Numerical Approximation by Quantization of Control Problems in Finance under Partial Observations

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Abstract

We study numerical solutions to discrete time control problems under partial observation when the state of the system is described by (X, Y, V^{α}) with X signal process, Y observation process and V^{α} controlled process. The control processes α is required to be adapted with respect to the observation filtration. The structure of the control problem is motivated with a view towards financial applications. In particular we consider the problem of hedging a future liability in the context of incomplete information. To cope with difficulties arising from partial information, stochastic filtering is used and the filter process is discretized in order to obtain a feasible numerical solution. This is done by performing a quantization of the pair process filter-observation. Dynamic programming is finally applied to solve the approximated filtered control problem. Convergence results are given and numerical applications are presented and discussed for the problem of hedging an European put (and call) option with unobservable volatility.

Key words : hedging, partial observation, volatility, filtering, quantization numerical method, dynamic programming.

1 Introduction

This paper concerns numerically feasible approximations to discrete time stochastic control problems under partial observation. Such problems arise naturally in financial market models where some model coefficients (volatility, drift, ...) may depend on stochastic factors that are not observable. They were investigated in numerous papers, mostly from a theoretical viewpoint. However, numerical tests are rarely performed due to computational difficulties, especially when observations are multiplicative noises and non gaussian, like in unobservable stochastic volatility models.

Here we consider a discrete time model where the signal process X is a Markov chain, which may not be observable and takes value in a set E consisting of a finite number of points $\{x^1, ..., x^m\}$. The observation process Y takes values in $\mathbb{R}^d$ and is such that the pair (X, Y) is a Markov chain. The control process, denoted by α, is adapted with respect to the observation filtration, and V^α is the controlled process.

The structure of our model is motivated with a view towards financial applications. Think for example of the case where Y is the price of a risky asset, X is its unobservable volatility or drift and V^α is the wealth process. The investment strategy is represented by a control process α, which gives the number of risky asset shares held in the portfolio. Denoting by $\mathbb{F}^Y = (\mathcal{F}_k^Y)_k$ the filtration generated by the observation process Y, the filter process Π is given by :

$$\Pi_k^i := \mathbb{P}\big[X_k = x^i | \mathcal{F}_k^Y\big], \quad k \in \mathbb{N}, \quad i = 1, ..., m.$$

By using the filter process, the original control problem under partial observation is transformed into an equivalent one under complete observation with observed state process given by the filter Π instead of the unobservable signal X, and we may apply dynamic programming method, see e.g. [1].

The numerical difficulty of this procedure concerns the filtered problem dimension because the number of values taken by the filter is infinite even though the process X has only a finite number of states. More precisely, as the state space E consists of a finite number m of points $\{x_1, ..., x_m\}$, the filter is characterized by an m-vector with components $\Pi_k^i := \mathbb{P}[X_k = x^i | \mathcal{F}_k^Y]$ and it takes values in the m simplex K_m of $\mathbb{R}^m$. Therefore, in order to numerically solve the problem, the filter has to be approximated with another process taking only a finite number of values in K_m. A classical approach (see for example [2]) is to discretize the observation process Y by a process $\hat{Y}$ taking a finite number N of values and then approximate for each k the filter Π_k by the filter of X_k given $\hat{Y}_1, ..., \hat{Y}_k$. The numerical drawback of this approach is that the number of possible values taken by the approximating filter grows exponentially with the time step; in fact at time n the approximated filter is identified by a random vector taking N^n possible values.

In this paper, we suggest an alternative approach, which has been recently developed to numerically solve optimal stopping time problems under partial observations (see [14]). The method consists in approximating the Markov pair process (Π, Y) by a process $(\hat{\Pi}, \hat{Y})$ taking at each time step k a finite number of values N_k that is arbitrarily assigned. This approach relates to the field of quantization methods, recently developed in numerical probability and applied to solve various financial problems (see [12], [14], [13], [11]). In particular, by using results from [14], it is possible to make an optimal quantization, which for each time step k minimizes the quantity :

$$\mathbb{E}\Big[\big|(Y_k,\Pi_k)-(\hat{Y}_k,\hat{\Pi}_k)\big|^2\Big]$$

called quantization error or distorsion. The implementation of this optimal quantization is based on a stochastic gradient descent method combined with Monte-carlo simulations of the pair (Π, Y). Once the problem has been discretized, we can solve it numerically by using dynamic programming and we prove that when N_k grows the approximated solution converges towards the real solution with rate dominated by the quantization error.

Finally we apply the method described above in order to solve a specific financial problem, which consists in the hedging of a European put (and call) option. Since we are in an incomplete market setting, it is not possible to obtain a self financing and perfect hedging strategy, and we consider as hedging criterion the expected value of a convex function applied to the residual hedging error. In particular we will focus on the case of the quadratic criterion (see [6]) and the shortfall risk criterion (see [5]).

The outline of the paper is as follows. In Section 2, we formulate the partial observation discrete-time control problem. In Section 3, stochastic filtering is used to transform the original control problem into a complete observation one that can be studied via the dynamic programming method. We describe in Section 4 the numerical approximation by quantization to this control problem, and we prove some convergence results. The financial application is presented in Section 5 where we study the problem of hedging a European put (and call) option with unobservable volatility. Some numerical tests are finally performed and discussed.

Notations

In the sequel, we denote by $|.|_1$ the l^1 norm on $\mathbb{R}^l$, by $|.|$ the euclidean norm on $\mathbb{R}^l$ and, for any random variable X taking values in $\mathbb{R}^l$ we denote:

$$\|X\|_2 := \left(\mathbb{E}|X|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|X\|_1 := \mathbb{E}|X|_1.$$

For any measurable function g from $D \subset \mathbb{R}^l$ into $\mathbb{R}$, we define :

$$[g]_{Sup} := \sup_{x\in D} |g(x)| \tag{1.1}$$

and

$$[g]_{Lip} := \sup_{x,y\in D; x\neq y} \frac{|g(x)-g(y)|}{|x-y|_1}. \tag{1.2}$$

2 Problem setup

Let us consider a discrete time dynamical system over a horizon $\{0,...,n\}$ with n fixed and with state at time k ($k = 0,...,n$) described by the variables (X_k, Y_k, V_k^α). In particular $(X_k)_k$ represents the signal process which may not be observable, $(Y_k)_k$ is the observation process and $(V_k^\alpha)_k$ is the process controlled by a process α adapted with respect to $(\mathcal{F}_k^Y)$ the filtration generated by $(Y_k)_k$.

In a financial setting we can think of the case where Y is the price of a risky asset, X is its unobservable volatility or drift and V^α is the wealth process. The investment strategy is represented by a control process α representing the number of risky asset shares held in the portfolio, and based on the information derived from the prices observations.

We assume that the process $(X_k)_k$ is a finite-state Markov chain taking values in the space $E = \{x^1, ..., x^m\}$. Its probability transition P_k (from the period $k-1$ to the period k) and initial law μ are defined by :

$$\begin{aligned} \mu^i &= \mathbb{P}[X_0 = x^i], \quad i = 1, ..., m, \\ P_k^{ij} &= \mathbb{P}[X_k = x^j | X_{k-1} = x^i], \quad i = 1, ..., m. \end{aligned}$$

The process $(Y_k)_k$ takes values in $\mathbb{R}^d$ and is such that the pair $(X_k, Y_k)_k$ is a Markov chain and the conditional law of Y_k given (X_{k-1}, Y_{k-1}, X_k) admits a (known) bounded density:

$$y' \to g_k(X_{k-1}, Y_{k-1}, X_k, y').$$

For simplicity, we assume that Y_0 is a known deterministic constant, fixed equal to y_0. The control process is denoted by $(\alpha_k)_{k\geq 0}$, takes values in $A \subset \mathbb{R}^l$, and is supposed to be adapted with respect to the filtration $(\mathcal{F}_k^Y)_k$ generated by (Y_k). We denote by $\mathcal{A}$ the set of control processes. The controlled process $(V_k^\alpha)_k$ takes values in $\mathbb{R}$ and is governed by a dynamics of the form:

$$V_{k+1}^\alpha = H(V_k^\alpha, \alpha_k, Y_k, Y_{k+1}), \tag{2.1}$$

where H is a measurable function.

We are given a running (measurable) cost function f on $E \times \mathbb{R}^d \times \mathbb{R} \times A$, and a terminal (measurable) cost function h on $E \times \mathbb{R}^d \times \mathbb{R}$. Given an initial value v_0 for the controlled process, an admissible control $\alpha \in \mathcal{A}$, the expected cost function is defined by :

$$J(v_0, \alpha) = \mathbb{E}\left[\sum_{k=0}^{n-1} f(X_k, Y_k, V_k^\alpha, \alpha_k) + h(X_n, Y_n, V_n^\alpha)\right] \tag{2.2}$$

and the goal is to choose a control process in order to minimize the cost J up to the time horizon n :

$$J_{opt}(v_0) = \inf_{\alpha\in\mathcal{A}} J(\alpha). \tag{2.3}$$

Financial Example

A typical financial example corresponds to the case where Y represents the price of a risky asset and X is its unobservable volatility. Assume that a riskless n-maturity bond is available for trading, yielding constant interest rate $r = 0$ (for simplicity). We consider an economic agent over an investment time horizon n. At time $k = 0$ the agent starts with an initial wealth v and then at each instant $k = 1, ..., n$ he rebalances his portfolio holdings by choosing the investment allocations in the bond and in the risky asset. Under the assumption of self-financing the wealth process V satisfies

$$V_{k+1}^\alpha = V_k^\alpha + \alpha_k \left[Y_{k+1} - Y_k\right] \tag{2.4}$$

where α_k represents the number of shares of risky asset held in the portfolio at time k. The process $(\alpha_k)_{k=1,\dots,n}$ is supposed to be adapted with respect to the filtration generated by the price process Y, i.e. the investment strategy is selected only on the basis of past observations of the security prices.

Given a loss function $\ell : \mathbb{R} \to \mathbb{R}$, the hedging criterion for a derivative asset $h(Y_n)$ of maturity n, consists in minimizing the expected loss :

$$\mathbb{E}\Big[\ell(h(Y_n) - V_n^\alpha)\Big]$$

over all admissible portfolios $\alpha = (\alpha_k)_{k=0,\dots,n}$.

In order to prove convergence results, we shall make some technical assumptions :

H1 The set A is compact;

H2 H is continuous, and there exists some positive constant $[H]_{Lip}$ s.t. for all (v, a, y, y') and $(\hat{v}, a, \hat{y}, \hat{y}') \in \mathbb{R} \times A \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\big|H(v, a, y, y') - H(\hat{v}, a, \hat{y}, \hat{y}')\big|_1 \le [H]_{Lip}\left(|v - \hat{v}| + |y - \hat{y}|_1 + |y' - \hat{y}'|_1\right)$$

H3 Functions f and h are bounded and Lipschitz;

H4 There exists some positive constant L_g such that for all $k = 1, \dots, n$:

$$\sum_{i,j=1}^{m} P_k^{ij} \int \big|g_k(x^i, y, x^j, y') - g_k(x^i, \hat{y}, x^j, y')\big|\, dy' \le L_g |y - \hat{y}|_1 \quad \forall y, \hat{y} \in \mathbb{R}^d.$$

Remark 2.1. *The hypothesis* ***H2*** *is verified by (2.4) in the previous example. Concerning the hypothesis* ***H4*** *we will see that it is satisfied for the model analyzed in the numerical application given in the last section.*

3 Filtering and dynamic programming

Recalling that the state space of (X_k) consists of a finite number of points and denoting by $(\mathcal{F}_k^Y)$ the filtration generated by the observation process (Y_k), the filter is defined as follows:

$$\Pi_k^i = \mathbb{P}[X_k = x^i | \mathcal{F}_k^Y] \quad i = 1, ..., m \text{ and } \quad k = 1, ..., n$$

and is a random vector process, which takes values in the m-simplex K_m in $\mathbb{R}^m$:

$$K_m = \left\{ \pi = (\pi^i) \in \mathbb{R}^m : \pi^i \ge 0 \text{ and } \quad |\pi|_1 = \sum_{i=1}^m \pi^i = 1 \right\}.$$

By using Bayes' formula the filter process can be calculated in a recursive way as follows (see [9]):

$$\begin{aligned}\Pi_0 &= \mu \\ \Pi_k &= \bar{G}_k\left(\Pi_{k-1}, Y_{k-1}, Y_k\right) = \frac{GP_k(Y_{k-1}, Y_k)^{\intercal}\,\Pi_{k-1}}{|GP_k(Y_{k-1}, Y_k)^T\Pi_{k-1}|_1}, \quad k \geq 1\end{aligned} \tag{3.1}$$

where $GP_k(Y_{k-1}, Y_k)$ is a $m \times m$ random matrix given by :

$$GP_k(Y_{k-1}, Y_k)_{ij} = g_k(x^i_{k-1}, Y_{k-1}, x^j_k, Y_k)P^{ij}_k, \quad 1 \leq i, j \leq m,$$

and $^{\intercal}$ is the transpose. One can also show (see e.g. [14]) that the pair $(\Pi_k, Y_k)_k$ is a Markov chain with respect to the filtration $(\mathcal{F}^Y_k)_k$ and the conditional law Q_k of Y_k given (Π_{k-1}, Y_{k-1}) admits a density given by :

$$y' \to q_k(\Pi_{k-1}, Y_{k-1}, y') := \sum_{i,j=1}^{m} g_k(x^i, Y_{k-1}, x^j, y')P^{ij}_k \Pi^i_{k-1} \tag{3.2}$$

Relations (3.1)-(3.2) show that, although the probability transition of the Markov chain (Π_k, Y_k) is not explicitly known, it can be simulated. This point is important when one needs Monte-Carlo simulations of (Π_k, Y_k), see paragraph 4.1.1.

By using the law of iterated conditional expectations, we can rewrite the expected cost function (2.2) as follows:

$$\begin{aligned}J(v_0, \alpha) &= \mathbb{E}\left[\sum_{k=0}^{n-1} \mathbb{E}\left[f(X_k, Y_k, V^\alpha_k, \alpha_k)|\mathcal{F}^Y_k\right] + \mathbb{E}\left[h(X_n, Y_n, V^\alpha_n)|\mathcal{F}^Y_n\right]\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{n-1}\sum_{i=1}^{m} f(x^i, Y_k, V^\alpha_k, \alpha_k)\Pi^i_k + \sum_{i=1}^{m} h(x^i, Y_n, V^\alpha_n)\Pi^i_n\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{n-1} \hat{f}(\Pi_k, Y_k, V^\alpha_k, \alpha_k) + \hat{h}(\Pi_n, Y_n, V^\alpha_n)\right]\end{aligned}$$

where

$$\begin{aligned}\hat{f}(\pi, y, v, a) &:= \int f(x, y, v, a)\pi(dx) = \sum_{i=1}^{m} f(x^i, y, v, a)\pi^i \\ \hat{h}(\pi, y, v) &:= \int h(x, y, v)\pi(dx) = \sum_{i=1}^{m} h(x^i, y, v)\pi^i\end{aligned}$$

The original problem (2.3) can now be formulated as a problem under full observation with state variables (Π_k, Y_k, V_k) :

$$J_{opt}(v_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\sum_{k=0}^{n-1} \hat{f}(\Pi_k, Y_k, V^\alpha_k, \alpha_k) + \hat{h}(\Pi_n, Y_n, V^\alpha_n)\right]. \tag{3.3}$$

Actually, recalling (2.1) and following the Dynamic Programming Algorithm (see e.g. [3]), for solving the filtered problem (3.3), we define the sequence of functions :

$$
\text{(DP)} \quad \begin{cases} u_n(\pi, y, v) &= \hat{h}(\pi, y, v) \\ \\ u_k(\pi, y, v) &= \inf_{a \in A} \Big\{ \hat{f}(\pi, y, v, a) \\ & \quad + \mathbb{E}\Big[u_{k+1}(\Pi_{k+1}, Y_{k+1}, H(v, a, y, Y_{k+1})) \Big| (\Pi_k, Y_k) = (\pi, y) \Big] \Big\} \\ & \quad k = 0, \ldots, n-1. \end{cases}
$$

The following result shows that this backward procedure gives the solution for $k = 0$ to the original problem (2.3).

Proposition 3.1. *Assume* **H1**, **H2**, *and* **H3**. *Then the algorithm (DP) provides the solution of problem (2.3), i.e.:*

$$
u_0(\mu, y_0, v_0) \;=\; J_{opt}(v_0).
$$

Proof. See Appendix A.

4 Approximation by quantization and error analysis

4.1 The numerical approximation method

From a numerical viewpoint the formula given by the (DP) algorithm is still untractable since the state variable $Z_k^\alpha := (\Pi_k, Y_k, V_k^\alpha)$ takes values in a continuous state space. In order to obtain a numerical solution, the basic idea is to approximate at each time step k the continuous state variable Z_k^α by a discrete state variable $\hat{Z}_k^\alpha$ taking a finite number of values. The main concern is how to discretize in an efficient and feasible way the variables Z_k^α that depend on the control α?

We deal separately with the approximation of the pair filter-observation $W := (\Pi, Y)$ that does not depend on the control, and the approximation of the controlled state variable V^α. The approximation of (Π, Y) is obtained following an optimal quantization method as in [14]. The approximation of V^α is obtained by a classical uniform space discretization similar to the Markov chain method as in [8].

4.1.1 Optimal quantization of the pair filter-observation

In a first step, we discretize for each k the pair (Π_k, Y_k) by approximating it by $(\hat{\Pi}_k, \hat{Y}_k)$ taking a finite number of values. The space discretization (or quantization) of the random vector $W_k = (\Pi_k, Y_k)$ valued in $K_m \times \mathbb{R}^d$ is constructed as follows. At initial time $k = 0$, recall that W_0 is a known deterministic vector equal to $w_0 = (\mu, y_0)$, so we start from the grid with one point in $K_m \times \mathbb{R}^d$:

$$
\Gamma_0 \quad = \quad \{ w_0 = (\mu, y_0) \} \, .
$$

At time $k \geq 1$, we are given a grid Γ_k of N_k points in $K_m \times \mathbb{R}^d$:

$$
\Gamma_k \quad = \quad \Big\{ w_k^1 = (\pi_k(1), y_k^1), \ldots, w_k^{N_k} = (\pi_k(N_k), y_k^{N_k}) \Big\} \, ,
$$

and we approximate the pair $W_k = (\Pi_k, Y_k)$ by $\hat{W}_k = (\hat{\Pi}_k, \hat{Y}_k)$ valued in Γ_k and defined as the closest neighbour projection :

$$\hat{W}_k = \text{Proj}_{\Gamma_k}(W_k) := \sum_{i=1}^{N_k} w_k^i 1_{C_i(\Gamma_k)}(W_k),$$

where the so-called Voronoi tesselations $C_1(\Gamma_k)$, ..., $C_{N_k}(\Gamma_k)$ are Borel partitions of $K_m \times \mathbb{R}^d$ satisfying :

$$C_i(\Gamma_k) \subset \left\{ w \in K_m \times \mathbb{R}^d \; : |w - w_k^i| = \min_{j=1,\ldots,N_k} |w - w_k^j| \right\}, \quad i = 1, \ldots, N_k.$$

The L^2-error induced by this projection, called L^2-quantization error, is equal at time k to : $\|W_k - \hat{W}_k\|_2$. As a function of the grid Γ_k identified with the N_k-tuple $(w_k^1, \ldots, w_k^{N_k})$ in $K_m \times \mathbb{R}^d$, the square of the L^2-quantization error, called distorsion, is written as :

$$D_{N_k}^{W_k}(\Gamma_k) = \|W_k - \text{Proj}_{\Gamma_k}(W_k)\|_2^2 = \mathbb{E}\Big[\min_{i=1,\ldots,N_k} |W_k - w_k^i|^2 \Big]. \tag{4.1}$$

Notice, by definition of the closest neighbour projection that the L^2-quantization error is the minimum of the L^2-error $\|W_k - U\|_2$ among all random variables U taking values in the grid Γ_k.

In a second step, we approximate the probability transitions of the Markov chain (W_k) by the following probability transition matrix :

$$\hat{r}_k^{ij} = \mathbb{P}\Big[\hat{W}_k = w_k^j \Big| \hat{W}_{k-1} = w_{k-1}^i \Big] = \frac{\mathbb{P}\Big[W_k \in C_j(\Gamma_k), W_{k-1} \in C_i(\Gamma_{k-1})\Big]}{\mathbb{P}\Big[W_{k-1} \in C_i(\Gamma_{k-1})\Big]}$$

for all $k = 1, \ldots, n$, $i = 1, \ldots, N_{k-1}$, $j = 1, \ldots, N_k$.

The grids Γ_k are optimally chosen so as to minimize at each time k the distorsion $D_{N_k}^{W_k}(\Gamma_k)$. This relies on the property that the distorsion is differentiable, with a gradient obtained by formal differentiation in (4.1) :

$$\nabla D_{N_k}^{W_k}(\Gamma_k) = 2 \left(\mathbb{E}\Big[(w_k^i - W_k) 1_{W_k \in C_i(\Gamma_k)} \Big] \right)_{1 \le i \le N_k}. \tag{4.2}$$

The optimal grids and the associated probability transition matrix are then processed and estimated by a stochastic gradient descent method, known in this context as the Kohonen Algorithm, and based on the integral representation (with respect to the probability law of W_k) (4.2). This is achieved by Monte-Carlo simulations of the Markov chain $(W_k)_k = (\Pi_k, Y_k)_k$ through the following simulation procedure : starting from (Π_{k-1}, Y_{k-1}),

- simulate Y_k according to the density given in (3.2)
- compute Π_k by the formula (3.1).

We refer to [14] for the details and the practical implementation of the optimal grids.

4.1.2 Space discretization of the controlled variable

We fix a bounded uniform grid on the state space $\mathbb{R}$ for the controlled process V^α. Namely, we set

$$\Gamma^V := (2\nu)\mathbb{Z} \cap [-R, R],$$

where ν is the spatial step and R is the grid size. We denote by Proj_{Γ^V} the projection on the grid Γ^V according to the closest neighbor rule. Recalling the dynamics (2.1) of the controlled process, we approximate it as follows : given a control $\alpha \in \mathcal{A}$, we discretize $(\hat{V}_k^\alpha)_k$ by the controlled process valued in Γ^V, and evolving according to the dynamics :

$$\hat{V}_{k+1}^\alpha = \mathrm{Proj}_{\Gamma^V}(H(\hat{V}_k^\alpha, \alpha_k, \hat{Y}_k, \hat{Y}_{k+1})). \tag{4.3}$$

Here $\hat{Y}_k$ is the quantization of Y_k obtained in the previous paragraph.

4.1.3 Approximation of the control problem

We approximate the sequence of functions (u_k) by the sequence of function $\hat{u}_k$ defined on $\Gamma_k \times \Gamma^V$, $k = 0, \ldots, n$, by a dynamic programming type formula :

$$\begin{aligned} \hat{u}_n(\pi, y, v) &= \hat{h}(\pi, y, v) \\ \hat{u}_k(\pi, y, v) &= \inf_{a \in A} \Big\{ \hat{f}(\pi, y, v, a) \\ &\quad + \mathbb{E}\big[\hat{u}_{k+1}\big(\hat{\Pi}_{k+1}, \hat{Y}_{k+1}, \mathrm{Proj}_{\Gamma^V}(H(v, a, y, \hat{Y}_{k+1}))\big) \big| (\hat{\Pi}_k, \hat{Y}_k) = (\pi, y)\big] \Big\} \end{aligned}$$

From an algorithmic viewpoint, this is computed explicitly as follows :

$$\begin{aligned} \hat{u}_n(w_n^i, v) &= \hat{h}(w_n^i, v), \quad w_n^i = (\pi_n(i), y_n^i) \in \Gamma_n, i = 1, \ldots, N_n, \ v \in \Gamma^V \\ \hat{u}_k(w_k^i, v) &= \inf_{a \in A} \Big\{ \hat{f}(w_k^i, v, a) \\ &\quad + \sum_{j=1}^{N_{k+1}} \hat{r}_{k+1}^{ij} \hat{u}_{k+1}\big(\hat{w}_{k+1}^j, \mathrm{Proj}_{\Gamma^V}(H(v, a, y_k^i, y_{k+1}^j))\big) \Big\} \\ &\quad w_k^i = (\pi_k(i), y_k^i) \in \Gamma_k, i = 1, \ldots, N_k, \ v \in \Gamma^V. \end{aligned} \tag{4.4}$$

For $v_0 \in \Gamma^V$, the solution $J_{opt}(v_0) = u_0(\mu, y_0, v_0)$ to our control problem is then approximated by

$$\hat{J}_{quant}(v_0) = \hat{u}_0(\mu, y_0, v_0).$$

Moreover, this backward dynamic programming scheme allows us to compute at each step $k = 0, \ldots, n-1$, an approximate optimal control $\hat{\alpha}_k(w, v)$, $w = (\pi, y) \in \Gamma_k$, $v \in \Gamma^V$, by taking the infimum in (4.4).

4.2 Error analysis and rate of convergence

We state an error estimation between the optimal cost function J_{opt} and the approximated cost function $\hat{J}_{quant}$, in terms of :

- the quantization errors $\Delta_k = \|W_k - \hat{W}_k\|_2$ for the pair $W_k = (\Pi_k, Y_k)$, $k = 0, \ldots, n$
- the spatial step ν and the grid size R for V_k^α, $k = 0, \ldots, n$.

Theorem 4.1. *Under* **H1**, **H2**, **H3** *and* **H4**, *we have for all* $v_0 \in \Gamma^V$ *:*

$$\Big|J_{opt}(v_0) - \hat{J}_{quant}(v_0)\Big| \;\; \leq \;\; C_1(n) \sum_{k=0}^{n} \sum_{j=0}^{k} \Psi^k \Big(\nu + \frac{C_2}{R} + \Delta_{k-j}\Big), \tag{4.5}$$

where $C_1(n) = \sqrt{m+d+1}\big[2\left(n\bar{L}_g\bar{f} + \bar{M} + 3\bar{L}_g\bar{h}\right)\frac{(2\bar{L}_g)^n}{2\bar{L}_g - 1} + \bar{f} + \bar{h}\big]$, $\bar{f} = \max([f]_{Sup}, [f]_{Lip})$, $\bar{h} = \max([h]_{Sup}, [h]_{Lip})$, $\bar{L}_g = \max(L_g, 1)$, $\bar{M} = \max([H]_{Lip}, 1)$, C_2 *is the maximum value of* H *over* $\Gamma^V \times A \times \cup_k \Gamma_k \times \cup_k \Gamma_k$, *and* $\Psi = (2d+1)[H]_{Lip}$.

Proof. See Appendix B.

Convergence of the approximation

As a consequence of Zador's theorem (see [7]), which gives the asymptotic behavior of the optimal quantization error, when the number of grid points goes to infinity, we can derive the following estimation on the optimal quantization error for the pair filter-observation (see [14]) :

$$\limsup_{N_k \to \infty} N_k^{\frac{2}{m-1+d}} \min_{|\Gamma_k| \leq N_k} \|W_k - \hat{W}_k\|_2^2 \;\leq\; C_k(m,d),$$

where $C_k(m,d)$ is a constant depending on m, d and the marginal density of Y_k. Therefore, the estimation (4.5) provides a rate of convergence for the approximation of J_{opt} of order

$$n^2 \Psi^n C_1(n) \Big(\nu + \frac{1}{R} + \frac{1}{N^{\frac{1}{m-1+d}}}\Big),$$

when $N_k = N$ is the number of points at each grid Γ_k used for the optimal quantization of $W_k = (\Pi_k, Y_k)$, $k = 1, \ldots, n$. We then get the convergence of the approximated cost function $\hat{J}_{quant}$ to the optimal cost function J_{opt} when ν goes to zero, and N, R go to infinity. Moreover, by extending the approximate control $\hat{\alpha}_k$, $k = 0, \ldots, n-1$, to the continuous state space $K_m \times \mathbb{R}^d \times \mathbb{R}$ by :

$$\hat{\alpha_k}(\pi, y, v) \;=\; \hat{\alpha_k}(\mathrm{Proj}_{\Gamma_k}(\pi, y), \mathrm{Proj}_{\Gamma^V}(v)), \qquad \forall\, (\pi, y, v) \in K_m \times \mathbb{R}^d \times \mathbb{R},$$

and by setting (by abuse of notation) : $\hat{\alpha}_k = \hat{\alpha}_k(\Pi_k, Y_k, \hat{V}_k^{\hat{\alpha}})$, we get an approximate control $\hat{\alpha} = (\hat{\alpha}_k)_k$ in $\mathcal{A}$, which is ε-optimal for the original control problem (see [15]) in the sense that for all $\varepsilon > 0$:

$$J(v_0, \hat{\alpha}) \quad \leq \quad J_{opt}(v_0) + \varepsilon,$$

whenever N, R are large enough, and ν is small enough.

5 Financial application : European option hedging in a partially observed stochastic volatility model

In this section we apply the methodologies described above in order to study the problem of hedging an European put (or call) option in the context of incomplete information on the underlying price evolution model. Since we are in an incomplete market setting, the perfect replication of the claim is not possible and as hedging criterion we choose the expected value of a convex loss function applied to the hedging error. In particular we will consider the case of the quadratic criterion and that of the shortfall risk criterion.

5.1 The model

We consider a stochastic volatility model where for simplicity we have only one risky asset with observable price (S_k) whose dynamics is given by :

$$\begin{aligned} S_{k+1} &= S_k \exp\left[\left(r - \frac{1}{2}X_k^2\right)\delta + X_k\sqrt{\delta}\epsilon_{k+1}\right], \quad k = 1, ..., n \\ S_0 &= s_0 > 0 \end{aligned}$$

where $(\epsilon_k)_k$ is a Gaussian white noise sequence, X_k is the unobservable volatility process, $\delta = 1/n$ represents the discretization time step over the interval $[0, 1]$, and r is the riskless interest rate per unit of time.

We denote by S^0 the riskless asset price with dynamics :

$$S_{k+1}^0 = S_k^0 e^{r\delta}.$$

Notice that the conditional law of S_{k+1} given (X_k, S_k) has a density given by:

$$g(X_k, S_k, s') = \frac{1}{s'\sqrt{2\pi\delta X_k^2}} \exp\left[-\frac{\left(\ln s' - \ln S_k - (r - \frac{1}{2}X_k^2)\delta\right)^2}{2X_k^2\delta}\right], \quad s' > 0,$$

and notice that, as the first derivative of g with respect to s' is bounded, the hypothesis **H4** is satisfied.

The volatility (X_k) is described by a Markov chain taking three possible values $x^b < x^m < x^h$ in $(0, \infty)$. Its probability transition matrix is given by :

$$P_k = \begin{pmatrix} 1 - (p_{bm} + p_{bh})\delta & p_{bm}\delta & p_{bh}\delta \\ p_{mb}\delta & 1 - (p_{mb} + p_{mh})\delta & p_{mh}\delta \\ p_{hb}\delta & p_{hm}\delta & 1 - (p_{hb} + p_{hm})\delta \end{pmatrix}. \quad (5.1)$$

The volatility (X_k) is a Markov-chain approximation à la Kushner (see [8]) of a mean-reverting process :

$$dX_t = \lambda(x_0 - X_t)dt + \eta dW_t.$$

Denoting by $\Delta > 0$ the spatial step, this corresponds to a probability transition matrix of the form (5.1) with :

$$x^b = x_0 - \Delta, \quad x^m = x_0, \quad x^h = x_0 + \Delta,$$

and

$$
\begin{aligned}
p_{bm} &= \lambda + \frac{\eta^2}{2\Delta^2}, \qquad p_{bh} = 0 \\
p_{mb} &= \frac{\eta^2}{2\Delta^2}, \qquad p_{mh} = \frac{\eta^2}{2\Delta^2} \\
p_{hb} &= 0, \qquad p_{hm} = \lambda + \frac{\eta^2}{2\Delta^2},
\end{aligned}
$$

with the condition that $1 - \lambda - \frac{\eta^2}{2\Delta^2} > 0$ and $1 - \frac{\eta^2}{\Delta^2} > 0$.

In order to hedge the European put option with strike K, we invest an initial capital v_0 in the risky asset following a self financing strategy. Recall that the wealth process is given by:

$$
V_{k+1}^{\alpha} = V_k^{\alpha} e^{r\delta} + \alpha_k \left[S_{k+1} - S_k e^{r\delta} \right] \tag{5.2}
$$

where α_k represents the number of shares of asset S_k held in the portfolio at time k. Observe that (5.2) verifies the hypothesis **H2** and recall that the control process (α_k) is adapted with respect to the filtration $(\mathcal{F}_k^S)$ generated by the observation process.

In what follows we will work with the log-price instead of the price and we set $Y_k = \ln S_k$.

5.2 Hedging of an European put option: quadratic criterion

Using a quadratic loss criterion (see [6]), an optimal strategy is a solution to the optimization problem :

$$
\inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\left(\left(K - e^{Y_n} \right)_+ - V_n^{\alpha} \right)^2 \right] \tag{5.3}
$$

where $\mathcal{A}$ is the control space.

Since the process $(X_k)_{k=1,\dots,n}$ is unobservable, the optimization problem described above is a control problem under partial information and can thus be studied by using stochastic filtering and approximation techniques as shown in the previous sections. An approximated solution is in particular obtained from the following steps:

1. Quantization. Denoting by Π_k the filter process, we discretize the pair (Π_k, Y_k) by performing an optimal quantization as explained in paragraph 4.1.1. This procedure provides, for all instants k:

1. A N_k-point grid Γ_k which is a discretization of the state space of (Π_k, Y_k). This discretization is optimal in the sense specified in [12].
2. A matrix $\{\hat{r}_k^{ij}, i = 1, \dots, N_{k-1}, j = 1, \dots, N_k\}$, which approximates the probability transition of the Markov chain (Π_k, Y_k).

The controlled one dimensional process (V_k^{α}) is discretized using a regular N^V-point grid of $\mathbb{R}$ given by:

$$
\Gamma^V = (2\nu)\mathbb{Z} \cap [V_{inf}, V_{Sup}]
$$

where ν is some discretization space step and V_{inf} and V_{Sup} are the bounds of the grid size.

2. Dynamic Programming. Once the problem has been discretized, we use the dynamic programming algorithm to calculate an approximated solution:

$$\begin{aligned}
\hat{u}_n(w_n^i, v) &= \left(v - (K - e^{y_n^i})_+\right)^2 \quad \forall w_n^i = (\pi_n(i), y_n^i) \in \Gamma_n, \quad \forall\, v \in \Gamma^V \\
\hat{u}_k(w_k^i, v) &= \inf_{a \in A} \sum_{j=1}^{N_{k+1}} \hat{r}_{k+1}^{ij} \hat{u}_{k+1}\Big(w_{k+1}^j, \mathrm{Proj}_{\Gamma^V}\left(ve^{r\delta} + a(e^{y_{k+1}^j} - e^{y_k^i} e^{r\delta})\right)\Big) \\
&\qquad \forall\, w_k^i = (\pi_k(i), y_k^i) \in \Gamma_k, \ \ \forall\, v \in \Gamma^V, \ \ k = 0, ..., n-1.
\end{aligned}$$

Numerical tests are performed by using the following parameter values :

- Price at time 0 : $S_0 = 110$;
- Strike of the European put option: $K = 110$;
- Riskless interest rate over the interval $[0, 1]$: $r = 0.05$;
- Volatility : $x_0 = 0,15$, $\Delta = 0,05$, $\lambda = 0,1$, $\eta = 0,1$.
- Quantization of (Π, Y) : grids have same size N for each time period with step $\delta = \frac{1}{n}$ and they are obtained by using 10^6 iterations of the procedure described in [14];
- Discretization of V^α : we use a N^V-point grid defined by $\Gamma^V = (2\nu)\mathbb{Z} \cap [V_{inf}, V_{Sup}]$ where ν, V_{inf} and V_{Sup}, determined by performing some preliminary tests, are given by:

$$\nu = \frac{35}{2(N^V - 1)}, \quad V_{inf} = -10, \quad V_{Sup} = 15;$$

- Approximation of the optimal control : golden search method (see [10]) on $A = [-1, 1]$
- When not specified the number of time steps is $n = 5$

In order to study the effects of the quantization grid size N and uniform grid size N^V we plot the graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((K - e^{Y_n})_+ - V_n^\alpha)^2)$ for different values of N and N^V (Figures 1 and 3).

As expected, the global shape of the graph is parabolic, due to the quadratic hedging criterion that we have used. The minimum is reached at v_{min} which can be considered as the "quadratic hedging price" of our European put option.

Corresponding hedging strategies at time $t = 0$ are given in Tables 1 and 2, and Figure 2 displays the graph of α_0 as a function of the initial wealth V_0. We can observe that the strategy is nearly constant for $V_0 \in [2, 4]$, where the non constant values may be due to numerical imprecision. This result can be explained[1] by observing that in our example the discounted price process

[1]For more details concerning the quadratic hedging in the martingale case see [5].

$\tilde{S}_k = S_k e^{-rk\delta}$, $k = 0 \dots, n$, is a martingale and by applying the Kunita Watanabe decomposition to the discounted option payoff $F = e^{-r}(K - e^{Y_n})_+$, we get:

$$F = \mathbb{E}[F] + \sum_{k=1}^{n} \alpha_k^F \Delta\tilde{S}_k + R_n^F, \tag{5.4}$$

where $\Delta\tilde{S}_k := \tilde{S}_{k+1} - \tilde{S}_k$, α^F is an admissible control process, and R^F is a martingale orthogonal to $\tilde{S}_k$, i.e. $E[R_k^F \Delta\tilde{S}_k] = 0$, $k = 0, \dots, n$. Recalling the dynamics (5.2) of the wealth V_n^α, we can write again the objective function as :

$$\mathbb{E}\Big[\Big((K - e^{Y_n})_+ - V_n^\alpha\Big)^2\Big] \;\;=\;\; e^{2r}\mathbb{E}\Big[\big(F - v_0 - \sum_{k=0}^{n} \alpha_k \Delta\tilde{S}_k\big)_+^2\Big]. \tag{5.5}$$

By combining (5.4) and (5.5) and by exploiting the orthogonality between R^F and $\tilde{S}$ we obtain

$$\mathbb{E}\Big[\Big((K - e^{Y_n})_+ - V_n^\alpha\Big)^2\Big] \;\;=\;\; e^{2r}\left\{\Big(\mathbb{E}[F] - v_0\Big)^2 + \mathbb{E}\left[\Big(\sum_{k=1}^{n}(\alpha_k^F - \alpha_k)\Delta\tilde{S}_k\Big)^2\right] + \mathbb{E}\Big[(R_n^F)^2\Big]\right\},$$

which shows that the optimal control is always $\alpha^{opt} = \alpha^F$ regardless of v.

In Figure 4 and in the Table 3, we compare the European put option price under partial and complete observation when we increase the number of observations (i.e. the time step δ decreases to zero). Denoting by $N_{\Pi,Y}$ the number of grid points used in the partial observation case to make an optimal quantization of the pair (Π, Y), by $(N_{X,Y})$ the number of grid points used in the total observation case to make an optimal quantization of the pair (X, Y), and by $R = V_{sup} - V_{inf}$ the grid size in the discretization of the controlled variable V^α, we recall that the discretization error is of order

$$\left(N_{\Pi,Y}^{\frac{-1}{d+m-1}} + \nu + \frac{1}{R}\right)$$

for the partial observation case. For the total observation case we have:

$$\left(\frac{1}{N_{X,Y}} + \nu + \frac{1}{R}\right)$$

where $N_{X,Y} = mN_Y$ (see [14]). So, in order to obtain comparable results, given the uniform grid discretizing the variable V^α, we perform an optimal quantization of (Π, Y) and (X, Y) by using grid sizes $N_{\Pi,Y}$ and $N_{X,Y} = mN_Y$ such that :

$$N_Y \simeq N_{\Pi,Y}^{\frac{1}{d+m-1}}$$

where $d = 1$ and $m = 3$. That is why we have chosen $N_{\Pi,Y} = 1500$ and $N_{X,Y} = 45$.

We notice that when the number of observations increase (i.e. $\delta \to 0$), the partial observation price converges to the complete observation price; this is due to the fact that with observation performed in continuous time we are able to calculate the volatility given by the quadratic variation of the price process (e^Y).

Figure 5 shows that by working in a total observation setting the quadratic risk associated to a given initial wealth is smaller than the corresponding value obtained in the partial observation case. This is consistent with the fact that the filtration generated by the observation price is included in the full information filtration, and consequently the corresponding optimal cost function in the partial information case is larger than the one in the full information case.

5.3 Hedging of an European put option: shortfall risk criterion

Using the shortfall risk criterion (see [5]), an optimal strategy is a solution to the optimization problem :

$$\inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\left(\left(K-e^{Y_n}\right)_+ - V_n^\alpha\right)_+\right] \tag{5.6}$$

where $\mathcal{A}$ is the control space.

The Figure 6 and the Table 4 are obtained by applying the procedure described in the previous section with

$$V_{inf} = -15 \qquad \text{and} \qquad V_{Sup} = 25$$

The Figure 6 shows the graph of $V_0 \mapsto \inf_{\alpha\in A}\mathbb{E}((K-e^{Y_n})_+ - V_T^\alpha)_+)$ in the partial and in the total observation case. Notice that, as expected, the shortfall risk given by $\inf_{\alpha\in A}\mathbb{E}((K - e^{Y_n})_+ - V_T^\alpha)_+)$ decreases with the initial capital and becomes zero for approximately the same value of V_0 in the partial and total observation case. Notice also that if we tolerate a little risk we can considerably reduce the requested initial capital.

Moreover, as in the quadratic hedging, for a given initial value V_0, the shortfall risk obtained in a partial observation setting is greater than the corresponding one in a context of total observation.

In Table 4 we compare the initial capital required to minimize the quadratic risk and the shortfall risk associated with our European put option. As expected the initial capital necessary to minimize the quadratic risk is bounded by the corresponding one in the shortfall risk case, which is actually the super-hedging price.

Finally, Figure 7 displays the quadratic and shortfall risk for various values of the initial capital.

5.4 Hedging of an European call option

Quadratic Hedging: an optimal strategy is a solution to the optimization problem :

$$\inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\left(\left(e^{Y_n}-K\right)_+ - V_n^\alpha\right)^2\right] \tag{5.7}$$

where $\mathcal{A}$ is the control space. The procedure described in the previous section has been applied by taking:

$$V_{inf} = -20 \qquad \text{and} \qquad V_{Sup} = 30$$

Figure 8 shows the graph of $V_0 \mapsto \inf_{\alpha\in A}\mathbb{E}((e^{Y_n}-K)_+ - V_n^\alpha)^2)$ i.e. the quadratic risk as a function of the initial capital V_0. Notice that as expected the global shape of the graph is parabolic; the initial capital corresponding to the minimum can be interpreted as the "quadratic hedging price" of the European call option. Notice also that as in the European put case, for a given initial wealth, the corresponding quadratic risk in the partial observation case is greater than in the total observation case.

Figure 9 displays the graph of the optimal strategy at time $t = 0$ as a function of the initial capital. As in the put case the optimal strategy is nearly constant.

Finally, in Table 5 we compare the initial capital requested to minimize the quadratic risk ("quadratic hedging price") with the call price obtained by using the put-call parity relation and the "quadratic-hedging" put price calculated in the previous section. We can observe that the two prices are very close thus justifying further the expression "quadratic hedging price".

Shortfall risk criterion: an optimal strategy is a solution to the optimization problem :

$$\inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\left(\left(e^{Y_n} - K\right)_+ - V_n^\alpha\right)_+\right] \tag{5.8}$$

where $\mathcal{A}$ is the control space.

Figure 10 and Table 6 are obtained by applying the procedure described in the previous sections with

$$V_{inf} = -25 \qquad \text{and} \qquad V_{Sup} = 35$$

In Figure 10 we observe that, as expected, the shortfall risk decreases with the initial capital and becomes zero for approximatively the same initial value V_0 for the total and the partial observation case. We also notice that if we tolerate a little risk, we can considerably reduce the requested initial capital. Moreover the shortfall risk associated to a given initial value V_0 is greater in the partial observation case with respect to the total observation case.

In Table 6, we compare the initial capital requested to minimize the quadratic risk and the shortfall risk associated to our European call option. As expected the initial capital necessary to minimize the quadratic risk is bounded by the corresponding one in the shortfall risk case, which is actually the super-hedging price.

N	European put price	Optimal control strategy α_0
300	3.04132	-0.2813
600	3.05965	-0.2813
1500	3.07098	-0.2813

Table 1: **Quadratic hedging of an European put:** European put price (defined as the initial capital minimizing the risk) and optimal control strategy calculated for different quantization grid sizes ($N = 300, 600, 1500$) and a fixed uniform grid size ($N^V = 400$)

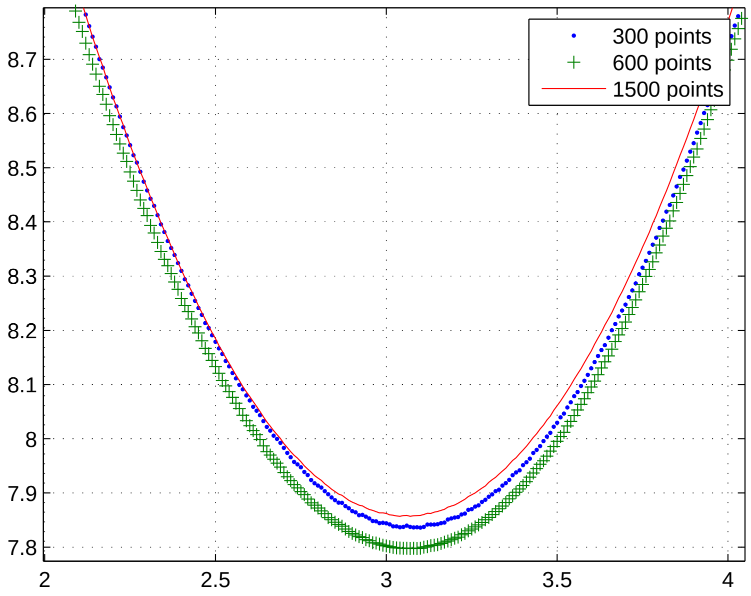

Figure 1: **Quadratic hedging of an European put:** graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((K - e^{Y_n})_+ - V_n^{\alpha})^2)$ for different quantification grid sizes ($N = 300, 600, 1500$) and a fixed uniform grid size ($N^V = 400$)

N^V	European put price	Optimal control strategy α_0
50	2.97501	-0.2813
100	3.04132	-0.2813
300	3.04132	-0.2813
400	3.04132	-0.2813

Table 2: **Quadratic hedging of an European put:** European put price (defined as the initial capital minimizing the risk) and optimal control strategy calculated for different fixed uniform grid sizes ($N^V = 50, 100, 200, 400$) and a fixed quantization grid size ($N = 300$)

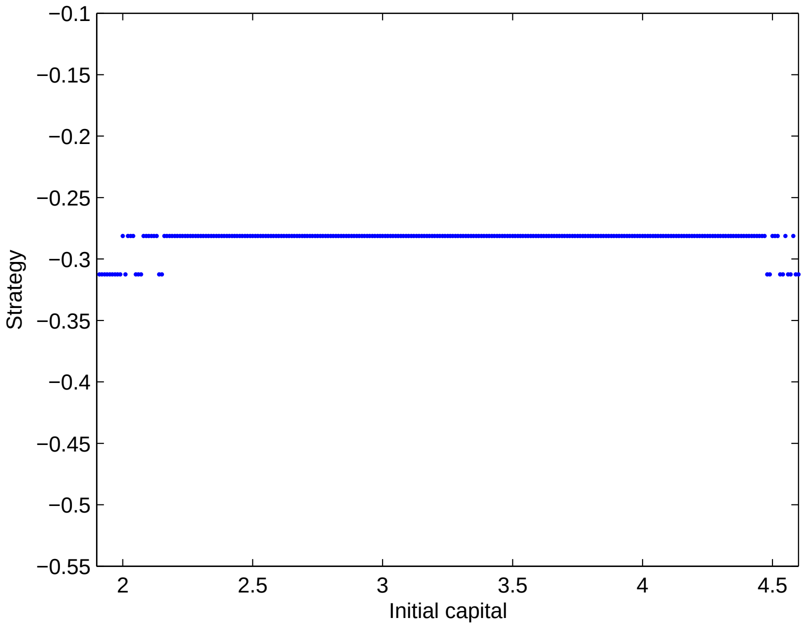

Figure 2: **Quadratic hedging of an European put:** graph of $V_0 \mapsto \alpha_0(V_0)$ for a quantization grid size of $N = 300$ and a fixed uniform grid size of $N^V = 400$

Time step δ	Partial observation price	Partial observation strategy	Total observation price	Total observation strategy
$1\backslash 5$	2.9933	-0.2813	3.24459	-0.2734
$1\backslash 10$	3.5255	-0.3013	3.65515	-0.2422
$1\backslash 20$	3.9501	-0.3215	4.02799	-0.3614

Table 3: **Quadratic hedging of an European put:** comparison between partial and total observation price (defined as the initial capital minimizing the quadratic risk) and strategies when we increase the number of observations and consequently the time step δ goes to 0.Size grid for $V^\alpha = 30$ points, size grid for $(e^Y, \Pi) = 1500$ points, size grid for $(e^Y, X) = 45$ points

Case	Quadratic hedging v_{min}	Shortfall hedging v_{min}	Quadratic hedging strategy	Shortfall hedging strategy
Total Observation	$3,5750$	~ 16	-0.2656	-0.98995
Partial Observation	3.07098	~ 17.8	-0.2813	-0.99187

Table 4: **European put option: comparison between quadratic hedging and shortfall hedging.** v_{min} is the initial capital requested to minimize the corresponding risk. Size grid for $V^\alpha = 100$ points, size grid for $(e^Y, \Pi) = 1500$ points, size grid for $(e^Y, X) = 45$ points

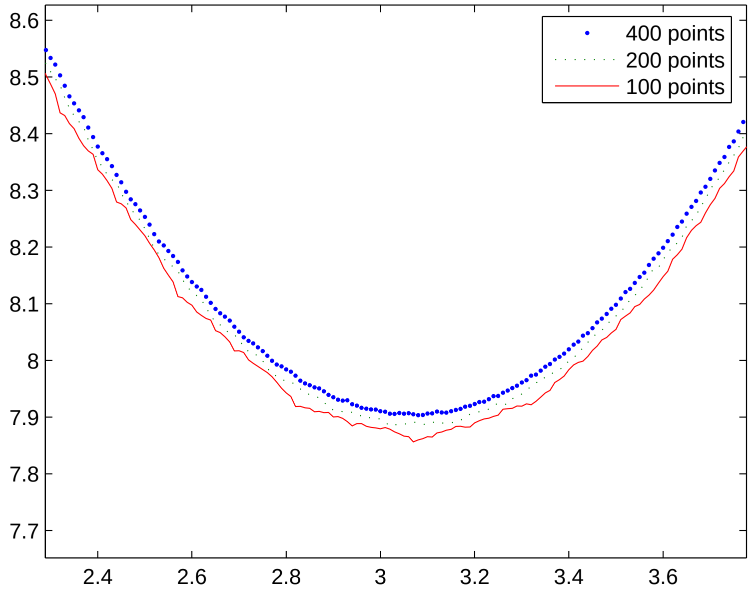

Figure 3: **Quadratic hedging of an European put:** graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((K - e^{Y_n})_+ - V_n^\alpha)^2)$ for different fixed uniform grid sizes ($N^V = 50, 100, 200, 400$) and a fixed quantization grid size ($N = 300$)

Case	Call price by quadratic hedging	Call price by call-put parity	Difference
Total Observation	8.74596	8.91377	0.1678
Partial Observation	8.55202	8.38009	0.1719

Table 5: **European call option: comparison between quadratic hedging and put-call parity.** Size grid for $V^\alpha = 100$ points, size grid for $(e^Y, \Pi) = 300$ points, size grid for $(e^Y, X) = 45$ points

Case	Quadratic hedging v_{min}	Shortfall hedging v_{min}	Quadratic hedging strategy	Shortfall hedging strategy
Total Observation	8.74596	~ 23.5	0.6973	0.6972
Partial Observation	8.55202	~ 24	0.6625	0.6250

Table 6: **European call option: comparison between quadratic hedging and shortfall hedging.** v_{min} is the initial capital requested to minimize the corresponding risk. Size grid for $V^\alpha = 100$ points, size grid for $(e^Y, \Pi) = 1500$ points, size grid for $(e^Y, X) = 45$ points

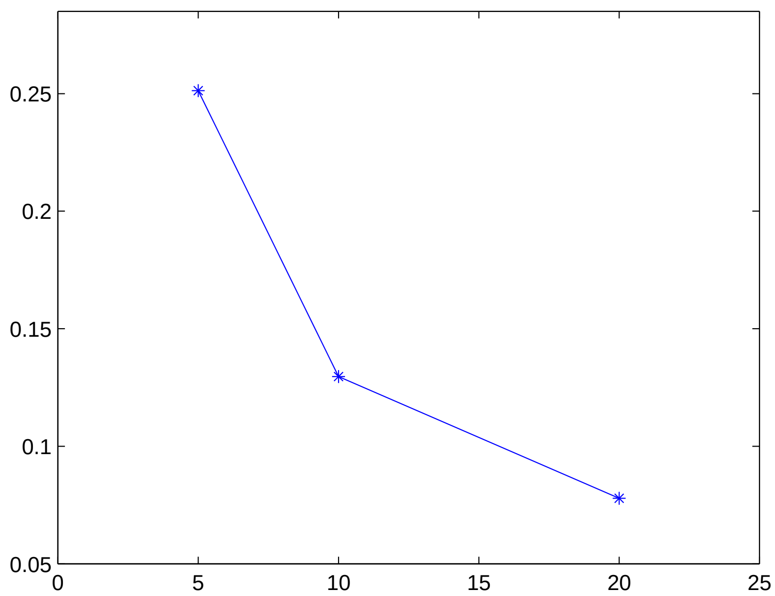

Figure 4: **Quadratic hedging of an European put:** distance between total and partial observation European put prices (defined as the initial capital minimizing the risk) when we increase the number of observations (axis of abscissae) and consequently the time step δ goes to 0. Size grid for $V^\alpha = 30$ points, size grid for $(e^Y, \Pi) = 1500$ points, size grid for $(e^Y, X) =$ 45 points

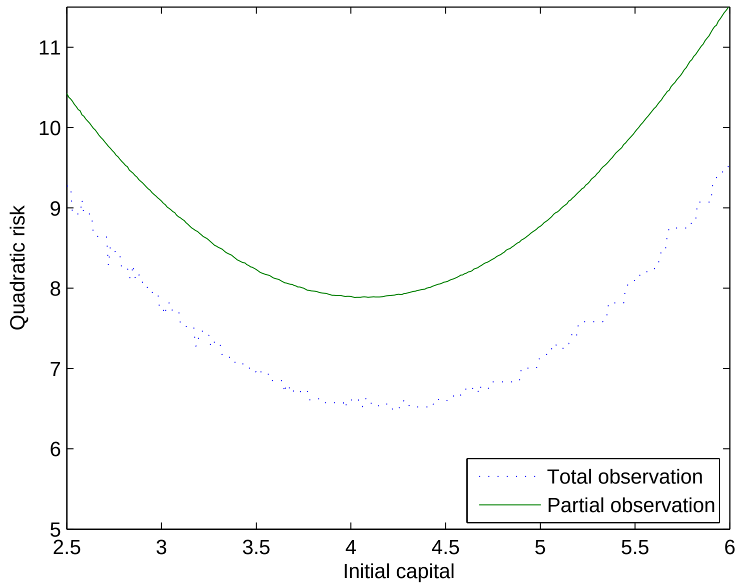

Figure 5: **Quadratic hedging of an European put**: graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((K - e^{Y_n})_+ - V_n)^2)$ in the partial and total observation case. Size grid for V $= 100$ points, size grid for $(e^Y, \Pi) = 1500$ points, size grid for $(e^Y, X) = 45$ points.

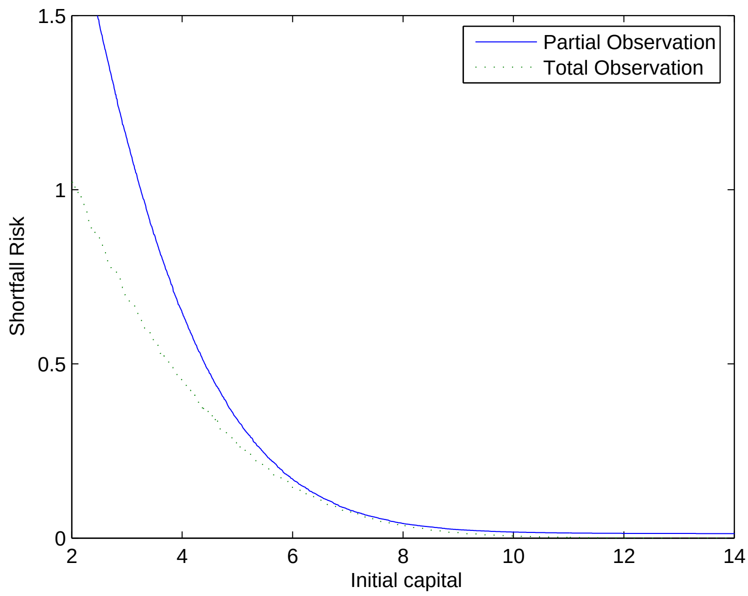

Figure 6: **European put option. Shortfall risk criterion:** graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((K - e^{Y_n})_+ - V_n)_+)$ in the partial and total observation case. Size grid for V $= 100$ points, size grid for $(e^Y, \Pi) = 600$ points, size grid for $(e^Y, X) = 45$ points.

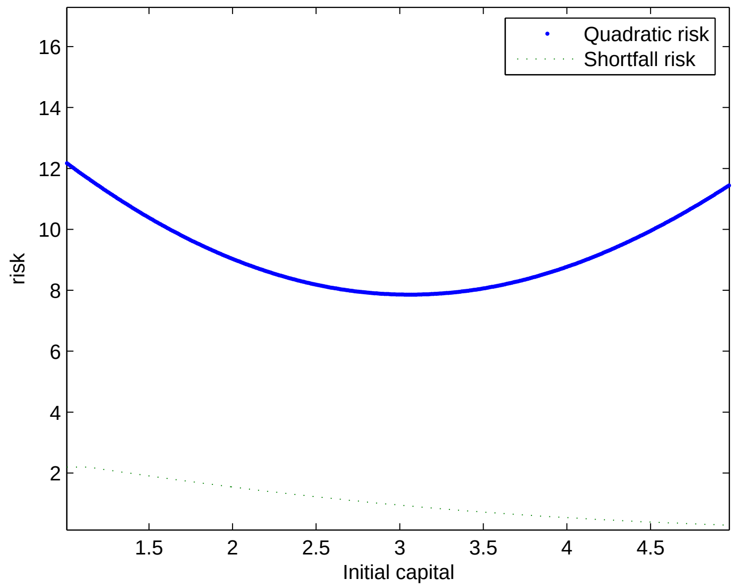

Figure 7: **European put option: comparison between quadratic hedging and shortfall hedging:** graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((K - e^{Y_n})_+ - V_n^\alpha)_+)$ in the partial observation case.Size grid for $V^\alpha = 100$ points, size grid for $(e^Y, \Pi) = 600$ points.

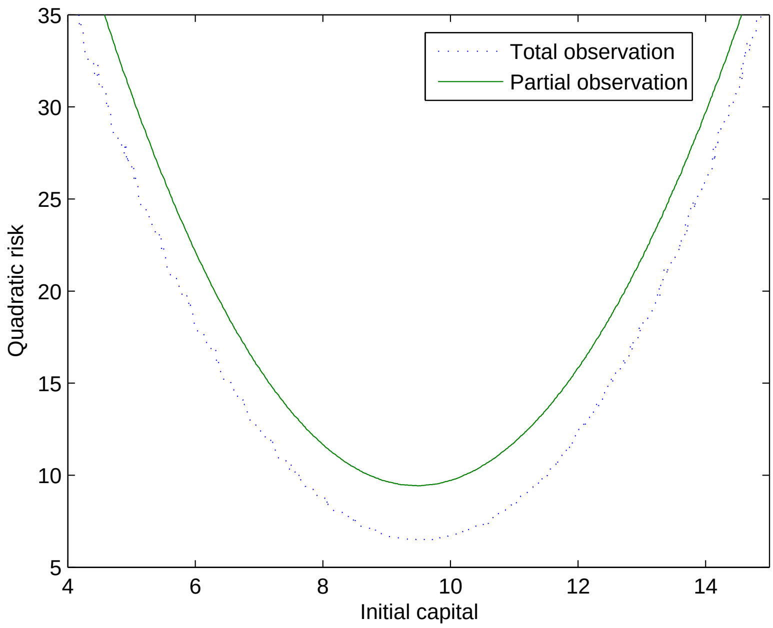

Figure 8: **Quadratic Hedging of an European call:** graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((e^{Y_n} - K)_+ - V_n^\alpha)^2)$. Size grid for $V^\alpha = 100$ points, size grid for $(e^Y, \Pi) = 600$ points

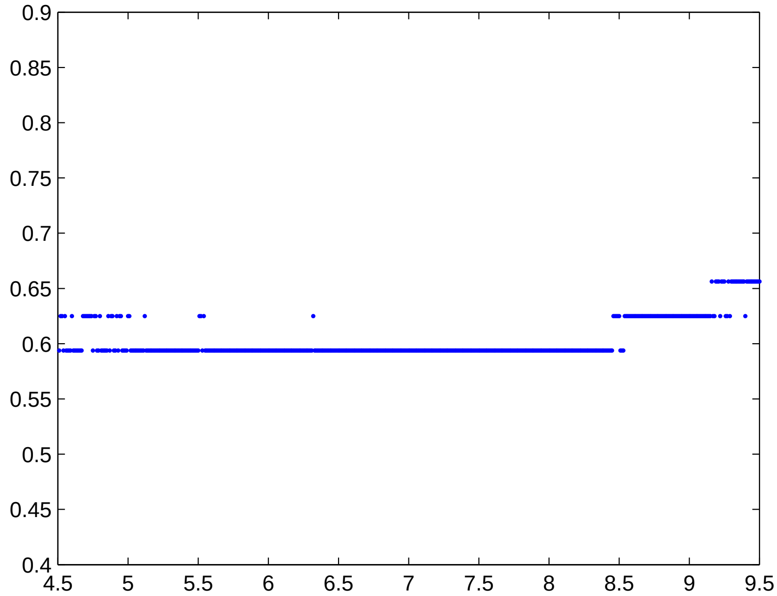

Figure 9: **Quadratic Hedging of an European call:** graph of $V_0 \mapsto \alpha_0(V_0)$. Size grid for $V^\alpha = 100$ points, size grid for $(e^Y, \Pi) = 600$ points.

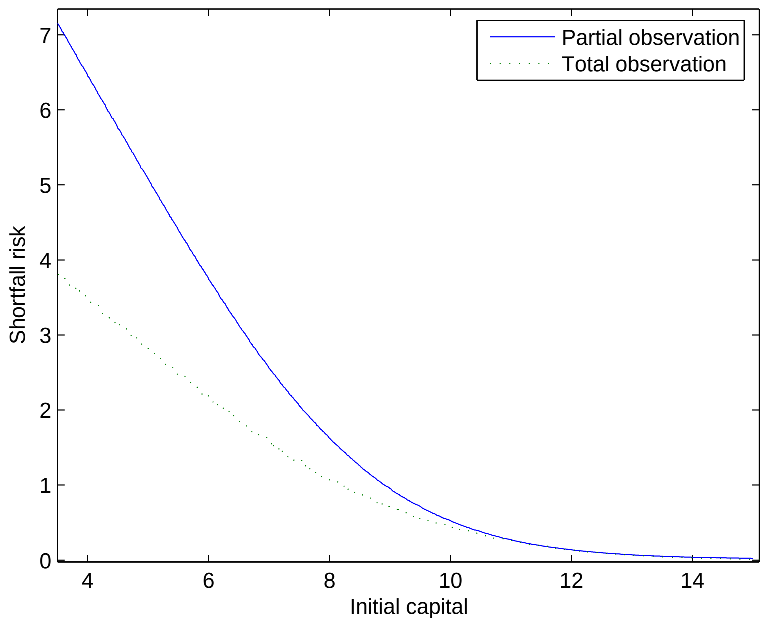

Figure 10: **European call option. Shortfall risk criterion:** graph of $V_0 \mapsto \inf_{\alpha \in A} \mathbb{E}((e^{Y_n} - K)_+ - V_n^\alpha)_+)$. Size grid for $V^\alpha = 100$ points, size grid for $(e^Y, \Pi) = 600$.

Appendix A: Proof of Proposition 3.1

We begin with a definition and a preliminary result:

Definition A.1. *Let* $\alpha = (\alpha_k)_k$ *be a fixed control process. Functions* u_k^α $(k = 0, ..., n)$ *are defined recursively by:*

$$\begin{cases} u_n^\alpha(\pi, y, v) & := & \hat{h}(\pi, y, v) \\ u_k^\alpha(\pi, y, v) & := & \hat{f}(\pi, y, v, \alpha_k) + \mathbb{E}\left[u_{k+1}^\alpha\left(\Pi_{k+1}, Y_{k+1}, H(v, \alpha_k, y, Y_{k+1})\right)\Big|(\Pi_k, Y_k) = (\pi, y)\right] \end{cases}$$

Lemma A.1. *Assume* **H1**, **H2** *and* **H3**. *Then there exists a control process* $\tilde{\alpha} = (\tilde{\alpha}_k)_k \in \mathcal{A}$*, such that for all* $k = 0, \ldots, n-1$*:*

$$u_k(\pi, y, v) = u_k^{\tilde{\alpha}}(\pi, y, v), \quad (\pi, y, v) \in K_m \times \mathbb{R}^d \times \mathbb{R}.$$

Proof. The function u_k is defined by:

$$u_k(\pi, y, v) = \inf_{a \in A}\left\{\hat{f}(\pi, y, v, a) + \mathbb{E}\left[u_{k+1}(\Pi_{k+1}, Y_{k+1}, H(v, a, y, Y_{k+1})\Big|(\Pi_k, Y_k) = (\pi, y)\right]\right\}$$

and we see that the terms in brackets are continuous functions with respect to (v, a, y). Indeed, $\hat{f}$ is Lipschitz, and the second term can be written as follows:

$$\begin{aligned} & F_k(\pi, y, v, a) := \\ = \;& \mathbb{E}\left[u_{k+1}\left(\Pi_{k+1}, Y_{k+1}, H(v, a, y, Y_{k+1})\right)\Big|(\Pi_k, Y_k) = (\pi, y)\right] \\ = \;& \mathbb{E}\left[u_{k+1}\left(\Pi_{k+1}, Y_{k+1}, V_{k+1}^\alpha\right)\big|\mathcal{F}_k^Y\right] \\ = \;& \mathbb{E}\left[u_{k+1}\left(\Pi_{k+1}, Y_{k+1}, H\left(v, a, y, Y_{k+1}\right)\right)\big|\mathcal{F}_k^Y\right] \\ = \;& \mathbb{E}\Big[\mathbb{E}\big[u_{k+1}\left(\Pi_{k+1}, Y_{k+1}, H\left(v, a, y, Y_{k+1}\right)\right)\big|\mathcal{F}_k\big]\Big|\mathcal{F}_k^Y\Big] \\ = \;& \mathbb{E}\Big[\int \sum_{i=1}^m u_{k+1}\left(\bar{G}_{k+1}\left(\pi, y, y'\right), y', H\left(v, a, y, Y_{k+1}\right)\right) g_{k+1}\left(X_k, y, x^j, y'\right) \\ & \mathbb{P}\big[X_{k+1} = x^j | X_k\big] dy'\Big|\mathcal{F}_k^Y\Big] \\ = \;& \int \sum_{i,j=1}^m u_{k+1}\left(\bar{G}_{k+1}\left(\pi, y, y'\right), y', H\left(v, a, y, Y_{k+1}\right)\right) g_{k+1}\left(x^i, y, x^j, y'\right) P_{k+1}^{ij}\Pi_k^i dy', \end{aligned}$$

which is a continuous function with respect to (π, y, v, a).

By exploiting this fact we build the requested control process following a backward recursion:

$$\begin{aligned} u_n(\pi, y, v) &= \hat{h}(\pi, y, v) \\ u_{n-1}(\pi, y, v) &= \inf_{a \in A}\left\{\hat{f}(\pi, y, v, a) + \mathbb{E}\left[u_n(\Pi_n, Y_n, H\left(v, a, y, Y_n\right))\Big|(\Pi_{n-1}, Y_{n-1}) = (\pi, y)\right]\right\} \\ &= \inf_{a \in A}\left[\hat{f}(\pi, y, v, a) + F_{n-1}(\pi, y, v, a)\right] \end{aligned}$$

Since A is a compact set and the argument of the infimum is a continuous function with respect to (π, y, v, a), we deduce the existence of

$$\tilde{\alpha}_{n-1}(\pi, y, v) \in \arg\min_{a \in A} \Big[\hat{f}(\pi, y, v, a) + F_{n-1}(\pi, y, v, a)\Big]$$

for almost every $(\pi, y, v) \in K_m \times \mathbb{R}^d \times \mathbb{R}^l$, which may be chosen to be Borel measurable by a classical measurable selection theorem (see proposition 7.33 in [4]). By using the same argument, at the generic time step k, we have :

$$\begin{aligned} u_k(\pi, y, v) &= \inf_{a \in A} \Big[\hat{f}(\pi, y, v, a) + \mathbb{E}\Big[u_{k+1}(\Pi_{k+1}, Y_{k+1}, H\,(v, a, y, Y_{k+1}))\Big|(\Pi_k, Y_k) = (\pi, y)\Big]\Big] \\ &= \hat{f}(\pi, y, v, \tilde{\alpha}_k(\pi, y, v)) + F_k(\pi, y, v, \tilde{\alpha}_k(\pi, y, v). \end{aligned}$$

Finally we define the $\mathcal{F}^Y$-adapted process $\tilde{\alpha}$ as follows:

$$\tilde{\alpha} := \big(\tilde{\alpha}_k(\Pi_k, Y_k, V_k)\big)_k$$

and we obtain by construction:

$$u_k(\pi, y, v) = u_k^{\tilde{\alpha}}(\pi, y, v) \quad \text{for all } k \geq 0$$

□

Proof of Proposition 3.1
We shall prove that

$$\inf_{\alpha \in \mathcal{A}} u_0^\alpha(\mu, y_0, v_0) \;=\; u_0(\mu, y_0, v_0) \;=\; J_{opt}(v_0). \tag{A.1}$$

First, we easily show by induction that :

$$u_k(\pi, y, v) \;\leq\; u_k^\alpha(\pi, y, v), \quad k = 0, \ldots, n, \;\; \alpha \in \mathcal{A}, \tag{A.2}$$

for all (π, y, v). Now, fix some arbitrary control $\alpha \in \mathcal{A}$. By taking expectation in the definition of u_k^α, we have:

$$\mathbb{E}[u_k^\alpha(\Pi_k, Y_k, V_k^\alpha)] = \mathbb{E}\Big[\hat{f}(\Pi_k, Y_k, V_k^\alpha, \alpha_k)\Big] + \mathbb{E}\Big[u_{k+1}^\alpha\big(\Pi_{k+1}, Y_{k+1}, V_{k+1}^\alpha\big)\Big], \quad k = 0, \ldots, n-1.$$

By adding up for k running from 0 to $n-1$, we get :

$$\begin{aligned} u_0^\alpha(\mu, y_0, v_0) &= \mathbb{E}\left[\sum_{k=0}^{n-1} \hat{f}\big(\Pi_k, Y_k, V_k^\alpha, \alpha_k\big) + u_n^\alpha\big(\Pi_n, Y_n, V_n^\alpha\big)\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{n-1} \hat{f}\big(\Pi_k, Y_k, V_k^\alpha, \alpha_k\big) + \hat{h}\big(\Pi_n, Y_n, V_n^\alpha\big)\right] \;=\; J(v_0, \alpha). \end{aligned} \tag{A.3}$$

From (A.2)-(A.3), we then get :

$$u_0(\mu, y_0, v_0) \;\leq\; \inf_{\alpha \in \mathcal{A}} J(v_0, \alpha) \;=\; J_{opt}(v_0). \tag{A.4}$$

Moreover, from Lemma A.1, there exists some $\tilde{\alpha} \in \mathcal{A}$ such that $u_0(\mu, y_0, v_0) = u_0^{\tilde{\alpha}}(\mu, y_0, v_0)$. Together with (A.3)-(A.4), this proves (A.1). □

Appendix B: Proof of Theorem 4.1

We first give some estimations on the functions u_k^α defined in (A.1).

Lemma B.1. *Assume* **H2***, then we have for all* $k = 0, \dots, n$*, and* $\alpha \in \mathcal{A}$ *:*

$$[u_k^\alpha]_{Sup} \leq (n-k)\bar{f} + \bar{h}$$

where

$$\bar{f} := \max([f]_{Sup}, [f]_{Lip}) \quad \textit{and} \quad \bar{h} := \max([h]_{Sup}, [h]_{Lip}).$$

Proof. By definition of u_k^α, we clearly have

$$[u_k^\alpha]_{Sup} \quad \leq \quad \bar{f} + [u_{k+1}^\alpha]_{Sup}$$

and so by induction :

$$[u_k^\alpha]_{Sup} \quad \leq \quad (n-k)\bar{f} + [u_n^\alpha]_{Sup} \leq (n-k)\bar{f} + \bar{h}$$

□

Lemma B.2. *Assume* **H2** *and* **H4** *, and set for all* $k = 0, \dots, n$*,* $(\pi, \hat{\pi}, y, \hat{y}, v, \hat{v}) \in K_m \times K_m \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$*,* $\alpha \in \mathcal{A}$ *:*

$$\begin{aligned} B_1(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha) \quad &= \quad \int \Big| u_{k+1}^\alpha\big(\bar{G}_{k+1}(\pi, y, y'), y', H(v, \alpha_k, y, y')\big) \\ &\qquad - u_{k+1}^\alpha\big(\bar{G}_{k+1}(\hat{\pi}, \hat{y}, y'), y', H(\hat{v}, \alpha_k, \hat{y}, y')\big)\Big| Q_{k+1}(\pi, y, dy') \end{aligned}$$

where $Q_k\left(\Pi_{k-1}, Y_{k-1}, dy'\right)$ *denotes the conditional law of* Y_k *given* (Π_{k-1}, Y_{k-1}) *and* $\bar{G}_k$ *is defined in (3.1). Then, we have*

$$\begin{aligned} B_1(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha) \quad &\leq \quad 2[u_{k+1}^\alpha]_{Lip}\left(L_g |y - \hat{y}]_1 + \|\pi - \hat{\pi}\|_1\right) \\ &\qquad + [H]_{Lip}\Big(|v - \hat{v}|_1 + |y - \hat{y}|_1\Big). \end{aligned}$$

Proof. Under assumption **H2**, we have :

$$\begin{aligned} B_1(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha) \quad &= \quad \int \Big| u_{k+1}^\alpha\big(\bar{G}_{k+1}(\pi, y, y'), y', H(v, \alpha_k, y, y')\big) \qquad \text{(B.1)} \\ &\qquad - u_{k+1}^\alpha\big(\bar{G}_{k+1}(\hat{\pi}, \hat{y}, y'), y', H(\hat{v}, \alpha_k, \hat{y}, y')\big)\Big| Q_{k+1}(\pi, y, dy') \\ &\leq \quad [u_{k+1}^\alpha]_{Lip} \int \Big|\bar{G}_{k+1}(\pi, y, y') - \bar{G}_{k+1}(\hat{\pi}, \hat{y}, y')\Big|_1 Q_{k+1}(\pi, y, dy') \\ &\qquad + \int \Big| H(v, \alpha_k, y, y') - H(\hat{v}, \alpha_k, \hat{y}, y')\Big|_1 Q_{k+1}(\pi, y, dy') \\ &\leq \quad [u_{k+1}^\alpha]_{Lip} \int \Big|\bar{G}_{k+1}(\pi, y, y') - \bar{G}_{k+1}(\hat{\pi}, \hat{y}, y')\Big|_1 Q_{k+1}(\pi, y, dy') \\ &\qquad + [H]_{Lip}\big(|v - \hat{v}|_1 + |y - \hat{y}|_1\big). \qquad \text{(B.2)} \end{aligned}$$

Now, from (3.1) and (3.2), we have :

$$
\begin{aligned}
& \int \Big|\bar{G}_{k+1}(\pi, y, y') - \bar{G}_{k+1}(\hat{\pi}, \hat{y}, y')\Big|_1 Q_{k+1}(\pi, y, dy') \\
= & \int \Big|\bar{G}_{k+1}(\pi, y, y') - \bar{G}_{k+1}(\hat{\pi}, \hat{y}, y')\Big|_1 q_{k+1}(\pi, y, y') dy'
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \sum_{i,j=1}^{m} \int \Big| \frac{g_{k+1}(x^i, y, x^j, y') P_k^{ij} \pi^i}{q_{k+1}(\pi, y, y')} - \frac{g_{k+1}(x^i, \hat{y}, x^j, y') P_k^{ij} \hat{\pi}^i}{q_{k+1}(\hat{\pi}, \hat{y}, y')} \Big| q_{k+1}(\pi, y, y') dy' \\
\leq\ & \sum_{i,j=1}^{m} P_k^{ij} \hat{\pi}^j \int \Big| \frac{g_{k+1}(x^i, y, x^j, y') q_{k+1}(\hat{\pi}, \hat{y}, y')}{q_{k+1}(\pi, y, y')} - \frac{g_{k+1}(x^i, \hat{y}, x^j, y') q_{k+1}(\pi, y, y')}{q_{k+1}(\hat{\pi}, \hat{y}, y')} \Big| dy' \\
& + \sum_{i=1}^{m} |\pi^i - \hat{\pi}^i| \\
\leq\ & \sum_{i,j=1}^{m} P_k^{ij} \int \big| g_{k+1}(x^i, y, x^j, y') - g_{k+1}(x^i, \hat{y}, x^j, y') \big| dy' \\
& + \int \big| q_{k+1}(\pi, y, y') - q_{k+1}(\hat{\pi}, \hat{y}, y') \big| dy' + \sum_{i=1}^{m} |\pi^i - \hat{\pi}^i| \\
\leq\ & 2 \sum_{i,j=1}^{m} P_k^{ij} \int \big| g_{k+1}(x^i, y, x^j, y') - g_{k+1}(x^i, \hat{y}, x^j, y') \big| dy' + 2 \sum_{i=1}^{m} |\pi^i - \hat{\pi}^i|. \qquad \text{(B.3)}
\end{aligned}
$$

Plugging (B.3) into (B.2), and using assumption **(H4)**, we get the required result. □

Lemma B.3. *Assume* **H4** *and set for all* $k = 0, \dots, n$, $(\pi, \hat{\pi}, y, \hat{y}, v, \hat{v}) \in K_m \times K_m \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$, $\alpha \in \mathcal{A}$ *:*

$$
\begin{aligned}
B_2(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha) \ = \ & \int \Big| u_{k+1}^{\alpha}\big(\bar{G}_{k+1}(\hat{\pi}, \hat{y}, y'), y', H(\hat{v}, \alpha_k, \hat{y}, y')\big) \Big| \\
& \Big[Q_{k+1}(\hat{\pi}, \hat{y}, dy') - Q_{k+1}(\pi, y, dy') \Big].
\end{aligned}
$$

Then, we have

$$
B_2(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha) \ \leq \ \big[u_{k+1}^{\alpha}\big]_{Sup} L_g |y - \hat{y}]_1 + \big[u_{k+1}^{\alpha}\big]_{Sup} |\pi - \hat{\pi}|_1.
$$

Proof. From (3.2), we have :

$$
\begin{aligned}
B_2(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha) \ \leq \ & [u_{k+1}^{\alpha}]_{Sup} \int \big| q_{k+1}(\hat{\pi}, \hat{y}, y') - q_{k+1}(\pi, y, y') \big| dy' \\
\leq \ & [u_{k+1}^{\alpha}]_{Sup} \sum_{i,j=1}^{m} P_k^{ij} \int \big| g_{k+1}(x^i, y, x^j, y') - g_{k+1}(x^i, \hat{y}, x^j, y') \big| dy' \\
& + [u_{k+1}^{\alpha}]_{Sup} \sum_{i=1}^{m} |\pi^i - \hat{\pi}^i|,
\end{aligned}
$$

and we conclude with **H4**. □

Lemma B.4. *Let* **H2**, **H3** *and* **H4** *hold. Then for all* $k = 0, \ldots, n$*, the function* u_k^α *is Lipschitz, uniformly with respect to* α *and*

$$[u_k^\alpha]_{Lip} \leq \mathcal{L}_k$$

where

$$\mathcal{L}_k := \left(\bar{L}_g \bar{f}(n-k) + \bar{M} + 3\bar{L}_g \bar{h}\right) \frac{\left(2\bar{L}_g\right)^{n-k}}{2\bar{L}_g - 1}$$

and $\bar{f} := \max([f]_{Sup}, [f]_{Lip})$, $\bar{h} := \max([h]_{Sup}, [h]_{Lip})$, $\bar{L}_g := \max(L_g, 1)$, $\bar{M} := \max([H]_{Lip}, 1)$.

Proof. We denote :

$$z \;:=\; (\pi, y, v), \quad \hat{z} := (\hat{\pi}, \hat{y}, \hat{v}), \quad Z_k^\alpha := (\Pi_k, Y_k, V_k^\alpha)$$

and we have :

$$\begin{aligned} [u_k^\alpha]_{Lip} &\leq [\hat{f}]_{Lip} + \left[\mathbb{E}\left(u_{k+1}^\alpha\left(Z_{k+1}^\alpha\right) \mid Z_k^\alpha = z\right)\right]_{Lip} \\ &= [\hat{f}]_{Lip} + [I_2]_{Lip} \end{aligned} \tag{B.4}$$

where

$$I_2 := \mathbb{E}\left[u_{k+1}^\alpha\left(Z_{k+1}^\alpha\right) \mid Z_k^\alpha = z\right].$$

We have for all $(\pi, \hat{\pi}, y, \hat{y}, v, \hat{y}, a) \in \times K_m \times K_m \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times A)$,

$$\begin{aligned} \left|\hat{f}(\pi, y, v, \alpha_k) - \hat{f}(\hat{\pi}, \hat{y}, \hat{v}, \alpha_k)\right| &= \left|\int f(x, y, v, \alpha_k)\pi(dx) - \int f(x, \hat{y}, \hat{v}, \alpha_k)\hat{\pi}(dx)\right| \\ &\leq \int |f(x, y, v, \alpha_k) - f(x, \hat{y}, \hat{v}, \alpha_k)|\, \pi(dx) \\ &\quad + \int |f(x, \hat{y}, \hat{v}, \alpha_k)|\, |\hat{\pi} - \pi|\, (dx) \\ &\leq [f]_{Lip}\left(|y - \hat{y}|_1 + |v - \hat{v}|_1\right) + [f]_{Sup}\, |\hat{\pi} - \pi|_1 \\ &\leq \bar{f}|z - \hat{z}|_1 \end{aligned}$$

where $\bar{f} := \max\left([f]_{Sup}, [f]_{Lip}\right)$. Therefore, $[\hat{f}]_{Lip} \leq \bar{f}$. Let us now consider now the **term** I_2.

By definition of Q_{k+1} and V^{α}_{k+1}, we have

$$
\begin{aligned}
& \Big| \mathbb{E}\big[u^{\alpha}_{k+1}(Z^{\alpha}_{k+1})\big|Z^{\alpha}_k = z\big] - \mathbb{E}\big[u^{\alpha}_{k+1}(Z^{\alpha}_{k+1})\big|Z^{\alpha}_k = \hat{z}\big] \Big| \\
= & \Big| \int u^{\alpha}_{k+1}\big(\bar{G}_{k+1}(\pi, y, y'), y', H(v, \alpha_k, y, y')\big) Q_{k+1}(\pi, y, dy') + \\
& - \int u^{\alpha}_{k+1}\big(\bar{G}_{k+1}(\hat{\pi}, \hat{y}, y'), y', H(\hat{v}, \alpha_k, \hat{y}, y')\big) Q_{k+1}(\hat{\pi}, \hat{y}, dy') \Big| \\
\leq & \int \Big| u^{\alpha}_{k+1}\big(\bar{G}_{k+1}(\pi, y, y'), y', H(v, \alpha_k, y, y')\big) \\
& \quad - u^{\alpha}_{k+1}\big(\bar{G}_{k+1}(\hat{\pi}, \hat{y}, y'), y', H(\hat{v}, \alpha_k, \hat{y}, y')\big) \Big| Q_{k+1}(\pi, y, dy') \\
& + \int \Big| u^{\alpha}_{k+1}\big(\bar{G}_{k+1}(\hat{\pi}, \hat{y}, y'), y', H(\hat{v}, \alpha_k, \hat{y}, y')\big) \Big| \Big[Q_{k+1}(\hat{\pi}, \hat{y}, dy') - Q_{k+1}(\pi, y, dy') \Big] \\
= & B_1(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha_k) + B_2(k, \pi, \hat{\pi}, y, \hat{y}, v, \hat{v}, \alpha_k). \qquad \text{(B.5)}
\end{aligned}
$$

By using lemmas B.2 and B.3, we then get

$$
\begin{aligned}
\Big| \mathbb{E}\big[u^{\alpha}_{k+1}(Z^{\alpha}_{k+1})\big|Z^{\alpha}_k = z\big] & - \mathbb{E}\big[u^{\alpha}_{k+1}(Z^{\alpha}_{k+1})\big|Z^{\alpha}_k = \hat{z}\big] \Big| \\
& \leq \Big(2[u^{\alpha}_{k+1}]_{Lip} + [u^{\alpha}_{k+1}]_{Sup} \Big) \Big(L_g \, |y - \hat{y}|_1 \, |\pi - \hat{\pi}|_1 \Big) \\
& \quad + [H]_{Lip} \Big(|v - \hat{v}|_1 + |y - \hat{y}|_1 \Big) \\
& \leq \Big(2[u^{\alpha}_{k+1}]_{Lip} + [u^{\alpha}_{k+1}]_{Sup} \Big) \, |z - \hat{z}|_1 \, \bar{L}_g + \bar{M} \, |z - \hat{z}|_1
\end{aligned}
$$

where $\bar{L}_g := \max(L_g, 1)$ and $\bar{M} := \max([H]_{Lip}, 1)$, and we deduce that :

$$
[I_2]_{Lip} \leq \Big(2[u^{\alpha}_{k+1}]_{Lip} + [u^{\alpha}_{k+1}]_{Sup} \Big) \bar{L}_g + \bar{M}. \qquad \text{(B.6)}
$$

Plugging (B.6) into (B.4) yields :

$$
[u^{\alpha}_k]_{Lip} \leq \bar{f} + \bar{M} + \bar{L}_g \Big(2[u^{\alpha}_{k+1}]_{Lip} + [u^{\alpha}_{k+1}]_{Sup} \Big)
$$

so that from lemma B.1 :

$$
\begin{aligned}
[u^{\alpha}_k]_{Lip} & \leq \bar{f} + \bar{M} + 2\bar{L}_g [u^{\alpha}_{k+1}]_{Lip} + \bar{L}_g \bar{h} + \bar{L}_g (n - k - 1) \\
& \leq 2\bar{L}_g [u^{\alpha}_{k+1}]_{Lip} + \bar{M} + \bar{L}_g \bar{h} + \bar{L}_g (n - k) \bar{f} \\
& \leq 2\bar{L}_g \Big\{ \bar{L}_g (n - k - 1) \bar{f} + \bar{M} + \bar{h} \bar{L}_g + 2\bar{L}_g \left[u^{\alpha}_{k+2} \right]_{Lip} \Big\} + \bar{M} + \bar{h} \bar{L}_g + \bar{L}_g (n - k) \bar{f} \\
& = \bar{L}_g \bar{f} \left[(n - k) + 2\bar{L}_g (n - k - 1) \right] + (\bar{M} + \bar{h} \bar{L}_g) \left[1 + 2\bar{L}_g \right] + \left(2\bar{L}_g \right)^2 \left[u^{\alpha}_{k+2} \right]_{Lip}.
\end{aligned}
$$

By induction, this yields :

$$\begin{aligned}
[u_k^\alpha]_{Lip} &\leq \bar{L}_g \bar{f} \sum_{i=0}^{n-k-1} \big(2\bar{L}_g\big)^i (n-k-i) + (\bar{M} + \bar{L}_g \bar{h}) \sum_{i=0}^{n-k-1} \big(2\bar{L}_g\big)^i + \big(2\bar{L}_g\big)^{n-k}\, \bar{h} \\
&\leq \Big(\bar{L}_g \bar{f}(n-k) + \bar{M} + \bar{L}_g \bar{h}\Big) \bigg(\frac{\big(2\bar{L}_g\big)^{n-k} - 1}{2\bar{L}_g - 1} \bigg) + \bar{h}\big(2\bar{L}_g\big)^{n-k} \\
&\leq \Big(\bar{L}_g \bar{f}(n-k) + \bar{M} + \bar{L}_g \bar{h} + \bar{h}(2\bar{L}_g - 1)\Big) \bigg(\frac{\big(2\bar{L}_g\big)^{n-k}}{2\bar{L}_g - 1} \bigg) \\
&\leq \Big(\bar{L}_g \bar{f}(n-k) + \bar{M} + 3\bar{L}_g \bar{h}\Big) \frac{\big(2\bar{L}_g\big)^{n-k}}{2\bar{L}_g - 1}.
\end{aligned}$$

Therefore,

$$[u_k^\alpha]_{Lip} \;\leq\; \Big(\bar{L}_g \bar{f}(n-k) + \bar{M} + 3\bar{L}_g \bar{h}\Big) \frac{\big(2\bar{L}_g\big)^{n-k}}{2\bar{L}_g - 1}$$

and the required result follows. □

We now study estimations for the approximated cost function. Similarly as in Definition A.1, we introduce the following sequence of functions :

Definition B.1. *Let $\alpha = (\alpha_k)_k$ be a control process in $\mathcal{A}$. Functions $\hat{u}_k^\alpha$, $k = 0, \ldots, n$, are defined recursively by :*

$$\begin{cases}
\hat{u}_n^\alpha(\pi, y, v) \;:=\; \hat{h}(\pi, y, v) \\[2ex]
\hat{u}_k^\alpha(\pi, y, v) \;:=\; \hat{f}(\pi, y, v, \alpha_k) + \mathbb{E}\left[\hat{u}_{k+1}^\alpha \Big(\hat{\Pi}_{k+1}, \hat{Y}_{k+1}, \hat{V}_{k+1}^\alpha \Big) \mid (\hat{\Pi}_k, \hat{Y}_k, \hat{V}_k^\alpha) = (\pi, y, v) \right].
\end{cases}$$

and we notice by same arguments as in Proposition 3.1 (see (A.1)) that

$$\inf_{\alpha \in \mathcal{A}} \hat{u}_0^\alpha(\mu, y_0, v_0) \;=\; \hat{u}_0(\mu, y_0, v_0) \;=\; \hat{J}_{quant}(v_0). \tag{B.7}$$

For any $\alpha \in \mathcal{A}$, we denote $Z_k^\alpha = (\Pi_k, Y_k, V_k^\alpha)$ and $\hat{Z}_k^\alpha = (\hat{\Pi}_k, \hat{Y}_k, \hat{V}_k^\alpha)$, $k = 0, \ldots, n$.

Lemma B.5. *Assume H1, H2, H3 and H4. Then, we have for all $k = 0, \ldots, n$, $\alpha \in \mathcal{A}$:*

$$\left\| u_k^\alpha\big(Z_k^\alpha\big) - \hat{u_k}^\alpha\big(\hat{Z}_k^\alpha\big) \right\|_1 \;\leq\; \mathcal{M}_k^{(\alpha)} \tag{B.8}$$

with:

$$\mathcal{M}_k^{(\alpha)} := \sqrt{m+d+q} \sum_{i=k}^{n} \left[2\left(\bar{L}_g \bar{f}(n-i) + \bar{M} + 3\bar{L}_g \bar{h}\right) \frac{\big(2\bar{L}_g\big)^{n-i}}{2\bar{L}_g - 1} + \bar{f} + \bar{h} \right] \left\| Z_i^\alpha - \hat{Z}_i^\alpha \right\|_2$$

Proof.

$$
\begin{aligned}
& \left\|u_k^\alpha(Z_k^\alpha) - \hat{u_k}^\alpha(\hat{Z}_k^\alpha)\right\|_1 \\
\le\ & \left\|u_k^\alpha(Z_k^\alpha) - u_k^\alpha(\hat{Z}_k^\alpha)\right\|_1 + \left\|u_k^\alpha(\hat{Z}_k^\alpha) - \mathbb{E}\left[u_k^\alpha(Z_k^\alpha)\middle|\hat{Z}_k^\alpha\right]\right\|_1 + \left\|\mathbb{E}\left[u_k^\alpha(Z_k^\alpha)\middle|\hat{Z}_k^\alpha\right] - \hat{u}_k^\alpha(\hat{Z}_k^\alpha)\right\|_1 \\
\le\ & 2\left\|u_k^\alpha(Z_k^\alpha) - u_k^\alpha(\hat{Z}_k^\alpha)\right\|_1 + \left\|\mathbb{E}\left[u_k^\alpha(Z_k^\alpha)\middle|\hat{Z}_k^\alpha\right] - \hat{u}_k^\alpha(\hat{Z}_k^\alpha)\right\|_1 \\
=\ & I_1 + I_2
\end{aligned}
\tag{B.9}
$$

with:

$$I_1 := 2\left\|u_k^\alpha(Z_k^\alpha) - u_k^\alpha(\hat{Z}_k^\alpha)\right\|_1$$

and

$$I_2 := \left\|\mathbb{E}\left[u_k^\alpha(Z_k^\alpha)\middle|\hat{Z}_k^\alpha\right] - \hat{u}_k^\alpha(\hat{Z}_k^\alpha)\right\|_1 .$$

Consider now the term I_2:

$$
\begin{aligned}
I_2 &= \left\|\mathbb{E}\left[\hat{f}(Z_k^\alpha,\alpha) + \mathbb{E}\left[u_{k+1}^\alpha\middle|Z_k^\alpha\right]\middle|\hat{Z}_k^\alpha\right] - \hat{f}(\hat{Z}_k^\alpha,\alpha) + \mathbb{E}\left[\hat{u}_{k+1}^\alpha\middle|\hat{Z}_k^\alpha\right]\right\|_1 \\
&= \left\|\mathbb{E}\left[\hat{f}(Z_k^\alpha,\alpha) - \hat{f}(\hat{Z}_k^\alpha,\alpha) + u_{k+1}^\alpha(Z_{k+1}^\alpha) - \hat{u}_{k+1}^\alpha(\hat{Z}_{k+1}^\alpha)\middle|\hat{Z}_k^\alpha\right]\right\|_1 \\
&\le \left\|\hat{f}(Z_k^\alpha,\alpha) - \hat{f}(\hat{Z}_k^\alpha,\alpha)\right\|_1 + \left\|u_{k+1}^\alpha(Z_{k+1}^\alpha) - \hat{u}_{k+1}^\alpha(\hat{Z}_{k+1}^\alpha)\right\|_1 \\
&\le \bar{f}\left\|\hat{Z}_k^\alpha - Z_k^\alpha\right\|_1 + \left\|u_{k+1}^\alpha(Z_{k+1}^\alpha) - \hat{u}_{k+1}^\alpha(\hat{Z}_{k+1}^\alpha)\right\|_1
\end{aligned}
\tag{B.10}
$$

Concerning the term I_1 we have :

$$I_1 \le 2\mathcal{L}_k\left\|Z_k^\alpha - \hat{Z}_k^\alpha\right\|_1 \tag{B.11}$$

where we have used the proposition B.4.

Plugging (B.11) and (B.10) into (B.9) yields :

$$
\begin{aligned}
& \left\|u_k^\alpha(Z_k^\alpha) - \hat{u_k}^\alpha(Z_k^\alpha)\right\|_1 \\
\le\ & \left(2\mathcal{L}_k + \bar{f}\right)\left\|Z_k^\alpha - \hat{Z}_k^\alpha\right\|_1 + \left\|u_{k+1}^\alpha(Z_{k+1}^\alpha) - \hat{u}_{k+1}^\alpha(\hat{Z}_{k+1}^\alpha)\right\|_1 \\
\le\ & \left(2\mathcal{L}_k + \bar{f}\right)\left\|Z_k^\alpha - \hat{Z}_k^\alpha\right\|_1 + \left(2\mathcal{L}_{k+1} + \bar{f}\right)\left\|Z_{k+1}^\alpha - \hat{Z}_{k+1}^\alpha\right\|_1 + \left\|u_{k+2}^\alpha(Z_{k+2}) - \hat{u}_{k+2}^\alpha(\hat{Z}_{k+2})\right\|_1 \\
\le\ & \sum_{i=k}^{n-1}\left\|Z_i^\alpha - \hat{Z}_i^\alpha\right\|_1\left(2\mathcal{L}_i + \bar{f}\right) + \bar{h}\left\|Z_n - \hat{Z}_n\right\|_1 \\
\le\ & \sum_{i=k}^{n}\left[2\left(\bar{L}_g\bar{f}(n-i) + \bar{M} + 3\bar{L}_g\bar{h}\right)\frac{\left(2\bar{L}_g\right)^{n-i}}{2\bar{L}_g - 1} + \bar{f} + \bar{h}\right]\left\|Z_i^\alpha - \hat{Z}_i^\alpha\right\|_1
\end{aligned}
$$

and the required result is proved by using the Cauchy-Schwarz inequality on $\left\|Z_i^\alpha - \hat{Z}_i^\alpha\right\|_1$. □

The term $\|Z_k^\alpha - \hat{Z}_k^\alpha\|_2$ represents the discretization error at time k and is bounded with the following estimation :

Lemma B.6. *Assume* **H2** *holds. Then, for each time step* $k = 0, \ldots, n$, *and* $\alpha \in \mathcal{A}$, *we have*

$$\|Z_k^\alpha - \hat{Z}_k^\alpha\|_2 \quad \leq \quad \Psi^k |v_0 - \textit{Proj}_{\Gamma^V}(v_0)| + \sum_{i=0}^{k} \Delta_{k-i}\Psi^i + \big(\nu + \frac{C_2}{R}\big) \sum_{i=0}^{k} \Psi^i \tag{B.12}$$

where $\Psi := [H]_{Lip}(2d+1)$, C_2 *is the maximum value of* H *over* $\Gamma \times A \times \cup_k \Gamma_k \times \cup_k \Gamma_k$ *and* Δ_k *is the* L^2 *quantization error at the time step* k *:*

$$\Delta_k \;=\; \big\|(\hat{\Pi}_k, \hat{Y}_k) - (\Pi_k, Y_k)\big\|_2.$$

Proof. By Minkowski's inequality, we have :

$$\|Z_k^\alpha - \hat{Z}_k^\alpha\|_2 \quad \leq \quad \big\|V_k^\alpha - \hat{V}_k^\alpha\big\|_2 + \Delta_k. \tag{B.13}$$

Recalling the dynamics (2.1) and (4.3), we have :

$$\big\|V_k^\alpha - \hat{V}_k^\alpha\big\|_2 \quad \leq \quad \big\|H_k^\alpha - \hat{H}_k^\alpha\big\|_2 + \big\|\hat{H}_k^\alpha - \mathrm{Proj}_\Gamma \hat{H}_k^\alpha\big\|_2, \tag{B.14}$$

where $H_k^\alpha := H(V_{k-1}^\alpha, \alpha_{k-1}, Y_{k-1}, Y_k)$ and $\hat{H}_k^\alpha := H(\hat{V}_{k-1}^\alpha, \alpha_{k-1}, \hat{Y}_{k-1}, \hat{Y}_k)$. Under **H2**, and by using Minkowski's and Cauchy-Schwarz' inequalities, we get :

$$\begin{aligned}
\big\|H_k^\alpha - \hat{H}_k^\alpha\big\|_2 \quad &\leq \quad [H]_{Lip}(2d+q)\Big(\big\|\hat{V}_{k-1}^\alpha - V_{k-1}^\alpha\big\|_2 + \big\|\hat{Y}_{k-1} - Y_{k-1}\big\|_2 + \big\|\hat{Y}_k - Y_k\big\|_2\Big) \\
&\leq \quad \Psi\Big(\big\|\hat{V}_{k-1}^\alpha - V_{k-1}^\alpha\big\|_2 + \Delta_{k-1} + \Delta_k\Big),
\end{aligned}$$

and so by (B.14) :

$$\begin{aligned}
\big\|V_k^\alpha - \hat{V}_k^\alpha\big\|_2 + \Delta_k \quad \leq \quad & \Psi\Big(\big\|\hat{V}_{k-1}^\alpha - V_{k-1}^\alpha\big\|_2 + \Delta_{k-1}\Big) \\
& + (\Psi+1)\Delta_k + \big\|\hat{H}_k^\alpha - \mathrm{Proj}_\Gamma \hat{H}_k^\alpha\big\|_2
\end{aligned}$$

Hence, a direct backward induction yields :

$$\begin{aligned}
\big\|V_k^\alpha - \hat{V}_k^\alpha\big\|_2 + \Delta_k \quad \leq \quad & \Psi^k|v_0 - \mathrm{Proj}_{\Gamma^V}(v_0)| + \sum_{i=0}^{k}\Delta_{k-i}\Psi^i \\
& + \sum_{i=0}^{k}\Psi^i\big\|\hat{H}_{k-i}^\alpha - \mathrm{Proj}_\Gamma \hat{H}_{k-i}^\alpha\big\|_2.
\end{aligned} \tag{B.15}$$

By noting that $|v - \mathrm{Proj}_{\Gamma^V}(v)| \leq \max(|v| - R, 0) + \nu$, for all $v \in \mathbb{R}$, we have

$$\begin{aligned}
\big\|\hat{H}_{k-i}^\alpha - \mathrm{Proj}_\Gamma \hat{H}_{k-i}^\alpha\big\|_2 \quad &\leq \quad \nu + \big\|\hat{H}_{k-i}^\alpha \mathbf{1}_{\{\hat{H}_{k-i}^\alpha \geq R\}}\big\|_2 &\text{(B.16)} \\
&\leq \quad \nu + \frac{1}{R}\big\|\hat{H}_{k-i}^\alpha\big\|_2 \\
&\leq \quad \nu + \frac{1}{R}C_2, &\text{(B.17)}
\end{aligned}$$

where we used Markov inequality. The requested result is proved by plugging (B.17) and (B.15) into (B.13). ☐

Proof of Theorem 4.1

This follows directly from the estimations (B.8) and (B.12) for $k = 0$, and from the relations (A.1) and (B.7).

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