boundaries a, b, α, β given in (42) and (43) depend on the quadratic form of the solution Φ of the HJB equation (32) in the continuation region.

5 A verification result

In this section we present a verification result proving that the solution described in sections 3 and 4 is indeed optimal for our Problem 1 if $a(t)$ and $b(t)$ as given in (43) satisfy suitable inequalities that involve also $\alpha(t)$ and $\beta(t)$ as given in (42). We start by proving the following lemma

Lemma 2. Let $\Phi(t, x)$ be a solution of the HJB equation (18)-(19) as derived in section 4.1 in the form of (34) and let $(u, \mathcal{T}, \xi)$ be a mixed classical-impulse control according to (12) as well as (14), (15), where $a(t), \alpha(t), \beta(t), b(t)$ satisfy (28)-(31) and are thus given by (42), (43). Then $V(t, x)$ defined according to (10) is the optimal value function for our Problem 1 and $(u, \mathcal{T}, \xi)$ is an optimal control provided $V(t, x)$ satisfies the conditions required in Definition 2, namely satisfies the weak QVI.

Proof. Let $(u, \mathcal{T}, \xi)$ be an admissible control and denote by X_t the trajectory corresponding to $(u, \mathcal{T}, \xi)$. Note that property (23) and the boundedness of V_x imply

$$\mathbb{E}\left\{ \int_0^T \left[e^{-\lambda t} X_t\, V_x(t, X_t)\right]^2 \mathrm{d}t \right\} < \infty. \tag{44}$$

We next derive an expression for $e^{-\lambda(t\wedge\tau_n)} V(t \wedge \tau_n, X_{(t\wedge\tau_n)+}) - V(0, x)$ for which, analogously to the proofs of the classical verification theorems, we apply the Itô formula to the function $e^{-\lambda t} V(t, X_t)$, but only between the stopping times τ_i and recalling that, for an admissible control, we have $X_t \in [a(t), b(t)]$ a.s. (see (17)). For each

$t \in [0,T]$ and $n \in \mathbb{N}$, we thus obtain

$$\begin{aligned} e^{-\lambda(t\wedge\tau_n)} & V(t\wedge\tau_n, X_{(t\wedge\tau_n)+}) - V(0,x) \\ &= \sum_{i=1}^{n} \left[\int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s} \left(V_t + \mathcal{L}^{u_s} V\right)(s, X_s) 1_{\mathbb{R}_+\smallsetminus\{a(s),b(s)\}}(X_s)\, \mathrm{d}s \right. \\ &\qquad \left. + \sigma \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s} X_s\, V_x(s, X_s) 1_{\mathbb{R}_+\smallsetminus\{a(s),b(s)\}}(X_s)\, \mathrm{d}W_s \right] \\ &\qquad + \sum_{i=1}^{n} 1_{\{\tau_i \le t\}} e^{-\lambda\tau_i} \big[V(\tau_i, X_{\tau_i+}) - V(\tau_i, X_{\tau_i})\big]. \end{aligned}$$

The inequality (7) implies that

$$\left(V_t + \mathcal{L}^{u_t} V\right)(t, X_t) \ge -f(X_t, u_t)$$

$\forall\, t \in [0,T]$ and a.a. values of X_t. The a.a. values of X_t that we shall exclude are, for each $t \in [0,T]$, the values $X_t = a(t)$ and $X_t = b(t)$. Furthermore, (8) in Definition 2 implies

$$V(\tau_i, X_{\tau_i+}) - V(\tau_i, X_{\tau_i}) = M\, V(\tau_i, X_{\tau_i}) - V(\tau_i, X_{\tau_i}) - g(\xi_i) \ge -g(\xi_i) \quad \forall\, i \in \mathbb{N}.$$

Therefore, for each $t \in [0,T]$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned} & e^{-\lambda(t\wedge\tau_n)} V(t\wedge\tau_n, X_{(t\wedge\tau_n)+}) - V(0,x) \\ & \quad \ge -\textstyle\sum_{i=1}^{n} \left[\int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s} f(X_s, u_s)\, \mathrm{d}s + \sigma \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s} X_s\, V_x(s, X_s)\, \mathrm{d}W_s\right] - \sum_{i=1}^{n} 1_{\{\tau_i \le t\}} e^{-\lambda\tau_i} g(\xi_i). \end{aligned}$$

This inequality is an identity for the QVI-control specified in the statement of the lemma. From condition (27) we deduce

$$e^{-\lambda(t\wedge\tau_n)} V(t\wedge\tau_n, X_{(t\wedge\tau_n)+}) \xrightarrow{n\to\infty} e^{-\lambda t} V(t, X_{t+}) \qquad \text{a.s.}.$$

Taking expectations, we obtain

$$\begin{aligned} & V(0,x) - \mathbb{E}\left\{e^{-\lambda t} V(t, X_{t+})\right\} \\ & \quad \le \mathbb{E}\left\{ \sum_{i=1}^{\infty} \left[\int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s} f(X_s, u_s)\, \mathrm{d}s - \sigma \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s} X_s\, V_x(s, X_s)\, \mathrm{d}W_s\right] + \sum_{i=1}^{\infty} 1_{\{\tau_i \le t\}} e^{-\lambda\tau_i} g(\xi_i) \right\}, \end{aligned}$$

with equality for the QVI control described in the statement. Inequality (44) implies

$$e^{-\lambda t} X_t\, V_x(t, X_t) \in \mathcal{L}^2([0,T] \times \Omega),$$

therefore

$$\mathbb{E}\left\{ \int_{t\wedge\tau_{i-1}}^{t\wedge\tau_i} e^{-\lambda s} X_s\, V_x(s, X_s)\, \mathrm{d}W_s \right\} = 0 \qquad \forall\, t \in [0,T],\ i \in \mathbb{N}.$$

Consequently

$$V(0,x) - \mathbb{E}\left\{e^{-\lambda t} V(t, X_{t+})\right\} \le \mathbb{E}\left\{ \int_0^t e^{-\lambda s} f(X_s, u_s)\, \mathrm{d}s + \sum_{i=1}^{\infty} 1_{\{\tau_i \le t\}} e^{-\lambda\tau_i} g(\xi_i) \right\},$$

with equality for the QVI control described in the statement. Finally, substituting T for t and recalling, see (7), that $V(T,x) = h(x)$, we get

$$\begin{aligned} V(0,x) &\le \mathbb{E}\left\{ \int_0^T e^{-\lambda s} f(X_s, u_s)\, \mathrm{d}s + e^{-\lambda T} h(X_T) + \sum_{i=1}^{\infty} 1_{\{\tau_i \le T\}} e^{-\lambda\tau_i} g(\xi_i) \right\} \\ &= \mathcal{J}(0, x; u, \mathcal{T}, \xi), \end{aligned}$$

where one has equality for the QVI-control described in the statement. The conclusion then follows. □

We now state and prove the verification theorem

Theorem 1. Let the function $V(t,x)$ satisfy (10), namely in $\Sigma \setminus \mathcal{C}$ it is given by (21), while in $\mathcal{C}$ it is supposed to satisfy (18), (19) and be given by (20) with the coefficients as determined in Section 4.1. Let the continuous functions $a(t), \alpha(t), \beta(t), b(t)$ satisfy (28)-(31) and thus be given by (42), (43). Let the control $(u, \mathcal{T}, \xi)$ be given by (12) as well as (14), (15) for $a(t), \alpha(t), \beta(t), b(t)$ as given by (42), (43). If for each $t \in [0,T]$,

$$a(t) \le \frac{1}{2}\left\{ c(\mu-\lambda) + 2\rho - \sqrt{\Big[c(\lambda-\mu) - 2\rho\Big]^2 - 4\,A(t)} \right\}, \tag{45}$$

where

$$A(t) = \Phi_t(t,\alpha(t)) + \Phi_x(t,\alpha(t))\alpha'(t) + c\,\alpha'(t) - \lambda\Big[\Phi(t,\alpha(t)) + C + c\,\alpha(t)\Big] + \rho^2$$

and

$$b(t) \ge \frac{1}{2}\left\{ d(\lambda-\mu) + 2\rho + \sqrt{\Big[d(\mu-\lambda) - 2\rho\Big]^2 - 4\,B(t)} \right\}, \tag{46}$$

where

$$B(t) = \Phi_t(t,\beta(t)) + \Phi_x(t,\beta(t))\beta'(t) - d\,\beta'(t) - \lambda\Big[\Phi(t,\beta(t)) + D - d\,\beta(t)\Big] + \rho^2$$

and, furthermore, the right hand sides in (45) and (46) are well defined, then $V(t,x)$ is the optimal value function of our Problem 1, namely it is optimal among the value functions that are quadratic in the continuation region and such that

$$V(0,x) = \inf\big\{\mathcal{J}(0,x;u,\mathcal{T},\xi) : (u,\mathcal{T},\xi) \in \mathcal{A}\big\}. \tag{47}$$

Furthermore, the above strategy is optimal in the sense that it achieves the infimum in (47).

Proof. If we show that $V(t,x)$ satisfies the conditions for the QVI of Definition 2, then Lemma 2 ensures that $V(t,x)$ given by (10) is the optimal value function and the optimal strategy is given by (14), (15) and (12)-(13). In fact, $V(t,x)$ as defined in (10) with $\Phi(t,x)$ given by (34) (see also (20)) is $\mathcal{C}^2$ except in the points $a(t)$ and $b(t)$ for each $t \in [0,T]$, where it is not even $\mathcal{C}^1$. For the QVI of Definition 2 it thus suffices to verify (7)-(9) and that the QVI control associated with V is admissible. For this latter purpose notice that, in the continuation region where $X_t \in [a(t), b(t)]$, the process $\hat{u}$, defined by

$$\hat{u}_t = -\frac{K}{2k}\,V'(X_t) = -\frac{K}{2k}\,(2pX_t + q)\,,$$

is bounded and so $E\left\{\int_0^T e^{-\lambda t}\hat{u}_t^2\,\mathrm{d}t\right\} < \infty$, implying that $\mathcal{J}(0,x;u,\mathcal{T},\xi)$ in (5) is finite.

Coming to (7)-(9) notice that, excluding $X_t = a(t)$ and $X_t = b(t)$, we have

$$\begin{aligned} &V_t(t,x) + \inf_{u\in\mathbb{R}}\big\{\mathcal{L}^u V(t,x) + f(x,u)\big\} \\ &\quad = V_t(t,x) + \frac{\sigma^2}{2}\,x^2\,V_{xx}(t,x) - \frac{K^2}{4k}\Big(V_x(t,x)\Big)^2 1_{(a(t),b(t))}(x) + \mu\,x\,V_x(t,x) - \lambda\,V(t,x) + (x-\rho)^2 \end{aligned}$$

$$= \begin{cases} \Phi_t(t,\alpha(t)) + \Phi_x(t,\alpha(t))\alpha'(t) + c\,\alpha'(t) - c\mu x - \lambda\Big[\Phi(t,\alpha(t)) + C + c(\alpha(t) - x)\Big] + (x-\rho)^2 & \text{if } x < a(t) \\ 0 & \text{if } a(t) < x < b(t) \\ \Phi_t(t,\beta(t)) + \Phi_x(t,\beta(t))\beta'(t) - d\,\beta'(t) + d\mu x - \lambda\Big[\Phi(t,\beta(t)) + D + d(x - \beta(t))\Big] + (x-\rho)^2 & \text{if } x > b(t) \end{cases}.$$

For $x < a(t)$ it holds that

$$\begin{aligned} &V_t(t,x) + \inf_{u\in\mathbb{R}}\big\{\mathcal{L}^u V(t,x) + f(x,u)\big\} \\ &\quad = x^2 + \Big[c(\lambda-\mu) - 2\rho\Big]\,x + \Phi_t(t,\alpha(t)) + \Phi_x(t,\alpha(t))\alpha'(t) + c\,\alpha'(t) - \lambda\Big[\Phi(t,\alpha(t)) + C + c\,\alpha(t)\Big] + \rho^2, \end{aligned}$$

with roots

$$x_{1,2} = \frac{1}{2}\left\{c(\mu-\lambda)+2\rho \pm \sqrt{\Big[c(\lambda-\mu)-2\rho\Big]^2 - 4A(t)}\right\}.$$

Condition (45) then ensures

$$V_t(t,x) + \inf_{u\in\mathbb{R}} \left\{\mathcal{L}^u V(t,x) + f(x,u)\right\} \geq 0, \tag{48}$$

for each $x < a(t)$. Analogously, one can see that the condition (46) implies (48) for each $x > b(t)$ and so (7) and (9) are satisfied with (9) holding as an equality in the continuation region $\mathcal{C}$.

On the other hand, the minimum cost operator is given by

$$Mv(t,x) = \begin{cases} \Phi(t,\alpha(t)) + C + c(\alpha(t)-x) & \text{if } x \leq \alpha(t) \\ \Phi(t,x) + \min(C,D) & \text{if } \alpha(t) < x < \beta(t) \\ \Phi(t,\beta(t)) + D + d(x-\beta(t)) & \text{if } x \geq \beta(t) \end{cases}, \tag{49}$$

where, to obtain Mv in the interval $(\alpha(t),\beta(t))$, we used the fact that, by the way in which α,β are defined and by the convexity of Φ, we have $-c < v_x(t,x) < d \quad \forall\,(t,x) : \alpha(t) < x < \beta(t)$.

Therefore, $v(t,x) - M\,v(t,x)$ is zero in the intervention region $\{(t,x) : x \in (-\infty, a(t)] \cup [b(t),\infty)\}$ and is negative in the continuation region $\{(t,x) : x \in (a(t),b(t))\}$. In fact, it suffices to notice that, by the way in which α and β have been defined,

$$\begin{aligned} v_x(t,x) &\leq -c \qquad && \forall\,(t,x) : a(t) \leq x \leq \alpha(t), \\ v_x(t,x) &\geq d && \forall\,(t,x) : \beta(t) \leq x \leq b(t), \end{aligned}$$

This leads to (8) as well as (9) for what concerns the intervention region thus concluding the proof. □

6 Numerical illustrations

In this section, we give some numerical illustrations in order to get a better understanding of our theoretical results. In the following subsections we show, for various situations, the shapes of the functions $a(t)$, $b(t)$, $\alpha(t)$ and $\beta(t)$ for all $t \in [0,T]$, as well as the value function and the optimal interest rate at selected time points.

6.1 Maturity $T = 1$: Case 1

In this subsection, we show the behavior of $a(t)$, $b(t)$, $\alpha(t)$ and $\beta(t)$, and the shapes of the value function and of the optimal interest rate at the time points $t = 0,\ 0.5,\ 0.9$ using the parameters in Table 1. For those parameters the conditions (45) and (46) in Theorem 1 are satisfied. Note that the sign of K does not affect a possible trend for the exchange rate because, given our continuous strategy $\hat{u}_t = -\frac{K}{2k}\Phi_x(t,\hat{X}_t)$, it is only the sign of $\Phi_x(t,\hat{X}_t)$ that influences it.

From Figure 1, we can see the behavior of $a(t)$, $b(t)$, $\alpha(t)$ and $\beta(t)$ for $t \in [0,T]$. The functions $a(t)$ and $\alpha(t)$ are both decreasing and $b(t)$ and $\beta(t)$ are both increasing. The reason is that the strength of the penalty with respect to the terminal condition of the exchange rate, namely the size of $\ell = 1$, is the same as the one with respect to the running cost, that is the coefficient 1 of the first term in $f(x,u)$. Therefore, as a policy, the Central Bank does not strongly try to reach its target exchange rate ρ at the maturity $T = 1$, so at first it narrows the exercise boundaries to the level of the target exchange rate $\rho = 1.3$ since it has much residual time, but towards maturity the power is weakened. Note, however, that this monotonicity does not always hold, it depends on the size of the parameters, especially μ, K, k, ℓ and ρ. In Figure 1, there are some asymmetries, namely the distance from the target exchange rate ρ to the upper boundary $b(t)$ is larger than that from ρ to the lower boundary $a(t)$, and the distance between $b(t)$ and $\beta(t)$ is larger than the difference between $a(t)$ and $\alpha(t)$. This behavior is mainly determined by the parameter values for the impulse penalties, namely C, c, D

Table 1: parameters (case 1)

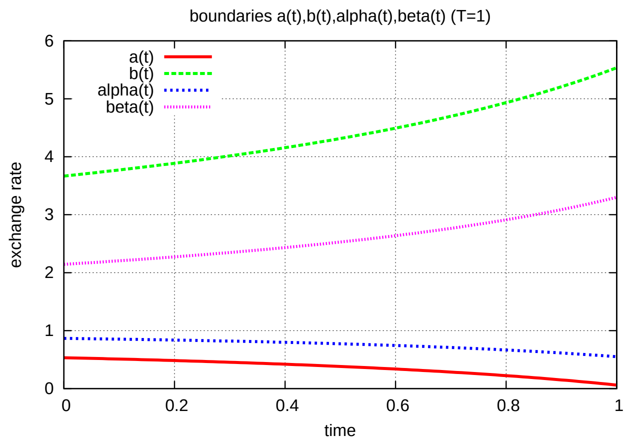
parameter	notation	size
drift	μ	0.05
target interest rate	$\bar{r}$	0.04
coefficient of continuous control	K	-0.2
volatility	σ	0.2
level of exchange rate	ρ	1.3
coefficient of running cost	k	2
penalty of impulse (fixed cost for lower bound)	C	0.24
penalty of impulse (proportional cost for lower bound)	c	1.5
penalty of impulse (fixed cost for upper bound)	D	5
penalty of impulse (proportional cost for upper bound)	d	4
coefficient of terminal condition	ℓ	1
discount factor	λ	0.01
maturity	T	1

and d and that for the continuous control, namely k. In fact, $C = 0.24 < D = 5$ and $c = 1.5 < d = 4$, that is, the intervention at the upper boundary is more costly than that at the lower boundary. Hence the Central Bank does not want to intervene too often at the upper boundary, and so the upper boundary is positioned farther from the target exchange rate ρ. The differences $\alpha(t) - a(t)$ and $b(t) - \beta(t)$ are due to similar reasons. As mentioned above, the Central Bank does not want to intervene often at the upper boundary due to the cost; therefore, when it decides to intervene, it does it decisively at a single time and the difference $b(t) - \beta(t)$ becomes large.

In Figure 2 we show the relationship between the exchange rate and the optimal interest rate $\hat{r}_t = \bar{r}\exp(\hat{u}_t)$ in the continuation region at the time points $t = 0,\ 0.5,\ 0.9$. They are increasing functions with exponential growth. In the drift term of our exchange rate model (1), we use u_t which is defined as $u_t = \log \frac{r_t}{\bar{r}}$, and our running cost with respect to u_t in the value function is the quadratic function ku_t^2 which is a symmetric function. Because of the asymmetry of the logarithmic function, when the interest rate is less than the target interest rate $\bar{r}$, the Central Bank is penalized much more than in the case when it is larger than $\bar{r}$. In Figure 2 we find that, for all time points, the optimal interest rate does not fall much below the target rate $\bar{r} = 0.04$, but exceeds it by far more. On the other hand, the gradients of the optimal interest rate are becoming small as the residual time is decreasing since, as mentioned above, now the Central Bank does not strongly try to reach its target exchange rate ρ at the maturity $T = 1$ and does not care anymore about the market towards the maturity avoiding that the interest rate value moves far from its target rate.

From Figure 3 we can find that the value function at time $t = 0$ is quadratic in the interval $(a(0), b(0))$, is linear outside of $(a(0), b(0))$, is continuous on $\mathbb{R}$, but is not smooth at the exercise boundaries $a(0)$ and $b(0)$. The gradients of the value function at $\alpha(0)$ and $\beta(0)$ are the same as the slopes c and d respectively. They show the typical shape of our value function in Theorem 1, and the value functions in the following subsections have a similar shape.

From Figure 4, we can see that our value function varies with t, that is, it depends on time since our optimization problem has a finite horizon. The minimizers of the value functions tend to the level of the target exchange rate $\rho = 1.3$. This is not too easy to see from Figure 4 but under different parameter values, as those for Figures 8 and 12 below, one can clearly see it. The shapes of the value functions are similar to the one at time $t = 0$ in Figure 3. Again, it is hard to see it from Figure 4, but the value functions do not always decrease monotonically for each exchange rate. We can clearly see it from Figure 8 below.



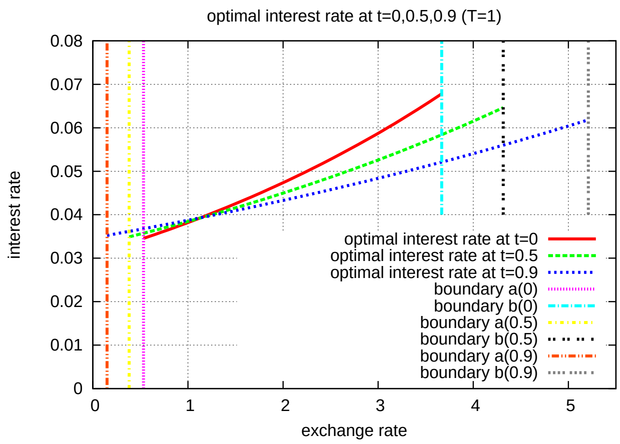
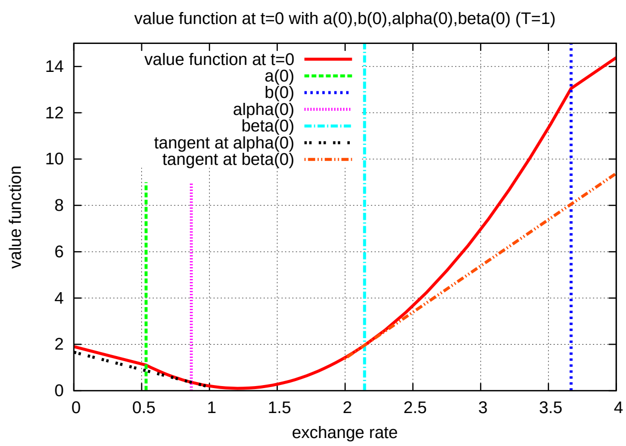


Figure 1: Behavior of optimal boundaries $a(t)$, $b(t)$, $\alpha(t)$, $\beta(t)$ (case 1)

Figure 2: Optimal interest rate at time $t = 0, 0.5, 0.9$ (case 1)



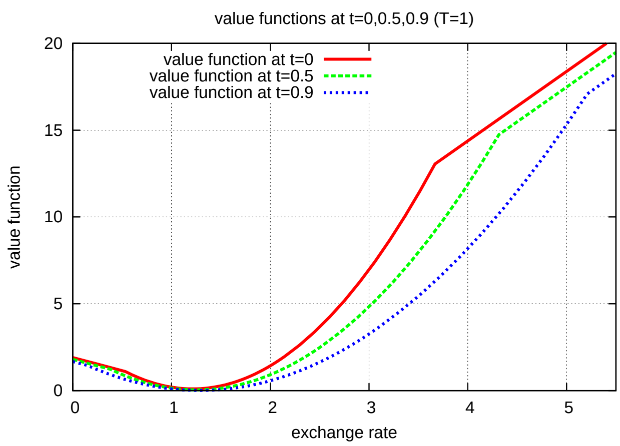


Figure 3: Value function at time $t = 0$ with boundaries $a(0)$, $b(0)$, $\alpha(0)$, $\beta(0)$ (case 1)

Figure 4: Value functions at time $t = 0,\ 0.5,\ 0.9$ (case 1)

6.2 Maturity $T = 1$: Case 2

In this subsection we present analogous diagrams to the previous subsection by using different parameter values as shown in Table 2 for which the conditions (45) and (46) in Theorem 1 are again satisfied.

Table 2: parameters (case 2)

parameter	notation	size
drift	μ	0.05
target interest rate	$\bar{r}$	0.04
coefficient of continuous control	K	-0.2
volatility	σ	0.2
level of exchange rate	ρ	1.3
coefficient of running cost	k	0.24
penalty of impulse (fixed cost for lower bound)	C	1.5
penalty of impulse (proportional cost for lower bound)	c	1
penalty of impulse (fixed cost for upper bound)	D	1.1
penalty of impulse (proportional cost for upper bound)	d	1.2
coefficient of terminal condition	ℓ	5
discount factor	λ	0.01
maturity	T	1

From Figure 5, we can see the behavior of the functions $a(t)$, $b(t)$, $\alpha(t)$ and $\beta(t)$ in the case of Table 2. The strength of the penalty with respect to the terminal condition of the exchange rate, namely the size of $\ell = 5$, is larger than the one with respect to the running cost, that is the coefficient 1 of the first term in $f(x, u)$. This situation denotes that, as a policy, the Central Bank wants strongly to reach its target exchange rate ρ at the maturity $T = 1$, so it narrows the exercise boundaries to the level of the target exchange rate ρ towards maturity. For this reason, in Figure 5, $a(t)$ is increasing and $b(t)$ is decreasing. These features are the opposite of those in the previous subsection. On the other hand, $\alpha(t)$ and $\beta(t)$ are both increasing. Furthermore, here the fixed cost for the intervention at the upper boundary, namely $D = 1.1$, is less than the one in the previous subsection (before it was $D = 5$), so the upper exercise boundary $b(t)$ is closer to the target exchange rate ρ than before. For the lower boundary the cost $C = 1.5$ is now larger than the one in the previous subsection (before it was $C = 0.24$), so at first the lower exercise boundary $a(t)$ is farther away from ρ than before, but towards maturity it becomes closer to the target than before due to the strength of the penalty with respect to the terminal condition of the exchange rate, namely $\ell = 5$. The proportional costs with respect to intervention $c = 1$ and $d = 1.2$ are both smaller than those in the previous subsection (before $c = 1.5$ and $d = 4$). If the proportional cost is small, then the Central Bank tries to push the exchange rate closer to the target exchange rate ρ. Therefore, $\alpha(t)$ and $\beta(t)$ are now closer to ρ than those in the previous subsection.

In Table 2, the strength of the penalty with respect to the running cost of the interest rate, namely $k = 0.24$, is smaller than before. Thus the Central Bank uses the interest rate to control the exchange market even if the interest rate is far away from the target value. We can see it from Figure 6 and the ranges of the optimal interest rate on $[a(t), b(t)]$ are $[3.78 \times 10^{-3}, 0.347]$ at time $t = 0$, $[3.33 \times 10^{-3}, 0.387]$ at time $t = 0.5$ and $[2.84 \times 10^{-3}, 0.444]$ at time $t = 0.9$. In contrast with the previous subsection, the gradients of the optimal interest rate are becoming large around the upper boundary and small around the lower boundary as the residual time is decreasing because of the opposite goals. Now the Central Bank wants to strongly reach its target exchange rate ρ at the maturity $T = 1$, so the Central Bank strongly intensifies the market control by using the interest rate up to the limit values towards the maturity. It is also one of the purposes of this example to show that, even if the parameters satisfy the requirements of Theorem 1, due to the balance between the intervention costs one might end up with unrealistically large values of the interest rate. In fact, for small values of the penalty due to using control values of the interest rate that deviate from the target value, the Central Bank might be induced to use the interest rate up to the limit values in order to push the exchange rate to its goal even if the interest rate itself differs considerably from its target value.

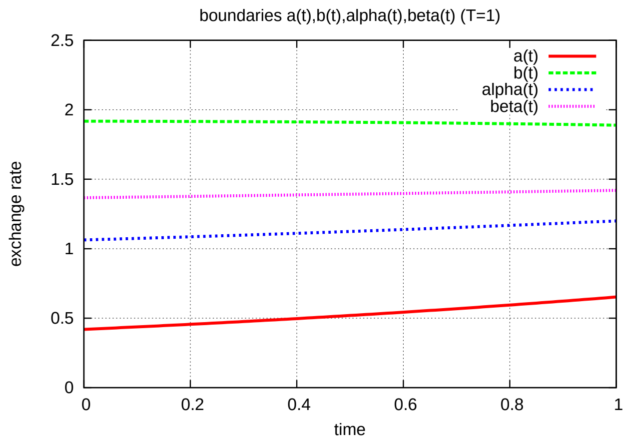


Figure 5: Behavior of optimal boundaries $a(t)$, $b(t)$, $\alpha(t)$, $\beta(t)$ (case 2)

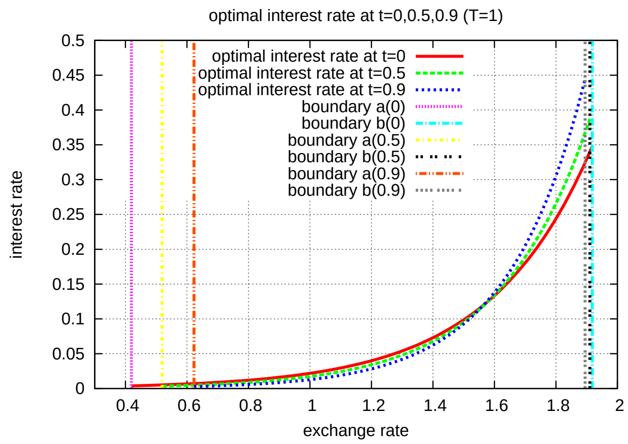


Figure 6: Optimal interest rate at time $t = 0, 0.5, 0.9$ (case 2)

From Figure 7 we can find that the value function is quadratic in $(a(0), b(0))$, is linear outside of $(a(0), b(0))$, and not smooth at the exercise boundaries $a(0)$ and $b(0)$. This figure shows the same behavior as Figure 3 in the previous subsection. From Figure 8 we see again that our value function varies with t, that is, it depends on time since our problem is a finite horizon problem. The minimizers of the value functions tend again to the level of the target exchange rate ρ. This tendency is analogous to that in Figure 4 in the previous subsection. Finally, from Figure 8 we can find that the value functions do not always decrease monotonically for each exchange rate.

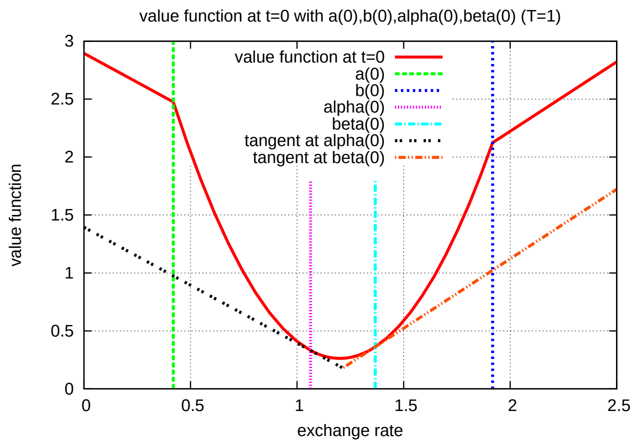


Figure 7: Value function at time $t = 0$ with boundaries $a(0)$, $b(0)$, $\alpha(0)$, $\beta(0)$ (case 2)

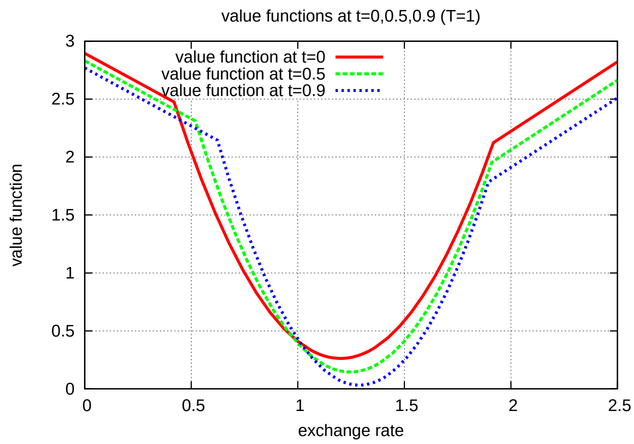


Figure 8: Value functions at time $t = 0,\ 0.5,\ 0.9$ (case 2)

6.3 Long Maturity $T = 5$

In this subsection we show behavior of the functions $a(t)$, $b(t)$, $\alpha(t)$ and $\beta(t)$, as well as the shapes of the value function and of the optimal interest rate at the time points $t = 0,\ 2.5,\ 4$ using the parameters in Table 1 except for the maturity $T = 5$. Again, for those parameter values the conditions (45) and (46) in Theorem 1 are satisfied.

From Figure 9 we can see the behavior of $a(t)$, $b(t)$, $\alpha(t)$ and $\beta(t)$. Their features are similar to those in Section 6.1, but here the boundaries $a(t)$ and $\alpha(t)$ are not monotonic functions. Note how the shape of the boundaries between times 4 and 5 in Figure 9 is completely analogous to that from time 0 to 1 in Figure 1 because of the time homogeneous coefficients in our model.

Figure 10 shows the relationship between the exchange rate and the optimal interest rate at the time points $t = 0,\ 2.5,\ 4$. Again, their behavior is similar to that in Section 6.1 and the optimal strategy of the exchange rate at time $t = 4$ is completely analogous to that at time $t = 0$ in Figure 2.

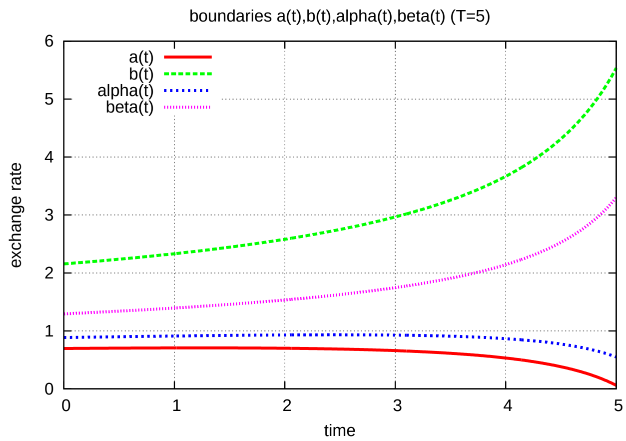


Figure 9: Behavior of optimal boundaries $a(t)$, $b(t)$, $\alpha(t)$, $\beta(t)$ (case $T = 5$)

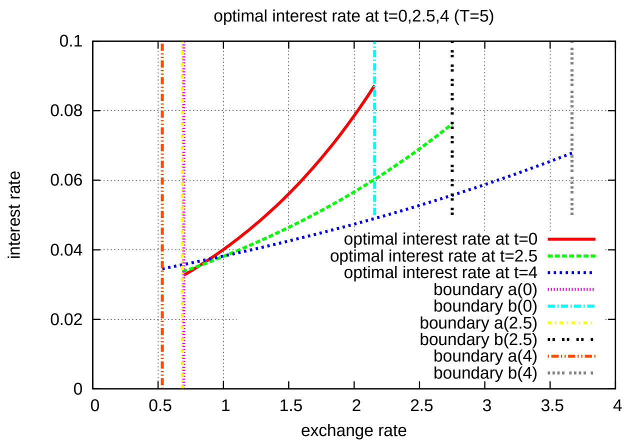


Figure 10: Optimal interest rate at time $t = 1, 2.5, 4$ (case $T = 5$)

From Figure 11, we can see that the value function is quadratic in $(a(0), b(0))$, is linear outside of $(a(0), b(0))$, and not smooth at the exercise boundaries $a(0)$ and $b(0)$. This figure is similar to Figure 3 and Figure 7 in the previous subsections.

From Figure 12 we can again see that the value function changes over time because of the finite horizon and shifts their minimizers towards the target exchange rate ρ as before. Note also that the value function at time 4 is completely analogous to the one in Figure 3 owing to the time homogeneous coefficients as above.

7 Appendix

In this Appendix we derive the solutions of the ODEs (35) and (36).

7.1 The solution of equation (35)

Consider the Riccati equation

$$p'(t) - \frac{K^2}{k}\big(p(t)\big)^2 + \big(-\lambda + 2\mu + \sigma^2\big)p(t) + 1 = 0, \tag{50}$$

with the corresponding terminal condition $p(T) = \ell$.

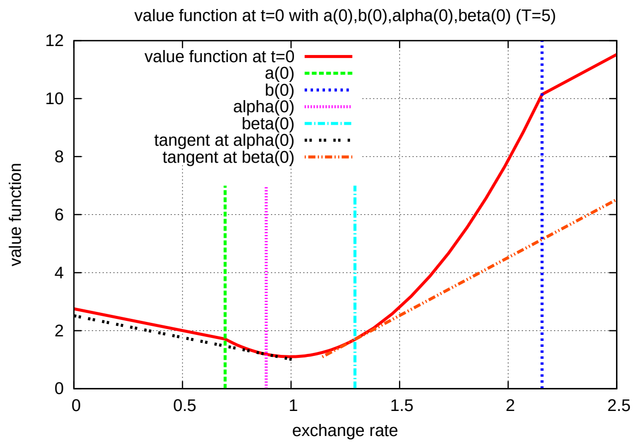


Figure 11: Value function at time $t = 0$ with boundaries $a(0)$, $b(0)$, $\alpha(0)$, $\beta(0)$ (case $T = 5$)

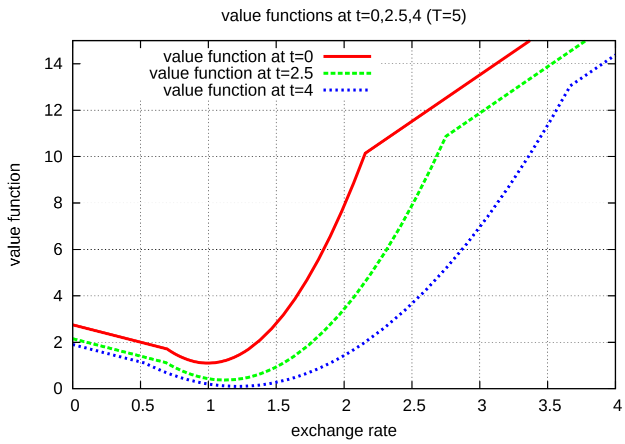


Figure 12: Value functions at time $t = 0$, 2.5, 4 (case $T = 5$)

We shall use two general results on ODEs, which can be found in sections 1.2.2-2(24) and 2.1.2-2(11) of [10]. Given an ODE $y'(t) + a\,\big(y(t)\big)^2 + b\,y(t) + c = 0$, by the transformation $y(t) = \frac{\omega'(t)}{a\,\omega(t)}$ we obtain a linear second order equation of the form $\omega''(t) + b\,\omega'(t) + ac\,\omega(t) = 0$.

The second general result is the following: given an ODE $\omega''(t) + a\,\omega'(t) + b\,\omega(t) = 0$ such that $\Lambda^2 \doteq a^2 - 4b > 0$, we have the solution

$$\omega(t) = e^{-\frac{1}{2}at}\left(L_1 e^{\frac{1}{2}\Lambda t} + L_2 e^{-\frac{1}{2}\Lambda t}\right).$$

Put now

$$p(t) = -\frac{k}{K^2}\,\frac{\omega'(t)}{\omega(t)},$$

for a function $\omega \in \mathcal{C}^2([0,T];\mathbb{R})$. The derivative p' is

$$p'(t) = \frac{k}{K^2}\,\frac{\big(\omega'(t)\big)^2}{\big(\omega(t)\big)^2} - \frac{k}{K^2}\,\frac{\omega''(t)}{\omega(t)}.$$

Substituting p and p' in (50) we obtain

$$\frac{k}{K^2}\,\frac{\big(\omega'(t)\big)^2}{\big(\omega(t)\big)^2} - \frac{k}{K^2}\,\frac{\omega''(t)}{\omega(t)} - \frac{K^2}{k}\,\frac{k^2}{K^4}\,\frac{\big(\omega'(t)\big)^2}{\big(\omega(t)\big)^2} - \big(-\lambda + 2\mu + \sigma^2\big)\,\frac{k}{K^2}\,\frac{\omega'(t)}{\omega(t)} + 1 = 0$$

$$\Longleftrightarrow \qquad \omega''(t) + \big(-\lambda + 2\mu + \sigma^2\big)\omega'(t) - \frac{K^2}{k}\omega(t) = 0.$$

Putting

$$\Delta = \sqrt{\big(-\lambda + 2\mu + \sigma^2\big)^2 + 4\frac{K^2}{k}},$$

the solution of the latter equation is

$$\omega(t) = e^{\frac{1}{2}(\lambda - 2\mu - \sigma^2)t}\left(L_1\,e^{\frac{1}{2}\Delta t} + L_2\,e^{-\frac{1}{2}\Delta t}\right) \qquad \forall\, L_1, L_2 \in \mathbb{R},$$

from which we obtain for the first order derivative

$$\omega'(t) = \frac{\lambda - 2\mu - \sigma^2}{2}\,\omega(t) + e^{\frac{1}{2}(\lambda - 2\mu - \sigma^2)t}\left(\frac{L_1}{2}\Delta\,e^{\frac{1}{2}\Delta t} - \frac{L_2}{2}\Delta\,e^{-\frac{1}{2}\Delta t}\right).$$

We have thus the following solution for (50):

$$
\begin{aligned}
p(t) = -\frac{k}{K^2}\,\frac{\omega'(t)}{\omega(t)} &= \frac{k}{K^2}\left(\frac{-\lambda+2\mu+\sigma^2}{2} - \frac{\frac{L_1}{2}\Delta\, e^{\frac{1}{2}\Delta t} - \frac{L_2}{2}\Delta\, e^{-\frac{1}{2}\Delta t}}{L_1\, e^{\frac{1}{2}\Delta t} + L_2\, e^{-\frac{1}{2}\Delta t}}\right)\\
&= \frac{k}{K^2}\left(\frac{-\lambda+2\mu+\sigma^2-\Delta}{2} + \frac{L_2\Delta\, e^{-\frac{1}{2}\Delta t}}{L_1\, e^{\frac{1}{2}\Delta t} + L_2\, e^{-\frac{1}{2}\Delta t}}\right)\\
&= \frac{k}{K^2}\left(\frac{-\lambda+2\mu+\sigma^2-\Delta}{2} + \frac{\Delta}{C_1\, e^{\Delta t}+1}\right).
\end{aligned}
$$

where the constant C_1 is determined by the terminal condition $p(T)=\ell$. It follows that

$$
\begin{aligned}
&\frac{\Delta}{C_1\, e^{\Delta T}+1} = \frac{K^2}{k}\ell + \frac{\lambda-2\mu-\sigma^2+\Delta}{2}\\
\iff\quad & C_1 = \left(\frac{2\Delta}{2\frac{K^2}{k}\ell+\lambda-2\mu-\sigma^2+\Delta} - 1\right) e^{-\Delta T}.
\end{aligned}
$$

7.2 The solution of equation (36)

Consider the ordinary differential equation

$$
q'(t) + \left(-\frac{K^2}{k}p(t) + \mu - \lambda\right) q(t) - 2\rho = 0, \tag{51}
$$

with the terminal condition

$$
q(T) = -2\ell\rho.
$$

Let us use the following general result: for an ODE

$$
y'(t) + a(t)\,y(t) - b(t) = 0,
$$

we have the following solution

$$
y(t) = e^{-\int_0^t a(s)\,\mathrm{d}s}\left[\int_0^t b(s)\, e^{\int_0^s a(\eta)\,\mathrm{d}\eta}\mathrm{d}s + C\right].
$$

The solution of (51) is then of the form

$$
q(t) = \exp\left((\lambda-\mu)t + \frac{K^2}{k}\int_0^t p(s)\,\mathrm{d}s\right)\left[2\rho\int_0^t \exp\left((\mu-\lambda)s - \frac{K^2}{k}\int_0^s p(\eta)\,\mathrm{d}\eta\right)\,\mathrm{d}s\, + C_2\right]. \tag{52}
$$

Let us first of all compute $\int_0^t p(s)\,\mathrm{d}s$:

$$
\int_0^t p(s)\,\mathrm{d}s = \frac{k}{K^2}\left(\frac{-\lambda+2\mu+\sigma^2-\Delta}{2}\,t + \Delta\int_0^t \frac{1}{C_1\, e^{\Delta s}+1}\,\mathrm{d}s\right). \tag{53}
$$

To compute explicitly the integral in (53), we perform the change of variable $e^{\Delta s} = \eta$:

$$
\begin{aligned}
\int_0^t \frac{1}{C_1\, e^{\Delta s}+1}\,\mathrm{d}s &= \int_1^{e^{\Delta t}} \frac{1}{(C_1\,\eta+1)}\,\frac{1}{\Delta\eta}\,\mathrm{d}\eta\\
&= \frac{1}{\Delta}\int_1^{e^{\Delta t}}\left(\frac{-C_1}{C_1\eta+1} + \frac{1}{\eta}\right)\mathrm{d}\eta\\
&= \frac{1}{\Delta}\left[\Delta t + \log\left(\frac{C_1+1}{C_1 e^{\Delta t}+1}\right)\right].
\end{aligned}
$$

Observe that the argument of the logarithm is always positive. In fact, $C_1 e^{\Delta t} + 1 \geq C_1 + 1 > 0$. To see the last inequality, it suffices to notice that, if

$$\frac{2\Delta}{2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta} - 1 \geq 0,$$

then obviously $C_1 + 1 > 0$ for each $t \in [0, T]$. On the other hand if

$$\frac{2\Delta}{2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta} - 1 < 0$$

we obtain

$$\begin{aligned} C_1 + 1 &= \left(\frac{2\Delta}{2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta} - 1 \right) e^{-\Delta T} + 1 \\ &\geq \frac{2\Delta}{2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta}, \end{aligned}$$

and this quantity is positive since

$$2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \Delta = 2\frac{K^2}{k}\ell + \lambda - 2\mu - \sigma^2 + \sqrt{\big(-\lambda + 2\mu + \sigma^2\big)^2 + 4\frac{K^2}{k}} > 0.$$

We may thus write

$$\int_0^t p(s)\,\mathrm{d}s = \frac{k}{K^2}\left[\frac{-\lambda + 2\mu + \sigma^2 + \Delta}{2} t + \log\left(\frac{C_1 + 1}{C_1 e^{\Delta t} + 1} \right) \right].$$

At this point we have an explicit expression for the first factor in (52), namely

$$\exp\left((\lambda - \mu)t + \frac{K^2}{k} \int_0^t p(s)\,\mathrm{d}s \right) = \frac{C_1 + 1}{C_1 e^{\Delta t} + 1}\, e^{\frac{1}{2}(\lambda + \sigma^2 + \Delta)t}. \tag{54}$$

Next we compute the integral in the second factor in (52),

$$\begin{aligned} &\int_0^t \exp\left((\mu - \lambda)s - \frac{K^2}{k}\int_0^s p(\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \\ &\quad = \int_0^t \frac{C_1 e^{\Delta s} + 1}{C_1 + 1}\, e^{\frac{1}{2}(-\lambda - \sigma^2 - \Delta)s}\,\mathrm{d}s \\ &\quad = \frac{2}{C_1 + 1}\left[C_1 \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 + \Delta)t} - 1}{-\lambda - \sigma^2 + \Delta} + \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 - \Delta)t} - 1}{-\lambda - \sigma^2 - \Delta} \right]. \end{aligned}$$

We thus obtained

$$q(t) = \frac{C_1 + 1}{C_1 e^{\Delta t} + 1}\, e^{\frac{1}{2}(\lambda + \sigma^2 + \Delta)t} \left\{ \frac{4\rho}{C_1 + 1}\left[C_1 \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 + \Delta)t} - 1}{-\lambda - \sigma^2 + \Delta} + \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 - \Delta)t} - 1}{-\lambda - \sigma^2 - \Delta} \right] + C_2 \right\}.$$

Finally we determine the constant C_2 on the basis of the terminal condition $q(T) = -2\ell\rho$, namely

$$\frac{4\rho}{C_1 + 1}\left[C_1 \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 + \Delta)T} - 1}{-\lambda - \sigma^2 + \Delta} + \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 - \Delta)T} - 1}{-\lambda - \sigma^2 - \Delta} \right] + C_2 = -2\ell\rho \frac{C_1 e^{\Delta T} + 1}{C_1 + 1}\, e^{-\frac{1}{2}(\lambda + \sigma^2 + \Delta)T}$$

from which

$$C_2 = -\frac{2\rho}{C_1 + 1}\left\{ 2\left[C_1 \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 + \Delta)T} - 1}{-\lambda - \sigma^2 + \Delta} + \frac{e^{\frac{1}{2}(-\lambda - \sigma^2 - \Delta)T} - 1}{-\lambda - \sigma^2 - \Delta} \right] + \ell\left(C_1 e^{\Delta T} + 1 \right) e^{-\frac{1}{2}(\lambda + \sigma^2 + \Delta)T} \right\}.$$

Acknowledgements: The authors are grateful to Mihail Zervos and to anonymous Referee for helpful comments relating to our results. We also thank Tiziano Vargiolu for having pointed out an incorrect Remark in a previous version.

References

[1] A. Cadenillas and F. Zapatero, Optimal Central Bank Intervention in the Foreign Exchange Market. *J. Econ. Theory* 87 (1999), pp. 218-242.

[2] A. Cadenillas and F. Zapatero, Classical and Impulse Stochastic Control of the Exchange Rate Using Interest Rates and Reserves, *Mathematical Finance* 10 (2000), pp. 141-156.

[3] A. Jack and M. Zervos, Impulse and Absolutely Continuous Ergodic Control of One-dimensional Itô Diffusions, In: *From Stochastic Analysis to Mathematical Finance*, Festschrift for Albert Shiryaev (Y. Kabanov, R. Liptser and J. Stoyanov, eds.),(2006), Springer, pp.295-314.

[4] M. Jeanblanc-Picqué, Impulse Control Method and Exchange Rate, *Mathematical Finance* 3 (1993), pp. 161-177.

[5] R. Korn, Optimal Impulse Control when Control Actions have Random Consequences, *Mathematics of Operations Research* 22 (1997), pp. 639-667.

[6] D. Melas and M. Zervos, An Ergodic Impulse Control Model with Applications, In: *Modern Trends in Controlled Stochastic Processes: Theory and Applications* (A.B. Piunovskiy, ed), Luniver Press, (2010), pp. 161-180.

[7] G. Mundaca and B. Øksendal, Optimal Stochastic Intervention Control with Application to the Exchange Rate, *J. Math. Econ.* 29 (1998), pp. 223-241.

[8] G. Peskir, A Change-of-Variable Formula with Local Time on Curves, *J. Theoret. Probab.* 18, no.3 (2005), pp. 499-535.

[9] G. Peskir and A. Shiryaev, *Optimal Stopping and Free-Boundary Problems*, Lectures in Mathematics, ETH Zürich (2006), Birkhäuser.

[10] A.D. Polyanin and V.F. Zaitsev, *Handbook of exact solutions for ordinary differential equations.* Chapman & Hall/CRC, London, 2003.