Classical and Restricted Impulse Control for the Exchange Rate under a Stochastic Trend Model

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Abstract

Building on [4] and [2] we consider the problem faced by a Central Bank to optimally control the exchange rate, whereby the control is composed of a direct impulse control intervention and an indirect, continuously acting intervention given by the control of the domestic interest rate. Similarly to [4] and [2] we formulate the problem as a mixed classical-impulse control problem and the approach is based on a quasi-variational inequality by considering a specific class of the optimal value functions and controls. As in [2], but differently from [4], we consider a finite horizon that makes the problem time inhomogeneous and we do not have to impose a smooth fit condition so that a fully analytical solution is possible. With respect to [2] we generalize the problem by letting, more realistically, the drift in the dynamics of the exchange rate to be time varying or even unobservable so that it has to be filter-estimated from observable data. Numerical illustrations are presented as well.

Keywords: Exchange rate control, partial information, stochastic filtering, impulse control, quasi-variational inequalities.

JEL Classification: C61, D81, F31, G15, E58

1 Introduction

The control of the foreign exchange rate by a Central Bank has been the object of several studies in the literature, in particular also in the stochastic control literature. The setting is that of a so-called managed float or dirty float regime. The actual exchange rate fluctuates from day to day and may achieve unacceptable levels. One of the purposes of a Central Bank is therefore to intervene in order to keep the exchange rate at an acceptable level. Following [4] and [3], which generalize previous studies on the subject, we assume that the Central Bank can control the exchange rate by two non-excluding tools: direct intervention in the foreign exchange market by buying and selling currencies, and indirect intervention through determination of the domestic interest rate level. Interest rates have in fact an effect on the exchange rate through the attraction or deflection of foreign capital. In choosing the intervention, the Central Bank has to aim at achieving a specified goal. According to the idea of a target zone regime, the Central Bank has to guarantee that the exchange rate, as well as the domestic interest rate, stay within a given band or, more specifically, stay as close as possible to a given target that is usually established at higher levels of authority as result of negotiations or for political reasons and is considered to be sustainable for a given period. Consequently it is reasonable to assume, as we shall do it here, that these target levels are given as known constants or known time varying functions. On the other hand, any type of intervention is costly with the cost that is increasing with the level of the intervention. Since, in order to be effective, actual interventions are not frequent but of relatively large size (see e.g. [5]), as in [3] and [4]

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we consider both fixed and proportional costs with parameters that can be assumed to be relatively easy to identify and to quantify.

The control of the interest rate is of the type of a continuously acting control, while the direct intervention on the exchange rate is of the type of an impulse control. The problem thus concerns a mixed classical-impulse stochastic control problem and the authors in [4] aim at determining this control in order to balance the purpose of keeping the exchange rate as close as possible to the given target level and on the other hand to minimize the expected total cost of the intervention. The control of the exchange rate as a purely impulse control problem had been studied previously. One of the first such studies appears to be [8] which was then further developed in [9]. Within a more general context of impulse control, the problem had more recently also been studied in [11] (see [7] as well) over an infinite horizon and with a long-term average cost criterion. An additional indirect intervention by using the control also of the interest rate, as we do it here, has been considered in [12].

As mentioned, without interventions the exchange rate fluctuates from day to day. Assuming small investors, whose decisions do not affect the evolution of the exchange rate, following [15] and [3] we model its evolution between intervention times as a geometric Brownian motion (see (1) below) with three parameters: i) the parameter σ in the stochastic volatility that we assume to be known (can be estimated from historical data). ii) The parameter K that represents the influence of the interest rate on the level of the exchange rate. In this paper we assume it to be a given exogenous constant; our procedure works however equally well (see Remark 1 below) if it is a known time varying function and also if it is random and expressed as a known (time varying) function of past and present values of the exchange rate. iii) There is finally the drift parameter μ_t that, at the generic time t, represents the exogenous economic pressure on the level of the exchange rate and that in the literature has so far been considered as constant and known. Here we consider it, as it appears to be more natural, to be time varying or even not directly observable. In the latter case we apply filtering techniques whereby the unobserved drift is filter-estimated on the basis of the past and present observations of the exchange rate itself. In both cases one has thus a time varying drift, which complicates the approach in the sense that the strategic boundaries that, as we shall show, determine the impulse control interventions, will depend not only on time, but also on the current values (either observed or estimated) of the drift. Furthermore, the state space in the optimization problem becomes three-dimensional (time, exchange rate, filtered drift) whereby the pair (exchange rate, filtered drift) is Markov. In setting up the filter model, we shall for the sake of generality introduce a prior dynamic evolution model for the drift μ in the form of a diffusion process with given drift and diffusion parameters. We may however put all these parameters equal to zero so that the drift becomes an unknown parameter and the filter reduces to a Bayesian-type updating of this unknown parameter (for a pure impulse control problem with Bayesian parameter updating see e.g. [1]). The model thus depends on various parameters. Some are assumed to be exogenously given and a justification for this has been presented above; others are determined endogenously. Determining a solution for a given set of exogenously determined parameter values, can then also allow for a sensitivity analysis of the solution with respect to the individual parameter values. It also allows to investigate various scenarios and to understand up to what degree of precision one should determine the values of the exogenous parameters.

The approach in [4] builds on the notion of a quasi-variational inequality (QVI). The control horizon in [4] is infinite, which leads to a time-homogeneous solution and the authors search for such a solution within a specific class, for which the value function has to be $\mathcal{C}^1$ throughout and $\mathcal{C}^2$ except for the boundary points. They end up with six conditions on four parameters, which makes the problem difficult to solve analytically, but they nevertheless come up with a solution by using partly a numerical approach. The results in [4] have been generalized in [2] in two directions:

i) The value function is not required to be $\mathcal{C}^1$ throughout (the numerical results in section 6 below show in fact that on the boundary of the continuation region it is in general not $\mathcal{C}^1$). This allows one to avoid a smooth fit condition and to end up with only four conditions on the four parameters thus leading to an analytic solution.

ii) The horizon is finite, which appears to be more natural in economic-financial applications, where one wants to achieve a goal within a finite, foreseeable amount of time; at the end of the planning period the problem may have to be reformulated anew. This is for example relevant in our context where, as it was done mostly in the relevant literature, we limit ourselves to two currencies: the given domestic one and one specific foreign currency. This specific foreign currency may in fact be of interest only over a given horizon, after which the interest may shift to another one thus changing the problem. On the other hand a finite horizon causes the problem to become time inhomogeneous and therefore more complex.

As in [4], also in [2] the solution is obtained within a specific class of value functions and controls. It consists in an indirect intervention through a continuously acting control of the domestic interest rate when the exchange rate is in a specific interval, namely the *continuation region* that in the finite-horizon case depends on the current time. On the other hand, the impulse control intervenes when the exchange rate reaches either one of the two boundaries of that interval, namely when it enters the *intervention region*. At that point the Central Bank is supposed to intervene directly by pushing the exchange rate to yet another interval, the *preferred region*.

Here we follow this latter line and consider a restricted form of impulse control where the determination of the control action reduces thus to the determination of the boundaries of the continuation region and of the amount by which the exchange rate is pushed back when it reaches either one of the two boundaries. More precisely, the solution is supposed to be given by four continuous functions $a(\cdot) < \alpha(\cdot) < \beta(\cdot) < b(\cdot)$ with $\beta(\cdot) > 0$, where $a(\cdot)$ and $b(\cdot)$ represent the boundaries of the continuation region, while $\alpha(\cdot)$ and $\beta(\cdot)$ represent the boundaries of the preferred region, namely the values to which the exchange rate is shifted when it reaches $a(\cdot)$ or $b(\cdot)$ respectively. We shall call these functions "strategic boundaries". For an infinite horizon setup as in [4] these functions reduce to constants, in the finite horizon setup of [2] they become time varying functions. Notice also that in much of the previous target zone literature the boundaries were chosen exogenously, while here they will be determined endogenously.

Finding a fully optimal solution within the specific class mentioned above is still not possible in a purely analytical way and this is why in [4] the solution was obtained by a combination with a numerical approach. In the search for a completely analytical solution, in [2] the authors make a further assumption, namely that the value function is given, within the continuation region, by a quadratic solution of the implied HJB equation and is linear outside this region. They also show that this assumption is well motivated, in particular it is consistent with the solution obtained in [4] for the infinite-horizon case. In the present paper we also make this quadratic Ansatz for the solution of the implied HJB equation. We want to point out however that, by restricting the solution of the HJB equation within the continuation region to be a quadratic function, implies that our optimal solution may actually be only an upper bound to the true optimal solution; it leads however to an explicitly computable value function as well as controls. The true optimal solution, even with controls in the specific class, may have a very complicated structure within the continuation region and we do not know of any result to this effect in the literature.

In the next Section 2 we formulate the model and the problem, building mainly on [4] and [2]. In Section 3 we describe the specific class of value functions and controls. In Section 4 we compute the optimal value function and controls within the specific class by assuming that the solution satisfies a given weak quasi-variational inequality (QVI). Finally, in section 5 we prove the optimality, within the given class, of the solution derived in section 4. This is done in two steps: in subsection 5.1 we prove, via a verification theorem, that the candidate solution is an optimal solution if it is a weak QVI solution. Then, in subsection 5.2 we show that the candidate is indeed a weak QVI solution. Finally, some numerical illustrations are provided in section 6.

2 The model and problem setting

Our model corresponds to that in [4] and [2] that we recall in this section. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and W be a one dimensional $\mathcal{F}_t$-Brownian motion. Consider a given foreign currency (e.g. the Dollar) and a domestic currency (e.g. the Euro) and define, for each $t \in \mathbb{R}_+$ the exchange rate at time t as

$$X_t := \text{units of domestic currency for one unit of foreign currency.}$$

Suppose that $X = (X_t)$ is an adapted stochastic process satisfying the following dynamics that are slightly different from [4] and [2], namely

$$X_t = x + \int_0^t (\mu_s + K\, u_s)\, X_s \,\mathrm{d}s + \sigma \int_0^t X_s \,\mathrm{d}W_s + \sum_{n=1}^{\infty} 1_{\{\tau_n \leq t\}} \xi_n, \tag{1}$$

where the various quantities have the following meaning: x is the initial value of the exchange rate. The process $\mu = (\mu_t)$ represents the exogenous economic pressure on the level of the exchange rate so that $\mu_t > 0$ indicates a pressure towards a devaluation of the domestic currency, while $\mu_t < 0$ indicates the opposite situation. The continuously acting control $u_t := (X_t)^{-1} \log \frac{r_t}{\bar{r}}$, where r_t is the domestic interest rate and the constant $\bar{r}$ is its target, measures the ratio with respect to the current value X_t of the log-relationship between the interest rate level set by the Central Bank and the target $\bar{r}$. We shall consider the control u_t to be admissible if (see also Definition 1 below) it is adapted and mean square integrable (in Lemma 7 below we shall show that the optimal continuously acting control $\hat{u}_t$ is indeed mean square integrable). The parameter $K \in (-\infty, 0)$ represents the influence of the interest rate on the level of the exchange rate. As mentioned in the Introduction, here we assume it exogenously given (see however Remark 1 below). Finally, the constant $\sigma \in (0, \infty)$ is the exogenous volatility of the exchange rate. Coming to the impulse controls, τ_n is the time of the n-th intervention of the Central Bank and ξ_n represents the amount of the n-th intervention. Below we shall work with the process $x_t = \log X_t$ (see (2)) so that $X_t > 0$ *a.s.* implying that, if $r_t > \bar{r}$, then u_t is positive thus pushing X_t downwards while, if $r_t < \bar{r}$, then the opposite happens. The justification for this is given by the fact that $\bar{r}$ is perceived as a natural equilibrium rate on the long run (long-term sustainable rate) and deviations from this rate make the domestic assets more or less attractive.

One difference of the above model with respect to [4] and [2] is that u_t is here the ratio of $\log \frac{r_t}{\bar{r}}$ with respect to X_t. In this way X_t becomes a factor in the drift in (1) which implies that the solution X_t of (1) is positive between two successive impulse times. We can thus pass to logarithms of X_t thus simplifying some of the derivations below and allowing the filtering problem to be solved more easily. The main difference however is that we allow the drift μ_t to be time-varying or even not directly observable. We shall thus consider two possible situations for μ_t according to

Assumption 1. The process μ_t is supposed to be either one of the following

$$\mu_t \text{ is } \begin{cases} i) & \text{an observable adapted process} \\ ii) & \text{unobservable and has to be estimated from observable data.} \end{cases}$$

It is in fact realistic to assume that the drift μ_t may not be directly observable. In view of estimating (filtering) μ_t from observations of the exchange rate X_t itself, consider the dynamics of the latter between two successive intervention times by putting $x_t := \log X_t$, namely

$$dx_t = \left(\left[\mu_t - \frac{\sigma^2}{2} \right] + K\, u_t \right) dt + \sigma\, dW_t := (m_t + K\, u_t)\, dt + \sigma\, dW_t \tag{2}$$

and thereby defining implicitly $m_t := \mu_t - \frac{\sigma^2}{2}$. Whether observable or not, we assume that the factor process m_t is a diffusion given by

$$dm_t = (A\, m_t + a)\, dt + \Lambda\, dB_t\,, \quad m_0 \text{ a given Gaussian r.v.} \tag{3}$$

where $\Lambda = (\Lambda_1, \Lambda_2)$ and $B_t = (\beta_t, W_t)'$ with β_t a scalar $\mathcal{F}_t-$Wiener process, independent of W_t, and so there is correlation between the two driving Wiener processes W_t and B_t. As already mentioned in the Introduction, in our filtering context (see the filter model (5) below), that can be interpreted as a dynamic version of Bayesian statistics, the dynamics in (3) can be seen as prior dynamics for m_t (the posterior dynamics are given by the dynamics of $\hat{m}_t$ in (6) below) with the coefficients some prior constants that are exogenously given. It is not necessary to assign a full prior dynamics for m_t; by putting all the parameters in (3) equal to zero, the prior process m_t reduces to the random variable m_0, whose Gaussian distribution becomes (from a Bayesian point of view) the prior distribution of the constant but unknown drift (even if m_t is a priori a constant, a posteriori it is always the process $\hat{m}_t$, adapted to the filtration generated by the observations x_t).

With some abuse of notation we shall now denote by the same ξ_n the amount of intervention on $x_t = \log X_t$, i.e. $\xi_n = x_{\tau_n+} - x_{\tau_n}$ and by the same x the initial value x_0. Rather than on (1) we shall thus base ourselves on the following dynamic model

$$x_t = x + \int_0^t \big(m_s + K\,u_s\big)\,\mathrm{d}s + \sigma\,W_t + \sum_{n=1}^{\infty} 1_{\{\tau_n \le t\}}\xi_n. \tag{4}$$

In the case when m_t is observable we can determine the strategy u_t, τ_n and ξ_n using information coming from observing x_t and m_t. When μ_t, and thus m_t is not directly observable, but only x_t is observable then, defining $\mathcal{F}_t^x := \sigma\{x_s\,,\, 0 \le s \le t\}$, we need to decide our strategy u_t, τ_n and ξ_n on the basis of the information $\mathcal{F}_t^x$. This means that u_t has to be an $\mathcal{F}_t^x$-adapted process, τ_n an $\mathcal{F}_t^x$-stopping time and ξ_n an $\mathcal{F}_{\tau_n}^x$-measurable random variable. To distinguish for the moment the control u_t in the full and partial information setting, we denote it by $\bar{u}_t$ in the latter case. We can now consider the following filter model

$$\begin{cases} dm_t &= (A\,m_t + a)\,dt + \Lambda\,dB_t, \quad m_0 \text{ a given r.v.} \\ dx_t &= (m_t + K\,\bar{u}_t)\,dt + \sigma\,dW_t\,, \quad x_0 = x. \end{cases} \tag{5}$$

Given the assumption of mean square integrability of u_t, the above system is a well defined stochastic system. Always with this assumption, the system (5) becomes a linear, conditionally Gaussian filter system to which one can apply an extended form of the Kalman filter to obtain the dynamics of the conditional mean $\hat{m}_t := E\{m_t \mid \mathcal{F}_t^x\}$, and conditional variance $\Gamma_t := E\left\{(m_t - \hat{m}_t)^2 \mid \mathcal{F}_t^x\right\}$. Theorem 12.1 in [10] leads in fact to the following solution

$$\begin{cases} d\hat{m}_t &= (A\,\hat{m}_t + a)\,dt + \left(\Lambda_2 + \frac{\Gamma_t}{\sigma}\right)\,d\nu_t\,, \quad \hat{m}_0 = m_0 \\ d\Gamma_t &= \left(2\left[A - \frac{\Lambda_2}{\sigma}\right]\,\Gamma_t + {\Lambda_1}^2 - \frac{{\Gamma_t}^2}{\sigma^2}\right)\,dt\,, \quad \Gamma_0 = Var(m_0), \end{cases} \tag{6}$$

where the filter variance Γ_t can be computed off-line and therefore considered to be a deterministic time function, while

$$d\nu_t := \sigma^{-1}\,\left[dx_t - (\hat{m}_t + K\,\bar{u}_t)\,dt\right] \tag{7}$$

is an $\mathcal{F}_t^x \subset \mathcal{F}_t$ Wiener process called *innovations process.*

Remark 1. An important fact is that, in our setup, the filter solution does not depend on the continuously acting control u_t. According to Theorem 12.1 in [10] we could apply the extended Kalman filter for the conditionally Gaussian case to our model also when the parameter K is unobservable with a given prior Gaussian distribution. Instead of the dynamics only for the posterior filtered process $\hat{m}_t$, we would then obtain also a dynamic equation for $\hat{K}_t := E\{K \mid \mathcal{F}_t^x\}$. The problem with this is that then the entire filter solution becomes dependent on the continuously acting control u_t which is not acceptable since the control in turn depends on the filter (see (19) and (22), (23) below). This problem arises in general when the control affects the observations; it does not happen in the case of an impulse control when, as in our case, after the impulse the filter can be re-initialized at its pre-impulse value (see the second bullet of Remark 4 below). An analogous situation occurs also in [1] where the control is only of the impulsive type. While we cannot allow K to be unobservable, we could however allow it to be an $\mathcal{F}_t^x$-adapted process that is observable; also in this latter more general case the filter would be independent of the control.

From the preceding development and considering also Section 3 in [14], we have the following immediate

Lemma 1. The filtered drift process $\hat{m}_t$ and the observed log-exchange rate process x_t satisfy

$$\begin{cases} d\hat{m}_t &= (A\,\hat{m}_t + a)\,dt + \left(\Lambda_2 + \frac{\Gamma_t}{\sigma}\right)\,d\nu_t\,, \quad \hat{m}_0 = m_0 \\ dx_t &= (\hat{m}_t + K\,\bar{u}_t)\,dt + \sigma\,d\nu_t\,, \quad x_0 = x. \end{cases} \tag{8}$$

Notice that the models (5) and (8) are completely analogous, only the volatility for the factor process changes. In what follows we shall therefore study in a possibly unified way both the cases when m_t is observable and when it is not, where in the latter case it is replaced by $\hat{m}_t$. For this purpose we shall use the notation m_t^j with $j \in \{f, p\}$, where f stands for "full information" (of m_t) and p for partial information, so that $m_t^f = m_t$ and $m_t^p = \hat{m}_t$. Also set $u_t^f = u_t$ and $u_t^p = \bar{u}_t$. Similarly, we shall also denote various other quantities with a sub- or super-script j depending on whether they refer to the situation when m_t is observable and when it is not (when it matters we shall also write x_t^j).

Definition 1. A *mixed classical-impulse stochastic control* with $j = f, p$ is a triple

$$(u, \mathcal{T}, \xi)^j = (u^j; \tau_1^j, \tau_2^j, \tau_3^j, \ldots, \tau_n^j, \ldots; \xi_1^j, \xi_2^j, \xi_3^j, \ldots, \xi_n^j, \ldots).$$

Here u^j is as described previously and represents a classical stochastic control, namely $u^j : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is an $\mathcal{F}_t^j$-adapted stochastic process that we have already assumed to be mean square integrable and where $\mathcal{F}_t^f = \sigma(x_s, m_s, 0 \le s \le t)$ and $\mathcal{F}_t^p = \sigma(x_s, 0 \le s \le t)$. Furthermore, the pair $(\mathcal{T}, \xi)^j$ is an impulse control, namely $0 \le \tau_1^j < \tau_2^j < \tau_3^j < \ldots < \tau_n^j \ldots$ is an increasing sequence of stopping times and $\{\xi_n^j\}_{n \in \mathbb{N}}$ is a sequence of random variables such that $\xi_n^j : \Omega \to \mathbb{R}$ is $\mathcal{F}_{\tau_n}^j$-measurable. The Central Bank (the controller) decides to act at time τ_n^j adding the quantity ξ_n^j to the value of the log-exchange rate at that moment of time, namely $x_{\tau_n+}^j = x_{\tau_n}^j + \xi_n^j$, where x_t^j is the process x_t with control $(u, \mathcal{T}, \xi)^j$.

Remark 2. From the results below it will follow that the optimal controls are of the Markovian type, namely functions of (x_t^j, m_t^j). It then follows from (5) and (8) that, with Markovian controls, both $(x_t^f, m_t^f) = (x_t^j, m_t)$ and $(x_t^p, m_t^p) = (x_t^j, \hat{m}_t)$ are Markov.

Next we introduce a cost functional to be minimized with respect to the controls $(u, \mathcal{T}, \xi)^j$. We consider a finite planning horizon $T > 0$ so that $\tau_i^j \le T$ and call *generically admissible* the mixed classical-impulse stochastic control $(u, \mathcal{T}, \xi)^j$ as specified so far and denote their class by $\mathcal{A}^{g,j}$. For the given horizon $T > 0$, and assuming that we stand at a generic time $t \in [0, T]$ with values (x^j, m^j) of the Markov process (x_t^j, m_t^j), define as cost-to-go function for a given control $(u, \mathcal{T}, \xi)^j$ the following

$$\begin{aligned} V_j^{(u,\mathcal{T},\xi)^j}(t, x, m) := \mathbb{E}_{t,x,m} \Bigg\{ & \int_t^T e^{-\lambda(s-t)} f(x_s^j, u_s^j)\, \mathrm{d}s + e^{-\lambda(T-t)} h(x_T^j) \\ & + \sum_{n=1}^{\infty} e^{-\lambda(\tau_n^j - t)} g(\xi_n^j)\, 1_{\{t \le \tau_n^j \le T\}} \Bigg\}, \end{aligned} \tag{9}$$

where $j \in \{f, p\}$, $\lambda > 0$ is a discount factor, $f(x, u)$ is a running cost penalizing both the quadratic deviation of x_t^j from a given target $\rho > 0$ as well as u_t^j in the square, namely

$$f(x, u) := (x - \rho)^2 + k\, u^2 \quad (k \ge 0). \tag{10}$$

Analogously, $h(x)$ is a terminal cost, also of the quadratic form $h(x) = \ell\,(x - \rho)^2$, and $g(\xi)$ represents a cost for the amount ξ of the intervention that, similarly to [4] and [2], we assume to be of the form

$$g(\xi) := \begin{cases} C + c\,\xi & \text{if } \xi > 0 \\ \min(C, D) & \text{if } \xi = 0 \\ D - d\,\xi & \text{if } \xi < 0 \end{cases}, \tag{11}$$

where $k, \ell, \lambda, \rho, C, c, D, d$ are positive constants. Notice that k penalizes the use of the continuous control u_t; the fact that in (10) we consider its square, implies that deviations of r_t from $\bar{r}$ are penalized equally, whether they exceed or fall short of $\bar{r}$. On the other hand, C and D denote fixed intervention costs for a push upwards or downwards respectively and c, d are proportional costs. Notice, furthermore, that a penalization by $(X - \bar{\rho})^2$, with X the actual exchange rate, is not equivalent to $(x - \rho)^2$, with $x = \log X$, even if we put $\rho = \log \bar{\rho}$. In a sense, $(x - \rho)^2$ may be preferable since, when x and ρ are close to one, as it may often be the case, then the squared difference $(X-\rho)^2$ becomes negligible and therefore not significant, while $(\log X - \log\rho)^2$ is not. Finally, as pointed out in [7], an exchange rate is not an asset and so the functions f, h, g do not represent a tangible cost leaving a discount factor without a clear economic interpretation. We still keep the discount factor here just for the sake of generality.

Remark 3. Since all the entries in the cost-to-go function in (9) are $\mathcal{F}_t^x$-adapted, the value function for the full and partial information cases, i.e. for both $j = f, p$, are the same and this further justifies the possibility of studying the full and partial information problem in a unified way.

An optimal control is now a control $(u^*, \mathcal{T}^*, \xi^*)^j$, for the moment in $\mathcal{A}^{g,j}$, for which at the initial time $t = 0$ with values x, m of the Markov process (x_t^j, m_t^j)

$$\mathcal{J}_j(0,x,m) = V_j^{(u^*,\mathcal{T}^*,\xi^*)^j}(0,x,m) := \inf_{(u,\mathcal{T},\xi)^j} V_j^{(u,\mathcal{T},\xi)^j}(0,x,m)\,; \quad j \in \{f,p\}. \tag{12}$$

In its full generality as described above, the problem of choosing $(u,\mathcal{T},\xi)^j$ in $\mathcal{A}^{g,j}$ to minimize $V_j^{(u,\mathcal{T},\xi)^j}(0,x,m)\,, j = f,p$ is very difficult to solve and so, as in [2], we shall look for an optimal solution within a subclass of strategies and value functions that is however still rather general, contains the optimal solution in [4], and above all admits an analytical solution. This subclass is the subject of the next section.

3 Specific class

3.1 The specific class of value functions and admissible controls

As suggested by the results in [4] and generalizing to the model of the present paper the class considered in [2], we conjecture that, for $j = f, p$, the optimal value function

$$V^j(t,x,m) = \inf_{(u,\mathcal{T},\xi)^j} V_j^{(u,\mathcal{T},\xi)^j}(t,x,m) \tag{13}$$

has the following structure: for given continuous functions

$$a^j(t,m) < \alpha^j(t,m) < \beta^j(t,m) < b^j(t,m) \tag{14}$$

that below we shall call "strategic boundaries" let

$$V^j(t,x,m) = \begin{cases} \Phi^j(t,\alpha^j(t,m),m) + C + c(\alpha^j(t,m) - x) & \text{if } x \le a^j(t,m) \\ \Phi^j(t,x,m) & \text{if } a^j(t,m) < x < b^j(t,m) \\ \Phi^j(t,\beta^j(t,m),m) + D + d(x - \beta^j(t,m)) & \text{if } x \ge b^j(t,m) \end{cases} \tag{15}$$

where, for $j = f, p$, the $\Phi^j(t,x,m)$ satisfy the HJB-type equation

$$\begin{aligned} &\big(\Phi_t^j + \mathcal{L}_j^{\hat{u}^j}\Phi^j\big)(t,x,m) + f\big(x,\hat{u}^j\big) = \Phi_t^j\big(t,x,m\big) + \inf_{u\in\mathbb{R}}\Big\{\mathcal{L}_j^u\Phi^j\big(t,x,m\big) + f\big(x,u\big)\Big\} = 0 \\ &\Phi^j(T,x,m) = h(x) = \ell\,(x-\rho)^2 \end{aligned} \tag{16}$$

with $\hat{u}^j$ a control achieving the inf on the right hand side in (16). According to (5) (respectively (8)), the generator $\mathcal{L}_j^u$ is, for a sufficiently regular function $\Phi^j : [0,T]\times\mathbb{R}^2 \to \mathbb{R}_+$, given by

$$\begin{aligned} \mathcal{L}_j^u\Phi^j(t,x,m) :=& \big(m + Ku\big)\Phi_x^j(t,x,m) + \big(A\,m + a\big)\Phi_m^j(t,x,m) + \tfrac{\sigma^2}{2}\Phi_{xx}^j(t,x,m) \\ &+\Sigma_1^j(t)\,\Phi_{xm}^j(t,x,m) + \Sigma_2^j(t)\,\Phi_{mm}^j(t,x,m) - \lambda\,\Phi^j(t,x,m) \end{aligned} \tag{17}$$

where the subscripts in $\Phi^j(\cdot)$ denote partial derivatives and where

$$\Sigma_1^j(t) := \begin{cases} \sigma\,\Lambda_2 & \text{for } j=f \\ \sigma\,\Lambda_2 + \Gamma_t & \text{for } j=p \end{cases}, \quad \Sigma_2^j(t) := \begin{cases} \frac{\Lambda_1^2+\Lambda_2^2}{2} & \text{for } j=f \\ \frac{(\Lambda_2+\frac{\Gamma_t}{\sigma})^2}{2} & \text{for } j=p \end{cases} \tag{18}$$

Due to the quadratic structure of $f(x,u)$ the optimal continuous control $\hat{u}^j$ exists and is, for $j=f,p$ and for each $t\in[0,T]$, given by

$$\hat{u}_t^j = -\frac{K}{2k}\,\Phi_x^j\big(t,x_t^j,m_t^j\big) \tag{19}$$

where x_t^j has here to be considered as the process x_t in (2) corresponding to the continuous control $\hat{u}_t^j$. Consequently, the corresponding optimal interest rate is

$$\hat{r}_t^j = \bar{r}\,\exp\left(-\frac{K\,\hat{x}_t^j}{2k}\,\Phi_x^j\big(t,x_t^j,m_t^j\big)\right), \quad j=f,p \tag{20}$$

With $u=\hat{u}^j$ as in (19), the HJB-equation (16) becomes

$$\begin{aligned} &\Phi_t^j(t,x,m) + m\,\Phi_x^j(t,x,m) + \big(A\,m+a\big)\Phi_m^j(t,x,m) + \tfrac{\sigma^2}{2}\Phi_{xx}^j(t,x,m) - \tfrac{K^2}{4k}\left(\Phi_x^j(t,x,m)\right)^2 \\ &\qquad + \Sigma_1^j(t)\,\Phi_{xm}^j(t,x,m) + \Sigma_2^j(t)\,\Phi_{mm}^j(t,x,m) - \lambda\,\Phi^j(t,x,m) + (x-\rho)^2 = 0, \\ &\Phi^j(T,x,m) = \ell(x-\rho)^2. \end{aligned} \tag{21}$$

Concerning the impulse control, given always the continuous functions $a^j(\cdot)<\alpha^j(\cdot)<\beta^j(\cdot)<b^j(\cdot)$, we let the impulse part $(\mathcal{T},\xi)$ of the control be, for $j=f,p$, of the form

$$\tau_n^j = \inf\left\{t>\tau_{n-1}^j \mid x_t^j \notin \big(a^j(t,m_t^j),b^j(t,m_t^j)\big)\right\}, \quad (\tau_0^j=0) \tag{22}$$

$$x_{\tau_n^j+}^j = x_{\tau_n^j}^j + \xi_n^j = \beta^j(\tau_n^j,m_{\tau_n^j}^j)\,\mathbf{1}_{\{b^j(\tau_n^j,m_{\tau_n^j}^j)\}}(x_{\tau_n^j}^j) + \alpha^j(\tau_n^j,m_{\tau_n^j}^j)\,\mathbf{1}_{\{a^j(\tau_n^j,m_{\tau_n^j}^j)\}}(x_{\tau_n^j}^j). \tag{23}$$

Since, modulo a slight adjustment, the optimal solution obtained by [4] belongs to the class described above, we shall restrict the original class of generic admissible controls $\mathcal{A}^{g,j}$ to the subclass $\mathcal{A}^j$ of *admissible* controls according to the above specifications and search for an optimal solution within this class, namely

Definition 2. We say that for each $j=f,p$, a mixed classical-impulse stochastic control $(u,\mathcal{T},\xi)^j$ is admissible if u^j is of the form as in (19) and there exist four continuous functions $a^j(\cdot)<\alpha^j(\cdot)<\beta^j(\cdot)<b^j(\cdot)$ such that (22) and (23) are satisfied. Furthermore, $V_j^{(u,\mathcal{T},\xi)^j}(t,x,m)$ defined in (9) has to be finite. We shall denote by $\mathcal{A}^j$ this class of admissible mixed classical-impulse stochastic controls for the process x_t^j.

We shall show below (see Remark 6) that, for our model and the class of controls as specified above, $V_j^{(u,\mathcal{T},\xi)^j}(t,x,m)$ is indeed finite.

Limiting ourselves to the class of solutions as specified above, we shall consider, for each $j=f,p$, as "continuation region" the subset $\mathcal{C}^j$ of the state space $\Sigma := \{(t,x,m)\in[0,T]\times\mathbb{R}^2\}$ defined as

$$\mathcal{C}^j = \big\{(t,x,m)\in\Sigma \,:\, a^j(t,m)<x<b^j(t,m)\big\} \tag{24}$$

and, consequently, the intervention region $\mathcal{I}^j$ is then its complement in Σ.

Given the quadratic form of the cost functions we shall make a quadratic Ansatz for $\Phi^j(t,x,m)$ as solution of the HJB equation (21), more precisely we make the following

Assumption 2. The optimal value function $V^j(t,x,m)$ is supposed to be given, for each $j=f,p$ and within the continuation region $\mathcal{C}^j$, by $V^j(t,x,m)=\Phi^j(t,x,m)$ where $\Phi^j(t,x,m)$ is a quadratic function of the type

$$\Phi^j(t,x,m) = p_{xx}^j(t)\,x^2 + p_{xm}^j(t)xm + p_{mm}^j(t)\,m^2 + p_x^j(t)\,x + p_m^j(t)\,m + p^j(t) \tag{25}$$

where, here, the pedices do not denote partial derivatives, but are simply indexes.

Remark 4.

- Thanks to the particular structure of our class of admissible impulse controls, namely by defining them through the strategic boundaries $a^j(\cdot) < \alpha^j(\cdot) < \beta^j(\cdot) < b^j(\cdot)$ as well as to the structure of the value function in (15), in the proof of the verification result in Proposition 1 below we do not need to use Ito's formula on the entire state space. We only need to apply it in the open interval $(a^j(t, m_t^j), b^j(t, m_t^j))$, which allows us to avoid having regularity throughout as required by the smooth fit condition in general stopping and impulse control problems (see e.g. [13]).
- In the case of partial information we apply the filter in (6) that is initialized by $\hat{m}_0 = m_0^p = m_0,\ \Gamma_0 = Var(m_0)$. This filter is run up to the first impulse intervention time τ_1^p, where $x^p_{\tau_1^p}$ is pushed to $x^p_{\tau_1^p} + \xi_1^p$. Both τ_1^p and ξ_1^p are $\mathcal{F}_t^p-$adapted with $\mathcal{F}_t^p = \mathcal{F}_t^x$ the observation filtration, with respect to which the filter is determined. In particular, by (22) and (23), they depend on $\mathcal{F}_t^p$ via m_t^p that, see (6), between impulse times does not depend on the continuously acting control u_t. The control u_t does therefore not change the observation filtration $\mathcal{F}_t^p$ and so at every intervention time τ_n^p we therefore re-initialize the filter with the values $(\hat{m}_{\tau_n^p}, \Gamma_{\tau_n^p})$ that are obtained from (6) at time τ_n^p.

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3.2 (Weak) Quasi-variational inequality

As in [4], also our methodology builds on the notion of a quasi-variational inequality (QVI) that we use here in a weaker form. First we introduce the minimal cost operator M that we define for each continuous function $\phi : \Sigma \to \mathbb{R}$ as

$$M\phi(t, x, m) := \inf\{\phi(t, x + \eta, m) + g(\eta) \mid \eta \in \mathbb{R}\}, \tag{26}$$

where $g(\cdot)$ is the cost function for the impulse interventions as defined in (11).

Definition 3. We say that, for $j = f, p$, a function $v : \Sigma \longrightarrow \mathbb{R}_+$ satisfies the (weak) quasi-variational inequality (QVI) for the problem specified in Section 2 if

$$\begin{array}{l} i)\ v_t(t,x,m) + \inf_{u\in\mathbb{R}} \left\{\mathcal{L}_j^u v(t,x,m) + f(x,u)\right\} \geq 0 \\ \quad \forall t \in [0,T),\ m \in \mathbb{R},\ x \in (a^j(t,m), b^j(t,m)), \\ \quad \text{and } v(T,x,m) = h(x) \ \ \forall m \in \mathbb{R},\ x \in (a^j(T,m), b^j(T,m)), \\ ii)\ Mv(t,x,m) - v(t,x,m) \geq 0 \quad \text{everywhere.} \end{array} \tag{27}$$

Furthermore, at least one of the two inequalities in i) and ii) holds as an equality.

Notice that equality in point i) may also hold in the intervention region $\mathcal{I}^j$.

We shall choose the strategic boundaries (see Section 4.2 below) in such a way that the optimal value function $V^j(t, x, m)$ satisfies QVI. The continuation region $\mathcal{C}^j$ in (24) can then, for $j = f, p$, also be expressed as

$$\mathcal{C}^j = \left\{(t,x,m) \in \Sigma \mid V^j(t,x,m) < MV^j(t,x,m),\ V_t^j(t,x,m) + \inf_{u\in\mathbb{R}} \left\{\mathcal{L}_j^u V^j(t,x,m) + f(x,u)\right\} = 0\right\},$$

while the intervention region then becomes

$$\mathcal{I}^j = \left\{(t,x,m) \in \Sigma \mid V^j(t,x,m) = MV^j(t,x,m)\right\}.$$

In sub-section 5.1 below we shall show by a verification result that a certain value function $V^j(t, x, m)$ is optimal by assuming that it satisfies the QVI of Definition 3. We shall thus also have to verify that the candidate solution satisfies QVI and this will be done in sub-section 5.2; on the other hand the QVI property turns out to be useful in determining the optimal strategic boundaries as will be shown in the next Section 4.

4 Computing the optimal value function and strategic boundaries

We first study the solution $\Phi^j(t,x,m)$, $j=f,p$ of the HJB equation (21) that will give us (see (15) and Assumption 2) the optimal value function in $\mathcal{C}^j$. This will be the subject of subsection 4.1. In subsection 4.2 we shall then determine the strategic boundaries so that the value function $V^j(t,x,m)$ in (15) is a QVI solution. Finally, in subsection 4.3 we derive some estimates that will be crucial in the verification results in section 5 and will also guarantee that $V_j^{(u,\mathcal{T},\xi)^j}(0,x,m)$ is finite for all admissible strategies $(u,\mathcal{T},\xi)^j$.

4.1 Solution of the HJB equation

Recall from sub-section 3.1 that, for the optimal continuous control as given in (19), the function $\Phi^j(t,x,m)$ has to satisfy, for each $j=f,p$, the Cauchy problem in (21). On the other hand, by Assumption 2 it follows that the coefficients in (25) have to satisfy the system of equations in (28) below that is obtained from computing the various derivatives of $\Phi^j(\cdot)$ in the form of (25), plugging them into (21) and putting equal to zero the coefficients of the resulting 2nd order polynomial in (x,m). We have in fact

$$\begin{cases} \dot p^j_{xx}(t) - \frac{K^2}{k}(p^j_{xx}(t))^2 - \lambda p^j_{xx}(t) + 1 = 0, & p^j_{xx}(T)=\ell, \\ \dot p^j_{xm}(t) + \left(A-\lambda-\frac{K^2}{k}p^j_{xx}(t)\right)p^j_{xm}(t) + 2p^j_{xx}(t) = 0, & p^j_{xm}(T)=0, \\ \dot p^j_{mm}(t) + (2A-\lambda)p^j_{mm}(t) + p^j_{xm}(t) - \frac{K^2}{4k}(p^j_{xm}(t))^2 = 0, & p^j_{mm}(T)=0, \\ \dot p^j_{x}(t) - \left(\lambda+\frac{K^2}{k}p^j_{xx}(t)\right)p^j_{x}(t) + ap^j_{xm}(t) - 2\rho = 0, & p^j_{y}(T)=-2\ell\rho, \\ \dot p^j_{m}(t) + (A-\lambda)\,p^j_{m}(t) + p^j_{x}(t) + 2ap^j_{mm}(t) - \frac{K^2}{2k}p^j_{xm}(t)p^j_{x}(t) = 0, & p^j_{m}(T)=0, \\ \dot p^j(t) - \lambda p^j(t) + ap^j_m(t) + \sigma^2 p^j_{xx}(t) - \frac{K^2}{4k}(p^j_x(t))^2 + \Sigma^j_1(t)p^j_{xm}(t) + 2\Sigma^j_2(t)p^j_{mm}(t) + \rho^2 = 0, & p^j(T)=\ell\rho^2. \end{cases} \tag{28}$$

These are first order linear ODEs, for which an analytic solution may be obtained as described in Section 4.1 of [2]. However, for our purposes below and, in particular, for the numerical results it suffices to obtain a numerical solution after a discretization of the various equations. We shall nevertheless recall here from Section 4.1 of [2] the solution of the first, a Riccati equation, namely

$$p^j_{xx}(t) = \frac{k}{K^2}\left(\frac{-\lambda-\Delta}{2} + \frac{\Delta}{C_1 e^{\Delta t}+1}\right), \tag{29}$$

where $\Delta := \sqrt{\lambda^2 + 4\frac{K^2}{k}}$ and $C_1 := \left(\frac{2\Delta}{2\frac{K^2}{k}\ell+\lambda+\Delta} - 1\right)e^{-\Delta T}$. For the given solution one can obtain a lower bound that is derived in the next lemma and that will be useful later on.

Lemma 2. One has

$$p^j_{xx}(t) > \min\left[\ell, \frac{1}{\Delta}\right] > 0\,, \quad j=f,p. \tag{30}$$

where ℓ corresponds to the terminal cost $h(x)=\ell\,(x-\rho)^2$ and Δ is as in (29).

Proof. First of all notice that $C_1>-1$. If $C_1\geq 0$ then $p^j_{xx}(t)$ is non-increasing in t so that $p^j_{xx}(t)\geq p^j_{xx}(T)=\ell$. If $C_1\in(-1,0]$, then $p^j_{xx}(t)$ is non-decreasing so that, for $t\in[0,T]$,

$$p^j_{xx}(t) \geq p^j_{xx}(0) = \frac{k}{K^2}\left(\frac{-\lambda-\Delta}{2}+\frac{\Delta}{C_1+1}\right) = \frac{k}{K^2}\,\frac{\Delta-\lambda}{2} = \frac{k}{K^2}\,\frac{\Delta^2-\lambda^2}{2\,(\Delta+\lambda)} = \frac{2}{\Delta+\lambda} \geq \frac{2}{2\,\Delta} = \frac{1}{\Delta}$$

where we have used the definition of Δ, which also implies that $\Delta > \lambda$. Combining the two cases of positive and negative C_1 we obtain the result. $\square$

4.2 Determining the strategic boundaries

We have already seen that, in order that the optimal value function $V^j(t,x,m)$ is a QVI solution, in the intervention region $\mathcal{I}^j$ it has to satisfy, for $j = f, p$, point ii) of Definition 3 as an equality which implies that

$$V^j(t,x,m) = \inf\{V^j(t,x+\eta,m) + g(\eta),\ \ \eta \in \mathbb{R}\}. \tag{31}$$

On the other hand, from (15) and (22) as well as (23) we have that, for $x \leq a^j(t,m)$, the inf in (31) is obtained for $x + \eta = \alpha^j(t,m)$, while for $x \geq b^j(t,m)$ it is obtained for $x + \eta = \beta^j(t,m)$. This allows us to determine the optimal strategic boundaries as shown in the next two lemmas.

Lemma 3. For any $t > 0$ and $m \in \mathbb{R}$, the strategic boundaries $\alpha^j(t,m)$ and $\beta^j(t,m)$ have to satisfy, for $j = f, p$,

$$V_x^j(t,x,m) = \begin{cases} -c & \text{for} \quad x \leq a^j(t,m) \\ d & \text{for} \quad x \geq b^j(t,m) \end{cases} \tag{32}$$

and, given the quadratic Ansatz for $\Phi^j(t,x,m)$ in (25) of Assumption 2, they are explicitly given by

$$\alpha^j(t,m) = -\frac{c + p_x^j(t) + m\, p_{xm}^j(t)}{2p_{xx}^j(t)}, \quad \beta^j(t,m) = \frac{d - p_x^j(t) - m\, p_{xm}^j(t)}{2p_{xx}^j(t)}, \quad j = f, p. \tag{33}$$

Proof. As mentioned before the statement of the lemma, for $x \leq a^j(t,m)$, the minimal value of $x+\eta$ is obtained for $x + \eta = \alpha^j(t,m)$, while for $x \geq b^j(t,m)$ it is obtained for $x + \eta = \beta^j(t,m)$. Both lie in the interior of the continuation region $\mathcal{C}^j$, where $V^j(t,x,m) = \Phi^j(t,x,m)$ and is thus differentiable. A necessary condition is therefore that, for all $x \leq a^j(t,m)$,

$$0 = \frac{\partial}{\partial\eta}\left[V^j(t,x+\eta,m) + g(\eta)\right]_{|\eta = \alpha^j(t,m) - x} = V_x^j(t, \alpha^j(t,m), m) + c,$$

where we have taken into account the structure of the cost function $g(\eta)$ in (11). Analogously for $\beta^j(t,m)$. This and the structure of $V^j(t,x,m)$ in (15) then gives us (32). Form here and, again, from the structure of $V^j(t,x,m)$ as well as from the quadratic Ansatz in Assumption 2 one then obtains immediately also (33). $\square$

Lemma 4. The optimal boundary functions $a^j(t,m)$ and $b^j(t,m)$ are, for $j = f, p$, given by

$$\begin{aligned} a^j(t,m) &= -\tfrac{c + p_x^i(t) + m\, p_{xm}^j(t) + 2\sqrt{C p_{xx}^j(t)}}{2p_{xx}^j(t)} = \alpha^j(t,m) - \sqrt{\tfrac{C}{p_{xx}^j(t)}}, \\ b^j(t,m) &= \tfrac{d - p_x^j(t) + m\, p_{xm}^j(t) + 2\sqrt{D p_{xx}^j(t)}}{2p_{xx}^j(t)} = \beta^j(t,m) + \sqrt{\tfrac{D}{p_{xx}^j(t)}}. \end{aligned} \tag{34}$$

Proof. The structure given by (15) to $V^j(t,x,m)$ leads, for $j = f, p$, to

$$\begin{cases} \Phi^j(t, a^j(t,m), m) &= \Phi^j(t, \alpha^j(t,m), m) + C + c(\alpha^j(t,m) - a^j(t,m)), \\ \Phi^j(t, b^j(t,m), m) &= \Phi^j(t, \beta^j(t,m), m) + D + d(b^j(t,m) - \beta^j(t,m)). \end{cases} \tag{35}$$

Since the case of $b^j(t,m)$ is completely analogous to that of $a^j(t,m)$, we present the details only for the latter. For convenience we also put

$$G^j(t,m) := p_x^j(t) + m\, p_{xm}^j(t).$$

Due to the quadratic structure in (25) of $\Phi^j(t,x,m)$, the first relation in (35) then implies that

$$(a^j(t,m))^2\, p^j_{xx}(t) + a^j(t,m)\, G^j(t,m) + a^j(t,m)\, c$$
$$-(\alpha^j(t,m))^2\, p^j_{xx}(t) - \alpha^j(t,m)\, G^j(t,m) - \alpha^j(t,m)\, c - C = 0.$$

For given $\alpha^j(t,m)$ this then leads to the following two possible expressions for $a^j(t,m)$ that are the solutions of the above quadratic equation

$$a^j_{1,2}(t,m) = \frac{-(G^j(t,m)+c) \pm \sqrt{[(G^j(t,m)+c) + 2\alpha^j(t,m) p^j_{xx}(t)]^2 + 4C\, p^j_{xx}(t)}}{2p^j_{xx}(t)}.$$

Substituting for $\alpha^j(t,m)$ its expression from (33), we end up with

$$a^j_{1,2}(t,m) = \frac{-(G^j(t,m)+c) \pm 2\sqrt{C\, p^j_{xx}(t)}}{2p^j_{xx}(t)} = \alpha^j(t,m) \pm \sqrt{\frac{C}{p^j_{xx}(t)}}, \tag{36}$$

which is well defined since C and (see (30)) $p^j_{xx}(t)$ are positive. To conform with the requirement that $a^j(t,m) < \alpha^j(t,m)$, one has to choose the minus sign on the right of (36). □

Remark 5. The relations (33) and (34) determine completely the optimal strategic boundaries, which are continuous as assumed in (14). Notice also that the analytic form of the strategic boundaries is a consequence of the Assumption 2 of a quadratic Ansatz for the solution of the HJB equation (21).

4.3 Resulting estimates

Having obtained the solution of the HJB equation and the analytic expressions for the strategic boundaries, we shall now derive in the form of lemmas some estimates that will be important for the verification result in the next Section 5. In the lemmas below we use the fact that all the coefficients in the quadratic expression (25) as well as the solution Γ_t of (6) are continuous and thus bounded on $[0,T]$. For simplicity we shall also write generically x_t and u_t instead of x^j_t and u^j_t.

Lemma 5. For the process m^j_t satisfying (5) for $j=f$ and (8) for $j=p$ we have

$$E\left\{\int_0^T e^{-\lambda t} \left|m^j_t\right|^k dt\right\} \le E\left\{\int_0^T \left|m^j_t\right|^k dt\right\} \le C_k\left(1 + E\left\{\left|m^j_0\right|^k\right\}\right) < \infty$$

for all $\lambda > 0$, all positive integers k for which for the initial condition $m^j_0 = m_0$ one has $E\left\{|m^j_0|^k\right\} < \infty$ and where C_k is a constant depending on k and T as well as on the constant A in equations (5) and (8).

Proof. Since the drift in equations (5) and (8) has linear growth, from Theorem 4.2 in Chapter 5 of [6] we have

$$E\left\{\left|m^j_t\right|^k\right\} \le \bar{C}_k\left(1 + E\left\{\left|m^j_0\right|^k\right\}\right),$$

where $\bar{C}_k$ depends on k, T and A. This implies that $E\left\{\int_0^T \left|m^j_t\right|^k dt\right\}$ is finite for each k for which $E\left\{\left|m^j_0\right|^k\right\} < \infty$. Since $|m^j_t| > 0$, *a.s.*, we can apply Fubini's Theorem concluding that

$$E\left\{\int_0^T e^{-\lambda t} \left|m^j_t\right|^k dt\right\} \le E\left\{\int_0^T \left|m^j_t\right|^k dt\right\} \le T\bar{C}_k\left(1 + E\left\{\left|m^j_0\right|^k\right\}\right)$$

and thus obtaining the statement of the lemma for $C_k := T\,\bar{C}_k$. □

Below we shall assume that Lemma 5 holds at least for $k \leq 4$. It is in fact reasonable to assume that $\hat{m}_0 = m_0$ has finite moments up to at least $k = 4$.

Lemma 6. We have $E\left\{\int_0^T x_t^2 dt\right\} < \infty$.

Proof. By (34), (30) and the fact that $p_x^j(t)$, $p_{xm}^j(t)$ are continuous in t, we have for all $t \in [0,T]$ and $j \in \{f,p\}$

$$|a^j(t, m_t^j)| \leq \left|\frac{c + p_x^j(t)}{2p_{xx}^j(t)}\right| + |m_t^j| \left|\frac{p_{xm}^j(t)}{2p_{xx}^j(t)}\right| + \sqrt{\frac{C}{p_{xx}^j(t)}} \leq \bar{K}|m_t^j| + \bar{H} \quad a.s.$$

for some positive constants $\bar{K}$ and $\bar{H}$ and, analogously, for $|b^j(t, m_t^j)|$. Since $x_t \in \left[a^j(t, m_t^j), b^j(t, m_t^j)\right]$, it then also follows that

$$|x_t| \leq \bar{K}\,|m_t^j| + \bar{H} \quad a.s. \tag{37}$$

so that, for suitable and positive constants K_1, K_2, H_1 one has $E\{x_t^2\} \leq K_2 E\{(m_t^j)^2\} + K_1 E\{|m_t^j|\} + H_1$. The result then follows from Lemma 5 using again Fubini's theorem. □

Lemma 7. For the optimal control $\hat{u}_t$ given in (19), we have $E\left\{\int_0^T \hat{u}_t^2 dt\right\} < \infty$.

Proof. The continuous control $\hat{u}_t$ is acting only in the continuation region $\mathcal{C}^j$, $j = f, p$, where

$$|\hat{u}_t| = \left|-\frac{K}{2k}\,\Phi_x^j\left(t, \hat{x}_t^j, m_t^j\right)\right| = \frac{K}{2k}\left|2\hat{x}_t^j p_{xx}^j(t) + m_t^j p_{xm}^j(t) + p_x^j(t)\right| \leq K_3|\hat{x}_t^j| + K_4|m_t^j| + K_0 \quad a.s.$$

for $K_3,\ K_4 > 0$, so that

$$E\left\{\int_0^T \hat{u}_t^2 dt\right\} \leq 3K_3^2 E\left\{\int_0^T (\hat{x}_t^j)^2 dt\right\} + 3K_4^2 E\left\{\int_0^T |\hat{m}_t|^2\, dt\right\} + TK_0^2$$

which is bounded by Lemmas 5 and 6 (in this latter lemma for $x_t = x_t^j$ with $j = f, p$). □

Notice that by Lemmas 6 and 7 we also have

$$E\left\{\int_0^T e^{-\lambda t} x_t^2 dt\right\} < \infty\,; \quad E\left\{\int_0^T e^{-\lambda t} \hat{u}_t^2 dt\right\} < \infty.$$

Remark 6. From the definitions of the cost function in (10) we have

$$|f(x,u)| \leq 2(x^2 + \rho^2) + ku^2\,, \quad |h(x)| \leq 2\ell(x^2 + \rho^2)$$

Furthermore, since ξ_n is either equal to $\alpha^j(\tau_n^j, m_{\tau_n^j}^j) - a^j(\tau_n^j, m_{\tau_n^j}^j)$ or $b^j(\tau_n^j, m_{\tau_n^j}^j) - \beta^j(\tau_n^j, m_{\tau_n^j}^j)$, from (11) and (34) and using the inequality (30) we obtain

$$|g(\xi)| \leq \max[C,D] + \max[c,d]\,|\xi| \leq \max[C,D] + \max[c,d]\sqrt{\frac{\max[C,D]}{\min\left[\ell, \frac{1}{\Delta}\right]}}.$$

Finally, from Lemma 8 below we have $E\left\{\sum_{n=1}^{\infty} e^{-\lambda\tau_n}\mathbf{1}_{\{\tau_n \leq T\}}\right\} < \infty$. Lemmas 6 and 7 then imply that for all controls in the specific class, which in Definition 2 we have denoted by $\mathcal{A}^j$, we have that $V_j^{(u,\mathcal{T},\xi)^j}(t,x,m)$ in (9) is finite. The class $\mathcal{A}^j$ consists therefore indeed of admissible controls according to the Definition 2.

Lemma 8. We have $E\left\{\sum_{n=1}^{\infty} e^{-\lambda\tau_n^j}\mathbf{1}_{\{\tau_n^j\le T\}}\right\}<\infty$ for $j=f,p$.

Proof. Recall that the generic $n-$th impulse time τ_n^j was defined in (22) as

$$\tau_n^j=\inf\left\{t>\tau_{n-1}^j \mid x_t^j\notin\left(a^j(t,m_t^j),b^j(t,m_t^j)\right)\right\},\quad(\tau_0^j=0)$$

where the optimal strategic boundaries $a^j(t,m_t^j),b^j(t,m_t^j)$ are as given in (33) of Lemma 2 and can therefore be represented in the form

$$a^j(t,m_t^j)=\underline{K}^j(t)m_t^j+\underline{H}^j(t)\quad;\quad b^j(t,m_t^j)=\bar{K}^j(t)m_t^j+\bar{H}^j(t)$$

for suitable functions $\underline{K}^j(t),\underline{H}^j(t),\bar{K}^j(t),\bar{H}^j(t)$. We can then rewrite τ_n^j as

$$\tau_n^j=\inf\left\{t>\tau_{n-1}^j \mid x_t^j\le\underline{K}^j(t)m_t^j+\underline{H}^j(t)\text{ or }x_t^j\ge\bar{K}^j(t)m_t^j+\bar{H}^j(t)\right\}$$

Consider next, for $j=f,p$, the stopping time

$$\tau^j=\inf\left\{s>0 \mid x_s^j\le\underline{K}^j(s)m_s^j+\underline{H}^j(s)\text{ or }x_s^j\ge\bar{K}^j(s)m_s^j+\bar{H}^j(s)\right\}$$

Then, since x_t^j and m_t^j are diffusion processes, we have $\gamma_{t,T}^j:=P\{t\le\tau^j\le T\}<1$ for all $t\in[0,T]$. And, $\gamma_{t_2,T}^j\le\gamma_{t_1,T}^j\le\gamma_{0,T}^j<1$ for $t_1<t_2$.

Returning to the sequence of impulse times $0<\tau_1^j<\tau_2^j<\cdots<\tau_n^j<\cdots$, we then have $P\{\tau_1^j\le T\}=\gamma_{0,T}^j<1$ and

$$P\{\tau_2^j\le T\}=E\left\{\mathbf{1}_{\{\tau_2^j\le T\}}\right\}=E\left\{E\left\{\mathbf{1}_{\{\tau_2^j\le T\}}\mid\tau_1^j\right\}\right\}=E\left\{E\left\{\mathbf{1}_{\{\tau_1^j<\tau_2^j\le T\}}\mid\tau_1^j\right\}\right\}$$

$$=E\left\{\gamma_{\tau_1^j,T}^j\mathbf{1}_{\{\tau_1^j\le T\}}\right\}\le\gamma_{0,T}^jP\{\tau_1^j\le T\}=\left(\gamma_{0,T}^j\right)^2$$

Proceeding by induction, assume that $P\{\tau_{n-1}^j\le T\}\le\left(\gamma_{0,T}^j\right)^{n-1}$. We then have

$$P\{\tau_n^j\le T\}=E\left\{E\left\{\mathbf{1}_{\{\tau_n^j\le T\}}\mid\tau_{n-1}^j\right\}\right\}=E\left\{E\left\{\mathbf{1}_{\{\tau_{n-1}^j<\tau_n^j\le T\}}\mid\tau_{n-1}^j\right\}\right\}$$

$$=E\left\{\gamma_{\tau_{n-1}^j,T}^j\mathbf{1}_{\{\tau_{n-1}^j\le T\}}\right\}\le\gamma_{0,T}^j\left(\gamma_{0,T}^j\right)^{n-1}=\left(\gamma_{0,T}^j\right)^n$$

so that, for all positive integers n, we have $P\{\tau_n^j\le T\}\le\left(\gamma_{0,T}^j\right)^n$ with $\gamma_{0,T}^j<1$. It follows that

$$\lim_{n\to\infty}P\{\tau_n^j\le T\}=\lim_{n\to\infty}\left(\gamma_{0,T}^j\right)^n=0$$

and that

$$E\left\{\sum_{n=1}^{\infty}e^{-\lambda\tau_n^j}\mathbf{1}_{\{\tau_n^j\le T\}}\right\}\le E\left\{\sum_{n=1}^{\infty}\mathbf{1}_{\{\tau_n^j\le T\}}\right\}=\sum_{n=1}^{\infty}P\{\tau_n^j\le T\}<\sum_{n=0}^{\infty}\left(\gamma_{0,T}^j\right)^n=\frac{1}{1-\gamma_{0,T}^j}<\infty$$

thus concluding the proof of the lemma. □

5 Verification Theorem and optimality of the solution

In this section we present a verification result proving that the solution described in sections 3 and 4 is indeed optimal for our problem. In subsection 5.1 we present some preliminaries and show a first verification result assuming that the candidate solution is a QVI solution in the weak sense of Definition 3. In subsection 5.2 we then present the main verification theorem showing that the candidate solution is indeed such a QVI solution.

5.1 A preliminary verification result

We start with a couple of preliminary lemmas.

Lemma 9. For $j = f, p$, we have $E\left\{\int_0^T \left(V_x^j(t, x_t^j, m_t^j)\right)^2 dt\right\} < \infty$.

Proof. By the boundedness, over $[0, T]$, of $p_{xx}^j(t)$, $p_{xm}^j(t)$, $p_x^j(t)$ and the fact that in $\mathcal{C}^j$ one has $V_x^j(\cdot) = \Phi_x^j(\cdot)$, from (25) one first has $|V_x(t, x_t, m_t^j)| \le c_1|x_t| + c_2|m_t^j| + c_3,\ c_i > 0$ so that, using also (37),

$$\left(V_x^j(t, x_t, m_t^j)\right)^2 \le (c_1|x_t| + c_2|m_t^j| + c_3)^2 \le \left[c_1(\bar{K}\,|m_t^j| + \bar{H}) + c_2|m_t^j| + c_3\right]^2 \le L(m_t^j)^2 + M|m_t^j| + N$$

for suitable constants $L, M, N > 0$. The proof can now be concluded by using Lemma 5 □

Lemma 10. For $j = f, p$ we have

$$E\left\{\int_0^T \left(\bar{\sigma}_t^j V_m^j(t, x_t, m_t^j)\right)^2 dt\right\} < \infty,$$

where

$$E\left\{\int_0^T \left(\bar{\sigma}_t^j V_m^j(t, x_t, m_t^j)\right)^2 dt\right\} = \begin{cases} E\left\{\int_0^T \left(V_m^f(t, x_t, m_t^f)\right)^2 (\Lambda_1^2 + \Lambda_2^2)\, dt\right\} & \text{for} \quad j = f, \\ E\left\{\int_0^T \left(V_m^p(t, x_t, m_t^p)\left(\Lambda_2 + \dfrac{\Gamma_t}{2}\right)\right)^2 dt\right\} & \text{for} \quad j = p. \end{cases}$$

Proof. Analogously to the proof of the previous Lemma 9, since in $\mathcal{C}^j$ we have $V_m^j(\cdot) = \Phi_m^j(\cdot)$, from (25) it follows that $|V_m^j(t, x_t, m_t^j)| \le \bar{d}_1|x_t| + \bar{d}_2|m_t^j| + \bar{d}_3$, $\quad d_i > 0\ (i = 1, 2, 3)$ with (see (37)) $|x_t| \le \bar{K}\,|m_t^j| + \bar{H}$. Furthermore, since Γ_t is solution of a Riccati equation and therefore continuous in $[0, T]$, we have $\Gamma_t \le \Gamma$ for $t \in [0, T]$ and with $\Gamma > 0$. One can thus conclude that there exist positive constants d_1, d_2, d_3 such that

$$E\left\{\int_0^T \left(\bar{\sigma}_t^j V_m^j(t, x_t, m_t^j)\right)^2 dt\right\} \le E\left\{d_1 \int_0^T |m_t^j|^2 dt + d_2 \int_0^T |m_t^j| dt\right\} + d_3\,, \quad j = f, p,$$

where the right hand is finite by Lemma 5. □

We come now to the preliminary verification result, namely

Proposition 1. For $j = f, p$, let $\Phi^j(t, x, m)$ be a solution of the HJB equation (21) and let $(u, \mathcal{T}, \xi)^j$ be a mixed classical-impulse control according to (19) as well as (22), (23), where $a^j(\cdot), \alpha^j(\cdot), \beta^j(\cdot), b_j(\cdot)$ satisfy (33) and (34). Then $V^j(t, x, m)$ defined according to (15) is the optimal value function for our problem and $(u, \mathcal{T}, \xi)^j$ is an optimal control provided $V^j(t, x, m)$ is a weak QVI solution according to Definition 3.

Proof. Given an arbitrary integer $n > 0$ and $t \in [0, T]$, we first derive, for $j = f, p$, an expression for $e^{-\lambda(t\wedge\tau_n^j)} V^j(t \wedge \tau_n^j, x_{(t\wedge\tau_n^j)+}, m^j_{(t\wedge\tau_n^j)+}) - V(0, x, m_0^j)$ for which, analogously to the proofs of the classical verification theorems, we apply the Itô formula to the function $e^{-\lambda t} V^j(t, x_t^j, m_t^j)$, but only between two successive impulse shifts, thereby obtaining

$$\begin{aligned} & V^j(t \wedge \tau_n^j, x_{(t\wedge\tau_n^j)+}, m^j_{(t\wedge\tau_n^j)}) - V(0, x, m_0^j) \\ & = \sum_{i=1}^n \left[\int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} \left(V_t^j + \mathcal{L}_j^{u_s} V^j\right)(s, x_s^j, m_s^j)\, ds + \sigma \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} V_x^j(s, x_s^j, m_s^j)\, dw_s^j \right. \\ & \left. + \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} \bar{\sigma}_s^j V_m(s, x_s^j, m_s^j)\right] + \sum_{i=1}^n 1_{\{\tau_i^j \le t\}} \left[V^j(\tau_i^j, x_{\tau_i^j+}, m^j_{\tau_i^j}) - V(\tau_i^j, x^j_{\tau_i^j}, m^j_{\tau_i^j})\right], \end{aligned} \tag{38}$$

where

$$w_t^j = \begin{cases} W_t & \text{for} \quad j=f \\ \nu_t & \text{for} \quad j=p \end{cases}, \quad b_t^j = \begin{cases} B_t & \text{for} \quad j=f \\ \nu_t & \text{for} \quad j=p \end{cases}, \quad \bar{\sigma}_t^j = \begin{cases} \Lambda & \text{for} \quad j=f \\ \Lambda_2 + \frac{\Gamma_t}{\sigma} & \text{for} \quad j=p \end{cases}$$

with B_t and W_t the original Wiener processes in (5) and ν_t the innovations Wiener process in (7). Next we use the assumption that $V^j(\cdot)$ is a QVI-solution. Inequality $i)$ in Definition 3 implies

$$\left(V_t^j + \mathcal{L}_j^{u_t} V^j\right)(t, x_t^j, m_t^j) \geq -f(x_t^j, u_t^j) \quad \text{on} \quad (a^j(t, m^j), b^j(t, m^j))$$

Furthermore, inequality $ii)$ in Definition 3 implies

$$V^j(\tau_i^j, x^j_{\tau_i^j+}, m^j_{\tau_i^j+}) - V(\tau_i^j, x_{\tau_i^j}, m^j_{\tau_i^j}) = M\, V^j(\tau_i^j, x^j_{\tau_i^j}, m^j_{\tau_i^j})) - V^j(\tau_i^j, x^j_{\tau_i^j}, m^j_{\tau_i^j})) - g(\xi_i^j) \geq -g(\xi_i^j) \quad \forall i \in \mathbb{N}.$$

Therefore, for each $t \in [0,T]$ and $n \in \mathbb{N}$ we obtain

$$\begin{aligned} &e^{-\lambda(t\wedge\tau_n^j)} V^j(t\wedge\tau_n^j, x^j_{(t\wedge\tau_n^j)+}, m^j_{(t\wedge\tau_n^j)+}) - V(0, x, m_0^j) \\ &\geq -\sum_{i=1}^{n} \Bigg[\int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} e^{-\lambda s} f(x_s^j, u_s^j)\,\mathrm{d}s + \sigma \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} V_x^j(s, x_s^j, m_s^j)\, dw_s^j \\ &\quad + \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} \bar{\sigma}_s^j V_m^j(s, x_s^j, m_s^j)\, db_s^j \Bigg] - \sum_{i=1}^{n} \mathbf{1}_{\{\{\tau_i^j \leq t\}\}} e^{-\lambda\tau_i^j} g(\xi_i^j). \end{aligned}$$

This inequality is an equality for the control specified in the statement. From Lemma 8 we deduce that

$$e^{-\lambda(t\wedge\tau_n^j)} V^j(t\wedge\tau_n^j, x_{(t\wedge\tau_n^j)+}, m^j_{(t\wedge\tau_n^j)+}) \xrightarrow{n\to\infty} e^{-\lambda t} V^j(t, x_{t+}^j, m_{t+}^j) \qquad \text{a.s..}$$

Taking expectations, we obtain

$$\begin{aligned} &V^j(0, x, m_0^j) - \mathbb{E}\left\{e^{-\lambda t} V^j(t, x_{t+}^j, m_{t+}^j)\right\} \leq \mathbb{E}\Bigg\{ \sum_{i=1}^{\infty} \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} e^{-\lambda s} f(x_s^j, u_s^j)\,\mathrm{d}s \\ &+\sigma \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} e^{-\lambda s} V_x^j(s, x_s^j, m_s^j)\, dw_s^j + \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} e^{-\lambda s} \bar{\sigma}_s^j V_m^j(s, x_s^j, m_s^j)\, db_s^j - \sum_{i=1}^{n} \mathbf{1}_{\{\{\tau_i^j \leq t\}\}} e^{-\lambda\tau_i^j} g(\xi_i^j) \Bigg\}. \end{aligned}$$

with equality for the control described in the statement. Lemmas 9 and 10 imply that both $e^{-\lambda t} V_x^j(t, x_t^j, m_t^j)$ as well as $e^{-\lambda t} \bar{\sigma}_t^j V_m^j(t, x_t^j, m_t^j)$ are in $\mathcal{L}^2([0,T] \times \Omega)$. Therefore

$$E\left\{ \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} e^{-\lambda s} V_x^j(s, x_s^j, m_s^j)\, dw_s^j \right\} = E\left\{ \int_{t\wedge\tau_{i-1}^j}^{t\wedge\tau_i^j} e^{-\lambda s} \bar{\sigma}_s^j V_m^j(s, x_s^j, m_s^j)\, db_s^j \right\} = 0$$

Consequently

$$V^j(0, x, m_0^j) - \mathbb{E}\left\{e^{-\lambda t} V^j(t, x_{t+}^j, m_{t+}^j)\right\} \leq \mathbb{E}\left\{ \int_0^t e^{-\lambda s} f(x_s^j, u_s^j)\,\mathrm{d}s + \sum_{i=1}^{\infty} \mathbf{1}_{\{\{\tau_i^j \leq t\}\}} e^{-\lambda\tau_i^j} g(\xi_i^j) \right\},$$

with equality for the control described in the statement. Finally, substituting T for t and recalling that $V^j(T, x, m) = h(x)$, we get

$$V^j(0, x, m_0^j) \leq \inf \mathbb{E}\left\{ \int_0^T e^{-\lambda s} f(x_s^j, u_s^j)\,\mathrm{d}s + e^{-\lambda T} h(x_T^j) + \sum_{i=1}^{\infty} \mathbf{1}_{\{\{\tau_i^j \leq T\}\}} e^{-\lambda\tau_i^j} g(\xi_i^j) \right\} = \mathcal{J}_j(0, x, m_0^j),$$

with $\mathcal{J}_j(0, x, m_0^j)$ as in (12) and where one has equality for the control described in the statement. The conclusion then follows. □

5.2 Main verification result

Before mentioning our main theorem, we introduce some more notations: based on (33) let

$$\alpha^j(t,m) = -\frac{c + p_x^j(t) + m p_{xm}^j(t)}{2p_{xx}^j(t)} =: \alpha_1^j(t) + \alpha_2^j(t)m, \quad \beta^j(t,m) = \frac{d - p_x^j(t) - m p_{xm}^j(t)}{2p_{xx}^j(t)} =: \beta_1^j(t) + \beta_2^j(t)m,$$

We can now state and prove our main verification result, namely

Theorem 1. Let, for $j = f, p$, the function $V^j(t,x,m)$ satisfy (15). Let the continuous and differentiable functions $a^j(t,m)$, $\alpha^j(t,m)$, $\beta^j(t,m)$, $b^j(t,m)$ be given by (33) and (34). Let the control $(u, \mathcal{T}, \xi)$ be given by (19) as well as (22), (23). Then $V^j(t,x,m^j)$ is, for $j = f,p$, the optimal value function of our problem, namely it is optimal among the value functions that are quadratic in the continuation region and such that (see (12) and (13))

$$V^j(0,x,m^j) = \inf\left\{V_j^{(u,\mathcal{T},\xi)^j}(0,x,m^j) : (u,\mathcal{T},\xi) \in \mathcal{A}^j\right\}. \tag{39}$$

Furthermore, the above strategy is optimal in the sense that it achieves the infimum in (39).

Proof. (In the proof we shall simply write m for m^j). Given Proposition 1, we have to show that $V^j(t,x,m)$, $j = f,p$, satisfies the QVI conditions i.e., see (27) in Definition 3, that for all $t \in [0,T]$, $m \in \mathbb{R}$ and all $x \in (a^j(t,m), b^j(t,m))$,

$$V_t^j(t,x,m) + \inf_u \left[\mathcal{L}^u V^j(t,x,m) + f(x,u)\right] \geq 0 \tag{40}$$

as well as

$$MV^j(t,x,m) - V^j(t,x,m) \geq 0 \quad \forall (t,x,m) \tag{41}$$

We start from (40) recalling that, for $x \in \mathcal{C}^j = (a^j(t,m), b^j(t,m))$ we have that $V^j(t,x,m) = \Phi^j(t,x,m)$ and so it satisfies the HJB equation (21), namely

$$\begin{aligned} V_t^j(t,x,m) + \inf_u \left[\mathcal{L}^u V^j(t,x,m) + f(x,u)\right] &= V_t^j(\cdot) + \tfrac{\sigma^2}{2} V_{xx}(\cdot) + \Sigma_1^j V_{xm}^j(\cdot) + \Sigma_2^j V_{mm}^j(\cdot) \\ -\tfrac{K^2}{4k}(V_x^j)^2(\cdot) + m\, V_x^j(\cdot) &+ (A\,m + a)\, V_m^j(\cdot) - \lambda\, V^j(\cdot) + (x-\rho)^2 = 0 \end{aligned} \tag{42}$$

where $(\cdot)$ stands for (t,x,m).

It remains to show that also (41) holds. To this effect notice that, by the convexity of $\Phi^j(t,x,m)$ in x for each (t,m) and and by (32) in Lemma 3, we have that

$$\Phi_x^j(t,x,m) \begin{cases} \leq -c & \text{for } x \leq \alpha^j(t,m) \\ \in (-c,d) & \text{for } x \in (\alpha^j(t,m), \beta^j(t,m)) \\ \geq d & \text{for } x \geq \beta^j(t,m). \end{cases} \tag{43}$$

Given that for x in the continuation region $\mathcal{C}^j = (a^j(t,m), b^j(t,m))$ we have $V^j(t,x,m) = \Phi^j(t,x,m)$, this also implies that, for the minimal cost operator $MV(t,x,y)$ (see (26)), we have

$$MV^j(t,x,m) = \begin{cases} \Phi^j(t,\alpha^j(t,m),m) + C + c(\alpha^j(t,m) - x) & \text{if } x \leq \alpha^j(t,m) \\ \Phi^j(t,x,m) + \min(C,D) & \text{if } x \in (\alpha^j(t,m), b^j(t,m)) \\ \Phi^j(t,\beta^j(t,m),m) + D + d(x - \beta^j(t,m)) & \text{if } x \geq \beta^j(t,m) \end{cases} \tag{44}$$

Taking then into account (15) and (44), we immediately have that (41) holds outside of the intervals $(a^j(t,m), \alpha^j(t,m))$ and $(\beta^j(t,m), b^j(t,m))$. Concerning these two intervals, we consider here just the case of $(a^j(t,m), \alpha^j(t,m))$ since the case of $(\beta^j(t,m), b^j(t,m))$ is completely analogous.

We first recall that on $(a^j(t,m), \alpha^j(t,m))$, which is part of the continuation region, the function $V^j(t,x,m)$ and the operator $MV^j(t,x,m)$ are, for $j = f,p$, given by (see (15), (25))

$$V^j(t,x,m) = x^2 p_{xx}^j(t) + m^2 p_{mm}^j(t) + xm\, p_{xm}^j(t) + x\, p_x^j(t) + m\, p_m^j(t) + p^j(t),$$

$$MV^j(t,x,m) = (\alpha^j(t,m))^2 p^j_{xx}(t) + m^2 p^j_{mm}(t) + \alpha^j(t,m) m\, p^j_{xm}(t) + \alpha^j(t,m)\, p^j_x(t) \\ + m\, p^j_m(t) + p^j(t) + C + c(\alpha^j(t,m) - x).$$

It follows that

$$V^j(t,x,m) - MV^j(t,x,m) = (x^2 - (\alpha^j(t,m))^2)\, p^j_{xx}(t) + (m\, p^j_{xm}(t) + p^j_x(t) + c)(x - \alpha^j(t,m)) - C$$

$$= (x - \alpha^j(t,m))^2 p^j_{xx}(t) + (2\alpha^j(t,m)\, p^j_{xx}(t) + m\, p^j_{xm}(t) + p^j_x(t) + c)(x - \alpha^j(t,m)) - C$$

$$:= (z^j)^2 p^j_{xx}(t) + z^j\, (2\alpha^j(t,m)\, p^j_{xx}(t) + m\, p^j_{xm}(t) + p^j_x(t) + c) - C$$

where we have put $z^j := x - \alpha^j(t,m)$. Taking now into account the expressions for $\alpha^j(t,m)$ and $a^j(t,m)$ in (33) and (34), by which $a^j(t,m) - \alpha^j(t,m) = -\sqrt{\frac{C}{p^j_{xx}(t)}}$, we have that the range of z^j is $z^j \in \left[-\sqrt{\frac{C}{p^j_{xx}(t)}}, 0\right]$ and that $2\alpha^j(t,m)\, p^j_{xx}(t) + m\, p^j_{xm}(t) + p^j_x(t) + c = 0$. It follows that, on $(a^j(t,m), \alpha^j(t,m))$, the difference $V^j(t,x,m) - MV^j(t,x,m)$ can be expressed as the function

$$\bar{V}^j(t, z^j) = (z^j)^2 p^j_{xx}(t) - C$$

This latter function is defined on $z^j \in \left[-\sqrt{\frac{C}{p^j_{xx}(t)}}, 0\right]$, where the derivative is negative, and so its maximum is achieved in $-\sqrt{\frac{C}{p^j_{xx}(t)}}$. It thus follows that

$$V^j(t,x,m) - MV^j(t,x,m) \le \bar{V}^j\left(t, -\sqrt{\frac{C}{p^j_{xx}(t)}}\right) = 0$$

showing that (41) also holds in the interval $(a^j(t,m), \alpha^j(t,m))$ and, by analogy, also in $(\beta^j(t,m), b^j(t,m))$. Finally, since in the continuation region $\mathcal{C}^j = (a^j(t,m), b^j(t,m))$ we have

$$V^j_t(t,x,m) + \inf_u \left[\mathcal{L}^u V^j(t,x,m) + f(x,u)\right] = 0$$

and in the intervention region $\mathcal{I}^j$ (its complement) we have $M\, V^j(t,x,m) = V^j(t,x,m)$, at least one among the (40) and (41) hold as equality. □

6 Numerical Illustrations

In this section we present some numerical results to illustrate the characteristics of the optimal solution obtained from our approach. The results are grouped into four sub-sections. Subsections 6.1 and 6.2 are intended to convey an idea of the behavior of the optimal strategic boundaries and of the optimal continuous control respectively. Subsections 6.3 and 6.4 then describe the behavior of the value function and of the optimal log-exchange rate as well as of the drift (factor) process for the cases when the drift m_t is fully observed and when it is not and thus replaced by its filtered value $\hat{m}_t$. This allows one to see differences that arise between the full and partial information cases.

For the numerical calculations we had to choose numerical values for the various parameters and they are given in Table 1 below. In our simulations we use a (backward) Euler approximation for a uniform time discretization with step 0.01 to obtain numerical solutions of the ODEs (28) with their terminal conditions. Furthermore, to simulate the sample paths in subsection 6.4, we use the Euler approximation for ODEs (28) and the Euler-Maruyama approximation for the SDEs (5) with time discretization step 0.005.

6.1 Optimal strategic boundaries

First we discuss our strategic boundaries $a^j(t,m),\ \alpha^j(t,m),\ \beta^j(t,m), b^j(t,m)$ given by (33) and (34). As one can easily see from (33), (34) and (28), the structures of these optimal strategic boundaries are completely

Table 1: parameters

parameter	notation	size
target interest rate	$\bar{r}$	0.04
coefficient of continuous control	K	-0.51
volatility of exchange rate	σ	1
drift of factor	A	-0.1
drift of factor	a	0.01
volatility of factor	$\Lambda = (\Lambda_1, \Lambda_2)$	$(1.5, 2)$
target level of log-exchange rate	ρ	2
coefficient of running cost	k	6
penalty of impulse (fixed cost for lower bound)	C	0.24
penalty of impulse (proportional cost for lower bound)	c	1.5
penalty of impulse (fixed cost for upper bound)	D	0.5
penalty of impulse (proportional cost for upper bound)	d	3
coefficient of terminal condition	ℓ	1.5
discount factor	λ	0.05
maturity	T	1
variance of initial value of factor	$Var(m_0)$	0.5

the same in the two cases of full and partial information. In fact, for $j = f, p$, all coefficients of the first five equations in (28) are the same while the coefficients $\Sigma_1^j(t)$ and $\Sigma_2^j(t)$ of the last equation in (28) are different in the two cases. However, the solution of the last equation in (28) is not used for the strategic boundaries $a^j(t,m)$, $\alpha^j(t,m)$, $\beta^j(t,m), b^j(t,m)$ given in (33) and (34). Hence the values of the optimal strategic boundaries differ in the two cases only because the value of m is different. In the case of full information we insert the observed factor value for m, in the case of partial information we insert instead its filtered value $\hat{m}$.

The features of the strategic boundaries are shown in Figure 1. One of typical characteristics of the strategic boundaries is that for a fixed t, they are linear and parallel in the factor value m. This can be seen from the Figure but it follows also from (33) and (34). Another characteristic is that, since the coefficient $\ell = 1.5$ in the terminal cost is not very large, the central bank does not try too hard to push the exchange rate to its target level by using the interventions. The width between $a^j(t,m)$ and $b^j(t,m)$ becomes thus wider as time progresses to the maturity $t = 1$.

Figures 2, 3, 4 are extracts of a part of Figure 1, namely for $m = -1,\ 0,\ 1$ respectively. From these Figures we find that, when the factor values are neutral, i.e. $m = 0$, the upper strategic boundary $b^j(t,m)$ is further away from the target level of the exchange rate ρ than the lower strategic boundary $a^j(t,m)$ and this is due to the fact that the fixed cost D for an upper intervention is larger than that for the lower intervention, namely C. However, this is not always true when the factor values are not zero, that is, the exchange rate has a trend. For example, in Figure 4, the upper strategic boundary $b^j(t,m)$ is closer to the target level of the exchange rate than the lower strategic boundary $a^j(t,m)$ since the trend of the exchange rate is strongly positive and, in order to keep the values close to the target level of the exchange rate, the central bank needs to intervene more often in the market when the exchange rate becomes larger. From Figures 2, 3, 4, one might guess that the strategic boundaries have a monotonicity property, but this is not always the case.

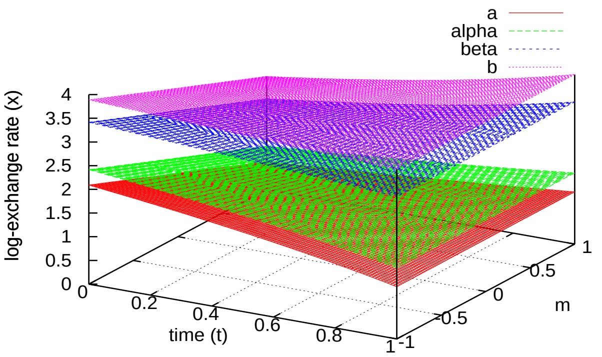

Figure 1: Strategic boundaries $a(t,m)$, $b(t,m)$, $\alpha(t,m)$, $\beta(t,m)$

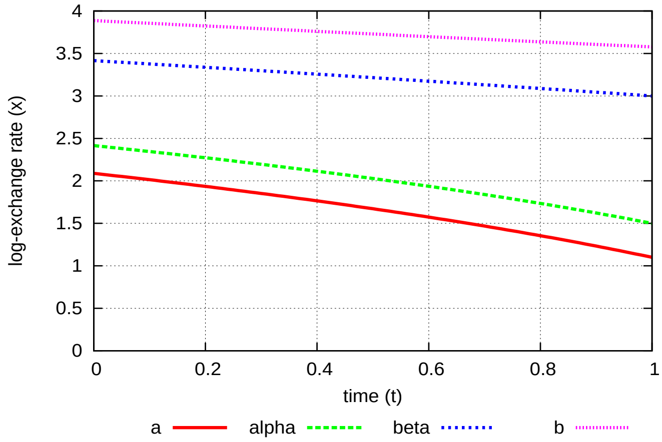

Figure 2: Strategic boundaries $a(t,-1)$, $b(t,-1)$, $\alpha(t,-1)$, $\beta(t,-1)$ at $m=-1$

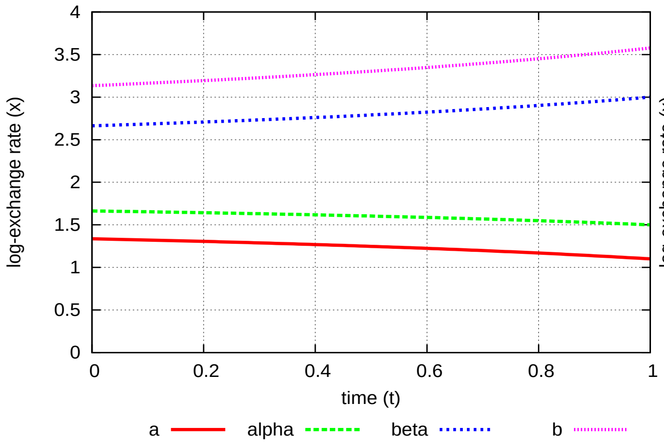

Figure 3: Strategic boundaries $a(t,0)$, $b(t,0)$, $\alpha(t,0)$, $\beta(t,0)$ at $m=0$

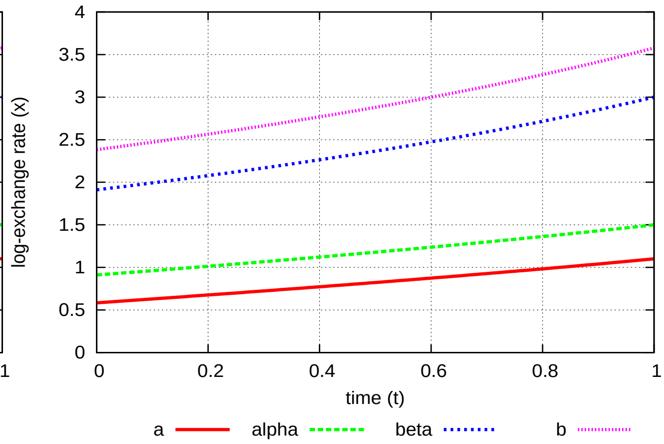

Figure 4: Strategic boundaries $a(t,1)$, $b(t,1)$, $\alpha(t,1)$, $\beta(t,1)$ at $m=1$

6.2 Optimal continuous control

In this subsection, we show some characteristics of our optimal continuous control $\hat{u}_t^j$ given in (19) with (25): when $a^j(t, m_t^j) < x_t^j < b^j(t, m_t^j)$,

$$\hat{u}_t^j = -\frac{K}{2k}\left(2p_{xx}^j(t)x_t^j + p_{xm}^j(t)m_t^j + p_x^j(t)\right). \tag{45}$$

Analogously to the behavior of the strategic boundaries as functions of t and m, also the structure of the optimal continuous control is completely the same in the two cases of full and partial observation of the drift factor. As shown in subsection 6.4, the differences will however appear from the values of the controlled process x_t^j and of the factor process given by the observed drift m_t or its filtered value $\hat{m}_t$ respectively. Figures 5, 6, 7 show graphs of u_t^j as functions in t and x when $m = -1,\ 0,\ 1$ respectively. From the graphs and (45), we find that the function is linear in x, but non-linear in t. Also from (45) this function turns out to be linear in m. We furthermore find that the slope in x at the maturity $t = 1$ is not sharper than the one at the beginning in $t = 0$ in all the figures since, as mentioned in the previous sub-section, the coefficient of the terminal cost $\ell = 1.5$ for the exchange rate is not large in comparison with the running cost, and so we find that the central bank does not try to control too strongly the exchange rate towards the target level neither by using the continuous control.

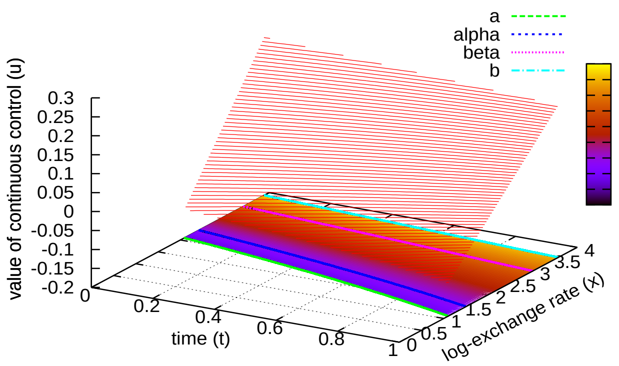

Figure 5: Optimal continuous control at $m = -1$

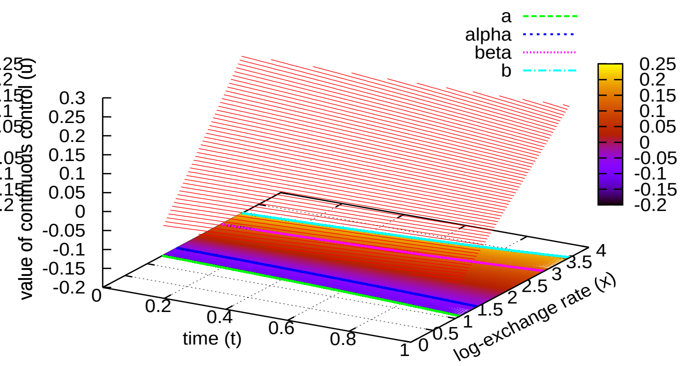

Figure 6: Optimal continuous control at $m = 0$

6.3 Value functions

In this subsection, we consider the shapes of our value functions. Unlike the previous two sub-sections, the value functions in the two cases of full and partial information have some differences since they depend on the function $p^j(t)$ for which in equation (28) the parameters $\Sigma_1^j(t)$ and $\Sigma_2^j(t)$ are different in the two cases. Figures 8, 10, 12 show the features of the value functions under full information at the times $t = 0,\ 0.5,\ 0.9$ respectively. Figures 9, 11, 13 show instead the features of the value functions under partial information at the same times $t = 0,\ 0.5,\ 0.9$ respectively. In all the figures, the features of the value functions outside of the strategic boundaries $a^j(t, m)$ and $b^j(t, m)$ are, in line with their formulas, linear in x, while between two boundaries, they have quadratic forms in x.

The slope on the outside of the upper strategic boundary is shaper than that on the outside of the lower strategic boundary since in our parameter set the proportional cost d for the upper intervention is larger than that for lower intervention, namely c. Comparing the two value functions at each time t (it might be difficult to see it from the following figures), the value function under full information is naturally lower than that under partial information. As time progresses, the values of each value function become gradually smaller at each point.

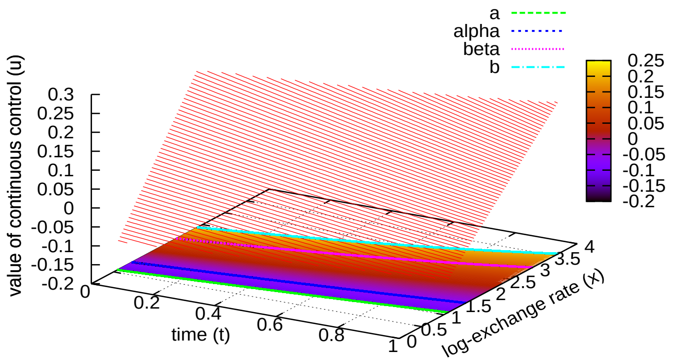

Figure 7: Optimal continuous control at $m = 1$

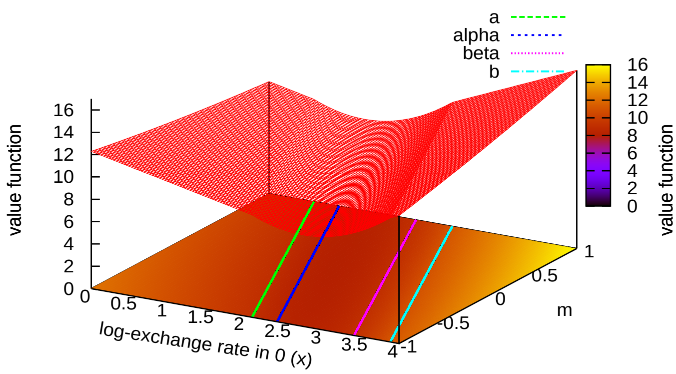

Figure 8: Value function at time $t = 0$ under full information

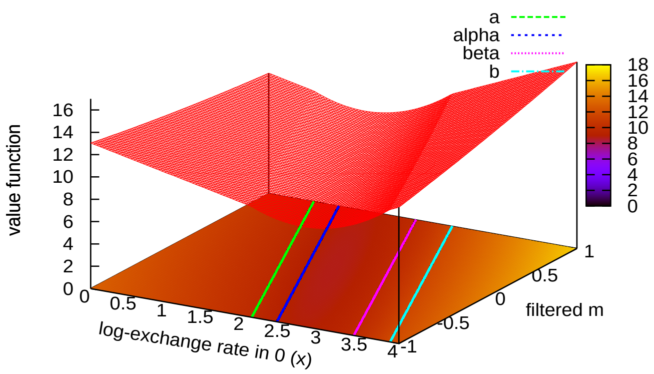

Figure 9: Value function at time $t = 0$ under partial information

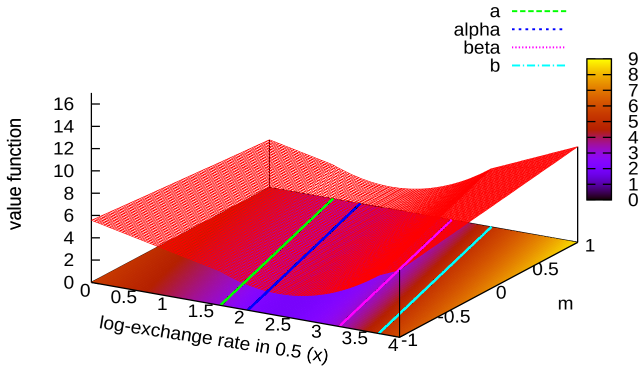

Figure 10: Value function at time $t = 0.5$ under full information

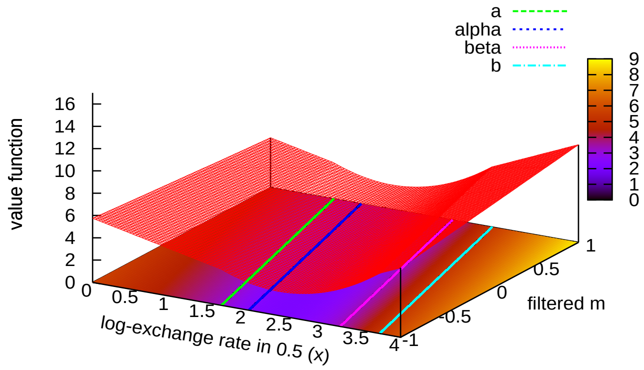

Figure 11: Value function at time $t = 0.5$ under partial information

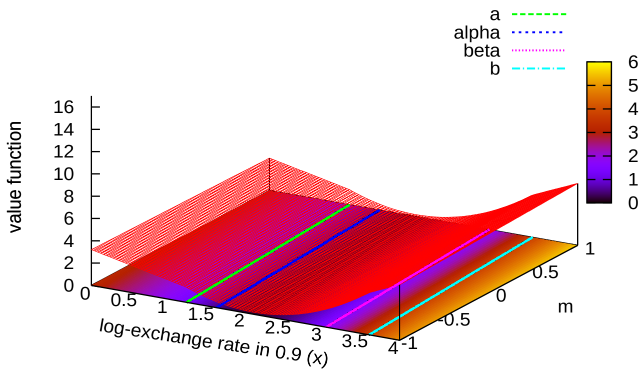

Figure 12: Value function at time $t = 0.9$ under full information

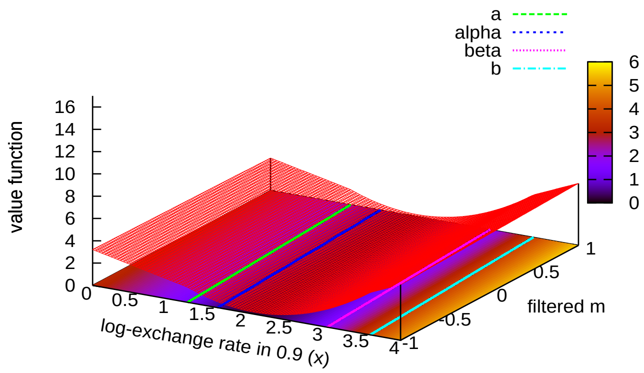

Figure 13: Value function at time $t = 0.9$ under partial information

6.4 Sample paths

Finally, in this subsection we present sample paths of the log-exchange rate x_t, of the strategic boundaries, of the factor process, namely the drift of x_t, and of the optimal continuous control in the two cases of full and partial information and look at their fluctuations. Figure 14 shows sample paths of the log-exchange rate and of the four strategic boundaries under full information together with the points where an intervention occurs. Figure 15 shows the same for the partial information case. Figure 16 shows sample paths of the factor process, namely of the drift m_t and of the corresponding filtered process $\hat{m}_t$. Figure 17 shows sample paths of the continuous controls in both cases.

From Figure 14 we find that, in the case of full information, the fluctuations of the strategic boundaries are larger than those of the exchange rate due to the values of each volatility. Note that the fluctuations of the factor process can be seen from Figure 16. On the other hand, in Figure 15, the fluctuations of the strategic boundaries under partial information are small due to small fluctuations in the filtered values given in Figure 16. Due to the difference of the controls, the fluctuations of the exchange rate in the two cases are rather different and also the intervention times are different. From Figure 16, we find that the fluctuations of the filtered process are smaller than those of the factor process since the filtered value is given as the conditional expectation of the factor process, so that their fluctuations are reduced. This reduction of fluctuations has an impact on the fluctuations of the strategic boundaries in Figure 15 and of the continuous controls in Figure 17. In Figure 16 and 17, the fluctuations of the factor process and of the filtered process as well as the shape of the fluctuations of the continuous controls in the two cases are very different, however their values do not have great differences. The filtered values and the continuous controls under partial information seem to be the averaged values of the factor values and the continuous control under full information respectively.

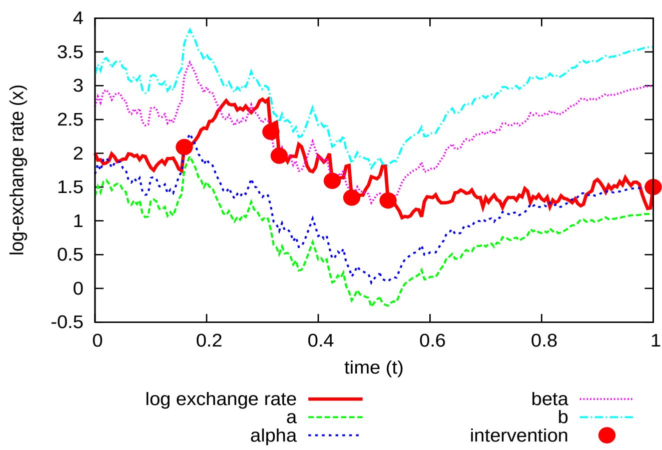

Figure 14: Sample paths of exchange rate and boundaries under full information

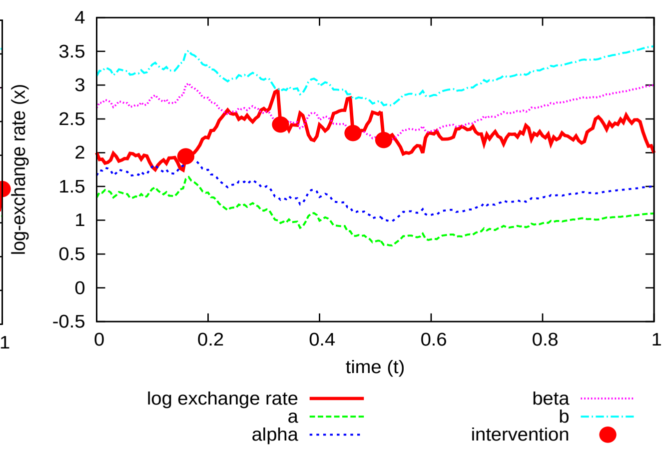

Figure 15: Sample paths of exchange rate and boundaries under partial information

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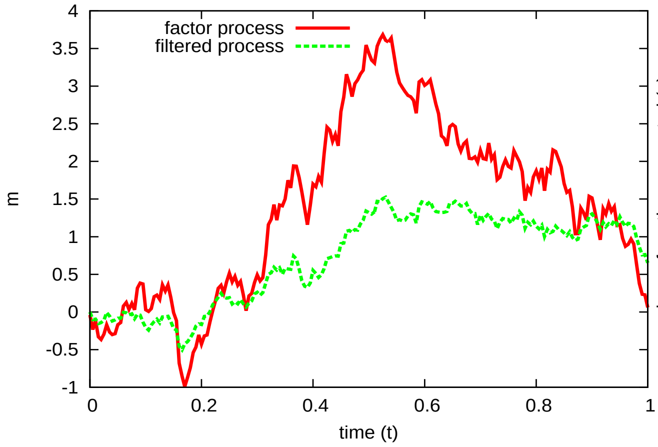

Figure 16: Sample paths of factor process and filtered process

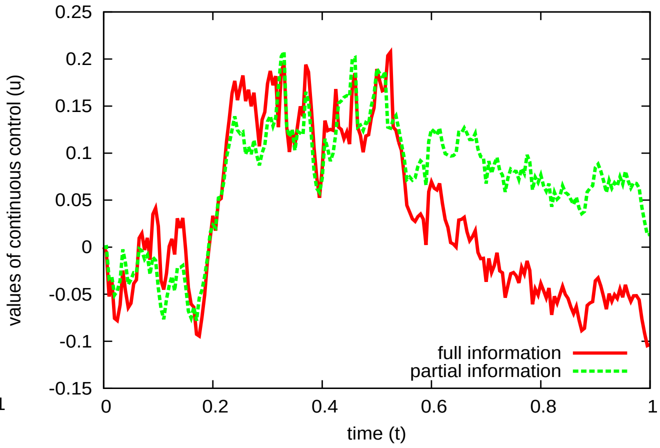

Figure 17: Sample paths of continuous control under the both cases

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