A Bayesian adaptive control approach to risk management in a binomial model

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Abstract. We consider the problem of shortfall risk minimization when there is uncertainty about the exact stochastic dynamics of the underlying. Starting from the general discrete time model and the approach described in Runggaldier and Zaccaria (1999), we derive explicit analytic solutions for the particular case of a binomial model when there is uncertainty about the probability of an "up-movement". The solution turns out to be a rather intuitive extension of that for the classical Cox-Ross-Rubinstein model.

1. Introduction

In incomplete markets the superhedging criterion allows one to eliminate the risk completely, but it requires in general too much initial capital; it corresponds in fact to a min-max-type criterion. One may then ask by how much one can lower the initial cost if one is willing to accept some risk or, dually, what is the risk corresponding to an initial capital less than what is required for superhedging. The shortfall risk minimization approach allows one to deal with these issues.

Given a market with a non-risky and a certain number of risky assets, let H_T be a liability to be hedged at some fixed future time T. Denote by $V_T(\varphi)$ the value at T of a portfolio corresponding to a self-financing investment strategy φ, possibly satisfying some additional constraints such as a shortselling prohibition. The problem is to find

$$J_0(S_0, V_0) := \inf_{\varphi} E^P_{S_0,V_0} \left\{ \ell\left([H_T - V_T(\varphi)]^+ \right) \right\} \tag{1}$$

for a given initial value S_0 of the asset(s) in the portfolio, for a given initial capital V_0 and where $\ell(\cdot)$ is a suitable increasing function such that $\ell(0) = 0$ and $\ell(x) > 0$ for all $x > 0$. For $\ell(z) = 1_{\{z>0\}}$, the right hand side in (1) corresponds to the smallest shortfall probability. Problems of the type (1) have recently attracted

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considerable attention (see e.g. [2, 3, 4, 5, 7, 8, 9, 10, 16]). Let, for a given S_0,

$$V_0^*(S_0) := \inf\{V_0 \mid J_0(S_0, V_0) = 0\}. \tag{2}$$

Since $J_0(S_0, V_0) = 0$ means that $H_T \leq V_T(\varphi^*)$ P-almost surely (where φ^* is the optimal strategy in (1)), $V_0^*(S_0)$ is the minimal initial capital needed to *superhedge* the claim. It follows that if $V_0 \geq V_0^*(S_0)$ then $V_T(\varphi^*) \geq H_T$, P-a.s. For superhedging, the choice of the underlying probabilistic model for the evolution of the risky assets is thus irrelevant as long as it induces probability measures that are equivalent. However for the more general problem of risk minimization in (1) the probabilistic structure of the underlying model matters, but the true model is almost never known exactly. A possibility is then to use a min-max-type criterion as in [4] trying to find

$$\inf_{\varphi} \sup_{P \in \mathcal{P}} E^P_{S_0,V_0} \left\{ \ell\left([H_T - V_T(\varphi)]^+ \right) \right\},$$

where $\mathcal{P}$ is a family of "real world probability measures". However, such a criterion does not allow one to incorporate additional information on the underlying model as it becomes successively available.

Thus, an adaptive approach, corresponding to a Bayesian-type criterion, appears more appropriate. Such adaptive approaches have already been dealt with in the literature (see e.g. [2, 3, 4, 5] and, in the context of portfolio optimization, in [1, 11, 12, 13]). In all these papers the uncertainty is only in the stock appreciation rate. The tools are mainly probabilistic in nature (involving also measure transformation) and are based on convex duality. An explicit solution is essentially possible only in simpler cases and transaction costs are not taken into account.

In the present paper we base ourselves on [17] following an approach along the lines of discrete time stochastic adaptive control. In that work the authors give a general description of this approach and they apply it, in particular, to a multinomial model for the risky assets where the probabilities are not known; for the specific case of a binomial model some numerical results are also presented. Here we focus our attention on the binomial model. By neglecting transaction costs (the portfolio is rebalanced only at discrete dates, which limits their impact) and imposing the self-financing requirement as the only constraint on the investment strategies, we succeed in deriving explicit solutions for the optimal investment strategy and for the corresponding minimal value of the shortfall risk in the case $\ell(x) = x$. We do this both for the case when the probability p of an "up-movement" is known as well as when it is unknown and, according to the Bayesian approach, treated as a random variable with a Beta-type distribution. We obtain an analytic solution, that turns out to be an interesting variant of the Cox-Ross-Rubinstein (CRR) solution (see e.g. [15]), when the initial capital is insufficient for a perfect hedge.

The paper is organised as follows. In Section 2 we briefly recall some facts from the CRR binomial model, that are relevant for the sequel. In Section 3 we present a solution method to our problem which is based on backwards *Dynamic*

Programming (DP). In Section 4, by assuming $\ell(x) = x$, we compute the minimizing admissible strategy as well as an explicit evaluation formula for the minimal discounted shortfall risk in the case when p is known. In Section 5, again by assuming $\ell(x) = x$, we compute the minimizing strategy as well as the minimal discounted shortfall risk in the case when p is unknown.

2. The Cox-Ross-Rubinstein binomial model

We consider a discrete-time market model with the set of dates $0, 1, \ldots, N$, and with two primary traded securities: a risky asset (a *stock*) S and a risk-free investment (a *bond*) B. We assume that the value of the bond is constantly equal to 1 through time, and that the stock price process S satisfies

$$S_{n+1} = S_n \omega_n, \quad n = 0, \ldots, N, \tag{3}$$

where $S_0 > 0$ is a given constant and $\{\omega_n\}_{n=0,\ldots,N}$ is a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, taking only two real values d and u satisfying $0 < d < 1 < u$, with probability law

$$p := P\{\omega_n = u\} = 1 - P\{\omega_n = d\}, \quad n = 0, \ldots, N.$$

We can assume without loss of generality $B_n \equiv 1$ by letting S_n be the discounted prices of the asset. Let us then denote by $\varphi_n = (\eta_n, \psi_n)$, $n = 0, \ldots, N$, an *investment strategy*, where η_n stands for the amount of the nonrisky asset and ψ_n stands for the number of units of the risky asset that are held in the portfolio in period n. We assume that φ_n is adapted to the observation σ-algebra $\mathcal{F}_n^S := \sigma\{S_m, \ m \leq n\}$, for all $n = 0, \ldots, N$, and that $\varphi = \{\varphi_n\}_{n=0,\ldots,N}$ satisfies the self-financing property

$$\begin{aligned} V_0 &= \eta_0 + \psi_0 S_0 \\ V_{n+1} &:= \eta_{n+1} + \psi_{n+1} S_{n+1} = \eta_n + \psi_n S_{n+1}, \quad n = 0, \ldots, N-1, \end{aligned}$$

where V_0 is a given constant, representing the initial value of the portfolio. We shall denote by $\mathcal{A}_{\mathrm{ad}}$ the set of all self-financing strategies (the *admissible investment strategies*).

It is a rather classical result that V follows the dynamics

$$V_{n+1} = V_n + \psi_n S_n (\omega_n - 1) =: V_{n+1}(V_n, S_n, \omega_n, \psi_n),$$

so that one can restrict oneself to just the decision variable ψ_n.

Consider a European contingent claim $H(S_N)$ and let P^* be the martingale measure for our model. It is well known that P^* corresponds to

$$P^*\{\omega_n = u\} = p^* := \frac{1-d}{u-d}, \quad P^*\{\omega_n = d\} = 1 - p^* = \frac{u-1}{u-d}, \quad n = 0, \ldots, N,$$

(see e.g. [15, Ch. 2]). The arbitrage free price V_n^* of $H(S_N)$ at time n, where $n = 0, \ldots, N-1$, is given by the Cox-Ross-Rubinstein (CRR) evaluation formula

$$V_n^*(S_n) = E^*[H(S_N) | S_n], \tag{4}$$

where by E^* we denote the expectation with respect to P^*. In particular, $V_0^*(S_0)$ is the minimal initial value of the portfolio needed to replicate the claim defined in (2). At time n, $n = 0, \ldots, N-1$, the replicating strategy ψ_n is given by

$$\psi_n = \frac{V_{n+1}^*(S_n u) - V_{n+1}^*(S_n d)}{S_n(u-d)}. \tag{5}$$

If $V_0 < V_0^*(S_0)$, then the replication of the terminal payoff is not possible. In this situation, an investor may be interested in analyzing the *shortfall risk* defined as the expectation of the terminal deficit weighted by some loss function. Let us then introduce this problem in more detail.

Denote by $V_N(\varphi) = \eta_N + \psi_N S_N$ the value of the portfolio at time N corresponding to an admissible investment strategy φ. The minimal *shortfall risk* is defined as

$$J_0(S_0, V_0) := \min_{\varphi \in \mathcal{A}_{\mathrm{ad}}} E^P_{S_0,V_0} \left\{ \ell \left([H(S_N) - V_N(\varphi)]^+ \right) \right\}, \tag{6}$$

for a given initial value S_0 of the risky asset in the portfolio and a given initial capital $V_0 < V_0^*(S_0)$, where $\ell(\cdot) \geq 0$ is a suitable *loss function*, that is an increasing function such that $\ell(0) = 0$ and $\ell(x) > 0$ for all $x > 0$.

In this paper we consider the optimization problem (6) by assuming either that the probability p is known or that it is not. For the case when p is unknown, we adopt a Bayesian-type approach which allows us to incorporate additional information on the underlying model as it becomes successively available.

3. The dynamic programming algorithm

In this section we provide a DP algorithm to compute a solution to our problem (6) both for the case when p is known and when it is not (see [6] for an analogous algorithm for the case of superhedging with transaction costs). In the case where p is unknown, adopting the Bayesian point of view, we use the ordinary Bayes formula to successively update the initial (prior) density $h(p)$ of p on the basis of $\{\mathcal{F}_n^S\}_{n=0,\ldots,N}$. This leads to what is called the Bayesian DP algorithm (see e.g. [14, 18]).

3.1. DP algorithm when p is known

The DP algorithm proceeds backwards according to the following steps:

$$\begin{aligned} J_N(s,v) &= \ell((H(s)-v)^+), \\ J_{n-1}(S_{n-1}, V_{n-1}) &= \inf_{\psi_{n-1}} E_{S_{n-1},V_{n-1}}\{J_n(S_n, V_n)\} = \\ &= \inf_{\psi_{n-1}} \{pJ_n(S_{n-1}u, V_{n-1} + \psi_{n-1}S_{n-1}(u-1)) \\ &\quad +(1-p)J_n(S_{n-1}d, V_{n-1} + \psi_{n-1}S_{n-1}(d-1))\}, \end{aligned} \tag{7}$$

for $n = N, \ldots, 1$.

3.2. DP algorithm when p is unknown

The Bayesian DP algorithm proceeds similarly to the DP algorithm:

$$\begin{aligned} J_N(s,v) &= \ell((H(s)-v)^+), \\ J_{n-1}(S_{n-1},V_{n-1}) &= \inf_{\psi_{n-1}} E^{P_{n-1}}_{S_{n-1},V_{n-1}}\{J_n(S_n,V_n)\}. \end{aligned} \tag{8}$$

Now p too is a random variable, and its distribution depends on the information $\mathcal{F}^S_n$ up to time n. We incorporate this information in the probability measure P_n, that depends also on the distribution of p. Since $P\{\omega_n|p\}$ is the Binomial distribution, a conjugate family of distributions of p is that of the Beta distributions. With a prior density

$$h_0(p) \propto p^{\alpha_0}(1-p)^{\beta_0},$$

with $\alpha_0, \beta_0 \geq 0$, the posterior density in period n becomes

$$h_n(p) \propto p^{\alpha_n}(1-p)^{\beta_n},$$

where, denoting by u_n the total number of "up-movements" ($u_0 := 0$) cumulated up to time n,

$$\alpha_n = \alpha_0 + u_n\ , \quad \beta_n = \beta_0 + n - u_n.$$

In particular, for $n = 0$ and $\alpha_0 = \beta_0 = 0$ the prior density $h_0(p)$ becomes the uniform density.

Since the values of p enter the DP recursions linearly, by the "smoothing property" of conditional expectations it is easily seen that, also in the present case, the DP recursions are given by the previous steps, except that p has to be replaced in $J_{n-1}(S_{n-1}, V_{n-1})$, $n = N, \ldots, 1$, by $E_{\alpha_{n-1},\beta_{n-1}}[p] := E\left[p|\mathcal{F}^S_{n-1}\right]$ and $1-p$ by $1 - E_{\alpha_{n-1},\beta_{n-1}}[p]$, where

$$E_{\alpha,\beta}[p] = \frac{\alpha+1}{\alpha+\beta+2}.$$

4. Explicit solutions when p is known

In this section we are concerned with the evaluation of the minimal *discounted shortfall risk* (6) and the corresponding strategy in the case when $\ell(x) = x$. Due to the possibility of making direct calculations on the DP algorithm steps, we can derive explicit evaluation formulas. These evaluation formulas are simple and meaningful, showing explicitly what was to be expected: that is, the shortfall risk is decreasing with respect to the initial capital (we will show that such dependence is linear), and it is always equal to zero when the level of the initial capital is greater than or equal to $V_0^*(S_0)$. We first consider the case when there is complete information on the underlying market and we compare our results with the well-known results on *perfect hedging* of a European contingent claim.

In the following theorem we give some formulas for the optimal discounted shortfall risk and for the corresponding investment strategy.

Theorem 4.1. *Consider a European contingent claim H on a stock whose price S is assumed to follow the CRR binomial model (3). Let $V_n^*(S_n)$, where $n = 0, \dots, N-1$, be the arbitrage free price at time n defined by the CRR evaluation formula (4). Assume the loss function $\ell(\cdot)$ in (6) is the identity function $\ell(x) = x$. Then*

i) *if $p > p^*$, then*

$$J_n(S_n, V_n) = \left(\frac{1-p}{1-p^*} \right)^m [V_n^*(S_n) - V_n]^+ , \tag{9}$$

for $n = 0, \dots, N-1$. In particular, for $n = 0$ the minimal discounted shortfall risk is

$$J_0(S_0, V_0) = \left(\frac{1-p}{1-p^*} \right)^N [V_0^*(S_0) - V_0]^+ ,$$

where $V_0^*(S_0) = C_0^*(S_0)$. The minimizing investment strategy is given by

$$\psi_n^1 = \frac{V_{n+1}^*(S_n u) - V_n}{S_n(u-1)}, \tag{10}$$

for $n = 0, \dots, N-1$;

ii) *if $p < p^*$, then*

$$J_n(S_n, V_n) = \left(\frac{p}{p^*} \right)^m [V_n^*(S_n) - V_n]^+ ,$$

for $n = 0, \dots, N-1$. In particular, for $n = 0$ the minimal discounted shortfall risk is

$$J_0(S_0, V_0) = \left(\frac{p}{p^*} \right)^N [V_0^*(S_0) - V_0]^+ ,$$

where $V_0^*(S_0) = C_0^*(S_0)$. The minimizing investment strategy is given by

$$\psi_n^2 = \frac{V_{n+1}^*(S_n d) - V_n}{S_n(d-1)}, \tag{11}$$

for $n = 0, \dots, N-1$.

Proof. We start from $n = N-1$ by considering expression (7) of $J_{N-1}(S_{N-1}, V_{N-1})$. The function to be minimized in (7) is a linear combination of the two piecewise affine functions

$$[H(S_{N-1}u) - V_{N-1} - \psi_{N-1} S_{N-1}(u-1)]^+ \tag{12}$$

and

$$[H(S_{N-1}d) - V_{N-1} - \psi_{N-1} S_{N-1}(d-1)]^+ \tag{13}$$

The function in (12) is decreasing for ψ_{N-1} less than ψ_{N-1}^1 in Equation (10), since $u > 1$ and S_{N-1} is positive, and thereafter it is equal to zero, while the function in (13) is equal to zero for ψ_{N-1} less than ψ_{N-1}^2 in Equation (11), since $d < 1$ and S_{N-1} is positive, and from there on it is increasing. Therefore, if $\psi_{N-1}^1 \leq \psi_{N-1}^2$

or, equivalently, if $V_{N-1} \geq E^*[H(S_N)]$, then both strategies ψ^1_{N-1} and ψ^2_{N-1} are optimal as well as any admissible strategy between them. If $V_{N-1} < E^*[H(S_N)]$, then in order to establish the infimum in (7) it suffices to analyze the sign of the slope of

$$p[H(S_{N-1}u) - V_{N-1} - \psi_{N-1}S_{N-1}(u-1)]+ \\ +(1-p)[H(S_{N-1}d) - V_{N-1} - \psi_{N-1}S_{N-1}(d-1)],$$

which is given by the expression

$$S_{N-1}[p(d-u)+1-d]. \tag{14}$$

If (14) is less than zero, or, equivalently, if $p > p^*$, then the infimum in (7) is achieved at ψ^1_{N-1}. If this is the case, by putting ψ^1_{N-1} in (7) we obtain

$$J_{N-1}(S_{N-1}, V_{N-1}) = \frac{1-p}{1-p^*}\left[C^*_{N-1}(S_{N-1}) - V_{N-1}\right]^+.$$

Conversely, if $p < p^*$ then the infimum in (7) is achieved at ψ^2_{N-1}, and we obtain

$$J_{N-1}(S_{N-1}, V_{N-1}) = \frac{p}{p^*}\left[C^*_{N-1}(S_{N-1}) - V_{N-1}\right]^+.$$

This shows that formula (9) is true for $n = N-1$.

We now proceed by backward induction with respect to n. Assume $p > p^*$ from now on. Assume that equality (9) holds for n, where $n = N-1, \ldots, 1$, with the minimizing strategy given by (10). We show that it also holds for $n-1$, with the same strategy as (10) for $n-1$. From (7) of the DP algorithm we have

$$\begin{aligned} J_{n-1}(S_{n-1}, V_{n-1}) &= \inf_{\psi_{n-1}} \{p\, J_n\left(S_{n-1}u, V_n(V_{n-1}, S_{n-1}, u, \psi_{n-1})\right) + \\ &\quad +(1-p)\, J_n\left(S_{n-1}d, V_n(V_{n-1}, S_{n-1}, d, \psi_{n-1})\right)\}, \end{aligned}$$

which implies, by induction,

$$J_{n-1}(S_{n-1}, V_{n-1}) = \left(\frac{1-p}{1-p^*}\right)^m \inf_{\psi_{n-1}} \{p\, [V^*_n(S_{n-1}u) - V_{n-1} - \tag{15}$$
$$-\psi_{n-1}S_{n-1}(u-1)]^+ + (1-p)\, [V^*_n(S_{n-1}d) - V_{n-1} - \psi_{n-1}S_{n-1}(d-1)]^+\Big\}.$$

Using the same arguments as in the first step, and taking into account that the slope of the expression

$$p\,[V^*_n(S_{n-1}u) - V_{n-1} - \psi_{n-1}S_{n-1}(u-1)] + \\ +(1-p)\,[V^*_n(S_{n-1}d) - V_{n-1} - \psi_{n-1}S_{n-1}(d-1)]$$

is again given by (14) with S_{N-1} replaced by S_{n-1}, we have that, under the assumption $p > p^*$, the infimum in expression (15) is achieved at ψ^1_{n-1} given by (10). Putting ψ^1_{n-1} in (15) we easily obtain

$$J_{n-1}(S_{n-1}, V_{n-1}) = \left(\frac{1-p}{1-p^*}\right)^{m+1} \left[V^*_{n-1}(S_{n-1}) - V_{n-1}\right]^+.$$

This ends the proof of i). The proof of ii) can be obtained by using arguments similar to those of i). We only observe that, under the assumption $p < p^*$, the infimum in the DP algorithm steps is achieved at ψ_{n-1}, for $n = 0, \ldots, N-1$, satisfying

$$V_n^*(S_{n-1}d) - V_{n-1} - \psi_{n-1}S_{n-1}(d-1) = 0.$$

□

Remark 4.2. Notice that this approach is linked to the CRR model: in fact, by calculating the expected shortfall risk under the *historical* probability P, we arrive at an expression containing the expected price of the claim under the *risk-neutral* probability P^*. Moreover, the hedging strategy in this case is similar to (5), which is the one of the CRR model: in fact in the CRR model the hedging strategy ψ is equal to the ratio between the difference of the expected prices of the claim in the two possible future outcomes and the different prices of the underlying; here ψ is equal to the ratio between the difference of the expected price of the claim in one of the possible future outcomes and the value the portfolio would have if it were invested in the bond B, and the difference between the price of the underlying in the same future outcome considered before and its present price as if it were invested in the bond B. In other words, it is as if we were hedging a claim having a payoff that in each state of nature could be the one of the original claim or the money corresponding to the present value of the portfolio; in the same way, the underlying could either assume the value corresponding to the value of the derivative, or the value corresponding to a riskless investment.

Remark 4.3. Differently from [8], here we have not imposed that $V \geq 0$. This leads to different results: in fact, if we imposed $V \geq 0$, we would have obtained (as they do) an optimal strategy equal to the replicating strategy of a modified claim that is between zero and the original claim; we have a different strategy, that in general gives an optimal expected shortfall *lower* than they have. In particular, our strategy succeeds in replicating perfectly the claim in all the states of nature but the least probable one (see [7] for an explicit proof), so that the expected shortfall comes entirely from this state of nature. However, if V_0 is near V_0^*, then V remains positive at all times prior to the maturity N, so the two strategies that we obtain by imposing or not the constraint $V \geq 0$ coincide.

5. Explicit solutions when p is unknown

The formulas given in the following theorem for the optimal discounted shortfall risk are similar to those given in Theorem 4.1, and the minimizing investment strategies are the same. However, while in the previous case only two alternatives for the possible values of p^* were considered, i.e. $p^* > p$ and $p^* < p$, here at each step n we have to consider several alternatives according to the estimates of p, each of them leading to a different formula for $J_n(S_n, V_n)$.

Theorem 5.1. *Consider the assumptions of Theorem 4.1 for the case when p is unknown with a prior $h_0(p) \propto p^{\alpha_0}(1-p)^{\beta_0}$, with $\alpha_0, \beta_0 \geq 0$. Then*

i) *if $p^* < E_{\alpha_n,\beta_n+m-1}[p]$, then*

$$J_n(S_n, V_n) = \left(\prod_{j=0}^{m-1} \frac{1 - E_{\alpha_n,\beta_n+j}[p]}{1-p^*}\right)[V_n^*(S_n) - V_n]^+ . \tag{16}$$

The minimizing investment strategy is given by (10).

ii) *if $E_{\alpha_n+i,\beta_n+m-1-i}[p] < p^* < E_{\alpha_n+i+1,\beta_n+m-2-i}[p]$, where $i = 0, \dots, m-2$, then*

$$J_n(S_n, V_n) = \left(\prod_{j=0}^{m-2-i} \frac{1 - E_{\alpha_n,\beta_n+j}[p]}{1-p^*}\right) \times$$

$$\times \left(\prod_{j=0}^{i} \frac{E_{\alpha_n+j,\beta_n+m-1-i}[p]}{p^*}\right)[V_n^*(S_n) - V_n]^+ = \tag{17}$$

$$= \left(\prod_{j=0}^{i} \frac{E_{\alpha_n+j,\beta_n}[p]}{p^*}\right)\left(\prod_{j=0}^{m-2-i} \frac{1 - E_{\alpha_n+i+1,\beta_n+j}[p]}{1-p^*}\right)[V_n^*(S_n) - V_n]^+ \tag{18}$$

Both the strategies (10) and (11) are optimal as well as any admissible strategy between them.

iii) *if $p^* > E_{\alpha_n+m-1,\beta_n}[p]$, then*

$$J_n(S_n, V_n) = \left(\prod_{j=0}^{m-1} \frac{E_{\alpha_n+j,\beta_n}[p]}{p^*}\right)[V_n^*(S_n) - V_n]^+. \tag{19}$$

The minimizing investment strategy is given by (11).

Proof. We start from $n = N-1$ by considering the expression

$$\begin{aligned} &J_{N-1}(S_{N-1}, V_{N-1}) = \qquad (20)\\ &= \inf_{\psi_{N-1}} E_{\alpha_{N-1},\beta_{N-1}}[p]\,[H(S_{N-1}u) - V_{N-1} - \psi_{N-1}S_{N-1}(u-1)]^+ + \\ &\quad + \left(1 - E_{\alpha_{N-1},\beta_{N-1}}[p]\right)[H(S_{N-1}d) - V_{N-1} - \psi_{N-1}S_{N-1}(d-1)]^+ , \end{aligned}$$

whose infimum is achieved, by using arguments similar to those in the proof of Theorem 4.1, at ψ_{n-1}^1 in Equation (10) if $E_{\alpha_{N-1},\beta_{N-1}}[p] > p^*$ and at ψ_{n-1}^2 in Equation (11) if $E_{\alpha_{N-1},\beta_{N-1}}[p] < p^*$. Therefore, putting the minimizing strategy in (20) we have

$$J_{N-1}(S_{N-1}, V_{N-1}) = \frac{1 - E_{\alpha_{N-1},\beta_{N-1}}[p]}{1-p^*}\left[C_{N-1}^*(S_{N-1}) - V_{N-1}\right]^+$$

if $E_{\alpha_{N-1},\beta_{N-1}}[p] > p^*$, and

$$J_{N-1}(S_{N-1},V_{N-1}) = \frac{E_{\alpha_{N-1},\beta_{N-1}}[p]}{p^*}\left[C^*_{N-1}(S_{N-1}) - V_{N-1}\right]^+$$

if $E_{\alpha_{N-1},\beta_{N-1}}[p] < p^*$, showing that formulas (16)–(19) are true for $n = N-1$ (let us observe that the validity of ii) is trivial for $n = N-1$).

We now proceed by induction with respect to n. We assume that equalities (16)—(19) hold for n, where $n = 1,\ldots,N-1$, and we show that they also hold for $n-1$. As regards alternative ii), we shall only prove formula (17). Indeed, it is not difficult to check (we omit calculations) the validity of the equality

$$\left(\prod_{j=0}^{m-2-i}\frac{1-E_{\alpha_n,\beta_n+j}[p]}{1-p^*}\right)\left(\prod_{j=0}^{i}\frac{E_{\alpha_n+j,\beta_n+m-1-i}[p]}{p^*}\right) =$$
$$= \left(\prod_{j=0}^{i}\frac{E_{\alpha_n+j,\beta_n}[p]}{p^*}\right)\left(\prod_{j=0}^{m-2-i}\frac{1-E_{\alpha_n+i+1,\beta_n+j}[p]}{1-p^*}\right),$$

corresponding to, respectively, (17) and (18) of ii). Let us remark that, as we shall see below, formula (17) (respectively, (18)) is obtained by always choosing strategy (10) (respectively, (11)) at each step n where, for some $i \in \{0,\ldots,N-n-2\}$, we have

$$E_{\alpha_n+i,\beta_n+N-n-1-i}[p] < p^* < E_{\alpha_n+i+1,\beta_n+N-n-2-i}[p].$$

This choice will be possible since, under the above condition for p^*, both the strategies (10) and (11), as well as any admissible strategy between them, will be optimal. In fact, other representation formulas for J_n, different from (17) and (18), but equivalent to them, could be possible, each of them corresponding to a different procedure for selecting a minimizing strategy between (10) and (11).

From (8), we have

$$J_{n-1}(S_{n-1},V_{n-1}) = \inf_{\psi_{n-1}} E_{\alpha_{n-1},\beta_{n-1}}[p]\, J_n(S_{n-1}u, V_n(V_{n-1},S_{n-1},u,\psi_{n-1})) +$$
$$+(1-E_{\alpha_{n-1},\beta_{n-1}}[p])J_n\left(S_{n-1}d, V_n(V_{n-1},S_{n-1},d,\psi_{n-1})\right),$$

which implies, by induction,

$$J_{n-1}(S_{n-1},V_{n-1}) = \inf_{\psi_{n-1}} E_{\alpha_{n-1},\beta_{n-1}}[p]\, A(\alpha_{n-1}+1,\beta_{n-1}) \times$$
$$\times\left[V_n^*(S_{n-1}u) - V_n(V_{n-1},S_{n-1},u,\psi_{n-1})\right]^+ + \left(1-E_{\alpha_{n-1},\beta_{n-1}}[p]\right)\times$$
$$\times A(\alpha_{n-1},\beta_{n-1}+1)\left[V_n^*(S_{n-1}d) - V_n(V_{n-1},S_{n-1},d,\psi_{n-1})\right]^+,$$

where (choosing e.g. (17) when taking into account alternative *ii*))

$$
\begin{aligned}
A(\alpha,\beta) &= \left(\prod_{j=0}^{N-n-1} \frac{1-E_{\alpha,\beta+j}[p]}{1-p^*}\right) 1_{\{p^*<E_{\alpha,\beta+N-n-1}[p]\}}+ \\
&+\sum_{k=0}^{N-n-2}\left(\prod_{j=0}^{N-n-2-k} \frac{1-E_{\alpha,\beta+j}[p]}{1-p^*}\right)\left(\prod_{j=0}^{k} \frac{E_{\alpha+j,\beta+N-n-1-k}[p]}{p^*}\right)\times \\
&\times 1_{\{E_{\alpha+k,\beta+N-n-1-k}[p]<p^*<E_{\alpha+k+1,\beta+N-n-2-k}[p]\}}+ \\
&+\left(\prod_{j=0}^{N-n-1} \frac{E_{\alpha+j,\beta}[p]}{p^*}\right) 1_{\{p^*>E_{\alpha+N-n-1,\beta}[p]\}},
\end{aligned}
$$

Proof of part i). If $p^* < E_{\alpha_{n-1},\beta_{n-1}+N-n}[p]$ then it is not difficult to check that only the first indicator functions in both $A(\alpha_{n-1}+1,\beta_{n-1})$ and $A(\alpha_{n-1},\beta_{n-1}+1)$ are equal to one, while all the others are equal to zero. Therefore

$$
\begin{aligned}
J_{n-1}(S_{n-1},V_{n-1}) &= \inf_{\psi_{n-1}} E_{\alpha_{n-1},\beta_{n-1}}[p]\left(\prod_{j=0}^{N-n-1} \frac{1-E_{\alpha_{n-1}+1,\beta_{n-1}+j}[p]}{1-p^*}\right)\times \\
&\times\left[V_n^*(S_{n-1}u)-V_{n-1}-\psi_{n-1}S_{n-1}(u-1)\right]^+ + \\
&+\left(1-E_{\alpha_{n-1},\beta_{n-1}}[p]\right)\left(\prod_{j=0}^{N-n-1} \frac{1-E_{\alpha_{n-1},\beta_{n-1}+j+1}[p]}{1-p^*}\right)\times \\
&\times\left[V_n^*(S_{n-1}d)-V_{n-1}-\psi_{n-1}S_{n-1}(d-1)\right]^+ . \qquad (21)
\end{aligned}
$$

The sign of the slope of this linear expression in ψ_{n-1} is given by

$$
\begin{aligned}
&E_{\alpha_{n-1},\beta_{n-1}}[p]\left(\prod_{j=0}^{N-n-1} \frac{1-E_{\alpha_{n-1}+1,\beta_{n-1}+j}[p]}{1-p^*}\right)(1-u)+ \qquad (22)\\
&+\left(1-E_{\alpha_{n-1},\beta_{n-1}}[p]\right)\left(\prod_{j=0}^{N-n-1} \frac{1-E_{\alpha_{n-1},\beta_{n-1}+j+1}[p]}{1-p^*}\right)(1-d).
\end{aligned}
$$

After making elementary manipulations, one can rewrite (22) as the product of a suitable strictly positive term and $(p^* - E_{\alpha_{n-1},\beta_{n-1}+N-n}[p])$, which is less than zero by assumption. Therefore, using the same arguments as those in the proof of Theorem 4.1, we have that the infimum in (21) is achieved at ψ_{n-1}^1 given by formula (10). Putting ψ_{n-1}^1 in (21) we easily obtain

$$
J_{n-1}(S_{n-1},V_{n-1}) = \left(\prod_{j=0}^{N-n} \frac{1-E_{\alpha_{n-1},\beta_{n-1}+j}[p]}{1-p^*}\right)\left[V_{n-1}^*(S_{n-1})-V_{n-1}\right]^+,
$$

which gives formula (16) with n replaced by $n-1$. This ends the proof of *i*).

Proof of part iii). It is not difficult to argue that the proof of *iii)* proceeds along the same lines as the proof of *i)*. We only observe that if $p^* > E_{\alpha_{n-1}+N-n,\beta_{n-1}}[p]$ (alternative *iii)* for $n-1$) then only the last indicator functions in both $A(\alpha_{n-1}+1,\beta_{n-1})$ and $A(\alpha_{n-1},\beta_{n-1}+1)$ are equal to one, while all the others are equal to zero. Moreover, the sign of the slope of the linear expression which arises from the DP algorithm is positive, so that the infimum is achieved at ψ_{n-1} given by formula (11), i.e. satisfying

$$V_n^*(S_{n-1}d) - V_{n-1} - \psi_{n-1}S_{n-1}(d-1) = 0.$$

Proof of part ii). Now we only have to prove *ii)*. As we shall see, the linear expressions arising from the DP algorithm will have angular coefficients equal to zero, giving the possibility to choose, as minimizing strategy, any admissible strategy between (10) and (11). In particular, choosing strategy (10) we shall prove the validity of formula (17). If

$$E_{\alpha_{n-1}+i,\beta_{n-1}+N-n-i}[p] < p^* < E_{\alpha_{n-1}+i+1,\beta_{n-1}+N-n-1-i}[p],$$

for some $i \in \{0,\dots,N-n-1\}$, then the last indicator function in $A(\alpha_{n-1}+1,\beta_{n-1})$ and the first indicator function in $A(\alpha_{n-1},\beta_{n-1}+1)$ are equal to zero, i.e.

$$1_{\{p^*>E_{\alpha_{n-1}+N-n,\beta_{n-1}}[p]\}} \equiv 0$$

and

$$1_{\{p^*<E_{\alpha_{n-1},\beta_{n-1}+N-n}[p]\}} \equiv 0$$

Moreover, we have to distinguish between the following three alternatives:

1. If $i=0$, then we have

$$\begin{aligned}J_{n-1}(S_{n-1},V_{n-1}) &= \inf_{\psi_{n-1}} E_{\alpha_{n-1},\beta_{n-1}}[p]\left(\prod_{j=0}^{N-n-1}\frac{1-E_{\alpha_{n-1}+1,\beta_{n-1}+j}[p]}{1-p^*}\right)\times\\ &\times[V_n^*(S_{n-1}u)-V_{n-1}-\psi_{n-1}S_{n-1}(u-1)]^+ +\\ &+(1-E_{\alpha_{n-1},\beta_{n-1}}[p])\left(\prod_{j=0}^{N-n-2}\frac{1-E_{\alpha_{n-1},\beta_{n-1}+1+j}[p]}{1-p^*}\right)\times\\ &\times\frac{E_{\alpha_{n-1},\beta_{n-1}+N-n}[p]}{p^*}[V_n^*(S_{n-1}d)-V_{n-1}-\psi_{n-1}S_{n-1}(d-1)]^+.\end{aligned}$$

   Since the slope of this linear expression in ψ_{n-1} is equal to zero (we omit calculations), the infimum is achieved at both (10) and (11) as well as at any admissible strategy between them. Choosing e.g. (10) we obtain

$$\begin{aligned}J_{n-1}(S_{n-1},V_{n-1}) &= \left(\prod_{j=0}^{N-n-1}\frac{1-E_{\alpha_{n-1},\beta_{n-1}+j}[p]}{1-p^*}\right)\times\\ &\times\frac{E_{\alpha_{n-1},\beta_{n-1}+N-n}[p]}{p^*}\left[V_{n-1}^*(S_{n-1})-V_{n-1}\right]^+,\end{aligned}$$

which gives formula (17) with n replaced by $n-1$ and $i=0$.

2. If $i \in \{1, \ldots, N-n-2\}$, then we have

$$J_{n-1}(S_{n-1}, V_{n-1}) = \inf_{\psi_{n-1}} E_{\alpha_{n-1},\beta_{n-1}}[p] \left(\prod_{j=0}^{N-n-1-i} \frac{1 - E_{\alpha_{n-1}+1,\beta_{n-1}+j}[p]}{1-p^*} \right)$$
$$\times \left(\prod_{j=0}^{i-1} \frac{E_{\alpha_{n-1}+1+j,\beta_{n-1}+N-n-i}[p]}{p^*} \right) \times$$
$$\times [V_n^*(S_{n-1}u) - V_{n-1} - \psi_{n-1}S_{n-1}(u-1)]^+ +$$
$$+ \left(1 - E_{\alpha_{n-1},\beta_{n-1}}[p]\right) \left(\prod_{j=0}^{N-n-2-i} \frac{1 - E_{\alpha_{n-1},\beta_{n-1}+1+j}[p]}{1-p^*} \right) \times$$
$$\times \left(\prod_{j=0}^{i} \frac{E_{\alpha_{n-1}+j,\beta_{n-1}+N-n-i}[p]}{p^*} \right) [V_n^*(S_{n-1}d) - V_{n-1} - \psi_{n-1}S_{n-1}(d-1)]^+ .$$

The slope of this linear expression in ψ_{n-1} is again equal to zero. Choosing e.g. (10) as ψ_{n-1} we obtain

$$J_{n-1}(S_{n-1}, V_{n-1}) = \left(\prod_{j=0}^{N-n-1-i} \frac{1 - E_{\alpha_{n-1},\beta_{n-1}+j}[p]}{1-p^*} \right) \times$$
$$\times \left(\prod_{j=0}^{i} \frac{E_{\alpha_{n-1}+j,\beta_{n-1}+N-n-i}[p]}{p^*} \right) \left[V_{n-1}^*(S_{n-1}) - V_{n-1}\right]^+ ,$$

which gives formula (17) with n replaced by $n-1$ and $i \in \{1, \ldots, N-n-2\}$.

3. If $i = N-n-1$, then we have

$$J_{n-1}(S_{n-1}, V_{n-1}) = \inf_{\psi_{n-1}} E_{\alpha_{n-1},\beta_{n-1}}[p] \left(\frac{1 - E_{\alpha_{n-1}+1,\beta_{n-1}}[p]}{1-p^*} \right) \times$$
$$\times \left(\prod_{j=0}^{N-n-2} \frac{E_{\alpha_{n-1}+1+j,\beta_{n-1}+1}[p]}{p^*} \right) \times$$
$$\times [V_n^*(S_{n-1}u) - V_{n-1} - \psi_{n-1}S_{n-1}(u-1)]^+ + \left(1 - E_{\alpha_{n-1},\beta_{n-1}}[p]\right) \times$$
$$\times \left(\prod_{j=0}^{N-n-1} \frac{E_{\alpha_{n-1}+j,\beta_{n-1}+1}[p]}{p^*} \right) [V_n^*(S_{n-1}d) - V_{n-1} - \psi_{n-1}S_{n-1}(d-1)]^+ .$$

The slope of this linear expression in ψ_{n-1} is again equal to zero. Choosing e.g. (10) as ψ_{n-1} we obtain

$$J_{n-1}(S_{n-1}, V_{n-1}) = \frac{1 - E_{\alpha_{n-1},\beta_{n-1}}[p]}{1-p^*} \times$$
$$\times \left(\prod_{j=0}^{N-n-1} \frac{E_{\alpha_{n-1}+j,\beta_{n-1}+1}[p]}{p^*} \right) [V^*_{n-1}(S_{n-1}) - V_{n-1}]^+ .$$

The inductive step is complete, and so is the proof. □

Corollary 5.2. *Letting* $n = 0$ *in Theorem 5.1, the minimal discounted shortfall risk is*

i) *if* $p^* < E_{\alpha_0,\beta_0+N-1}[p]$, *then*

$$J_0(S_0, V_0) = \left(\prod_{j=0}^{N-1} \frac{1 - E_{\alpha_0,\beta_0+j}[p]}{1-p^*} \right) [V^*_0(S_0) - V_0]^+ ;$$

ii) *if* $E_{\alpha_0+i,\beta_0+N-1-i}[p] < p^* < E_{\alpha_0+i+1,\beta_0+N-2-i}[p]$, *where* $i = 0, \ldots, N-2$, *then*

$$J_0(S_0, V_0) = \left(\prod_{j=0}^{N-2-i} \frac{1 - E_{\alpha_0,\beta_0+j}[p]}{1-p^*} \right) \times$$
$$\times \left(\prod_{j=0}^{i} \frac{E_{\alpha_0+j,\beta_0+N-1-i}[p]}{p^*} \right) [V^*_0(S_0) - V_0]^+ =$$
$$= \left(\prod_{j=0}^{i} \frac{E_{\alpha_0+j,\beta_0}[p]}{p^*} \right) \left(\prod_{j=0}^{N-2-i} \frac{1 - E_{\alpha_0+i+1,\beta_0+j}[p]}{1-p^*} \right) [V^*_0(S_0) - V_0]^+ ;$$

iii) *if* $p^* > E_{\alpha_0+N-1,\beta_0}[p]$, *then*

$$J_0(S_0, V_0) = \left(\prod_{j=0}^{N-1} \frac{E_{\alpha_0+j,\beta_0}[p]}{p^*} \right) [V^*_0(S_0) - V_0]^+ .$$

The minimizing investment strategies are those of Theorem 5.1.

Remark 5.3. In the case when p is unknown we obtain formulas that are similar to those of the case when p is known, with the following difference: while in the case when p is known we know immediately whether $p > p^*$ or not, and that relation either holds at all times n or does not hold, when p is unknown we cannot know whether $p > p^*$ or not, so we must use the Bayes estimators of p. Obviously these estimators change over time, so we obtain products of different factors depending on p^* and on the Bayes estimators of p, while in the case when p is known these factors are all equal either to p/p^* or to $(1-p)/(1-p^*)$. We also notice that the investment strategies are equal to those when p is known; this happens because the Bayes estimators of p enter linearly in the DP algorithm and do not modify the optimum.

Remark 5.4. Also when p is unknown, if we impose $V \geq 0$, we obtain different results: in fact in this case too our strategy succeeds in replicating perfectly the claim in all the states of nature but one (see [7]); with the constraint $V \geq 0$, it may happen that this is not possible, so that the optimal solution gives a shortfall higher than in our case. However, also in this case (as in the case when p is known), if V_0 is near V_0^*, then V remains positive at all times prior to the maturity N, so the two strategies that we obtain by imposing or not the constraint $V \geq 0$ coincide.

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