On the Existence of Martingale Measures in Jump Diffusion Market Models*

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Abstract

In the context of jump-diffusion market models we construct examples that satisfy the weaker no-arbitrage condition of NA1 (NUPBR), but not NFLVR. We show that in these examples the only candidate for the density process of an equivalent local martingale measure is a supermartingale that is not a martingale, not even a local martingale. This candidate is given by the supermartingale deflator resulting from the inverse of the discounted growth optimal portfolio. In particular, we consider an example with constraints on the portfolio that go beyond the standard ones for admissibility.

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1 Introduction

The First Fundamental Theorem of Asset Pricing states the equivalence of the classical no-arbitrage concept of *No-Free-Lunch-With-Vanishing-Risk* (NFLVR) and the existence of an *equivalent* $\sigma-$*martingale measure* (EσMM) (see [8]). Since in this paper we shall consider only nonnegative price processes, we may restrict ourselves to the sub-class of *equivalent local martingale measures* (ELMM), which can be characterized by their density processes (see [6]). In real markets some forms of arbitrage however exist that are incompatible with FLVR. This is taken into account in the recent Stochastic Portfolio Theory (see the survey in [9]), where the NFLVR condition is not imposed as a normative assumption and it is shown that some arbitrage opportunities may arise naturally in a real market; furthermore, one of the roles of portfolio optimization is in fact also that of exploiting possible arbitrages. On the other hand the full strength of NFLVR is not necessarily needed to solve fundamental problems of valuation, hedging and portfolio optimization. In addition, the notion of NFLVR is not robust with respect to changes in the numeraire or the reference filtration and it is not easy to check it in real markets. In parallel, there is also the so-called *Benchmark approach to Quantitative Finance* (see e.g. [20]) that

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aims at developing a theory of valuation that does not rely on the existence of an ELMM. For the hedging problem in market models that do not admit an ELMM we refer to [22].

Weaker forms of no-arbitrage were thus introduced recently and it turns out that the equivalent notions of no-arbitrage given by the *No-Arbitrage of the First Kind* (NA1) (see [15], see also [18] for an earlier version of this notion) and *No Unbounded Profit with Bounded Risk* (NUPBR) (see [13]) represent the minimal condition that still allows one to meaningfully solve problems of pricing, hedging and portfolio optimization. It is thus of interest to develop economically significant market models, which satisfy NA1 (NUPBR), but not NFLVR. As long as one remains within continuous market models, there may not exist many models that satisfy NA1 (NUPBR) but not NFLVR. The classical examples are based on Bessel processes (see e.g. [7], [20], [11]). As hinted at in [4], discontinuous market models may offer many more possibilities. To this effect one may however point out that in [14] the author shows that for exponential Lévy models the various no-arbitrage notions weaker than NFLVR are all equivalent to NFLVR. This still leaves open the possibility of investigating the case of jump-diffusion models, which is the setting that we shall consider in this paper.

Working under NA1 (NUPBR) we cannot rely upon the existence of an ELMM, nor upon the corresponding density process. In fact, if there exists some form of arbitrage beyond NFLVR, then a possible candidate density of an EσMM (ELMM) turns out to be a strict local martingale. The density process can however be generalized to the notion of an *Equivalent Supermartingale Deflator* (ESMD) which, considering a finite horizon $[0,T]$, is a process D_t with $D_0 = 1$, $D_t \geq 0$, $D_T > 0$ $P-a.s.$ and such that $D\bar{V}$ is a supermartingale for all discounted value processes $\bar{V}_t$ of a self-financing admissible portfolio strategy (D_t is thus itself a supermartingale). Notice that, if there exists an *Equivalent Supermartingale Measure* (ESMM), namely a measure $Q \sim P$ under which all discounted self-financing portfolio processes are supermartingales, then the process $D_t := \left(\frac{dQ}{dP}\right)_{|\mathcal{F}_t}$ is an ESMD that is actually a martingale. An ESMD is however not necessarily a density process, it may not even be a (local) martingale. Instead of approaching directly the problem of constructing jump-diffusion market models that satisfy NA1 (NUPBR) but not NFLVR, in this paper we do it indirectly: for specific jump-diffusion market models that satisfy NA1 (NUPBR) we construct an ESMD that is not a martingale, not even a local martingale. We shall then show that for these models the so constructed ESMD is the only candidate for the density process of an ELMM (ESMM). If, thus, this only candidate is a supermartingale that is not a martingale, then there cannot exist an ELMM (ESMM). Since, furthermore, for these cases we shall show that the physical measure P cannot be an ELMM, the property of NFLVR fails to hold.

A basic tool to obtain an ESMD is via the *Growth Optimal Portfolio* (GOP), which is a portfolio that outperforms any other self-financing portfolio in the sense that the ratio between the two processes is a supermartingale. For continuous markets it can in fact be shown that, under local square integrability of the market price of risk process, if $\bar{V}_t^*$ denotes the discounted value process of the GOP, then $\hat{Z}_t := \frac{1}{\bar{V}_t^*}$ is an ESMD (see e.g. [10]). Furthermore, always in continuos markets and under local square integrability of the market price of risk, the undiscounted GOP process V_t^* is a *Numeraire Portfolio* in the sense that self-financing portfolio values, expressed in units of V_t^*, are supermartingales under the physical measure (in the continuous case they are actually positive local martingales so that NFLVR holds in the V_t^*-discounted market and P itself is an ELMM). This is however not true in general: the inverse of the discounted GOP may fail to be even a local martingale (see e.g. Example 5.1bis in [17]). In particular, this may happen when jumps are present (see e.g. [1] Example 6, and [4]). On the other hand, also for general semimartingale models one can show (see [13]) the equivalence of

i) Existence of a numeraire portfolio.

ii) Existence of an ESMD.

iii) Validity of NA1 (NUPBR).

Since, when the GOP exists, this GOP is a numeraire portfolio even if the inverse of its discounted value is not a local martingale (see e.g. [12]), by the previous equivalence there exists then an ESMD. In this case, namely when the inverse of the discounted GOP is not a local martingale, we then also have that the physical measure is not an ELMM when using the GOP as numeraire. In general this does not exclude that NFLVR may nevertheless hold (see [3], see also [24]). However, in the jump-diffusion case we shall show that the process $\hat{Z}_t$ is the only candidate for the density process of an ELMM (ESMM). Therefore, in this case the failure of the inverse of the discounted GOP to be a (local) martingale excludes the possibility of NFLVR.

In our quest for examples in the jump-diffusion setting, where the GOP exists and thus NA1 (NUPBR) holds, but the inverse of the discounted GOP may fail to be a local martingale, we shall start by studying the existence and the properties of the GOP, thereby focusing on the characteristics of the market price of risk vector and showing that the existence of an ELMM (ESMM) depends strictly on the relationship between the components of this vector and the corresponding jump intensities. We then prove that the inverse of the discounted GOP defines the only candidate to be the density process of an ELMM and provide examples in which the discounted GOP not only fails to be the inverse of a martingale density process, but is a supermartingale that is not a local martingale and this with and without constraints on the portfolio strategies.

The outline of the paper is as follows: In section 2 we describe our jump-diffusion market model and the admissible strategies. Section 3 is devoted to the existence and the properties of the GOP as well as its relation with ESMDs. Examples where the discounted GOP fails to be the inverse of a (super)martingale density process are then discussed in section 4. The model described in section 2 as well as the contents of subsection 3.1 are based on the first part of Chapter 14 in [20] (see also [5]). We recall them here to make the presentation self-contained thereby giving also some additional detail that will be needed later in the paper.

2 The Jump Diffusion Market Model

Let there be given a complete probability space $(\Omega, \mathcal{F}, P)$, with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of right-continuity and completeness. We consider a market containing $d \in \mathbb{N}$ sources of uncertainty. Continuous uncertainty is represented by an m-dimensional standard Wiener process $W = \{W_t = (W_t^1, \ldots, W_t^m)^\top,\ t \in [0,\infty)\}$. Event driven uncertainty on the other hand is modeled by an $(d\text{-}m)$-variate point process, identified by the $\mathcal{F}$-adapted counting process $N = \{N_t = (N_t^1, \ldots, N_t^{d-m})^\top,\ t \in [0,\infty)\}$, whose intensity $\lambda = \{\lambda_t = (\lambda_t^1, \ldots, \lambda_t^{d-m})^\top,\ t \in [0,\infty)\}$ is a given, predictable and strictly positive process satisfying $\lambda_t^k > 0$ and $\int_0^t \lambda_s^k ds < \infty$ almost surely for all $t \in [0,\infty)$ and $k \in \{1, 2, \ldots, d-m\}$. We shall denote by T_n the jump times of N_t. For each univariate point process we can define a corresponding jump martingale $M^k = \{M_t^k,\ t \in [0,\infty)\}$ which, following [20], we define via its stochastic differential

$$dM_t^k = \frac{dN_t^k - \lambda_t^k dt}{\sqrt{\lambda_t^k}}, \tag{1}$$

for all $t \in [0, \infty)$ and $k \in \{1, 2, \dots, d-m\}$. We assume that W and N are independent, generate all the uncertainty in the model and that pairwise the N^ks do not jump at the same time.

The financial market consists of $d+1$ securities S^j, for $j = 0, 1, \dots, d$, that model the evolution of wealth due to the ownership of primary securities, with all income and dividends reinvested. As usual, the first account S_t^0 is assumed to be locally risk free, which means that it is of finite variation and the solution of the differential equation

$$dS_t^0 = S_t^0 r_t dt$$

for $t \in [0, \infty)$, with $S_0^0 = 1$. The remaining assets S^j, for $j = 1, \dots, d$, are supposed to be risky and to be the solution to the jump diffusion SDE

$$dS_t^j = S_{t-}^j \left(a_t^j dt + \sum_{k=1}^{m} b_t^{j,k} dW_t^k + \sum_{k=m+1}^{d} b_t^{j,k} dM_t^{k-m} \right) \qquad S_0^j > 0 \tag{2}$$

for $t \in [0, \infty)$, where the short rate process r, the appreciation rate processes a^j, the generalized volatility processes $b^{j,k}$ and the intensity processes λ^k are almost surely finite and predictable. Assuming that these processes are such that a unique strong solution to the system of SDEs (2) exists, we obtain the following explicit expression for every $j = 1, \dots, d$

$$S_t^j = S_0^j \exp\left\{ \int_0^t \left(a_s^j - \frac{1}{2} \sum_{k=1}^{m} (b_s^{j,k})^2 \right) ds + \sum_{k=1}^{m} \int_0^t b_s^{j,k} dW_s^k \right\} \cdot$$
$$\cdot \prod_{k=m+1}^{d} \left[\exp\left\{ - \int_0^t b_s^{j,k} \sqrt{\lambda_s^{k-m}} ds \right\} \prod_{n=1}^{N_t^{k-m}} \left(\frac{b_{T_n}^{j,k}}{\sqrt{\lambda_{T_n}^{k-m}}} + 1 \right) \right]$$

To ensure non-negativity for each primary security account, and exclude jumps that would lead to negative values for S_t^j, we need to make the following assumption:

Assumption 2.1. *The condition*

$$b_t^{j,k} \geq -\sqrt{\lambda_t^{k-m}}$$

holds for all $t \in [0, \infty)$, $j \in \{1, 2, \dots, d\}$ *and* $k \in \{m+1, \dots, d\}$.

In what follows b_t will denote the *generalized volatility* matrix $[b_t^{j,k}]_{j,k=1}^d$ for all $t \in [0, \infty)$ and we make the further assumption:

Assumption 2.2. *The generalized volatility matrix* b_t *is invertible for Lebesgue-almost-every* $t \in [0, \infty)$.

The condition stated in Assumption 2.2 means that no primary security account can be formed as a portfolio of other primary security accounts, i.e. the market does not contain redundant assets.

We can now introduce the *market price of risk* vector

$$\theta_t = (\theta_t^1, \dots, \theta_t^d)^\top = b_t^{-1}[a_t - r_t \mathbf{1}], \tag{3}$$

for $t \in [0, \infty)$. Here $a_t = (a_t^1, \dots, a_t^d)^\top$ is the *appreciation rate* vector and $\mathbf{1} = (1, \dots, 1)^\top$ is the *unit* vector. Using (3), we obtain $a_t = b_t \theta_t + r_t \mathbf{1}$ so that we can rewrite the SDE (2) in the form

$$dS_t^j = S_{t-}^j \left(r_t dt + \sum_{k=1}^{m} b_t^{j,k} (\theta_t^k dt + dW_t^k) + \sum_{k=m+1}^{d} b_t^{j,k} (\theta_t^k dt + dM_t^{k-m}) \right), \tag{4}$$

for $t \in [0,\infty)$ and $j \in \{1,\dots,d\}$. For $k \in \{1,2,\dots,m\}$, the quantity θ_t^k denotes the market price of risk with respect to the k-th Wiener process W^k. In a similar way, if $k \in \{m+1,\dots,d\}$, then θ_t^k can be interpreted as the market price of the $(k-m)$-th event risk with respect to the counting process N^{k-m}.

The vector process $S = \{S_t = (S_t^0,\dots,S_t^d)^\top,\ t \in [0,\infty)\}$ characterizes the evolution of all primary security accounts. In order to rigorously describe the activity of trading in the financial market we now recall the concept of *trading strategy*. We emphasize that we only consider self-financing trading strategies which generate *positive portfolio processes*.

Definition 2.3. *(Admissible strategies)*

(a) *An admissible trading strategy with initial value s is a $\mathbb{R}^{d+1}$-valued predictable stochastic process $\delta = \{\delta_t = (\delta_t^0,\dots,\delta_t^d)^\top,\ t \in [0,\infty)\}$, where δ_t^j denotes the number of units of the j-th primary security account held at time $t \in [0,\infty)$ in the portfolio, and it is such that the Itô integral $\int_0^T \delta_t^j dS_t^j$ is well-defined for any $T > 0$ and $j \in \{0,1,\dots,d\}$. Furthermore, the value of the corresponding portfolio process at time t, which as in [20] we denote by $S_t^{s,\delta} = \sum_{j=0}^d \delta_t^j S_t^j$ with $S_0^{s,\delta} = s > 0$, is nonnegative for any $t \in [0,\infty)$. We let $S_t^\delta := S_t^{1,\delta}$ so that $S_t^{s,\delta} = s\, S_t^\delta$. Since the strategy is self-financing, we have also $dS_t^{s,\delta} = \sum_{j=0}^d \delta_t^j dS_t^j$.*

(b) *For any admissible trading strategy δ the value of the corresponding discounted portfolio process at time t is defined as $\bar{S}_t^{s,\delta} = \frac{S_t^{s,\delta}}{S_t^0}$.*

For a given strategy δ with strictly positive portfolio process $S^{s,\delta}$ we denote as usual by $\pi_{\delta,t}^j$ the fraction of wealth invested in the j-th primary security account at time t, that is $\pi_{\delta,t}^j = \delta_t^j \frac{S_{t-}^j}{S_{t-}^{s,\delta}}$ for $t \in [0,\infty)$ and $j \in \{0,1,\dots,d\}$. In terms of the vector of fractions $\pi_{\delta,t} = (\pi_{\delta,t}^0,\dots,\pi_{\delta,t}^d)^\top$ we obtain from (4), the self-financing property, and taking (1) into account, the following SDE for $S_t^{s,\delta}$

$$
\begin{aligned}
dS_t^{s,\delta} = S_{t-}^{s,\delta}\Bigg\{ \Bigg(& r_t + \sum_{k=1}^m \sum_{j=1}^d \pi_{\delta,t-}^j b_t^{j,k}\theta_t^k + \sum_{k=m+1}^d \sum_{j=1}^d \pi_{\delta,t-}^j b_t^{j,k}\left(\theta_t^k - \sqrt{\lambda_t^{k-m}}\right)\Bigg) dt \\
& + \sum_{k=1}^m \sum_{j=1}^d \pi_{\delta,t}^j b_t^{j,k} dW_t^k + \sum_{k=m+1}^d \sum_{j=1}^d \pi_{\delta,t}^j \frac{b_t^{j,k}}{\sqrt{\lambda_t^{k-m}}} dN_t^{k-m}\Bigg\}
\end{aligned} \tag{5}
$$

Therefore, the value at time t of the portfolio $S^{s,\delta}$ is

$$
\begin{aligned}
S_t^{s,\delta} = s\exp\Bigg\{ & \int_0^t \left(r_s + \sum_{k=1}^m \sum_{j=1}^d \pi_{\delta,s}^j b_s^{j,k}\theta_s^k - \frac{1}{2}\sum_{k=1}^m \left(\sum_{j=1}^d \pi_{\delta,s}^j b_s^{j,k}\right)^2 \right) ds \\
& + \sum_{k=1}^m \int_0^t \sum_{j=1}^d \left(\pi_{\delta,s}^j b_s^{j,k}\right) dW_s^k \Bigg\} \cdot \prod_{k=m+1}^d \Bigg[\exp\left\{ -\int_0^t \sum_{j=1}^d \pi_{\delta,s}^j b_s^{j,k} \sqrt{\lambda_s^{k-m}} ds \right\} \\
& \cdot \prod_{n=1}^{N_t^{k-m}} \left(\frac{\sum_{j=1}^d \pi_{\delta,T_n}^j b_{T_n}^{j,k}}{\sqrt{\lambda_{T_n}^{k-m}}} + 1 \right) \Bigg]
\end{aligned} \tag{6}
$$

From the last term on the right hand side in (6) it is immediately seen that a portfolio process remains strictly positive if and only if

$$\sum_{j=1}^{d} \pi_{\delta,t}^{j} b_t^{j,k} > -\sqrt{\lambda_t^{k-m}} \quad \text{a.s.} \tag{7}$$

for all $k \in \{m+1, \ldots, d\}$ and $t \in [0, \infty)$. This condition is guaranteed by Assumption 2.1.

3 The Growth Optimal Portfolio (GOP)

3.1 Derivation of the GOP and its dynamics

We start from

Definition 3.1. *For an admissible trading strategy δ, leading to a strictly positive portfolio process, the* **growth rate process** $g^\delta = (g_t^\delta)_{t\geq 0}$ *is defined as the drift term in the SDE satisfied by the process* $\log S^\delta = (\log S_t^\delta)_{t\geq 0}$. *An admissible trading strategy δ^* (and the corresponding portfolio process S^{δ^*}) is said to be growth-optimal if $g_t^{\delta^*} \geq g_t^\delta$ P-a.s. for all $t \in [0, \infty)$ for any admissible trading strategy δ. We shall use the acronym GOP to denote the growth-optimal portfolio.*

By applying Itô's formula and suitably adding and subtracting terms (see [19]) one immediately obtains the general expression for g_t^δ, namely

Lemma 3.2. *For any admissible trading strategy δ, the SDE satisfied by $\log S^\delta$ is*

$$d\log S_t^\delta = g_t^\delta dt + \sum_{k=1}^{m}\sum_{j=1}^{d} \pi_{\delta,t}^{j} b_t^{j,k} dW_t^k + \sum_{k=m+1}^{d} \log\left(1 + \sum_{j=1}^{d} \pi_{\delta,t}^{j} \frac{b_t^{j,k}}{\sqrt{\lambda_t^{k-m}}}\right)\sqrt{\lambda_t^{k-m}}\, dM_t^{k-m} \tag{8}$$

where g_t^δ is the growth rate given by

$$\begin{aligned} g_t^\delta = r_t &+ \sum_{k=1}^{m}\left[\sum_{j=1}^{d} \pi_{\delta,t}^{j} b_t^{j,k}\theta_t^k - \frac{1}{2}\left(\sum_{j=1}^{d} \pi_{\delta,t}^{j} b_t^{j,k}\right)^2\right] \\ &+ \sum_{k=m+1}^{d}\left[\sum_{j=1}^{d} \pi_{\delta,t}^{j} b_t^{j,k}\left(\theta_t^k - \sqrt{\lambda_t^{k-m}}\right) + \log\left(1 + \sum_{j=1}^{d} \pi_{\delta,t}^{j} \frac{b_t^{j,k}}{\sqrt{\lambda_t^{k-m}}}\right)\lambda_t^{k-m}\right] \end{aligned} \tag{9}$$

for $t \in [0, \infty)$.

In order to obtain the GOP dynamics we now maximize separately the two sums on the right hand side of (9) with respect to the *portfolio volatilities* $c_t^k := \sum_{j=1}^{d} \pi_{\delta,t}^{j} b_t^{j,k}$ for $k \in \{1, \ldots, d\}$. Note that for the first sum a unique maximum exists, because it is a negative definite quadratic form with respect to the portfolio volatilities. In order to guarantee the existence and the uniqueness of a maximum also in the second sum, we have to impose the following condition

Assumption 3.3. *The intensities and the market price of event risk components satisfy*

$$\sqrt{\lambda_t^{k-m}} > \theta_t^k$$

for all $t \in [0, \infty)$ and $k \in \{m+1, \ldots, d\}$.

This is because the first derivative of

$$c_t^k\left(\theta_t^k-\sqrt{\lambda_t^{k-m}}\right)+\log\left(1+\frac{c_t^k}{\sqrt{\lambda_t^{k-m}}}\right)\lambda_t^{k-m} \tag{10}$$

with respect to c_t^k, which is $\left(\theta_t^k-\sqrt{\lambda_t^{k-m}}\right)+\frac{\lambda_t^{k-m}}{\sqrt{\lambda_t^{k-m}}+c_t^k}$, is positive for all $c_t^k>-\sqrt{\lambda_t^{k-m}}$ if Assumption 3.3 does not hold. These, by virtue of (7), are precisely all the possible values of the *event driven volatility* $\sum_{j=1}^d \pi_{\delta,t}^j b_t^{j,k}$, $k\in\{m+1,\dots,d\}$. Therefore if Assumption 3.3 fails to hold there will not exist an optimal growth rate, since (10) tends to infinity as $c_t^k\to\infty$, for any $k\in\{m+1,\dots,d\}$.

This condition allows us to introduce the predictable vector process $c_t^*=(c_t^{*1},\dots,c_t^{*d})^\top$ which describes the optimal generalized portfolio volatilities. For the components with $k\in\{1,\dots,m\}$, we get from the first order condition that identifies the maximum growth rate the following

$$\theta_t^k-c_t^k=0\Longleftrightarrow c_t^k=\theta_t^k$$

For the last $(d-m)$ components we get, again from the first order condition, noticing that the function of c_t^k in (10) is strictly concave and that we must have $c_t^k>-\sqrt{\lambda_t^{k-m}}$,

$$\left(\theta_t^k-\sqrt{\lambda_t^{k-m}}\right)+\frac{\lambda_t^{k-m}}{\sqrt{\lambda_t^{k-m}}+c_t^k}=0\Longleftrightarrow c_t^k=\frac{\theta_t^k}{1-\theta_t^k(\lambda_t^{k-m})^{-\frac{1}{2}}}$$

Therefore the vector c_t^* has the following representation

$$c_t^{*k}=\begin{cases}\theta_t^k & \text{for}\quad k\in\{1,2,\dots,m\}\\ \frac{\theta_t^k}{1-\theta_t^k(\lambda_t^{k-m})^{-\frac{1}{2}}} & \text{for}\quad k\in\{m+1,\dots,d\}\end{cases} \tag{11}$$

for $t\in[0,\infty)$. Note that a very large jump intensity with $\lambda_t^{k-m}\gg 1$ or $\frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}}\ll 1$ causes the corresponding component c_t^{*k} to approach the market price of jump risk θ_t^k asymptotically for given $t\in[0,\infty)$ and $k\in\{m+1,\dots,d\}$. In this case the structure of the components $c_t^{*k}\approx\theta_t^k$ for $k\in\{m+1,\dots,d\}$ is similar to those obtained with respect to the Wiener processes. Intuitively this is because, when jumps occur more and more frequently, almost continuously, the jump martingales M^k become nearly indistinguishable from the continuous ones.

The above considerations lead immediately to the following (see also Corollary 14.1.5 in [20])

Lemma 3.4. *Under the Assumptions 2.1, 2.2 and 3.3 the fractions*

$$\pi_{\delta_*,t}=(\pi_{\delta_*,t}^1,\dots,\pi_{\delta_*,t}^d)=(c_t^{*\top}b_t^{-1})^\top \tag{12}$$

determine uniquely the GOP and the corresponding portfolio process $S^{\delta_*}=\{S_t^{\delta_*},\ t\in[0,\infty)\}$ satisfies the SDE

$$\begin{aligned}dS_t^{\delta_*}=S_{t-}^{\delta_*}\Bigg(&r_t dt+\sum_{k=1}^m\theta_t^k(\theta_t^k dt+dW_t^k)\\&+\sum_{k=m+1}^d\frac{\theta_t^k}{1-\theta_t^k(\lambda_t^{k-m})^{-\frac{1}{2}}}(\theta_t^k dt+dM_t^{k-m})\Bigg),\end{aligned} \tag{13}$$

for $t\in[0,\infty)$, with $S_0^{\delta_*}>0$. Note that Assumption 3.3 guarantees that the portfolio process S^{δ_*} is strictly positive.

By (9), (11) and (12) we obtain the optimal growth rate of the GOP in the form

$$
\begin{aligned}
g_t^{\delta_*} &= r_t + \sum_{k=1}^{m}\left[\left(\theta_t^k\right)^2 - \frac{1}{2}\left(\theta_t^k\right)^2\right] + \sum_{k=m+1}^{d}\left[\frac{\theta_t^k}{1-\theta_t^k(\lambda_t^{k-m})^{-\frac{1}{2}}}\left(\theta_t^k - \sqrt{\lambda_t^{k-m}}\right)\right.\\
&\quad \left. + \log\left(1+\frac{\theta_t^k}{1-\theta_t^k(\lambda_t^{k-m})^{-\frac{1}{2}}}\frac{1}{\sqrt{\lambda_t^{k-m}}}\right)\lambda_t^{k-m}\right]\\
&= r_t + \frac{1}{2}\sum_{k=1}^{m}\left(\theta_t^k\right)^2 + \sum_{k=m+1}^{d}\lambda_t^{k-m}\left(\log\left(1+\frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}-\theta_t^k}\right) - \frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}}\right)
\end{aligned}
$$

for $t \in [0,\infty)$.

3.2 GOP and Martingale Deflators

In what follows we consider a fixed finite time horizon $T < \infty$ and investigate whether the model introduced above represents a viable financial market, in particular we shall check whether properly defined arbitrage opportunities are excluded. First we recall the following

Definition 3.5. *An equivalent local martingale measure (ELMM) is a measure $Q \sim P$ such that all price processes, expressed in units of the risk-free asset (i.e. discounted prices), are local martingales. Analogously, $Q \sim P$ is an equivalent supermartingale measure (ESMM) if the price processes, expressed in units of the risk-free asset, are supermartingales.*

From [15] we also recall the

Definition 3.6. *An $\mathcal{F}_T$-measurable nonnegative random variable ξ is called arbitrage of the first kind if $P(\xi > 0) > 0$ and, for all initial values $s \in (0,\infty)$, there exists an admissible trading strategy δ such that $\bar{S}_T^{s,\delta} \geq \xi$ P-a.s. We say that the financial market is viable if there are no arbitrages of the first kind, i.e. the condition NA1 holds.*

We next show that a sufficient condition for the absence of arbitrages of the first kind is the existence of a *supermartingale deflator* which we describe as

Definition 3.7. *An equivalent supermartingale deflator (ESMD) is a real-valued nonnegative adapted process $D = (D_t)_{0\leq t\leq T}$ with $D_0 = 1$ and $D_T > 0$ P-a.s. and such that the process $D\bar{S}^\delta = (D_t\bar{S}_t^\delta)_{0\leq t\leq T}$ is a supermartingale for every admissible trading strategy δ. We denote by $\mathcal{D}$ the set of all supermartingale deflators. (By taking $\delta \equiv (1,0,\cdots,0)$, one has that D is itself a supermartingale).*

Proposition 3.8. *If $\mathcal{D} \neq \emptyset$ then there cannot exist arbitrages of the first kind.*

Proof. (adapted from [10]). Let $D \in \mathcal{D}$ and suppose that there exists a random variable ξ yielding an arbitrage of the first kind. Then, for every $n \in \mathbb{N}$, there exists an admissible trading strategy δ^n such that, $\bar{S}_T^{1/n,\delta^n} \geq \xi$ P-a.s. For every $n \in \mathbb{N}$, the process $D\bar{S}^{1/n,\delta^n} = (D_t\bar{S}_t^{1/n,\delta^n})_{0\leq t\leq T}$ is a supermartingale. So, for every $n \in \mathbb{N}$ one has $E[D_T\xi] \leq E[D_T\bar{S}_T^{1/n,\delta^n}] \leq E[D_0\bar{S}_0^{1/n,\delta^n}] = \frac{1}{n}$. Letting $n \to \infty$ gives $E[D_T\xi] = 0$ and hence $D_T\xi = 0$ P-a.s. Since, due to Definition 3.7, we have $D_T > 0$ P-a.s. this implies that $\xi = 0$ P-a.s., which contradicts the assumption that ξ is an arbitrage of the first kind. □

In the remaining part of this section we will derive a fundamental property of the GOP. Dealing with GOP-denominated portfolio processes, following [20] we first introduce the following notation.

Definition 3.9. *For any portfolio process* S^δ, *the process* $\hat{S}^\delta = \left(\hat{S}^\delta_t\right)_{0\le t\le T}$, *defined as* $\hat{S}^\delta_t := S^\delta_t / S^{\delta_*}_t$ *for* $t \in [0,T]$, *is called Benchmarked portfolio process.*

Remark 3.10. *We shall often refer to the inverse of the discounted GOP as* $\hat{Z}_t := \frac{1}{\bar{S}^{\delta_*}_t}$.

We begin with the following proposition.

Proposition 3.11. *Under Assumptions 2.1, 2.2 and 3.3 the discounted GOP process* $\bar{S}^{\delta_*} = \{\bar{S}^{\delta_*}_t,\ t \in [0,T]\}$ *is the inverse of a supermartingale deflator D. Equivalently,* $\hat{Z}_t$ *is a supermartingale deflator.*

Proof. It suffices to show that every benchmarked portfolio is a local martingale that, being nonnegative, is then a supermartingale by Fatou's lemma. According to the product formula (see Corollary II-2 in [21]) we first have that

$$d\left(\frac{S^\delta_t}{S^{\delta_*}_t}\right) = d\left(\frac{\bar{S}^\delta_t}{\bar{S}^{\delta_*}_t}\right) = \frac{d\bar{S}^\delta_t}{\bar{S}^{\delta_*}_{t-}} + d\left(\frac{1}{\bar{S}^{\delta_*}_{t-}}\right)\bar{S}^\delta_t + d\left[\bar{S}^\delta_t, \frac{1}{\bar{S}^{\delta_*}_t}\right]$$

For the term involving $d\bar{S}^\delta_t$ note that, from (5) and taking into account (1) as well as the definition of the investment ratios $\pi_{\delta,t}$, one obtains

$$\begin{aligned} d\bar{S}^\delta_t &= \bar{S}^\delta_{t-}\left\{\sum_{k=1}^m\left(\sum_{j=1}^d \delta^j_t \frac{S^j_t}{S^\delta_t} b^{j,k}_t\right)\left(\theta^k_t dt + dW^k_t\right) + \sum_{k=m+1}^d\left(\sum_{j=1}^d \delta^j_t \frac{S^j_{t-}}{S^\delta_{t-}} b^{j,k}_t\right)\left(\theta^k_t dt + dM^{k-m}_t\right)\right\} \\ &= \sum_{k=1}^m\left(\sum_{j=1}^d \delta^j_t \bar{S}^j_t b^{j,k}_t\right)\left(\theta^k_t dt + dW^k_t\right) + \sum_{k=m+1}^d\left(\sum_{j=1}^d \delta^j_t \bar{S}^j_{t-} b^{j,k}_t\right)\left(\theta^k_t dt + dM^{k-m}_t\right) \end{aligned} \tag{14}$$

Next, using Itô's formula, from (13) we obtain for the inverse of the discounted GOP

$$\begin{aligned} d\left(\frac{1}{\bar{S}^{\delta_*}_t}\right) = &\left[-\frac{1}{\bar{S}^{\delta_*}_t}\left(\sum_{k=1}^m\left(\theta^k_t\right)^2 + \sum_{k=m+1}^d \frac{\theta^k_t}{1-\frac{\theta^k_t}{\sqrt{\lambda^{k-m}_t}}}\left(\theta^k_t - \sqrt{\lambda^{k-m}_t}\right)\right)\right. \\ &\left. + \frac{1}{\bar{S}^{\delta_*}_t}\sum_{k=1}^m\left(\theta^k_t\right)^2\right]dt - \frac{1}{\bar{S}^{\delta_*}_t}\sum_{k=1}^m \theta^k_t dW^k_t \\ &+ \frac{1}{\bar{S}^{\delta_*}_{t-}}\sum_{k=m+1}^d\left[\left(1 + \frac{\theta^k_t}{1-\frac{\theta^k_t}{\sqrt{\lambda^{k-m}_t}}}\frac{1}{\sqrt{\lambda^{k-m}_t}}\right)^{-1} - 1\right]dN^{k-m}_t \end{aligned}$$

The terms of the last sum can be simplified in the following way

$$\frac{1}{S^{\delta_*}_{t-}}\left(\left(\frac{\sqrt{\lambda^{k-m}_t}}{\sqrt{\lambda^{k-m}_t} - \theta^k_t}\right)^{-1} - 1\right) = -\frac{\theta^k_t}{\bar{S}^{\delta_*}_{t-}\sqrt{\lambda^{k-m}_t}}$$

so that, by adding and subtracting $-\sum_{k=m+1}^{d} \frac{\theta_t^k \sqrt{\lambda_t^{k-m}}}{\bar{S}_t^{\delta_*}} dt$ and rearranging all the terms, we obtain

$$d\left(\frac{1}{\bar{S}_t^{\delta_*}}\right) = -\frac{1}{\bar{S}_t^{\delta_*}} \sum_{k=1}^{m} \theta_t^k dW_t^k - \frac{1}{\bar{S}_{t-}^{\delta_*}} \sum_{k=m+1}^{d} \theta_t^k dM_t^{k-m}. \tag{15}$$

Finally, the last term is just

$$d\left[\bar{S}_t^{\delta}, \frac{1}{\bar{S}_t^{\delta_*}}\right] = -\sum_{k=1}^{m}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_t^j b_t^{j,k}\right)\theta_t^k dt - \sum_{k=m+1}^{d}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_{t-}^j b_t^{j,k}\right)\frac{\theta_t^k}{\lambda_t^{k-m}} dN_t^{k-m}$$

Summing up all the components we obtain

$$\begin{aligned} d\left(\frac{\bar{S}_t^{\delta}}{\bar{S}_t^{\delta_*}}\right) = &\sum_{k=1}^{m}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_t^j b_t^{j,k} - \hat{S}_t^{\delta}\theta_t^k\right) dW_t^k + \sum_{k=m+1}^{d}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_{t-}^j b_t^{j,k} - \hat{S}_{t-}^{\delta}\theta_t^k\right) dM_t^{k-m} \\ &+ \sum_{k=m+1}^{d}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_t^j b_t^{j,k}\right)\theta_t^k dt - \sum_{k=m+1}^{d}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_{t-}^j b_t^{j,k}\right)\frac{\theta_t^k}{\lambda_t^{k-m}} dN_t^{k-m} \end{aligned}$$

from which, observing that

$$\begin{aligned} &\sum_{k=m+1}^{d}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_t^j b_t^{j,k}\right)\theta_t^k dt - \sum_{k=m+1}^{d}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_{t-}^j b_t^{j,k}\right)\frac{\theta_t^k}{\lambda_t^{k-m}} dN_t^{k-m} = \\ &= -\sum_{k=m+1}^{d}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_{t-}^j b_t^{j,k}\right)\frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}}\left(\frac{dN_t^{k-m} - \lambda_t^{k-m} dt}{\sqrt{\lambda_t^{k-m}}}\right) \end{aligned}$$

we obtain

$$\begin{aligned} d\left(\frac{\bar{S}_t^{\delta}}{\bar{S}_t^{\delta_*}}\right) = &\sum_{k=1}^{m}\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_t^j b_t^{j,k} - \hat{S}_t^{\delta}\theta_t^k\right) dW_t^k \\ &+ \sum_{k=m+1}^{d}\left(\left(\sum_{j=1}^{d} \delta_t^j \hat{S}_{t-}^j b_t^{j,k}\right)\left(1 - \frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}}\right) - \hat{S}_{t-}^{\delta}\theta_t^k\right) dM_t^{k-m} \end{aligned}$$

which, indeed, is a nonnegative local martingale and thus a supermartingale. □

From this proof it actually follows that $\hat{Z}_t$ is an equivalent local martingale deflator (ELMD) in the sense that $\hat{Z}_t \bar{S}^{\delta}$ are local martingales.

Generalizing the notion of Benchmarked portfolio process (see Definition 3.9) we recall the following

Definition 3.12. *An admissible portfolio process $S^{\tilde{\delta}} = \left(S_t^{\tilde{\delta}}\right)_{0\le t\le T}$ has the numeraire property if all admissible portfolio processes $S^{\delta} = \left(S_t^{\delta}\right)_{0\le t\le T}$, when denominated in terms of $S^{\tilde{\delta}}$, are supermartingales, i.e. if the process $S^{\delta}/S^{\tilde{\delta}} = \left(S_t^{\delta}/S_t^{\tilde{\delta}}\right)_{0\le t\le T}$ is a supermartingale for every admissible trading strategy δ.*

Remark 3.13. *As a corollary of Proposition 3.11 we have that the GOP has the numeraire property.*

In continuous financial markets it can be shown that the numeraire portfolio is unique (see e.g. [10]). The proof in [10] can be carried over rather straightforwardly to the jump-diffusion case (see [19]) so that we have

Proposition 3.14. *The numeraire portfolio process $S^{\tilde{\delta}} = \left(S_t^{\tilde{\delta}}\right)_{0\le t\le T}$ is unique (in the sense of indistinguishability). Furthermore, there exists an unique admissible trading strategy $\tilde{\delta}$ such that $S^{\tilde{\delta}}$ is the numeraire portfolio, up to a null subset of $\Omega \times [0,T]$.*

Remark 3.15. *Since, as we have seen, the GOP has the numeraire property and the numeraire portfolio is unique, the GOP is the unique numeraire portfolio.*

The numeraire property of the GOP plays a crucial role in the concept of *real world pricing* allowing one, also in the present jump-diffusion setting, to perform pricing of contingent claims in financial markets for which no ELMM may exist (see [20], [10]).

4 Supermartingale Deflators/Densities

Due to Propositions 3.11 and 3.14 (see Remarks 3.13 and 3.15), the GOP coincides with the unique numeraire portfolio and its discounted value is also the inverse of the supermartingale deflator $\hat{Z}_t$. This means that, if we express all price processes in terms of the GOP, the original probability measure P becomes an ESMM. We now investigate particular cases in which $\hat{Z}_t$ is not a martingale and so, since it will be shown to be also the only candidate for the density process of an ELMM, for these cases NFLVR fails to hold. Furthermore, when expressing the price processes in terms of the GOP, the physical measure P is not an ELMM.

We start by showing that for our financial market model the inverse of the discounted GOP is the only possible local martingale deflator; thus it is also the only candidate to be the Radon-Nikodym derivative (density process) of an ELMM. These concepts are in fact strictly linked to one another, since a supermartingale deflator D defines an ELMM if and only if D_T integrates to 1. On the other hand, the Radon-Nikodym derivative of an ELMM is of course a supermartingale deflator. We prove the claim of the uniqueness by studying changes of measure via the general Radon-Nikodym derivative when dealing with a jump-diffusion process, namely (see [2], [23])

$$
\begin{aligned}
L_t = \exp\left\{-\frac{1}{2}\sum_{k=1}^{m}\int_0^t \left(\varphi_s^k\right)^2 ds + \sum_{k=1}^{m}\int_0^t \varphi_s^k dW_s^k\right\}\cdot \\
\cdot \prod_{k=m+1}^{d}\left\{\exp\left[\int_0^t \left(1-\psi_s^{k-m}\right)\lambda_s^{k-m} ds\right]\prod_{n=1}^{N_t^{k-m}} \psi_{T_n}^{k-m}\right\}
\end{aligned}
\tag{16}
$$

where φ_t is a square integrable predictable process and ψ_t is a positive predictable process, integrable with respect to λ_t. We will show that these coefficients have to satisfy a linear system whose only solution leads to the same dynamics as for the inverse of the discounted GOP.

Proposition 4.1. *Under Assumptions 2.1, 2.2 and 3.3, the inverse of the discounted GOP, namely $\hat{Z}_t$, is the only candidate to be the Radon-Nikodym derivative of an ELMM. Equivalently, it is the only ELMD in the sense of what was specified after the statement of Proposition 3.11.*

Proof. We start by defining the Wiener and the Poisson martingales W_t^Q and M_t^Q under the new measure Q defined by L_t in (16):

$$\begin{cases} dW_t^{Q,k} &= dW_t^k \quad - \varphi_t^k dt & \text{for} \quad k \in \{1, 2, \dots, m\} \\ dM_t^{Q,k-m} &= dN_t^{k-m} - \psi_t^{k-m} \lambda_t^{k-m} dt & \text{for} \quad k \in \{m+1, \dots, d\} \end{cases}$$

thereby obtaining the SDEs satisfied by the primary security accounts

$$\begin{aligned} dS_t^j = S_{t-}^j \Bigg\{ \Bigg(r_t + \sum_{k=1}^{d} b_t^{j,k} \theta_t^k + \sum_{k=1}^{m} b_t^{j,k} \varphi_t^k + \sum_{k=m+1}^{d} b_t^{j,k} \psi_t^{k-m} \sqrt{\lambda_t^{k-m}} - \sum_{k=m+1}^{d} b_t^{j,k} \sqrt{\lambda_t^{k-m}} \Bigg) dt \\ + \sum_{k=1}^{m} b_t^{j,k} dW_t^{Q,k} + \sum_{k=m+1}^{d} \frac{1}{\sqrt{\lambda_t^{k-m}}} b_t^{j,k} dM_t^{Q,k-m} \Bigg\} \end{aligned} \tag{17}$$

for every $j \in \{1, \dots, d\}$ and $t \in [0, T]$. If L_t is the Radon-Nikodym derivative of an ELMM, the drift term in (17) must be equal to r_t for every $t \in [0, T]$, so that the coefficients φ_t and ψ_t must be the solution to the linear system defined by the following equations

$$\sum_{k=1}^{m} b_t^{j,k} \varphi_t^k + \sum_{k=m+1}^{d} b_t^{j,k} \psi_t^{k-m} \sqrt{\lambda_t^{k-m}} = -\sum_{k=1}^{m} b_t^{j,k} \theta_t^k - \sum_{k=m+1}^{d} b_t^{j,k} \theta_t^k + \sum_{k=m+1}^{d} b_t^{j,k} \sqrt{\lambda_t^{k-m}}$$

for every $j \in \{1, \dots, d\}$ and $t \in [0, T]$. Since, by virtue of the standing Assumption 2.2 the generalized volatility matrix b_t has full rank, the linear system admits an unique solution which is given by

$$\begin{cases} \varphi_t^k &= -\theta_t^k, & \text{for} \quad k \in \{1, 2, \dots, m\} \\ \psi_t^{k-m} &= 1 - \frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}}, & \text{for} \quad k \in \{m+1, \dots, d\} \end{cases}$$

Plugging these terms into (16) we see that the Radon-Nikodym derivative has thus the following expression

$$\begin{aligned} L_t = \exp \left\{ -\frac{1}{2} \sum_{k=1}^{m} \int_0^t \left(\theta_s^k \right)^2 ds - \sum_{k=1}^{m} \int_0^t \theta_s^k dW_s^k \right\} \\ \prod_{k=m+1}^{d} \left\{ \exp \left[\int_0^t \theta_t^k \sqrt{\lambda_s^{k-m}} ds \right] \prod_{n=1}^{N_t^{k-m}} \left(1 - \frac{\theta_{T_n}^k}{\sqrt{\lambda_{T_n}^{k-m}}} \right) \right\} \end{aligned} \tag{18}$$

which is precisely the inverse of the discounted GOP, since (see the equation (15) satisfied by $(\bar{S}_t^{\delta_*})^{-1}$)

$$dL_t = -L_{t-} \left(\sum_{k=1}^{m} \theta_t^k dW_t^k + \sum_{k=m+1}^{d} \theta_t^k dM_t^{k-m} \right)$$

□

Remark 4.2. *Since the vector ψ_t in (16) has to be positive for every $t \in [0, T]$, we note that the condition $1 - \frac{\theta_t^k}{\sqrt{\lambda_t^{k-m}}} \geq 0$ has to hold true for every $k \in \{m+1, \dots, d\}$. This means that if Assumption 3.3 is violated, and there is at least one $k \in \{m+1, \dots, d\}$ for which $\sqrt{\lambda_t^{k-m}} < \theta_t^k$, there cannot exist an ELMM.*

Remark 4.3. *Notice that, by analogy to the 2nd FTAP, for our complete market here we have uniqueness of the ELMD.*

We want to emphasize that, thanks to the proposition just proved, the financial market does not admit an ELMM as soon as the inverse of the discounted GOP is a strict supermartingale that is not a martingale. We therefore try to find some particular cases where the process $\hat{Z}_t$ is a supermartingale that is not a martingale (not even a local martingale). In the continuous case the most notorious example of a martingale deflator that does not yield an ELMM is the three-dimensional Bessel process β_t: the GOP is simply obtained by placing all the wealth in the stock, but it has infinite expected growth rate and $\frac{1}{\beta_t}$ fails to integrate to 1. In the jump diffusion market that we are considering we note that, as $\theta_t^k \to \sqrt{\lambda_t^{k-m}}$ for any $k \in \{m+1, \dots, d\}$, the GOP would explode and therefore $\hat{Z}_t$ would be close to zero as soon as a jump occurs (see (18)). In this case, $\hat{Z}_T$ will not integrate to 1, thus preventing the existence of an ELMM, and the GOP cannot be used as a numeraire.

4.1 Supermartingale Densities that are not (local) Martingale Densities

Let us see what happens if we violate Assumption 3.3, namely if we let $\sqrt{\lambda_t^{k-m}}$ be less than θ_t^k. For simplicity, in this entire section we will consider the particular case in which $d = 2$ and $m = 1$.

In order to be able to maximize the second sum in (9) we then have to impose a restriction on the possible trading strategies in the form of the following assumption.

Assumption 4.4. *There exists a positive real number ψ such that for any admissible trading strategy $\pi = \{\pi_t = (\pi_t^0, \pi_t^1, \pi_t^2)^\top,\ t \in [0,T]\}$ we have*

$$\pi_t^1 b_t^{1,2} + \pi_t^2 b_t^{2,2} \le \psi, \tag{19}$$

for all $t \in [0,T]$

Remark 4.5. *Assumption 4.4 can be seen as a convex constraint limiting the portfolio volatility belonging to the jump martingale M. This condition has a clearer financial interpretation when $b_t^2 := b_t^{1,2} = b_t^{2,2}$ for each $t \in [0,T]$. In this case we get that (19) is equivalent to a constraint on the minimum amount of wealth invested in the risk free asset $\pi_t^0 \ge 1 - \frac{\psi}{b_t^2}$. We emphasize that this applies also to the case in which $b_t^2 = -\sqrt{\lambda_t}$ for each $t \in [0,T]$, in which jumps are used to describe the default of the primary accounts.*

In this case the optimal generalized portfolio volatilities are described by the following predictable process

$$\tilde{c}_t^k = \begin{cases} \theta_t^1 & \text{for} \quad k = 1 \\ \psi & \text{for} \quad k = 2. \end{cases} \tag{20}$$

The first component $\tilde{c}_k^1$ follows from the first order conditions identifying the maximum growth rate, while $\tilde{c}_k^2$ is the maximum value obtainable in the constrained setting. From (5) we have that, for the case when c_t^k are given by (20), the discounted GOP must satisfy the following SDE

$$d\bar{S}_t^{\delta_*} = \bar{S}_{t-}^{\delta_*} \left\{ \theta_t^1 \left(\theta_t^1 dt + dW_t \right) + \psi \left(\theta_t^2 dt + dM_t \right) \right\}. \tag{21}$$

The convex constraints just introduced are the framework that enables us to provide a simple example of a market in which GOP denominated prices are strict supermartingales, as we show in the next theorem.

Theorem 4.6. *Under Assumptions 2.1, 2.2 and 4.4, the process $\hat{Z}_t$ and any benchmarked portfolio process are supermartingales which are not local martingales.*

Proof. By analogy to the proof of Proposition 3.11, we start by calculating the SDE for $\bar{S}_t^{\delta}/\bar{S}_t^{\delta_*}$, where δ_t is an arbitrary admissible trading strategy. According to the product formula we proceed by calculating separately the components $d\bar{S}_t^{\delta}$, $d\left(\frac{1}{\bar{S}_t^{\delta_*}}\right)$ and $d\left[\bar{S}_t^{\delta}, \frac{1}{\bar{S}_t^{\delta_*}}\right]$. The first component can be obtained by adjusting the general formula in (14) to the case in which $d = 2$ so that

$$\begin{aligned} d\bar{S}_t^{\delta} &= \left(\delta_t^1 \bar{S}_t^1 b_t^{1,1} + \delta_t^2 \bar{S}_t^2 b_t^{2,1}\right)\theta_t^1 dt + \left(\delta_t^1 \bar{S}_t^1 b_t^{1,2} + \delta_t^2 \bar{S}_t^2 b_t^{2,2}\right)\theta_t^2 dt \\ &\quad + \left(\delta_t^1 \bar{S}_t^1 b_t^{1,1} + \delta_t^2 \bar{S}_t^2 b_t^{2,1}\right) dW_t + \left(\delta_t^1 \bar{S}_{t-}^1 b_t^{1,2} + \delta_t^2 \bar{S}_{t-}^2 b_t^{2,2}\right) dM_t \end{aligned}$$

The second term comes from applying the Itô formula to (21), namely

$$\begin{aligned} d\left(\frac{1}{\bar{S}_t^{\delta_*}}\right) &= -\frac{1}{\bar{S}_t^{\delta_*}}\left(\left(\theta_t^1\right)^2 + \psi\left(\theta_t^2 - \sqrt{\lambda_t}\right)\right)dt + \frac{1}{\bar{S}_t^{\delta_*}}\left(\theta_t^1\right)^2 dt \\ &\quad - \frac{1}{\bar{S}_t^{\delta_*}}\theta_t^1 dW_t + \frac{1}{\bar{S}_{t-}^{\delta_*}}\left[\left(1 + \psi\frac{1}{\sqrt{\lambda_t}}\right)^{-1} - 1\right] dN_t \\ &= -\frac{1}{\bar{S}_t^{\delta_*}}\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right)dt - \frac{1}{\bar{S}_t^{\delta_*}}\theta_t^1 dW_t - \frac{1}{\bar{S}_{t-}^{\delta_*}}\frac{\psi}{\sqrt{\lambda_t}+\psi}dN_t \\ &= -\frac{1}{\bar{S}_t^{\delta_*}}\left(\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right) + \frac{\psi\lambda_t}{\sqrt{\lambda_t}+\psi}\right)dt - \frac{1}{\bar{S}_{t-}^{\delta_*}}\left(\theta_t^1 dW_t + \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}dM_t\right) \end{aligned} \tag{22}$$

Note that the drift term in the SDE satisfied by $\frac{1}{\bar{S}_t^{\delta_*}}$ is strictly negative as Assumption 3.3 does not hold and both ψ and λ_t are positive. This shows that the inverse of the discounted GOP, namely $\hat{Z}_t$, is a strict supermartingale. Finally,

$$d\left[\bar{S}_t^{\delta}, \frac{1}{\bar{S}_t^{\delta_*}}\right] = -\left(\delta_t^1 \hat{S}_t^1 b_t^{1,1} + \delta_t^2 \hat{S}_t^2 b_t^{2,1}\right)\theta_t^1 dt - \left(\delta_t^1 \hat{S}_{t-}^1 b_t^{1,2} + \delta_t^2 \hat{S}_{t-}^2 b_t^{2,2}\right)\frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}\frac{1}{\lambda_t}dN_t$$

On the other hand, for the benchmarked portfolios we have

$$\begin{aligned} d\left(\frac{\bar{S}_t^{\delta}}{\bar{S}_t^{\delta_*}}\right) &= \left(\delta_t^1 \hat{S}_t^1 b_t^{1,2} + \delta_t^2 \hat{S}_t^2 b_t^{2,2}\right)\theta_t^2 dt + \left(\delta_t^1 \hat{S}_t^1 b_t^{1,1} + \delta_t^2 \hat{S}_t^2 b_t^{2,1}\right) dW_t \\ &\quad + \left(\delta_t^1 \hat{S}_{t-}^1 b_t^{1,2} + \delta_t^2 \hat{S}_{t-}^2 b_t^{2,2}\right) dM_t - \left(\delta_t^1 \hat{S}_{t-}^1 b_t^{1,2} + \delta_t^2 \hat{S}_{t-}^2 b_t^{2,2}\right)\frac{\psi}{\sqrt{\lambda_t}+\psi}\frac{1}{\sqrt{\lambda_t}}dN_t \\ &\quad - \hat{S}_t^{\delta}\left(\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right) + \frac{\psi\lambda_t}{\sqrt{\lambda_t}+\psi}\right)dt - \hat{S}_{t-}^{\delta}\left(\theta_t^1 dW_t + \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}dM_t\right) \\ &= \left[\left(\delta_t^1 \hat{S}_t^1 b_t^{1,2} + \delta_t^2 \hat{S}_t^2 b_t^{2,2}\right)\left(\theta_t^2 - \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}\right) - \hat{S}_t^{\delta}\left(\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right) + \frac{\psi\lambda_t}{\sqrt{\lambda_t}+\psi}\right)\right]dt \\ &\quad + \left(\delta_t^1 \hat{S}_t^1 b_t^{1,1} + \delta_t^2 \hat{S}_t^2 b_t^{2,1}\right) dW_t + \left(\delta_t^1 \hat{S}_{t-}^1 b_t^{1,2} + \delta_t^2 \hat{S}_{t-}^2 b_t^{2,2}\right) dM_t \\ &\quad - \left(\delta_t^1 \hat{S}_{t-}^1 b_t^{1,2} + \delta_t^2 \hat{S}_{t-}^2 b_t^{2,2}\right)\frac{\psi}{\sqrt{\lambda_t}+\psi}\frac{dN_t - \lambda_t dt}{\sqrt{\lambda_t}} - \hat{S}_{t-}^{\delta}\left(\theta_t^1 dW_t + \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}dM_t\right) \end{aligned}$$

from which, being condition (19) equivalent to $\delta_t^1 \hat{S}_t^1 b_t^{1,2} + \delta_t^2 \hat{S}_t^2 b_t^{2,2} \le \psi \hat{S}_t^\delta$, we note that the drift term is negative, in fact

$$\begin{aligned}
&\left(\delta_t^1 \hat{S}_t^1 b_t^{1,2} + \delta_t^2 \hat{S}_t^2 b_t^{2,2}\right)\left(\theta_t^2 - \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}\right) - \hat{S}_t^\delta\left(\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right) + \frac{\psi\lambda_t}{\sqrt{\lambda_t}+\psi}\right) \le \\
&\le \hat{S}_t^\delta\left(\psi\left(\theta_t^2 - \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}\right) - \psi\left(\theta_t^2 - \sqrt{\lambda_t}\right) - \frac{\psi\lambda_t}{\sqrt{\lambda_t}+\psi}\right) = 0
\end{aligned}$$

This is because the factor $\theta_t^2 - \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi}$ is positive: the function $x \mapsto \frac{\psi x}{x+\psi}$ is always increasing, so that, since in this section we are violating Assumption 3.3, we have

$$\theta_t^2 - \frac{\psi\sqrt{\lambda_t}}{\sqrt{\lambda_t}+\psi} \ge \theta_t^2 - \frac{\psi\theta_t^2}{\theta_t^2+\psi} = \frac{\left(\theta_t^2\right)^2}{\theta_t^2+\psi} > 0$$

□

The above theorem shows that, if trading is restricted, the investor maximizing the expected logarithmic utility function (the GOP maximizes also the expected log-utility) goes as far as the admissibility constraints, expressed in Assumption 4.4, allow. Notice that, since $\hat{Z}_t$ is a supermartingale without being a local martingale, it cannot be the density process of an ELMM.

For the sake of completeness we now study the property of $\hat{Z}_t$ when Assumption 3.3 holds true but we enforce Assumption 4.4 as well. We shall see that $\hat{Z}_t$ is again a supermartingale that is not a local martingale and that $S_t^{\delta_*}$ is the numeraire portfolio.

Proposition 4.7. *Under Assumptions 2.1, 2.2, 3.3 and 4.4, if there exists $B \subset [0,T]$ with positive Lebesgue measure such that for $t \in B$ one has $\psi < \frac{\theta_t^2}{1-\frac{\theta_t^2}{\sqrt{\lambda_t}}}$, then any benchmarked portfolio process is a supermartingale which is not a local martingale.*

Proof. We start by noting that the condition $\psi < \frac{\theta_t^2}{1-\frac{\theta_t^2}{\sqrt{\lambda_t}}}$ simply requires that Assumption 4.4 imposes a real constraint on the investor, who would otherwise construct theoptimal portfolio as if the condition expressed in Assumption 4.4 was not present. Therefore, as soon as this condition holds, the discounted GOP dynamics is the same as (21), and the SDEs satisfied by the inverse of the discounted GOP, namely $\hat{Z}_t$, and by any benchmarked portfolio are the same as those obtained in the proof of Theorem 4.6. The only thing left to do is to check the negativity of the drift term in each of the dynamics. For the former, the drift term of $\hat{Z}_t$, we get from (22)

$$\begin{aligned}
-\hat{Z}_t\left(\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right) + \frac{\psi\lambda_t}{\sqrt{\lambda_t}+\psi}\right) &= -\hat{Z}_t\left(\frac{\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right)\left(\sqrt{\lambda_t}+\psi\right) + \psi\lambda_t}{\sqrt{\lambda_t}+\psi}\right) \\
&= -\hat{Z}_t\left(\frac{\psi\left(\psi\left(\theta_t^2 - \sqrt{\lambda_t}\right) + \theta_t^2\sqrt{\lambda_t}\right)}{\sqrt{\lambda_t}+\psi}\right)
\end{aligned}$$

which is negative since $\psi > 0$ and $\psi < \frac{\theta_t^2}{1-\frac{\theta_t^2}{\sqrt{\lambda_t}}}$. The latter follows in the same way as in the proof of Theorem 4.6, noting that $\theta_t^2 - \frac{\psi\sqrt{\lambda_t}}{\psi+\sqrt{\lambda_t}} > 0 \iff \psi < \frac{\theta_t^2}{1-\frac{\theta_t^2}{\sqrt{\lambda_t}}}$. □

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