A Benchmark Approach to Filtering in Finance

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Abstract. The paper proposes the use of the growth optimal portfolio for pricing and hedging in incomplete markets when there are unobserved factors that have to be filtered. The proposed filtering framework is applicable also in cases when there does not exist an equivalent risk neutral martingale measure. The reduction of the variance of derivative prices for increasing degrees of available information is measured.

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1 Introduction

In financial modeling it is typically the case that, in practice, not all quantities, which determine the dynamics of security prices, can be fully observed. Some of the factors that characterize the evolution of the market are hidden. For instance, correlations between driving Wiener processes often change quite randomly after certain periods of time. An example is the correlation between the fluctuations of specific and general market risk, see Platen & Stahl (2003). However, these unobserved factors and their correct calibration are essential to reflect in a financial market model correctly the market dynamics that one empirically observes. This leads naturally to a stochastic filtering problem. Given the available information, corresponding filter methods determine the distribution, called filter distribution, of the unobserved factors. For instance, this distribution allows then to compute the expectation of quantities that are dependent on unobserved factors, including derivative prices, optimal portfolio strategies and risk measures.

There is a growing literature in the area of filtering in finance. To mention a few recent publications let us list Elliott & van der Hoek (1997), Fischer, Platen & Runggaldier (1999), Elliott, Fischer & Platen (1999), Fischer & Platen (1999), Landen (2000), Gombani & Runggaldier (2001), Frey & Runggaldier (1999, 2001), Elliott & Platen (2001), Bhar, Chiarella & Runggaldier (2002, 2004) and Chiarella, Pasquali & Runggaldier (2001). These papers provide examples, where filter methods have been applied to dynamic asset allocation, interest rate term structure calibration, risk premia estimation, volatility estimation and hedging under partial observation.

A key problem that arises in most filtering applications in finance is the determination of a suitable pricing measure. The results depend often significantly on the assumptions made for choosing the pricing measure. Furthermore, it has been demonstrated in Heath & Platen (2002b), Platen (2004b) and Breymann, Kelly & Platen (2004) that any realistic parsimonious financial market model is unlikely to have an equivalent risk neutral martingale measure. Moreover, it is obvious that in filtering one has to deal with the real world probability measure to extract from the observations via filters estimates for the hidden factors. It is therefore highly important to explore methods that are purely based on the real world probability measure and allow consistent filtering under partial information for derivative pricing, portfolio optimization, risk measurement and other applications.

In this paper we extend the *benchmark approach* proposed in Platen (2002, 2004a, 2004b) to filtering, where the benchmark or numeraire is chosen as the *growth optimal portfolio* (GOP). This extends work by Long (1990) and Bajeux-Besnainou & Portait (1997) to the general case where no equivalent risk neutral measure exists and a jump diffusion financial market with hidden factors is considered. This paper provides therefore results for the case of partial information in an incomplete market framework.

The GOP has the economic interpretation of being the portfolio that maximizes expected logarithmic utility. It is highly relevant in portfolio optimization, see Korn & Schäl (1999), but also for derivative pricing, as we will see below. By using the GOP as numeraire or benchmark under a given information structure, one naturally obtains a *fair* derivative price system, where benchmarked derivative prices are martingales. This means, benchmarked derivative prices equal their expected future benchmarked values. Even if equivalent risk neutral martingale measures are assumed to exist, this avoids the delicate involvement of such measures under various information structures. In the special case of a complete market, with an equivalent risk neutral martingale measure, it will turn out that fair prices coincide with risk neutral prices. Furthermore, for an incomplete market, with a minimal equivalent martingale measure in the sense of Föllmer & Schweizer (1991), it will be shown that their prices coincide with the fair prices obtained under the benchmark approach. For incomplete jump diffusion markets under partial observation a much wider range of models is covered in this paper than under traditional approaches. It will be shown that all portfolios, when expressed in units of the GOP, turn out to be local martingales with respect to the given real world measure and under partial information. In this way, delicate issues that result from measure transformations under various information structures can not arise and major risk management tasks, such as hedging, portfolio optimization and risk measurement can be performed consistently under the real world probability measure. Moreover, in cases when no equivalent risk neutral measure exists, the benchmark approach allows to overcome problems arising under the risk neutral methodology, as, for instance, described in Delbaen & Schachermayer (1995, 1998).

The paper is structured in the following way. It summarizes in Section 2 the general filtering methodology for multi-factor jump diffusion models under partial observation. Section 3 describes the proposed filtered benchmark model. The fair pricing of derivatives is then studied in Section 4. This section also quantifies the reduction of the variance for derivative prices when additional information is available. The hedging under partial observation is subject of Section 5.

2 Filtered Multi-Factor Models

2.1 Factor Model

To build a financial market model with a sufficiently rich structure and high computational tractability we introduce a multi-factor model. We aim to model also market dynamics for which an equivalent risk neutral martingale measure does not exist. This is practically important since it has been indicated in Platen (2004d) that any realistic stock market model is unlikely to have an equivalent risk neutral martingale measure. Along these lines, detailed analysis of intraday data reveals, see Breymann, Kelly & Platen (2004), that there exist natural parameter processes that are only indirectly observable under some noise. For instance, the correlation process between Wiener processes that drive different factors, provide a typical example for such hidden quantities that are highly important for fund management and derivative pricing.

We consider a multi-factor model with $n \geq 2$ factors $z^1, z^2, \ldots, z^n$, forming the vector process

$$z = \left\{ z_t = \left(z_t^1, \ldots, z_t^k, z_t^{k+1}, \ldots, z_t^n \right)^\top, \, t \in [0,T] \right\}. \tag{2.1}$$

We shall assume that not all of the factors are observed. The unobserved factors will be treated via filtering methods. More precisely, only the first k factors are directly observed, while the remaining $n-k$ are not. Here k is an integer with $1 \leq k < n$ that we shall suppose to be fixed during most of this paper. However, in Section 4.3 we shall discuss the implications of a varying k, that is a varying degree of information. For fixed k we shall consider the following subvectors of z_t:

$$y_t = (y_t^1, \ldots, y_t^k)^\top = (z_t^1, \ldots, z_t^k)^\top \quad \text{and} \quad x_t = (x_t^1, \ldots, x_t^{n-k})^\top = (z_t^{k+1}, \ldots, z_t^n)^\top \tag{2.2}$$

with y_t representing the *observed* and x_t the *unobserved factors*. To be specific, we assume that y_t includes as components also the observed security prices. These are given by $d+1$ *primary security account* processes $S^{(0)}, S^{(1)}, \ldots, S^{(d)}$, $d \in \{1, 2, \ldots, k-1\}$. We assume that a primary security account holds only units of one security and the income or loss accrued from holding the units of this security is always reinvested. In the case of shares this models the usual ownership of productive units. Here $S^{(0)}$ is the *savings account* process $S^{(0)} = \{S_t^{(0)}, \, t \in [0,T]\}$, where T is a fixed horizon. Note that the market can be *incomplete*. We shall identify $S_t^{(j)}$ with y_t^j for $j \in \{1, 2, \ldots, d\}$ and the short rate r_t with y_t^{d+1} for $t \in [0,T]$. This means, we consider the short rate to be observable. Furthermore, the dynamics of y_t^j for $j \in \{1, 2, \ldots, d+1\}$ are assumed to be such that these observed quantities remain always nonnegative.

The following setup shall be relatively complex because we aim to provide a general framework where Markovian jump diffusion market models with hidden factors, driven by a finite number of Wiener and Poisson processes, are covered. Let there be given a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$, where $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0,T]}$ is a given filtration to which all the processes shall be adapted. We assume that the observed and unobserved factors satisfy the system of stochastic differential equations (SDEs)

$$\begin{aligned} dx_t &= a_t(z_t)\, dt + b_t(z_t)\, dw_t + g_{t-}(z_{t-})\, dm_t \\ dy_t &= A_t(z_t)\, dt + B_t(y_t)\, dv_t + G_{t-}(y_{t-})\, dN_t \end{aligned} \tag{2.3}$$

for $t \in [0,T]$ with given vector $z_0 = (y_0^1, \ldots, y_0^k, x_0^1, \ldots, x_0^{n-k})^\top$ of initial values. Here

$$w = \left\{ w_t = \left(w_t^1, \ldots, w_t^k, w_t^{k+1}, \ldots, w_t^n \right)^\top, \, t \in [0,T] \right\} \tag{2.4}$$

is an n-dimensional $(\underline{\mathcal{A}}, P)$-Wiener process and

$$v_t = \left(w_t^1, \dots, w_t^k\right)^\top \tag{2.5}$$

is the subvector of its first k components. The process $m = \{m_t = (m_t^1, \dots, m_t^k, m_t^{k+1}, \dots, m_t^n)^\top,\ t \in [0, T]\}$ is an n-dimensional $(\underline{\mathcal{A}}, P)$-jump martingale defined as follows: Consider n counting processes $N^1, \dots, N^n$ having no common jumps. These are at time $t \in [0, T]$ characterized by the corresponding vector of intensities $\lambda_t(z_t) = (\lambda_t^1(z_t), \dots, \lambda_t^n(z_t))^\top$, where

$$\lambda_t^i(z_t) = \tilde{\lambda}_t^i(y_t) \tag{2.6}$$

for $t \in [0, T]$ and $i \in \{1, 2, \dots, k\}$. This means, we assume without loss of generality that the jump intensities of the first k counting processes are observed. The ith $(\underline{\mathcal{A}}, P)$-jump martingale is then defined by the stochastic differential

$$dm_t^i = dN_t^i - \lambda_t^i(z_{t-})\, dt \tag{2.7}$$

for $t \in [0, T]$ and $i \in \{1, 2, \dots, n\}$. In (2.3) the vector

$$N_t = \left(N_t^1, \dots, N_t^k\right)^\top \tag{2.8}$$

denotes the vector of the first k counting processes at time $t \in [0, T]$. Concerning the coefficients in the SDE (2.3), we assume that the vectors $a_t(z_t)$, $A_t(z_t)$, $\lambda_t(z_t)$ and the matrices $b_t(z_t)$, $B_t(y_t)$, $g_t(z_t)$ and $G_t(y_t)$ are such that a unique strong solution of (2.3) exists that does not explode until time T, see Protter (1990). Furthermore, the components $y_t^1, \dots, y_t^{d+1}$ are assumed to be nonnegative for all $t \in [0, T]$. We shall also assume that the $k \times k$-matrix $B_t(y_t)$ is *invertible* for all $t \in [0, T]$. We furthermore assume that $A_t(z_t)$ and $B_t(y_t)$ in (2.3) are such that

$$\int_0^T E\left(|A_t(z_t)|\right) dt < \infty \quad \text{and} \quad \int_0^T B_t(y_t)\, B_t(y_t)^\top\, dt < \infty \tag{2.9}$$

P-a.s. Finally, $g_t(z_t)$ may be any bounded function and the $k \times k$-matrix $G_t(y_t)$ is assumed to be a given function of y_t that is *invertible* for each $t \in [0, T]$. This latter assumption implies that, since there are no common jumps among the components of N_t, by observing a jump of y_t we can establish which of the processes N^i, $i \in \{1, 2, \dots, k\}$, has jumped.

In addition to the filtration $\underline{\mathcal{A}}$, which represents the *complete information*, we shall also consider the subfiltration

$$\underline{\tilde{\mathcal{A}}}^k = (\tilde{\mathcal{A}}_t^k)_{t \in [0,T]} \subseteq \underline{\mathcal{A}}, \tag{2.10}$$

where $\tilde{\mathcal{A}}_t^k = \sigma\{y_s = (z_s^1, \dots, z_s^k)^\top,\ s \leq t\}$ represents the *observed information* at time $t \in [0, T]$. Thus $\underline{\tilde{\mathcal{A}}}^k$ provides the structure of the actually available information in the market, which depends on the specification of the *degree of available information k*.

We shall be interested in the conditional distribution of x_t, given $\tilde{\mathcal{A}}_t^k$, that we call, according to standard terminology, the *filter distribution* at time $t \in [0, T]$. There exist general filter equations for the dynamics described by the SDEs given in (2.3), see Liptser & Shiryaev (1977). It turns out that these are SDEs for the conditional expectations of integrable functions of the unobserved factors x_t, given $\tilde{\mathcal{A}}_t^k$. Notice that, in particular, $\exp[\imath \nu\, x_t]$ is, for given $\nu \in \Re^k$ and with $\imath$ denoting the imaginary unit, a bounded and thus integrable function of x_t. Its conditional expectation leads therefore to the conditional characteristic function of the distribution of x_t, given $\tilde{\mathcal{A}}_t^k$. The latter characterizes completely the entire filter distribution. Considering conditional expectations of integrable functions of x_t is thus not a restriction for the identification of filter equations.

The general case of filter equations is beyond the scope of this paper. These are, for instance, considered in Liptser & Shiryaev (1977). To keep the filter reasonably tractable we assume that the SDEs given in (2.3) are such that the corresponding filter distributions admit a representation of the form

$$P\left(z_t^{k+1} \leq z^{k+1}, \ldots, z_t^n \leq z^n \,\middle|\, \tilde{\mathcal{A}}_t^k\right) = F_{z_t^{k+1}, \ldots, z_t^n}\left(z^{k+1}, \ldots, z^n \,\middle|\, \zeta_t^1, \ldots, \zeta_t^q\right) \quad (2.11)$$

for all $t \in [0, T]$. This means that we have a *finite-dimensional filter*, characterized by the *filter state process*

$$\zeta = \left\{\zeta_t = \left(\zeta_t^1, \ldots, \zeta_t^q\right)^\top, \, t \in [0, T]\right\}, \quad (2.12)$$

which is an $\tilde{\mathcal{A}}_t^k$-adapted process with a certain finite dimension $q \geq 1$. We shall denote by $\tilde{z}_t^k$ the resulting $(k+q)$-vector of *observables*

$$\tilde{z}_t^k = \left(y_t^1, \ldots, y_t^k, \zeta_t^1, \ldots, \zeta_t^q\right)^\top, \quad (2.13)$$

which consists of the k observed factors and the q components of the filter state process. Furthermore, we assume that the filter state ζ_t satisfies an SDE of the form

$$d\zeta_t = C_t(\tilde{z}_t^k)\, dt + D_{t-}(\tilde{z}_{t-}^k)\, dy_t \quad (2.14)$$

with $C_t(\cdot)$ denoting a q-vector valued function and $D_t(\cdot)$ a $(q \times k)$-matrix valued function, $t \in [0, T]$.

There are various models of the type (2.3) that admit a finite-dimensional filter with ζ_t satisfying an equation of the form (2.14). In the following two subsections we recall two classical such models. These are the *conditionally Gaussian model*, which leads to a generalized Kalman-filter and the *finite-state jump model* for x, which is related to hidden Markov chain filters. Various combinations of these models have finite-dimensional filters and can be readily applied in finance, as demonstrated in the literature that we mentioned in the introduction.

Example 2.1 : Conditionally Gaussian Filter Model

Assume that in the system of SDEs (2.3) the functions $a_t(\cdot)$ and $A_t(\cdot)$ are linear in the factors and that $b_t(z_t) \equiv b_t$ is a deterministic function, while $g_t(z_t) \equiv G_t(y_t) \equiv 0$. This means that the model (2.3) takes the form

$$\begin{aligned} dx_t &= \left[a_t^0 + a_t^1\, x_t + a_t^2\, y_t\right] dt + b_t\, dw_t \\ dy_t &= \left[A_t^0 + A_t^1\, x_t + A_t^2\, y_t\right] dt + B_t(y_t)\, dv_t, \end{aligned} \tag{2.15}$$

for $t \in [0,T]$ with given deterministic initial values x_0 and y_0. Here a_t^0 and A_t^0 are column vectors of dimensions $(n-k)$ and k, respectively, and a_t^1, a_t^2, b_t, A_t^1, A_t^2, $B_t(y_t)$ are matrices of appropriate dimensions. Recall that w is an n-dimensional $(\underline{\mathcal{A}}, P)$-Wiener process and v the vector of its first k components.

In this case the filter distribution is a Gaussian distribution with vector mean $\mu_t = (\mu_t^1, \ldots, \mu_t^{n-k})^\top$, where

$$\mu_t^i = E\left(x_t^i \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{2.16}$$

and covariance matrix $c_t = [c_t^{\ell,i}]_{\ell,i \in \{1,2,\ldots,n-k\}}$, where

$$c_t^{\ell,i} = E\left(\left(x_t^\ell - \mu_t^\ell\right)\left(x_t^i - \mu_t^i\right) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{2.17}$$

for $t \in [0,T]$. The dependence of μ_t and c_t on k is for simplicity suppressed in our notation. The above filter can be obtained from a generalization of the well-known Kalman filter, see Liptser & Shiryaev (1977), namely

$$\begin{aligned} d\mu_t &= \left[a_t^0 + a_t^1\, \mu_t + a_t^2\, y_t\right] dt + \left[\bar{b}_t\, B_t(y_t)^\top + c_t\,(A_t^1)^\top\right]\,\left(B_t(y_t)\, B_t(y_t)^\top\right)^{-1} \\ &\qquad\qquad \cdot \left[dy_t - \left(A_t^0 + A_t^1\, \mu_t + A_t^2\, y_t\right) dt\right] \\ dc_t &= \Big(a_t^1\, c_t + c_t\,(a_t^1)^\top + (b_t\, b_t^\top) \\ &\quad - \left[\bar{b}_t\, B_t(y_t)^\top + c_t\,(A_t^1)^\top\right]\left(B_t(y_t)\, B_t(y_t)^\top\right)^{-1}\left[\bar{b}_t\, B_t(y_t)^\top + c_t\,(A_t^1)^\top\right]^\top\Big)\, dt, \end{aligned} \tag{2.18}$$

where $\bar{b}_t$ is the k-dimensional vector formed by the first k components of b_t, $t \in [0,T]$. We recall that $B_t(y_t)$ is assumed to be invertible.

Although for $t \in [0,T]$, c_t is defined as a conditional expectation, it follows from (2.18) that if $B_t(y_t)$ does not depend on the observable factors y_t, then c_t can be computed off-line. Notice that the computation of c_t is contingent upon the knowledge of the coefficients in the second equation of (2.18). These coefficients are given deterministic functions of time, except for $B_t(y_t)$ that depends also on observed factors. The value of $B_t(y_t)$ becomes known only at time t. However, this is sufficient to determine the solution of (2.18) at time t. The model (2.15) is in fact of the type of a *conditionally Gaussian filter model*, where the filter process ζ is given by the vector process $\mu = \{\mu_t,\, t \in [0,T]\}$ and the upper triangular array of the elements of the matrix process $c = \{c_t,\, t \in [0,T]\}$ with $q = (n-k)\,\frac{[3+(n-k)]}{2}$. Note by (2.17) that the matrix c_t is symmetric. Obviously, in the case when $B_t(y_t)$ does not depend on y_t for all $t \in [0,T]$, then we have a *Gaussian filter model.*

Example 2.2 : Finite-State Jump Model

Here we assume that the unobserved factors form a continuous time, $(n-k)$-dimensional jump process $x = \{x_t = (x_t^1, \ldots, x_t^{n-k})^\top,\, t \in [0,T]\}$, which can take a finite number M of values. More precisely, given an appropriate time t and z_t-dependent matrix $g_t(z_t)$, and an intensity vector $\lambda_t(z_t) = (\lambda_t^1(z_t), \ldots, \lambda_t^n(z_t))^\top$ at time $t \in [0,T]$ for the vector counting process $\bar{N} = \{\bar{N}_t = (N_t^1, \ldots, N_t^n)^\top,\, t \in [0,T]\}$, we consider the particular case of the model equations (2.3), where in the x_t-dynamics we have $a_t(z_t) = g_t(z_t)\lambda_t(z_t)$ and $b_t(z_t) \equiv 0$. Thus, by (2.3) and (2.7) we have

$$dx_t = g_{t-}(z_{t-})\, d\bar{N}_t \tag{2.19}$$

for $t \in [0,T]$. Notice that the process x of unobserved factors has here only jumps and is therefore piecewise constant. On the other hand, for the vector y_t of observed factors we assume that it satisfies the second equation in (2.3) with $G_t(y_t) \equiv 0$. This means that the process of observed factors y is only perturbed by continuous noise and does not jump.

In this example, the filter distribution is completely characterized by the vector of conditional probabilities $p_t = (p_t^1, \ldots, p_t^M)^\top$, where M is the number of possible states $\eta^1, \ldots, \eta^M$ of the vector x_t and

$$p_t^j = P\left(x_t = \eta^j \,\middle|\, \tilde{\mathcal{A}}_t^k\right), \tag{2.20}$$

for $t \in [0,T]$ and $j \in \{1, 2, \ldots, M\}$. Let $\bar{a}_t^{i,j}(y, \eta^h)$ denote the transition kernel for x at time t to jump from state i into state j given $y_t = y$ and $x_t = \eta^h$, see Liptser & Shiryaev (1977). The components of the vector p_t satisfy the following dynamics

$$\begin{aligned} dp_t^j &= \left(\tilde{a}_t(y_t, p_t)^\top p_t\right)^j dt + p_t^j \left[A_t(y_t, \eta^j) - \tilde{A}_t(y_t, p_t)\right] \left(B_t(y_t)\, B_t(y_t)^\top\right)^{-1} \\ &\quad \cdot \left[dy_t - \tilde{A}_t(y_t, p_t)\, dt\right], \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} \left(\tilde{a}_t(y_t, p_t)^\top p_t\right)^j &= \sum_{i=1}^M \left(\sum_{h=1}^M \bar{a}_t^{i,j}(y_t, \eta^h)\, p_t^h\right) p_t^i \\ A_t(y_t, \eta^j) &= A_t(y_t, x_t)\,\big|_{\,x_t = \eta^j} \\ \tilde{A}_t(y_t, p_t) &= \sum_{j=1}^M A_t(y_t, \eta^j)\, p_t^j \end{aligned} \tag{2.22}$$

for $t \in [0,T]$, $j \in \{1,2,\dots,M\}$. The filter state process $\zeta = \{\zeta_t = (\zeta_t^1 \dots, \zeta_t^q)^\top, t \in [0,T]\}$ for the finite state jump model is thus given by the vector process $p = \{p_t = (p_t^1, \dots, p_t^q)^\top, t \in [0,T]\}$ with $q = M-1$. Since the probabilities add up to one, we need only $M-1$ probabilities to characterize the filter.

2.2 Markovian Representation

As in the two previous examples we have, in general, in our filter setup to deal with the quantity $E(A_t(z_t) \,|\, \tilde{\mathcal{A}}_t^k)$, assuming that it exists. This is the conditional expectation of the coefficient $A_t(z_t) = A_t(y_t^1, \dots, y_t^k, x_t^1, \dots, x_t^{n-k})$, that appears in (2.3), with respect to the filter distribution at time t for the unobserved factors x_t. Since the filter is characterized by the filter state process ζ, we obtain for this conditional expectation the representation

$$\tilde{A}_t(\tilde{z}_t^k) = E\left(A_t(z_t) \,\middle|\, \tilde{\mathcal{A}}_t^k\right), \tag{2.23}$$

where the vector $\tilde{z}_t^k$ is as defined in (2.13). Note that we deal here in our financial market context with conditional expectations under the real world probability measure. This is an essential observation to understand the theoretical and practical benefits of the benchmark approach that we will pursue later on.

Notice that, in the case of Example 2.1, namely the conditionally Gaussian model, the expression $\tilde{A}_t(\tilde{z}_t^k)$ takes the particular form

$$\tilde{A}_t(\tilde{z}_t^k) = A_t^0 + A_t^1 \mu_t + A_t^2 y_t. \tag{2.24}$$

Furthermore, for Example 2.2, namely the finite-state jump model, $\tilde{\mathcal{A}}_t(\tilde{z}_t^k)$ can be represented as

$$\tilde{A}_t(\tilde{z}_t^k) = \tilde{A}_t(y_t, p_t) = \sum_{j=1}^{M} A_t(y_t, \eta^j)\, p_t^j \tag{2.25}$$

for $t \in [0,T]$, see (2.22).

In Appendix A we prove the following generalization of Theorem 7.12 in Liptser & Shiryaev (1977), which provides an important representation of the SDE for the observed factors.

Proposition 2.3 *Let $A_t(z_t)$ and the invertible matrix $B_t(y_t)$ in* (2.3) *be such that* (2.9) *holds. Then there exists a k-dimensional $\tilde{\mathcal{A}}^k$-adapted Wiener process $\tilde{v} = \{\tilde{v}_t, t \in [0,T]\}$ such that the process $y = \{y_t, t \in [0,T]\}$ of observed factors in* (2.3) *satisfies the SDE*

$$dy_t = \tilde{A}_t(\tilde{z}_t^k)\, dt + B_t(y_t)\, d\tilde{v}_t + G_{t-}(y_{t-})\, dN_t \tag{2.26}$$

with $\tilde{A}_t(\tilde{z}_t^k)$ as in (2.23).

Instead of the original factors $z_t = (y_t^1, \ldots, y_t^k, x_t^1, \ldots, x_t^{n-k})^\top = (z_t^1, \ldots, z_t^n)^\top$, where $x_t = (x_t^1, \ldots, x_t^{n-k})^\top$ is unobserved, we may now base our analysis on the components of the vector $\tilde{z}_t^k = (y_t^1, \ldots, y_t^k, \zeta_t^1, \ldots, \zeta_t^q)^\top$, see (2.13), that are all observed. Just as was the case with $z = \{z_t,\, t \in [0,T]\}$, also the vector process $\tilde{z}^k = \{\tilde{z}_t^k,\, t \in [0,T]\}$ has a Markovian dynamics. In fact, replacing dy_t in (2.14) by its expression resulting from (2.26), we obtain

$$\begin{aligned} d\zeta_t &= \Big[C_t(\tilde{z}_t^k) + D_t(\tilde{z}_t^k)\, \tilde{A}_t(\tilde{z}_t^k) \Big]\, dt + D_t(\tilde{z}_t^k)\, B_t(y_t)\, d\tilde{v}_t + D_{t-}(\tilde{z}_{t-}^k)\, G_{t-}(y_{t-})\, dN_t \\ &= \tilde{C}_t(\tilde{z}_t^k)\, dt + \tilde{D}_t(\tilde{z}_t^k)\, d\tilde{v}_t + \tilde{G}_{t-}(\tilde{z}_{t-}^k)\, dN_t, \end{aligned} \tag{2.27}$$

whereby we implicitly define the vector $\tilde{C}_t(\tilde{z}_t^k)$ and the matrices $\tilde{D}_t(\tilde{z}_t^k)$ and $\tilde{G}_t(\tilde{z}_t^k)$ for compact notation.

From equations (2.26) and (2.27) we immediately obtain the following result.

Corollary 2.4 *The dynamics of the vector $\tilde{z}_t^k = (y_t, \zeta_t)^\top$ can be expressed by the system of SDEs*

$$\begin{aligned} dy_t &= \tilde{A}_t(\tilde{z}_t^k)\, dt + B_t(y_t)\, d\tilde{v}_t + G_{t-}(y_{t-})\, dN_t \\ d\zeta_t &= \tilde{C}_t(\tilde{z}_t^k)\, dt + \tilde{D}_t(\tilde{z}_t^k)\, d\tilde{v}_t + \tilde{G}_{t-}(\tilde{z}_{t-}^k)\, dN_t. \end{aligned} \tag{2.28}$$

From Corollary 2.4 it follows that the process $\tilde{z}^k = \{\tilde{z}_t^k,\, t \in [0,T]\}$ is Markovian.

Due to the existence of a Markovian filter dynamics we have our original Markovian factor model, given by (2.3), projected into a Markovian model for the observed quantities. Here the driving observable noise $\tilde{v}$ is an $(\tilde{\mathcal{A}}^k, P)$-Wiener process and the observable counting process N is generated by the first k components $N^1, N^2, \ldots, N^k$ of the n counting processes.

For efficient notation, given k, we write for the vector of observables $\tilde{z}_t^k = \bar{z}_t = (\bar{z}_t^1, \bar{z}_t^2, \ldots, \bar{z}_t^{k+q})^\top$ the corresponding system of SDEs in the form

$$\begin{aligned} d\bar{z}_t^\ell &= \alpha^\ell(t, \bar{z}_t^1, \bar{z}_t^2, \ldots, \bar{z}_t^{k+q})\, dt + \sum_{r=1}^k \beta^{\ell,r}(t, \bar{z}_t^1, \bar{z}_t^2, \ldots, \bar{z}_t^{k+q})\, d\tilde{v}_t^r \\ &\quad + \sum_{r=1}^k \gamma^{\ell,r} \left(t-, \bar{z}_{t-}^1, \bar{z}_{t-}^2, \ldots, \bar{z}_{t-}^{k+q} \right) dN_t^r \end{aligned} \tag{2.29}$$

for $t \in [0,T]$ and $\ell \in \{1, 2, \ldots, k+q\}$. The functions α^ℓ, $\beta^{\ell,r}$ and $\gamma^{\ell,r}$ follow directly from $\tilde{A}$, B, G, $\tilde{C}$, $\tilde{D}$ and $\tilde{G}$, appearing in (2.28).

We also have as an immediate consequence of the Markovianity of $\tilde{z}^k = \bar{z}$, as well as property (2.11), the following result.

Corollary 2.5 *Any expectation of the form $E(u(t, z_t) \,|\, \tilde{\mathcal{A}}_t^k) < \infty$ for a given function $u : [0, T] \times \Re^n \to \Re$ and given $k \in \{1, 2, \ldots, n-1\}$ can be expressed as*

$$E\left(u(t, z_t) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) = \tilde{u}^k(t, \tilde{z}_t^k) = \tilde{u}^k(t, \bar{z}_t) \tag{2.30}$$

with a suitable function $\tilde{u}^k : [0, T] \times \Re^{k+q} \to \Re$.

Relation (2.30) in Corollary 2.5 is of significant practical importance, in particular, for contingent claim pricing, as we shall see later on.

3 Benchmark Model

3.1 Primary Security Accounts and Portfolios

Recall from Section 2.1 that we have in our Markovian jump-diffusion market model with observable and hidden factors, $d+1$ primary security account processes $S^{(0)}, \ldots, S^{(d)}$ with $d < k$, all of which are observable. This means that the vector process $S = \{S_t = (S_t^{(0)}, \ldots, S_t^{(d)})^\top,\, t \in [0, T]\}$ is $\tilde{\mathcal{A}}^k$-adapted. We have set in Section 2.1

$$y_t^j = \bar{z}_t^j = S_t^{(j)}$$

for $j \in \{1, 2, \ldots, d\}$ and

$$y_t^{d+1} = \bar{z}_t^{d+1} = r_t$$

for $t \in [0, T]$.

Since the $d+1$ primary security account processes coincide with the observable factors $y^1, \ldots, y^{d+1}$, we can write their dynamics in a form corresponding to (2.29). To this effect let by analogy to (2.7)

$$d\tilde{m}_t^i = \frac{1}{\sqrt{\tilde{\lambda}_{t-}^i(\bar{z}_{t-})}} \left(dN_t^i - \tilde{\lambda}_{t-}^i(\bar{z}_{t-})\, dt\right) \tag{3.1}$$

for $i \in \{1, 2, \ldots, k\}$ be the normalized compensated ith $(\tilde{\mathcal{A}}^k, P)$-jump martingale relative to the filtration $\tilde{\mathcal{A}}^k$. Here, with some abuse of notation, we have denoted by $\tilde{\lambda}_t^i(\bar{z}_t)$ the compensating jump intensity for N^i with respect to $\tilde{\mathcal{A}}^k$. For simplicity of notation, in what follows we shall often use $\bar{z}_t$ for $\tilde{z}_t^k$, see (2.29). Let us now rewrite (2.29) more concisely in vector form as

$$d\bar{z}_t = \bar{\alpha}(t, \bar{z}_t)\, dt + \beta(t, \bar{z}_t)\, d\tilde{v}_t + \gamma(t-, \bar{z}_{t-}) \sqrt{\tilde{\lambda}_{t-}(\bar{z}_{t-})}\; d\tilde{m}_t \tag{3.2}$$

with

$$\bar{\alpha}(t, \bar{z}_t) = \alpha(t, \bar{z}_t) + \gamma(t-, \bar{z}_{t-}) \sqrt{\tilde{\lambda}_{t-}(\bar{z}_{t-})}, \tag{3.3}$$

where $\tilde{m}$ and $\tilde{\lambda}$ are the k-vectors with components $\tilde{m}^i$ and $\sqrt{\tilde{\lambda}^i}$, respectively. Here $\alpha(t, \bar{z}_t)$ is a $(k+q)$-column vector and $\beta(t, \bar{z}_t)$ as well as $\gamma(t, \bar{z}_t)$ are $((k+q) \times k)$-matrices.

Since we have assumed $d < k$, the primary security accounts do not necessarily span the entire observable uncertainty of the market. Think, for instance, of asset price models with stochastic volatility, where the volatilities are driven by stochastic processes that are independent from those that directly drive the evolution of the asset prices. It is therefore reasonable to assume that among the driving random processes $\tilde{v}^i$ for $i \in \{1, 2, \ldots, k\}$ and $\tilde{m}^\ell$ for $\ell \in \{1, 2, \ldots, k\}$, those that directly drive the fluctuations of the asset prices $S_t^{(j)}$, $j \in \{1, 2, \ldots, d\}$, are exactly d in number. We shall thus assume that for any $j \in \{1, 2, \ldots, d\}$ the dynamics of the jth primary security account is given by the SDE

$$dS_t^{(j)} = \bar{\alpha}^j(t, \bar{z}_t)\, dt + \sum_{i=1}^{h_1} \beta^{j,i}(t, \bar{z}_t)\, d\tilde{v}_t^i + \sum_{\ell=1}^{h_2} \gamma^{j,\ell}(t-, \bar{z}_{t-}) \sqrt{\tilde{\lambda}_{t-}^\ell(\bar{z}_{t-})}\, d\tilde{m}_t^\ell \tag{3.4}$$

for $t \in [0, T]$, where $h_1 + h_2 = d$.

We assume that all model specifications are such that a unique strong solution of the system of SDEs (3.2) – (3.4) exists, see Protter (1990). For efficient and more transparent notation we now rewrite the SDE (3.4) in the form

$$dS_t^{(j)} = S_{t-}^{(j)} \left(r_t\, dt + \sum_{i=1}^{h_1} b_t^{j,i} \left(d\tilde{v}_t^i + \theta_t^i\, dt \right) + \sum_{\ell=h_1+1}^{d} b_{t-}^{j,\ell} \left(d\tilde{m}_t^{\ell-h_1} + \theta_{t-}^\ell\, dt \right) \right) \tag{3.5}$$

for $t \in [0, T]$ with $S_t^{(j)} > 0$, $j \in \{0, 1, \ldots, d\}$. Here we set $S_0^{(0)} = 1$ and $b_t^{0,i} = 0$ for $t \in [0, T]$ and $i \in \{0, 1, \ldots, d\}$, where r_t is the short rate. Above in (3.5) we have for $i \in \{1, 2, \ldots, h_1\}$ the *volatility*

$$b_t^{j,i} = \frac{\beta^{j,i}(t, \bar{z}_t)}{S_t^{(j)}} \tag{3.6}$$

and for $i \in \{h_1 + 1, \ldots, d\}$ the *jump coefficient*

$$b_{t-}^{j,i} = \frac{\gamma^{j,i-h_1}(t-, \bar{z}_{t-}) \sqrt{\tilde{\lambda}_{t-}^{i-h_1}(\bar{z}_{t-})}}{S_{t-}^{(j)}} \tag{3.7}$$

for $t \in [0, T]$ and $j \in \{1, 2, \ldots, d\}$. We assume that the matrix $b_t = [b_t^{j,i}]_{j,i=1}^d$ is *invertible* for all $t \in [0, T]$. This allows us to write the *market price for risk* vector $\theta_t = (\theta_t^1, \ldots, \theta_t^d)^\top$ in the form

$$\theta_t = b_t^{-1} [a_t - r_t\, \mathbf{1}] \tag{3.8}$$

for $t \in [0, T]$. Here $\mathbf{1} = (1, \ldots, 1)^\top$ is the corresponding unit vector and $a_t = (a_t^1, \ldots, a_t^d)^\top$ is the *appreciation rate* vector with

$$a_t^j = \frac{\bar{\alpha}^j(t, \bar{z}_t)}{S_t^{(j)}} \tag{3.9}$$

for $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$. Notice that in the dynamics (3.5) all the coefficients can be determined on the basis of the observables $\bar{z}_t$. The interest rate r_t was in fact identified with y_t^{d+1} and, by (3.6) – (3.9), the matrix b_t and the market price for risk vector θ_t are given functions of observables.

Let us form portfolios of primary security accounts. We say that an $\tilde{\mathcal{A}}^k$-predictable stochastic process $\delta = \{\delta_t = (\delta_t^0, \ldots, \delta_t^d)^\top,\ t \in [0,T]\}$ is a *self-financing strategy*, if δ is S-integrable, see Protter (1990), and the corresponding *portfolio value*

$$V_\delta(t) = \sum_{j=0}^{d} \delta_t^j\, S_t^j \tag{3.10}$$

at time t satisfies the SDE

$$dV_\delta(t) = \sum_{j=0}^{d} \delta_{t-}^j\, dS_t^j \tag{3.11}$$

for all $t \in [0,T]$. The jth component δ_t^j, $j \in \{0,1,\ldots,d\}$, of the self-financing strategy δ expresses the number of units of the jth primary security account held at time t in the corresponding portfolio. Under a self-financing strategy no outflow or inflow of funds occurs for the corresponding portfolio. All changes in the value of the portfolio are due to gains from trade in the primary security accounts. Since we will only deal with self-financing portfolios and strategies we omit in the following the phrase "self-financing".

As is shown in Platen (2004b), to avoid portfolios with infinite growth potential we need to assume that

$$\sqrt{\tilde{\lambda}_t^{\ell-h_1}(\bar{z}_t)} > \theta_t^\ell \tag{3.12}$$

for all $t \in [0,T]$ and $\ell \in \{h_1+1,\ldots,d\}$.

3.2 Growth Optimal Portfolio

In a financial market model it is advantageous for derivative pricing and other risk management tasks to choose an appropriate reference unit, numeraire or benchmark. Under the benchmark approach, see Platen (2002, 2004a, 2004b, 2004d), we use the *growth optimal portfolio* (GOP) as benchmark, which is the self-financing portfolio that achieves maximum expected logarithmic utility from terminal wealth. We denote by $V_{\underline{\delta}}(t)$ the value of the GOP at time $t \in [0,T]$. It has been shown in Platen (2004a, 2004c) that, under realistic assumptions, a global diversified portfolio is a good proxy for the GOP. This makes it a readily observable financial quantity, which can be used in various ways for risk management.

For the diffusion case without jumps the SDE for the GOP is well known, see for instance, Long (1990) or Karatzas & Shreve (1998). In the case with jumps the derivation of the SDE for the GOP is more involved, using first order conditions for the maximization of the drift of the logarithm of the portfolio, which leads to

$$
\begin{aligned}
dV_{\underline{\delta}}(t) &= V_{\underline{\delta}}(t-)\Bigg(r_t\,dt + \sum_{i=1}^{h_1}\theta_t^i\,(\theta_t^i\,dt + d\tilde{v}_t^i) \\
&\qquad + \sum_{i=h_1+1}^{d}\frac{\theta_{t-}^i}{1-\frac{\theta_{t-}^i}{\sqrt{\tilde{\lambda}_{t-}^{i-h_1}(\bar{z}_{t-})}}}\left(\theta_{t-}^i\,dt + d\tilde{m}_t^{i-h_1}\right)\Bigg)
\end{aligned}
\tag{3.13}
$$

for $t \in [0,T]$ with $V_{\underline{\delta}}(0) = 1$, as described in Platen (2004b). Here $\tilde{m}_t^i$ denotes the ith component of the jump martingale $\tilde{m}$ defined in (3.1).

In what follows we shall call *benchmarked prices* the prices when they are expressed in units of the GOP. This means that, for $j \in \{0,1,\ldots,d\}$, the jth benchmarked primary security account $\hat{S}^{(j)} = \{\hat{S}_t^{(j)},\, t \in [0,T]\}$ has at time t the value

$$
\hat{S}_t^{(j)} = \frac{S_t^j}{V_{\underline{\delta}}(t)},
\tag{3.14}
$$

which satisfies by (3.5), (3.13) and application of the Itô formula the SDE

$$
\begin{aligned}
d\hat{S}_t^{(j)} &= \hat{S}_{t-}^{(j)}\Bigg(\sum_{i=1}^{h_1}(b_t^{j,i}-\theta_t^i)\,d\tilde{v}_t^i \\
&\qquad + \sum_{i=h_1+1}^{d}\left[b_{t-}^{j,i}\left(1-\frac{\theta_{t-}^i}{\sqrt{\tilde{\lambda}_{t-}^{i-h_1}(\bar{z}_{t-})}}\right)-\theta_{t-}^i\right]d\tilde{m}_t^{i-h_1}\Bigg)
\end{aligned}
\tag{3.15}
$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$, see Platen (2004b).

Similarly, by application of the Itô formula it can be shown that the benchmarked portfolio $\hat{V}_\delta = \{\hat{V}_\delta(t),\, t \in [0,T]\}$ with

$$
\hat{V}_\delta(t) = \frac{V_\delta(t)}{V_{\underline{\delta}}(t)}
\tag{3.16}
$$

satisfies the SDE

$$
\begin{aligned}
d\hat{V}_\delta(t) &= \hat{V}_\delta(t-)\Bigg(\sum_{i=1}^{h_1}\left(\sum_{j=1}^{d}\frac{\delta_t^j\,\hat{S}_t^{(j)}}{\hat{V}_\delta(t)}\,b_t^{j,i}-\theta_t^i\right)d\tilde{v}_t^i \\
&\qquad + \sum_{i=h_1+1}^{d}\left[\left(\sum_{j=1}^{d}\frac{\delta_{t-}^j\,\hat{S}_{t-}^{(j)}}{\hat{V}_\delta(t-)}\,b_{t-}^{j,i}\right)\left(1-\frac{\theta_{t-}^i}{\sqrt{\tilde{\lambda}_{t-}^{i-h_1}(\bar{z}_{t-})}}\right)-\theta_{t-}^i\right]d\tilde{m}_t^{i-h_1}\Bigg)
\end{aligned}
\tag{3.17}
$$

for $t \in [0,T]$, see Platen (2004b). Notice that all the coefficients in (3.17) are given functions of observables.

Note furthermore that the jth benchmarked primary security account $\hat{S}^{(j)}$ and all benchmarked portfolios are driftless and thus $(\tilde{\mathcal{A}}^k, P)$-local martingales. Therefore, any nonnegative benchmarked portfolio process is an $(\tilde{\mathcal{A}}^k, P)$-supermartingale. This means that it is impossible for a nonnegative portfolio to generate, with strictly positive probability, strictly positive wealth from zero initial capital. This shows that the given benchmark framework does not permit arbitrage in the sense of Platen (2004a).

In the literature, there exist various mathematical definitions of arbitrage. The benchmark approach allows us to consider a more general class of models than is possible, for instance, under the *no free lunch with vanishing risk* concept developed in Delbaen & Schachermayer (1995, 1998), which links no-arbitrage directly to the existence of an equivalent risk neutral measure. Such a measure needs not to exist in our framework. In the given benchmark model a free lunch with vanishing risk arises, for instance, when the benchmarked savings account forms a strict $(\tilde{\mathcal{A}}^k, P)$-local martingale, as is, for instance, the case for models, described in Heath & Platen (2002a, 2002c) and Breymann, Kelly & Platen (2004).

4 Fair Pricing of Derivatives

4.1 Derivative Price Processes as Martingales

We emphasize that benchmarked security prices are in our framework, in general, not $(\tilde{\mathcal{A}}^k, P)$-martingales. However, we assume that any benchmarked derivative price process is *fair*, which means it is an $(\tilde{\mathcal{A}}^k, P)$-martingale. By choosing the GOP as numeraire, the real world probability measure becomes the unique pricing measure for derivatives. We stress the fact that, even if there does not exist an equivalent martingale measure, then it is possible to operate with the GOP as numeraire and with the real world probability measure as pricing measure, as will be explained in what follows.

We called above a price process $V = \{V(t),\, t \in [0,T]\}$ *fair* if its benchmarked value $\hat{V}(t) = \frac{V(t)}{V_{\underline{\delta}}(t)}$ forms an $(\tilde{\mathcal{A}}^k, P)$-martingale under the available information represented by $\tilde{\mathcal{A}}^k$. Recall that benchmarked nonnegative portfolios are $(\tilde{\mathcal{A}}, P)$-supermartingales and benchmarked primary security accounts can be strict supermartingales. Using the GOP as numeraire, we assumed that benchmarked derivative prices are fair and therefore $(\tilde{\mathcal{A}}^k, P)$-martingales. This puts buyers and sellers in comparable positions and generalizes the risk neutral approach, as we shall see below.

Note that $\tilde{\mathcal{A}}^k_t$ describes the information, which is available at time t, whereas $\mathcal{A}_t$ is the complete information at time t that determines the original model dynamics including also the unobserved factors. This means that observed derivative prices may, in general, not be $(\underline{\mathcal{A}}, P)$-martingales.

To provide an intuitive link between fair pricing and standard risk neutral pricing, let us study a candidate risk neutral probability measure P^k. We introduce its Radon-Nikodym derivative process $\Lambda = \{\Lambda_t,\ t \in [0,T]\}$ as the benchmarked savings account

$$\Lambda_t = \frac{dP^k}{dP}\bigg|_{\mathcal{A}_t} = \frac{S_t^{(0)}}{V_{\underline{\delta}}(t)} = \hat{S}_t^{(0)} \tag{4.1}$$

for $t \in [0,T]$. We can do this because if an equivalent risk neutral martingale measure P^k were to exist, then the standard risk neutral pricing methodology and (4.1) would yield the relations

$$\begin{aligned} S_t^{(j)} &= S_t^{(0)}\, E^{P^k}\left(\frac{S_\tau^{(j)}}{S_\tau^{(0)}}\,\bigg|\,\mathcal{A}_t\right) = S_t^{(0)}\,\frac{E\left(\Lambda_\tau \frac{S_\tau^{(j)}}{S_\tau^{(0)}}\,\Big|\,\mathcal{A}_t\right)}{E\left(\Lambda_\tau\,\big|\,\mathcal{A}_t\right)} \\ &= S_t^{(0)}\,\frac{E\left(\frac{S_\tau^{(j)}}{V_{\underline{\delta}}(\tau)}\,\Big|\,\mathcal{A}_t\right)}{\frac{S_t^0}{V_{\underline{\delta}}(t)}} = V_{\underline{\delta}}(t)\, E\left(\frac{S_\tau^{(j)}}{V_{\underline{\delta}}(\tau)}\,\bigg|\,\mathcal{A}_t\right) \end{aligned} \tag{4.2}$$

for $\tau \in [0,T]$, $t \in [0,\tau]$ and $j \in \{0,1,\ldots,d\}$, where E^{P^k} denotes expectation with respect to P^k. It turns out that the measure P^k above is under appropriate assumptions the *minimal equivalent martingale measure* in the sense of Föllmer & Schweizer (1991). However, since we do not assume that Λ is an $(\tilde{\mathcal{A}}, P)$-martingale and P^k may not be equivalent to P the first and second equalities in (4.2) may break down.

Notice that for any time instant $t \in [0,T]$, the value $V_{\underline{\delta}}(t)$ of the GOP represents that of a tradable portfolio, see (3.10), with the $\tilde{\mathcal{A}}^k$-predictable strategy $\underline{\delta}$. This portfolio invests in the primary security accounts that are all observable. It follows, as already mentioned earlier, that the GOP $V_{\underline{\delta}}(t)$ is $\tilde{\mathcal{A}}_t^k$-measurable. This implies that the Radon-Nikodym derivative Λ_t, see (4.1), is observable at time t and so, for the special case when P^k is an equivalent risk neutral martingale measure, $\Lambda = \{\Lambda_t,\, t \in [0,T]\}$ is not only an $(\underline{\mathcal{A}}, P)$- but also an $(\tilde{\mathcal{A}}^k, P)$-martingale. Relation (4.2) then holds with $\tilde{\mathcal{A}}_t^k$ replacing $\mathcal{A}_t$, that is

$$S_t^{(j)} = V_{\underline{\delta}}(t)\, E\left(\frac{S_\tau^{(j)}}{V_{\underline{\delta}}(\tau)}\,\bigg|\,\tilde{\mathcal{A}}_t^k\right) \tag{4.3}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$. In this special case it follows from (4.2) and the corresponding relation (4.3) with $\tilde{\mathcal{A}}_t^k$ instead of $\mathcal{A}_t$ that the triplets $(S^{(0)}, P^k, \mathcal{A})$ and $(V_{\underline{\delta}}, P, \mathcal{A})$ as well as $(S^{(0)}, P^k, \tilde{\mathcal{A}}^k)$ and $(V_{\underline{\delta}}, P, \tilde{\mathcal{A}}^k)$ define the same pricing systems, respectively. In our general situation this is not always the case. However, as we will show below, filtering and derivative pricing is still possible in a consistent manner under the benchmark approach.

4.2 Derivative Prices

In what follows denote by $\mathcal{T}_{t,T}$ the set of stopping times with values in $[t, T]$. For a given maturity date τ, which is assumed to be an $\tilde{\mathcal{A}}^k$-stopping time, we consider a *contingent claim* $U(\tau, y_\tau)$ as a nonnegative function of τ and the corresponding values of observed factors y_τ, where we assume that

$$E\left(\frac{U(\tau, y_\tau)}{V_{\underline{\delta}}(\tau)} \,\middle|\, \tilde{\mathcal{A}}_t^k\right) < \infty \tag{4.4}$$

for all $\tau \in \mathcal{T}_{t,T}$ and $t \in [0, T]$. There is no point to let the payoff function depend on any other than observed factors. Otherwise, the payoff would not be verifiable at time τ on the basis of available information.

Since, as mentioned in Section 4.1, $V_{\underline{\delta}}(\tau)$ is $\tilde{\mathcal{A}}_\tau^k$-measurable, it can be considered as a function of z_s for $s \leq \tau$. Furthermore, since y_τ is a subvector of z_τ and $z = \{z_t,\ t \in [0, T]\}$ is a Markov process, we can define the process $u = \{u(t, z_t),\ t \in [0, T]\}$ as

$$u(t, z_t) = E\left(\frac{V_{\underline{\delta}}(t)}{V_{\underline{\delta}}(\tau)} U(\tau, y_\tau) \,\middle|\, \mathcal{A}_t\right) \tag{4.5}$$

for $\tau \in \mathcal{T}_{t,T}$ and $t \in [0, T]$, which at time t exploits the complete information characterized by the σ-algebra $\mathcal{A}_t$. Next, we consider

$$\tilde{u}^k(t, \tilde{z}_t^k) = E\left(u(t, z_t) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{4.6}$$

for $t \in [0, T]$, which by Corollary 2.5 can be computed on the basis of the filtering results of Section 2. Combining (4.5) with (4.6) and using the fact that $V_{\underline{\delta}}(t)$ is $\tilde{\mathcal{A}}_t^k$-measurable, we obtain

$$\frac{\tilde{u}^k(t, \tilde{z}_t^k)}{V_{\underline{\delta}}(t)} = E\left(\frac{U(\tau, y_\tau)}{V_{\underline{\delta}}(\tau)} \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{4.7}$$

for $\tau \in \mathcal{T}_{t,T}$ and $t \in [0, T]$. This means that the benchmarked value $\frac{\tilde{u}^k(t, \tilde{z}_t^k)}{V_{\underline{\delta}}(t)}$, stopped at the maturity τ, forms for $t \in [0, T]$ a $(P, \tilde{\mathcal{A}}^k)$-martingale. Obviously, it is the only $(P, \tilde{\mathcal{A}}^k)$-martingale that coincides at time τ with $\frac{U(\tau, y_\tau)}{V_{\underline{\delta}}(\tau)}$. Thus $\tilde{u}^k(t, \tilde{z}_t^k)$ is the *fair price* at time $t \leq \tau$ of the claim $U(\tau, y_\tau)$ for the information represented by $\tilde{\mathcal{A}}_t^k$.

The above concept of fair pricing, which can be applied generally, see Platen (2004a), extends the well-known concept of risk neutral pricing and avoids not only the assumption on the existence of an equivalent risk neutral measure, see Platen (2002), but also some delicate issues that arise from measure changes under different filtrations in filtering applications, see Bhar, Chiarella & Runggaldier (2002). Therefore, under the benchmark approach we enter not only a richer modeling world but avoid also a number of technical issues that require typically particular assumptions.

To illustrate again the special case when there exists an equivalent martingale measure P^k then, corresponding to (4.2) and using (4.5), we have

$$S_t^{(0)}\, E^{P^k}\left(\frac{U(\tau,y_\tau)}{S_\tau^{(0)}}\,\middle|\,\mathcal{A}_t\right) = V_{\underline{\delta}}(t)\, E\left(\frac{U(\tau,y_\tau)}{V_{\underline{\delta}}(\tau)}\,\middle|\,\mathcal{A}_t\right) = u(t,z_t) \tag{4.8}$$

for $\tau \in \mathcal{T}_{t,T}$ and $t \in [0,T]$. In this special case the same arguments as for relation (4.3), with $\tilde{\mathcal{A}}_t^k$ replacing $\mathcal{A}_t$, lead then to

$$S_t^{(0)}\, E^{P^k}\left(\frac{U(\tau,y_\tau)}{S_\tau^{(0)}}\,\middle|\,\tilde{\mathcal{A}}_t^k\right) = V_{\underline{\delta}}(t)\, E\left(\frac{U(\tau,y_\tau)}{V_{\underline{\delta}}(\tau)}\,\middle|\,\tilde{\mathcal{A}}_t^k\right) = \tilde{u}^k(t,\tilde{z}_t^k), \tag{4.9}$$

for $\tau \in \mathcal{T}_{t,T}$ and $t \in [0,T]$, using (4.5) and (4.6). Therefore, if there exists an equivalent martingale measure P^k, this implies that corresponding to (4.7), we have

$$\frac{\tilde{u}^k(t,\tilde{z}_t^k)}{S_t^{(0)}} = E^{P^k}\left(\frac{U(\tau,y_\tau)}{S_\tau^{(0)}}\,\middle|\,\tilde{\mathcal{A}}_t^k\right) = E^{P^k}\left(\frac{u(t,z_t)}{S_t^{(0)}}\,\middle|\,\tilde{\mathcal{A}}_t^k\right) \tag{4.10}$$

for $\tau \in \mathcal{T}_{t,T}$ and $t \in [0,T]$. This means that the discounted derivative price process $\frac{\tilde{u}^k(t,\tilde{z}_t^k)}{S_t^{(0)}}$, stopped at the maturity τ, forms for $t \in [0,T]$ in this particular case a $(P^k, \tilde{\mathcal{A}}^k)$-martingale. Furthermore, the last equality in (4.10) implies, just as in (4.6), that we also have

$$\tilde{u}^k(t,\tilde{z}_t^k) = E^{P^k}\left(u(t,z_t)\,\middle|\,\tilde{\mathcal{A}}_t^k\right) \tag{4.11}$$

for $t \in [0,T]$ if an equivalent risk neutral martingale measure exists. We emphasize in this case that the expectations, under different measures, in (4.6) and (4.11) lead to the same result due to the fact that Λ is a martingale not only with respect to the filtration $\underline{\mathcal{A}}$ but also $\tilde{\mathcal{A}}^k$. In general, this does not hold under the described benchmark framework.

Notice that if we would not consider fair pricing using the GOP and could use an equivalent risk neutral martingale measure, then we may try to perform the computations on the basis of formula (4.11). Although similar to the right hand side of (4.6), the right hand side of (4.11) is considerably more difficult to compute. This is due to the fact that in order to filter the process z_t on the basis of the information contained in $\tilde{\mathcal{A}}_t^k$ we have in any case to work under the real world probability measure P. Fortunately, since by (4.9) the quantity $\tilde{u}^k(t,\tilde{z}_t^k)$ computed according to (4.11) is the same as that in (4.6), we can perform the computations according to fair pricing using (4.5) and (4.6), thereby obtaining the derivative price $\tilde{u}^k(t,\tilde{z}_t^k)$ under the information represented by $\tilde{\mathcal{A}}_t^k$. This shows that when it comes to actual computations the real world measure plays a crucial and dominant role. Therefore, we suggest in this paper to work in filtering in finance totally under the real world probability measure. Most importantly, this approach is still applicable in the case when (4.11) fails to hold due to the fact that one wants to use a realistic market model for which no equivalent martingale measure exists.

Note that the expression in (4.6) fits perfectly the one for the filtered factor model given in (2.30). The actual computation of the conditional expectation in (4.6) is therefore equivalent to the solution of the filtering problem for the unobserved factors.

4.3 Variance of Benchmarked Prices

From a financial modeling point of view it is important to be able to model and understand different degrees of available information. This is related to questions on insider trading but also for the valuation of information. As already mentioned in Section 2.1, the degree of available information is indexed by the parameter k. A larger value of k means that more factors are observed, providing thus more information in $\tilde{\mathcal{A}}^k$.

Let us now investigate the impact of varying degrees of information k concerning the factors $z_t = (z_t^1, \ldots, z_t^n)^\top$ that underly our model dynamics, see (2.2) – (2.3). We use now the notation $\tilde{z}_t^k$ for the vector of observables defined in (2.13), where we stress its dependence on k and recall that, by (2.28), the process $\tilde{z}^k$ is Markovian. Consider then a contingent claim

$$U(\tau, y_\tau) = U(\tau, y_\tau^1, y_\tau^2, \ldots, y_\tau^r) \tag{4.12}$$

for some fixed $r \in \{1, 2, \ldots, n-1\}$, where we assume that the number of observed factors that influence the claim equals r. For $k \in \{r, r+1, \ldots, n-1\}$ let $\tilde{u}^k(t, \tilde{z}_t^k)$ be the corresponding fair price at time t under the information $\tilde{\mathcal{A}}_t^k$, as given by (4.6). Recall that, by (4.6), $\tilde{u}^k(t, \tilde{z}_t^k)$ is the conditional expectation, under the real world probability measure, of $u(t, z_t)$ given $\tilde{\mathcal{A}}_t^k$. This implies that the corresponding *conditional variance*

$$\mathrm{Var}_t^k(u) = E\left(\left(u(t, z_t) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 \mid \tilde{\mathcal{A}}_t^k\right) \tag{4.13}$$

at time $t \in [0, T)$ is the minimal value of the, conditional on $\tilde{\mathcal{A}}_t^k$, mean square error corresponding to the deviation from $u(t, z_t)$ of any $\tilde{\mathcal{A}}_t^k$-measurable random variable. This conditional variance is computed under the real world probability measure. It would not make sense if computed under any other probability measure since the market participants are affected by the real difference between $u(t, z)$ and $\tilde{u}^k(t, \tilde{z}_t^k)$.

Note that for larger k we have more information available, which naturally should reduce the above conditional variance. We can prove the following practically relevant proposition, which quantifies the reduction in conditional variance. It can also be seen as a generalization of the celebrated Rao-Blackwell theorem towards filtering in incomplete markets under the benchmark approach.

Proposition 4.1 *For $m \in \{0, 1, \ldots, n-k\}$ and $k \in \{r, r+1, \ldots, n-1\}$ we have*

$$E\left(\mathrm{Var}_t^{k+m}(u) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) = \mathrm{Var}_t^k(u) - R_t^{k+m}, \tag{4.14}$$

where

$$R_t^{k+m} = E\left(\left(\tilde{u}^{k+m}(t, \tilde{z}_t^{k+m}) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 \,\middle|\, \tilde{\mathcal{A}}_t^k\right) \tag{4.15}$$

for $t \in [0, T)$.

Proof: For $t \in [0, T)$ and $k \in \{r, r+1, \ldots, n-1\}$ we have

$$\begin{aligned}\left(u(t, z_t) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 = {} & \left(u(t, z_t) - \tilde{u}^{k+m}(t, \tilde{z}_t^{k+m})\right)^2 + \left(\tilde{u}^{k+m}(t, \tilde{z}_t^{k+m}) - \tilde{u}^k(t, \tilde{z}_t^k)\right)^2 \\ & + 2\left(u(t, z_t) - \tilde{u}^{k+m}(t, \tilde{z}_t^{k+m})\right)\left(\tilde{u}^{k+m}(t, \tilde{z}_t^{k+m}) - \tilde{u}^k(t, \tilde{z}_t^k)\right).\end{aligned} \tag{4.16}$$

By taking conditional expectations with respect to $\tilde{\mathcal{A}}_t^k$ on both sides of the above equation it follows that

$$\begin{aligned}\mathrm{Var}_t^k(u) \;=\; & E\left(\mathrm{Var}_t^{k+m}(u) \,\middle|\, \tilde{\mathcal{A}}_t^k\right) + R_t^{k+m} + 2\,E\Big(\left(\tilde{u}^{k+m}(t, \tilde{z}_t^{k+m}) - \tilde{u}^k(t, \tilde{z}_t^k)\right) \\ & \cdot E\left(\left(u(t, z_t) - \tilde{u}^{k+m}(t, \tilde{z}_t^{k+m})\right) \,\middle|\, \tilde{\mathcal{A}}_t^{k+m}\right) \,\Big|\, \tilde{\mathcal{A}}_t^k\Big).\end{aligned} \tag{4.17}$$

Since the last term on the right hand side is equal to zero by definition, we obtain (4.14). □

5 Hedging Under Partial Observation

To determine in the given incomplete market a hedging strategy we have to use a hedging criterion. It turns out that the fair pricing and hedging concept, developed in Platen (2004b), can be generalized to our situation. Already under the existence of an equivalent risk neutral martingale measure it is known that there exist various hedging possibilities. In Platen (2004b) it has been pointed out that under the more general benchmark approach, even in a complete market setting, there may exist different self-financing hedge portfolios that replicate a contingent claim. In general, under the benchmark approach, nonnegative benchmarked portfolios are $(\tilde{\mathcal{A}}^k, P)$-supermartingales, see (3.17). The smallest possible supermartingale, which replicates the hedgable part, is known to be a martingale. Therefore, among possible hedge portfolios the fair portfolio process that replicates the hedgable part turns out to be special. It is the minimal portfolio that replicates the hedgable part because its benchmarked value forms a martingale. To see this in more detail, let us introduce the benchmarked pricing function

$$\hat{u}(t, \bar{z}_t^k) = \frac{\tilde{u}^k(t, \bar{z}_t^k)}{V_{\underline{\delta}}(t)} \tag{5.1}$$

for $t \in [0, T]$. We introduce for $i \in \{1, 2, \dots, k\}$ the operator

$$L^i \, \hat{u}(t, \bar{z}_t^k) = \sum_{\ell=1}^{k+q} \beta^{\ell,i}(t, \bar{z}_t^1, \dots, \bar{z}_t^{k+q}) \, \frac{\partial \hat{u}(t, \bar{z}_t^k)}{\partial \bar{z}^\ell} \tag{5.2}$$

and the jump operator

$$\begin{aligned} \Delta_{\hat{u}}^i \left(t-, \bar{z}_{t-}^k\right) \;=\; & \hat{u}\left(t, \bar{z}_{t-}^1 + \gamma^{1,i}\left(t-, \bar{z}_{t-}^1, \dots, \bar{z}_{t-}^{k+q}\right), \dots, \right. \\ & \left. \bar{z}_{t-}^{k+q} + \gamma^{k+q,i}\left(t-, \bar{z}_{t-}, \dots, \bar{z}_{t-}^{k+q}\right)\right) \\ & -\hat{u}\left(t-, \bar{z}_{t-}^1, \dots, \bar{z}_{t-}^{k+q}\right) \end{aligned} \tag{5.3}$$

with $\beta^{\ell,i}$ and $\gamma^{\ell,i}$, as in (2.29). For simplicity, we assume that the above benchmarked pricing function $\hat{u}(\cdot, \cdot)$ in (5.1) is differentiable with respect to time and twice differentiable with respect to the observables. Then we obtain with (2.29), (5.2) and (5.3) by the Itô formula for the $(\tilde{\mathcal{A}}^k, P)$-martingale $\hat{u} = \{\hat{u}(t, \bar{z}_t^k),\ t \in [0, \tau]\}$, where $\tau \in \mathcal{T}_{t,T}$ and $t \in [0, T]$, the following representation

$$\begin{aligned} \frac{U(\tau, y_\tau)}{V_{\underline{\delta}}(\tau)} &= \hat{u}(\tau, \bar{z}_\tau^k) \\ &= \hat{u}(t, \bar{z}_t^k) + \hat{I}_{t,\tau} + \hat{R}_{t,\tau}, \end{aligned} \tag{5.4}$$

with *hedgable part*, see (3.4),

$$\hat{I}_{t,\tau} = \sum_{\ell=1}^{h_1} \int_t^\tau L^\ell \, \hat{u}(s, \bar{z}_s^k) \, d\tilde{v}_s^\ell + \sum_{\ell=1}^{h_2} \int_t^\tau \Delta_{\hat{u}}^\ell(s-, \bar{z}_{s-}^k) \sqrt{\tilde{\lambda}_{s-}^\ell(\bar{z}_{s-})} \, d\tilde{m}_s^\ell \tag{5.5}$$

and *unhedgable part*

$$\hat{R}_{t,\tau} = \sum_{\ell=h_1+1}^{k} \int_t^\tau L^\ell \, \hat{u}(s, \bar{z}_s^k) \, d\tilde{v}_s^\ell + \sum_{\ell=h_2+1}^{k} \int_t^\tau \Delta_{\hat{u}}^\ell(s-, \bar{z}_{s-}^k) \sqrt{\tilde{\lambda}_{s-}^\ell(\bar{z}_{s-})} \, d\tilde{m}_s^\ell. \tag{5.6}$$

Note that (5.4) is an $(\tilde{\mathcal{A}}^k, P)$-martingale representation for the benchmarked contingent claim. Obviously, there is no way to hedge by the traded securities any fluctuations that arise in the unhedgable part. One can now search for a fair benchmarked portfolio process $\hat{V}_{\delta_U}$ with self-financing *hedging strategy* $\delta_U = \{\delta_U(t) = (\delta_U^0(t), \delta_U^1(t), \dots, \delta_U^d(t))^\top, t \in [0, \tau]\}$ that matches the hedgable part $\hat{I}_{t,\tau}$. This means that we compare the SDE (3.17) for $\hat{V}_{\delta_U}(t)$ with that of the hedgable part $\hat{I}_{t,\tau}$, see (5.5), where we use the jth *proportion*

$$\pi^j_{\delta_U}(t) = \frac{\delta^j_U(t)\,\hat{S}^{(j)}_t}{\hat{V}_{\delta_U}(t)} \tag{5.7}$$

of the value of the corresponding hedging portfolio that has to be invested into the jth primary security account, $j \in \{0, 1, \ldots, d\}$, provided $\tau \in \mathcal{T}_{t,T}$ at time t. By this comparison it follows that one needs to satisfy for $i \in \{1, 2, \ldots, h_1\}$ the equation

$$\sum_{j=1}^{d} \pi^j_{\delta_U}(t)\, b^{j,i}_t - \theta^i_t = \frac{L^i\, \hat{u}(t, \tilde{z}^k_t)}{\hat{V}_{\delta_U}(t)} \tag{5.8}$$

and for $i \in \{1, \ldots, h_2\}$ the relation

$$\left(\sum_{j=1}^{d} \pi^j_{\delta_U}(t-)\, b^{j,i}_{t-}\right)\left(1 - \frac{\theta^i_{t-}}{\sqrt{\tilde{\lambda}^{i-h_1}_{t-}(\bar{z}_{t-})}}\right) - \theta^i_{t-} = \frac{\Delta^i_{\hat{u}}(t-, \tilde{z}^k_{t-})}{\hat{V}_{\delta_U}(t-)}. \tag{5.9}$$

The equations (5.8) and (5.9) lead with $e_U(t) = (e^1_U(t), \ldots, e^d_U(t))^\top$, where

$$e^\ell_U(t-) = \begin{cases} \frac{L^\ell\, \hat{u}(t-, \tilde{z}^k_{t-})}{\hat{V}_{\delta_U}(t-)} + \theta^\ell_{t-} & \text{for} \quad \ell \in \{1, 2, \ldots, h_1\} \\ \frac{\sqrt{\tilde{\lambda}^{i-h_1}_{t-}(\bar{z}_{t-})}\left(\frac{\Delta^i_{\hat{u}}(t-, \tilde{z}^k_{t-})}{\hat{V}_{\delta_U}(t-)} + \theta^i_{t-}\right)}{\sqrt{\tilde{\lambda}^{i-h_1}_{t-}(\bar{z}_{t-})} - \theta^i_{t-}} & \text{for} \quad \ell = h_1 + i \in \{h_1 + 1, \ldots, d\} \end{cases} \tag{5.10}$$

and $\pi_{\delta_U}(t) = (\pi^1_{\delta_U}(t), \ldots, \pi^d_{\delta_U}(t))^\top$ to the vector equation

$$e_U(t-) = \left(\pi^\top_{\delta_U}(t-)\, b_{t-}\right)^\top. \tag{5.11}$$

Consequently, we can formulate the following statement.

Proposition 5.1 *The hedgable part of the contingent claim U can be replicated by the portfolio V_{δ_U} with proportions*

$$\pi_{\delta_U}(t) = \left(e_U(t)^\top\, b^{-1}_t\right)^\top \tag{5.12}$$

for $t \in [0, T]$ and $\tau \in \mathcal{T}_{t,T}$.

Notice that the elements of the vector $e_U(t)$ and the matrix b_t are given functions of observables. Since part of the latter are the results of filtering, this shows the usefulness of filtering also for hedging under partial observation.

Note furthermore that the driving martingales in the unhedgable part $\hat{R}_{t,\tau}$, see (5.6), are orthogonal to the martingales that drive the primary security accounts and thus orthogonal to the hedgable part $\hat{I}_{t,\tau}$, see (5.5). The above fair hedging strategy minimizes the quadratic variation of the resulting benchmarked profit and loss process under the real world probability measure. In Platen (2004e) it is derived as fluctuation minimization hedge. Obviously, to perform the fair hedge, the corresponding initial capital at time t needs to equal the fair price $\tilde{u}^k(t, \tilde{z}_t^k)$, see (4.6), of the contingent claim.

In the special case when an equivalent risk neutral martingale measure exists, the resulting hedging strategy equals the local risk minimizing strategy in the sense of Föllmer & Schweizer (1991) and the pricing measure is the minimal equivalent martingale measure, see Hofmann, Platen & Schweizer (1992). The martingale representation (5.4) is in this special case the corresponding benchmarked version of the Föllmer-Schweizer decomposition. We want to point out that, to the best of our knowledge, in the literature concerning local risk minimization under partial information, see Schweizer (1994), Fischer, Platen & Runggaldier (1999) and Frey & Runggaldier (1999), the authors assume from the outset that everything is defined under a risk neutral measure. This unpleasant assumption is avoided in our approach, which makes filtering more practicable.

Conclusions

We constructed a jump-diffusion financial market model with hidden variables and specified as benchmark the growth optimal portfolio. The random driving processes are Wiener and Poisson jump processes. For this incomplete market model with partial observation a consistent price system has been established without assuming the existence of an equivalent risk neutral martingale measure. Benchmarked fair derivative prices are obtained as martingales under the real world probability measure. Filtering has been described as an essential method for implementing fair pricing and hedging under partial information in the given incomplete market. The reduction of the conditional variance of fair derivative prices, when the available information increases, is quantified via a generalization of the Rao-Blackwell theorem.

A Appendix

Proof of Proposition 2.3

Denote by y^c the continuous part of the observation process y, that is

$$y_t^c = y_t - \sum_{\tau_j \leq t} G_{\tau_j -}(y_{\tau_j -})\, \Delta\, N_{\tau_j}, \tag{A.1}$$

where the τ_j denote the jump times of $N = \{N_t,\, t \in [0,T]\}$ and $\Delta\, N_{\tau_j} = N_{\tau_j} - N_{\tau_j -}$ is the vector $(\Delta\, N^1_{\tau_j -}, \ldots, \Delta\, N^k_{\tau_j -})^\top$. Let us now define the k-dimensional

$\tilde{\mathcal{A}}^k$-adapted process $\tilde{v} = \{\tilde{v}_t,\, t \in [0,T]\}$ by

$$B_t(y_t)\, d\tilde{v}_t = dy_t^c - \tilde{A}_t(\tilde{z}_t^k)\, dt. \tag{A.2}$$

From (2.3), (A.1) and (A.2) it follows that

$$d\tilde{v}_t = dv_t + B_t(y_t)^{-1} \left[A_t(z_t) - \tilde{A}_t(\tilde{z}_t^k) \right] dt. \tag{A.3}$$

From this we find, by the multi-variate Itô formula with $\nu \in \Re^k$ a row vector and $\imath$ the imaginary unit, that

$$\begin{aligned} \exp\left[\imath\nu\left(\tilde{v}_t - \tilde{v}_s\right)\right] \;&=\; 1 + \imath\nu \int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] dv_u \\ &\quad + \imath\nu \int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]\, B_u^{-1}(y_u) \left(A_u(z_u) - \tilde{A}_u(\tilde{z}_u^k) \right) du \\ &\quad - \frac{\nu\,\nu^\top}{2} \int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] du. \end{aligned} \tag{A.4}$$

Recalling that v is an $\tilde{\mathcal{A}}^k$-measurable Wiener process, notice that

$$E\left(\int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]\, dv_u \,\middle|\, \tilde{\mathcal{A}}_s^k \right) = 0 \tag{A.5}$$

and that, by our assumptions, the boundedness of $\exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]$ and the $\tilde{\mathcal{A}}_t^k$-measurability of $B_u^{-1}(y_n)$

$$\begin{aligned} & E\left(\int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right]\, B_u^{-1}(y_u) \left(A_u(z_u) - \tilde{A}_u(\tilde{z}_u^k) \right)\, du \,\middle|\, \tilde{\mathcal{A}}_s^k \right) \;= \\ & E\left(\int_s^t \exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] B_u^{-1}(y_u) E\left(\left(A_u(z_u) - \tilde{A}_u(\tilde{z}_u^k) \right) \,\middle|\, \tilde{\mathcal{A}}_u^k \right)\, du \,\middle|\, \tilde{\mathcal{A}}_s^k \right) \;=\; 0. \end{aligned} \tag{A.6}$$

Taking conditional expectations on the left and the right hand sides of (A.4) we end up with the equation

$$E\left(\exp\left(\imath\nu\left[\left(\tilde{v}_t - \tilde{v}_s\right)\right]\right) \,\middle|\, \tilde{\mathcal{A}}_s^k \right) = 1 - \frac{\nu\,\nu'}{2} \int_s^t E\left(\exp\left[\imath\nu\left(\tilde{v}_u - \tilde{v}_s\right)\right] \,\middle|\, \tilde{\mathcal{A}}_s^k \right) du, \tag{A.7}$$

which has the solution

$$E\left(\exp\left[\imath\nu\left(\tilde{v}_t - \tilde{v}_s\right)\right] \,\middle|\, \tilde{\mathcal{A}}_s^k \right) = \exp\left[-\frac{\nu\,\nu^\top}{2}\,(t-s) \right] \tag{A.8}$$

for $0 \le s \le t \le T$. We can conclude that $(\tilde{v}_t - \tilde{v}_s)$ is a k-dimensional vector of independent $\tilde{\mathcal{A}}_t^k$-measurable Gaussian random variables, each with variance $(t-s)$ and independent of $\tilde{\mathcal{A}}_s^k$. By Levy's theorem, $\tilde{v}$ is thus a k-dimensional $\tilde{\mathcal{A}}^k$-adapted standard Wiener process. □

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