Jump-Diffusion models

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Abstract

We discuss jump-diffusion type models for financial market as well as methods for pricing and hedging of contingent claims in such markets. We consider both, asset price and term structure models, and deal also with situations when there is a stochastic volatility correlated with the jumps and when one has very small time scales, i.e. high frequency data. To make the presentation possibly self-contained, in a preliminary section we reacall some basic notions from stochastic analysis for jump-diffusions.

Key words : jump-diffusions, Poisson point processes, marked point processes, Cox processes, hidden processes; martingale measures, market price of risk, pricing and hedging in incomplete markets, market completion, risk minimization; stochastic volatility, high frequency data, computing expectations of functionals of jump-diffusions.

1 Introduction

Most of the standard literature in Finance, in particular for pricing and hedging of contingent claims, is based on the assumption that the prices of the underlying assets follow a diffusion-type process, in particular a geometric Brownian motion (GBM).

Documentation from various empirical studies shows that such models are inadequate, both in relation to their descriptive power, as well as for the mispricing that they might induce. The contributions to the present volume deal with various generalizations of the basic GBM; here we concentrate on the fact that returns of various asset prices and interest rates may exhibit a jumping behaviour. We thus study possible superpositions of jump and diffusion processes, namely what is called *jump-diffusion processes*. Jump-diffusions form a particular class of Levy processes. Our purpose here is not to study the general case of a Levy driving process, but rather to concentrate on the specific aspects of the subclass of jump-diffusions. Jump-diffusion models have also some intuitive appeal in that they let prices and interest rates change continuously most of the time, but they also take into account the fact that from time to time larger jumps may occur that cannot be adequately modeled by pure diffusion-type processes.

Among the earlier empirical studies, documenting a jumping behaviour in prices and interest rates, one may quote [6], [35]. There are also studies, such as [4], putting forward specific sources of jumps in interest rates like moves by central banks. On the other hand, a first approach developing further the basic Black and Scholes (BS) model with the inclusion of jumps appears to be that of Merton [38]. Since the introduction of jumps in the BS model implies that derivative prices are no longer determined by the principle of absence of arbitrage alone, Merton solved the pricing problem by assuming that the jump risk was not systematic. This was later criticized by showing that such an assumption is equivalent to the existence of a market portfolio, that contains the underlying asset and that does not present a jumping behaviour (for a discussion on this point see e.g. [11]). Further studies then appeared showing that jumps in stock returns are indeed systematic. Another early approach is that in [14], where the market remains however complete since the authors consider just a simple jump-type process with fixed jump amplitude and thus with a single source of randomness. One of the major purposes of this paper is now to try to give an overview of the state of the art of jump-diffusion modeling in stock and bond markets as well as of the corresponding approaches for pricing and hedging.

It was further documented in empirical studies (see e.g. [5]) that a combination of jumps and stochastic volatility leads to even better fits and allows to avoid implied volatility skews. Stochastic volatility models are treated elsewhere and so in this paper we limit ourselves to stochastic volatility in conjunction with jump-diffusion modeling, also because empirical documentation gives evidence for a jump-type behaviour in the volatility and of a correlation between jumps in volatility and jumps in prices. In fact, as mentioned e.g. in [41], it is natural to expect that, if the volatility jumps, also the price should jump. A further purpose of the present paper is then to discuss issues related to such more general jump-diffusion- stochastic-volatility modeling.

On very small time scales actual prices do not really change continuously over time, but rather at discrete random points in time in reaction to trade and/or significant new information. We shall show that also such situations can be captured by models featuring a combination of diffusion and jump processes, that is however different from the canonical jump-diffusion processes.

The outline of the paper is as follows. In section 2 we recall some preliminary notions from stochastic analysis for jump-diffusion processes, such as a martingale representation result and generalized versions of the Ito formula as well as of the Girsanov measure transformation. We limit ourselves to those notions that will be used in the sequel. In section 3 we then describe various market models based on jump-diffusion representations. More precisely, in line with the introductory remarks above, we shall consider first canonical jump-diffusion models for stock and bond markets, then consider jump diffusions correlated with stochastic volatility and, finally, combinations of diffusions and jumps to describe high frequency data. In section 4 we discuss existence and uniqueness of martingale measures in a jump-diffusion setting, exhibiting also the market price of (jump-diffusion) risk. Some emphasis is given to the notion of *completion of the market* as a tool to obtain a unique martingale measure. In this context it is also pointed out that uniqueness of a martingale measure does not necessarily always imply completeness of the market in the sense of hedging, namely that every claim can be duplicated by a self financing portfolio. In section 5 we then concentrate on hedging in jump-diffusion market models having two goals in mind : first, investigating whether and when a market, that has been completed to yield a unique martingale measure, is also complete in the sense of hedging. Second, to study the hedging problem when the market cannot be completed or market completion is inappropriate. In such cases there is always some residual risk and so one may want to choose a strategy such as to minimize a criterion related to this risk. Finally, section 6 is devoted to the problem of pricing in jump-diffusion market models. With jumps and/or stochastic volatility, the market is incomplete. The principle of absence of arbitrage alone is then insufficient to define uniquely a price and so the preference structure of investors has to come into play to determine a *pricing measure*. From the point of view of pure pricing, the problem reduces formally to that of determining a specific martingale measure. In subsection 6.1 we mention various approaches to this effect, related to the literature, in particular approaches based on market completion and on the relationship between the choice of a hedging criterion and that of a martingale measure. Once a martingale measure has been chosen, there remains the problem of the actual computation of the expectation of the discounted claim and this is dealt with in subsection 6.2.

Unavoidably, this overview of the state of the art may turn out to be incomplete and reflects the specific interests and competences of the author. Among the topics that are not discussed here, we just mention the American-type options in a jump-diffusion setting (for this see e.g. [40], [43]) and Portfolio Optimization (see e.g. [20] and references therein). The same has to be said about the references to the literature : while we have tried to take into account a good deal of recent papers on the subject, we have only quoted a small selection of previous papers in order to keep the list within a reasonable size. Still, we hope to have succeeded in giving a sufficiently comprehensive account on models and methods related to jump-diffusions in financial markets.

2 Preliminaries

In this section we recall basic definitions and results needed for the study of jump-diffusion models, limiting ourselves to multivariate (univariate) and marked point processes and assuming that the reader is familiar with the corresponding notions concerning diffusion processes. In addition to the basic definitions we recall here a martingale representation result and discuss the Ito formula and Girsanov's measure transformation, generalized to jump-diffusion processes. The main reference for this section is [12], from which most of the contents of the section are taken.

2.1 Univariate point processes (Poisson jump processes)

A point process is intended to describe events that occur randomly over time. It can be represented as a sequence of nonnegative random variables

$$0 = T_0 < T_1 < T_2 < \cdots$$

where the generic T_n is the $n-$th instant of occurrence of an event. One makes the usual assumption of non-explosion, according to which

$$T_\infty = \lim \uparrow \ T_n = +\infty$$

The process may equivalently be represented via its associated counting process N_t where

$$N_t = n \quad \text{if} \quad t \in [T_n, T_{n+1}) \ , \ n \geq 0 \qquad \text{or, equivalently,} \quad N_t = \sum_{n \geq 1} 1_{\{T_n \leq t\}} \tag{1}$$

It counts the number of events up to and including time t. The nonexplosion condition becomes $N_t < \infty$ for $t \geq 0$. Both, T_n and N_t, are defined on some probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t$ to which N_t is adapted.

A point process N_t is called a Poisson point process if

i) $N_0 = 0$;

ii) N_t is a process with independent increments;

iii) $N_t - N_s$ is a Poisson random variable with a given parameter $\Lambda_{s,t}$.

Usually one assumes $\Lambda_{s,t} = \int_s^t \lambda_u du$ for a deterministic function λ_t; the latter is called the *intensity* of the Poisson point process N_t. If $\mathcal{F}_t$ is the filtration $\mathcal{F}_t^N$, generated by N_t, and $\lambda_t \equiv 1$, then N_t is called a *standard Poisson process*. It is also easily seen that, if N_t is a Poisson process with intensity $\lambda_t \equiv \lambda$, then $T_{n+1} - T_n$ are i.i.d., exponential random variables with parameter λ. A natural interpretation of the intensity and of this latter property comes from relating the above setup with the usual Poisson model, that is based on the following assumptions :

a) the probability of one change/jump in an interval of length Δ is $\lambda \cdot \Delta + o(\Delta)$;

b) the probability of two or more changes/jumps in an interval of length Δ is $o(\Delta)$;

c) the number of changes/jumps in nonoverlapping intervals are stochastically independent.

In this setup one can in fact consider the two "dually" related random variables :

1. A discrete random variable X describing the number of changes/jumps in a time interval of given length T and having as distribution the usual Poisson distribution, i.e.
$$P\{X=k\} = \frac{(\lambda T)^k}{k!}\, e^{-\lambda T}\ , \qquad k \in \mathbb{N}$$
2. A continuous random variable T describing the time that is needed to obtain k successive changes/jumps and for which the distribution is of the Gamma-type with density
$$f_T(t) = \frac{\lambda^k}{\Gamma(k)}\, t^{k-1}\, e^{-\lambda t}\ , \quad T>0$$

The parameter λ is the same in both cases and corresponds to the λ in assumption a) above.

It will be convenient to consider also the case when the intensity of a Poisson process is itself an adapted process being driven by some background process. This can be explained by a two-step randomization procedure : first one draws at random a trajectory of the background process, say Z_t; then one generates a Poisson process with intensity $\lambda_t = \lambda(t, Z_t)$, where the dependence also on t allows to incorporate seasonality effects. We now have a Poisson process N_t conditionally on Z_t and it is called a *doubly stochastic Poisson process*, or a *Cox process* (see [13]). Formally, we require that the random intensity λ_t is $\mathcal{F}_0-$measurable, i.e. $\mathcal{F}^Z_\infty \subset \mathcal{F}_0$. For additional details of the intensity of a Poisson process we refer to [12].

Notice that the above characterizations i)-iii) of a Poisson process parallel those of a Wiener process : both are processes with independent increments; the increments of a Wiener process are normally distributed, while those of a Poisson process are Poisson distributed. The Wiener process is the basic building block for processes with continuous trajectories, the Poisson process is a basic building block for processes with jumping trajectories. On the other hand, while the Wiener process is itself a martingale, a Poisson process as such is not. It becomes a martingale if one subtracts from N_t the process given by its mean. Indeed,
$$M_t := N_t - \int_0^t \lambda_s ds \tag{2}$$
is an $\mathcal{F}_t-$martingale by the $\mathcal{F}_0-$measurability of λ_t, assuming in addition that $E\left\{\int_0^t \lambda_u du\right\} < \infty$. By iii) one then has in fact
$$E\left\{N_t - N_s \mid \mathcal{F}_s\right\} = E\left\{\int_s^t \lambda_u du \mid \mathcal{F}_s\right\} \tag{3}$$
which implies that $E\{N_t\} < \infty$ and that M_t in (2) is an $\mathcal{F}_t-$martingale. Equality (3) admits a generalization in the form
$$E\left\{\int_0^\infty C_s dN_s\right\} = E\left\{\int_0^\infty C_s \lambda_s ds\right\} \tag{4}$$
that has to be valid for all nonnegative, $\mathcal{F}_t-$predictable processes C_t and as such characterizes a doubly stochastic Poisson process with intensity λ_t (see [12]).

2.2 Multivariate and marked point processes

Let T_n be a (univariate) point process and Y_n, $n \geq 1$ a sequence of random variables with values in $\{1, 2, \cdots, K\}$, all defined on the same $(\Omega, \mathcal{F}, P)$. For each $k = 1, \cdots, K$ we may then consider the counting process

$$N_t(k) := \sum_{n \geq 1} 1_{\{T_n \leq t\}} 1_{\{Y_n = k\}}$$

Each $N_t(k)$ is a univariate point process and the various $N_t(k)$'s have no common jumps, i.e. $\Delta N_t(k) \Delta N_t(h) = 0, \ \forall t \geq 0$ and all $k \neq h$. Analogously to the case of univariate point processes, here too we have now two equivalent representations, either as the double sequence $(T_n, Y_n)_{n \geq 1}$ or as the $K-$vector process $N_t = (N_t(1), \cdots, N_t(K))$ and this process is called a multivariate, more precisely a $K-$variate point process. As in the univariate case, here too we shall mainly use the representation as the $K-$vector process N_t and we have formula (2) with M_t a $K-$vector and λ_t the $K-$vector intensity process whose components are the individual intensisties of the components $N_t(k)$ of N_t.

Considering the representation (T_n, Y_n), we may interpret T_n as the $n-$th occurrence of some phenomenon and Y_n as an attribute or *mark* of this phenomenon. We may then speak of (T_n, Y_n) as a *marked point process*, or space-time point process and extend its definition to allow Y_n to take values in a general measurable *mark space* $(E, \mathcal{E})$. We synthesize the foregoing in the following

Definition 2.1 *An $E-$marked point process is a double sequence $(T_n, Y_n)_{n \geq 1}$ where*

i) *T_n is a (univariate) point process;*

ii) *Y_n is a sequence of $E-$valued random variables.*

Obviously, the univariate and multivariate point processes are special cases of a marked point process.

Generalizing the representation of a multivariate point process in the form of the $K-$vector process N_t, we associate to each $A \in \mathcal{E}$ the counting process

$$N_t(A) := \sum_{n \geq 1} 1_{\{T_n \leq t\}} 1_{\{Y_n \in A\}}$$

and let simply $N_t := N_t(E)$. Considering the filtration

$$\mathcal{F}_t^N := \sigma\{N_s(A);\, s \leq t,\, A \in \mathcal{E}\}$$

define the associated (random) counting measure

$$p\left((0, t],\, A\right) = N_t(A)\,, \ \ t \geq 0\,, \ \ A \in \mathcal{E} \tag{5}$$

which is $\sigma-$finite under the assumption of nonexplosion of T_n. This measure allows to obtain more concise expressions via integrals of the form

$$\int_0^t \int_E H(s, y)\, p(ds, dy) = \sum_{n \geq 1} H(T_n, Y_n)\, 1_{\{T_n \leq t\}} = \sum_{n=1}^{N_t} H(T_n, Y_n) \tag{6}$$

Again, we may represent an $E-$marked point process equivalently as the double sequence (T_n, Y_n) or as the counting measure $p(ds, dy)$.

To introduce now the intensity process in this more general setup, assume that for each $A \in \mathcal{E}$, the point process $N_t(A)$ admits the intensity $\lambda_t(A)$. This then leads to a measure-valued intensity $\lambda_t(dy)$ so that, generalizing (4), one has

$$E\left\{\int_0^\infty \int_E H(s,y)\, p(ds,dy)\right\} = E\left\{\int_0^\infty \int_E H(s,y)\, \lambda_s(dy)ds\right\} \tag{7}$$

that has to be valid for all nonnegative $\mathcal{F}_t-$predictable $E-$marked processes H (given a filtration $\mathcal{F}_t$ on Ω, $\mathcal{F}_t-$*predictability* here means measurability with respect to $\mathcal{P}(\mathcal{F}_t) \otimes \mathcal{E}$ where $\mathcal{P}(\mathcal{F}_t)$ is the predictable $\sigma-$field on $(0,\infty) \times \Omega$). We have also the generalization of (2) in the form

$$q(ds,dy) = p(ds,dy) - \lambda_s(dy)ds \tag{8}$$

where $q(ds, dy)$ is a (signed) measure-valued martingale in the sense that $\int_0^t \int_E H(s,y)\, q(ds,dy)$ is a $(P, \mathcal{F}_t)-$ martingale (local martingale) for each $\mathcal{F}_t-$predictable $E-$marked process H, satisfying appropriate integrability conditions. The most common form of intensity is

$$\lambda_t(dy) = \lambda_t\, m_t(dy) \tag{9}$$

where λ_t is nonnegative $\mathcal{F}_t-$predictable and represents the intensity of the Poisson process $N_t(E)$, while $m_t(dy)$ is a probability measure on E (typically, the Y_n will be i.i.d., independent of $N_t(E)$). The pair $(\lambda_t, m_t(dy))$ is called the $(P, \mathcal{F}_t)-$*local characteristics* of $p(ds, dy)$.

Notice finally that, as in the univariate case, we may let $\lambda_t(dy)$ depend on some driving $\mathcal{F}_0-$measurable random process Z_t, leading to a doubly stochastic marked point process. If, in the representation (9), λ_t is a deterministic time function, the marked point process is called a *marked Poisson process*.

2.3 Martingale representation

Martingale representation results are widely used in Finance, especially when it comes to solving hedging problems. For pure "Wiener-martingales" we have in fact the well-known result that every square integrable martingale with respect to the filtration generated by a Wiener process is, up to an additive constant, a stochastic integral of the Ito type. We shall now recall a corresponding result for point-process martingales that we formulate in the most general case of a marked point process. We have in fact the following theorem (see Thm. VIII, T8 in [12])

Theorem 2.2 *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a probability space satisfying the "usual assumptions" where $\mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{F}_t^p$ with $\mathcal{F}_t^p$ the filtration generated by a marked point process, represented by the counting measure $p(dt, dy)$. Then any $(P, \mathcal{F}_t)-$ martingale M_t admits the* *representation*

$$M_t = M_0 + \int_0^t \int_E H(s,y)\, q(ds,dy) \tag{10}$$

with $q(\cdot)$ as in (8) and H an integrable (with respect to $\lambda_t(dy)$) $\mathcal{F}_t-$predictable $E-$marked process. This representation is essentially unique.

In the case of a multivariate (and therefore also univariate) point process, the representation (10) becomes

$$M_t = M_0 + \sum_{k=1}^{K} \int_0^t H_s(k)\ (dN_s(k) - \lambda_s(k)\, ds) \tag{11}$$

where $[H_t(1), \cdots, H_t(K)]$ is $\mathcal{F}_t-$predictable with $H_t(k)$ integrable with respect to $\lambda_t(k)$.

This representation result can be generalized according to [29] to include martingales that are simultaneously "Wiener" and point-process martingales and that will have some relevance later on.

Theorem 2.3 *Given a Wiener process w_t and a marked point process $p(ds,dy)$, let*

$$\mathcal{F}_t := \sigma\left\{w_s,\, p((0,s],A), B\,;\ \ 0 \le s \le t\,,\ \ A \in \mathcal{E}\,,\ \ B \in \mathcal{N}\right\}$$

with $\mathcal{N}$ the collection of $P-$null sets from $\mathcal{F}$. Then any $(P, \mathcal{F}_t)-$local martingale M_t has the representation

$$M_t = M_0 + \int_0^t \phi_s dw_s + \int_0^t \int_E H(s,y)\ (p(ds,dy) - \lambda_s(dy)ds) \tag{12}$$

where ϕ_t is predictable and square integrable and H is an $\mathcal{F}_t-$predictable $E-$marked process, integrable with respect to $\lambda_t(dy)$.

2.4 Exponential formula; generalized Ito formula

With the definition of a marked point process and of integrals in the form of (6), we may now consider processes of the general type

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dw_s + \int_0^t \int_E \gamma(s,y)\, p(ds,dy) \tag{13}$$

that are called *jump-diffusion processes* and where the coefficients satisfy the implicit integrability conditions, β_t is adapted and $\gamma(t,y)$ is predictable in the sense as defined previously. As usual, we may rewrite (13) in differential form and consider, more specifically, differential equations of the type

$$dX_t = X_{t-}\left(\alpha_t dt + \beta_t dw_t + \int_E \gamma(t,y)\, p(dt,dy)\right) \tag{14}$$

where we write X_{t-} with $t-$ because of the predictability requirement in the last coefficient and where $\gamma(t,y) > -1$. Notice that (see (6)) the last term in (14) can also be written as

$$\int_E \gamma(t,y)\, p(dt,dy) = \gamma(t, Y_t)\, dN_t \tag{15}$$

where $N_t = N_t(E) = p((0,t], E)$ is the total number of jumps and Y_t denotes the piecewise constant, left-continuous time interpolation of the sequence Y_n. Notice also that, in the case of a multivariate (in particular univariate) point process, this last term in (14) takes the form

$$\int_E \gamma(t,y)\, p(ds,dy) = \sum_{k=1}^{K} \gamma_t(k)\, dN_t(k) \tag{16}$$

We shall not discuss here in detail equations of the form (14), in particular the uniqueness of their solutions, but limit ourselves to show that a solution to (14) is given by the following **Exponential formula**

$$\begin{aligned} X_t &= X_0 \exp\left[\int_0^t \left(\alpha_s - \frac{1}{2}\beta_s^2\right) ds + \int_0^t \beta_s dw_s + \int_0^t \log\left(1+\gamma(s,Y_s)\right)\, dN_s\right] \\ &= X_0 \exp\left[\int_0^t \left(\alpha_s - \frac{1}{2}\beta_s^2\right) ds + \int_0^t \beta_s dw_s\right] \prod_{n=1}^{N_t} \left(1+\gamma(T_n,Y_n)\right) \end{aligned} \tag{17}$$

While the diffusion part in this expression follows from the usual Ito formula, the jump part follows from the so-called *exponential formula* of Stieltjes-Lebesgue Calculus (see Theoren T4 of Appendix A4 in [12]), but it can also be obtained from the generalized Ito formula as we shall show next. For this purpose let a process X_t satisfy the general equation (13). Given a $\mathcal{C}^{1,2}-$function $F(t,X)$, we have the **generalized Ito formula**

$$\begin{aligned} dF(t,X_t) = F_t(\cdot)dt + F_X(\cdot)\alpha_t dt + \tfrac{1}{2}F_{XX}(\cdot)\beta_t^2 dt + F_X(\cdot)\beta_t dw_t \\ + \left[F\left(t, X_{t-} + \gamma(t,Y_t)\right) - F(t,X_{t-})\right] dN_t \end{aligned} \tag{18}$$

that, in the specific case of (14), becomes

$$\begin{aligned} dF(t,X_t) = F_t(\cdot)dt + F_X(\cdot)X_t\alpha_t dt + \tfrac{1}{2}F_{XX}(\cdot)X_t^2\beta_t^2 dt + F_X(\cdot)X_t\beta_t dw_t \\ + \left[F\left(t, X_{t-}(1 + \gamma(t,Y_t))\right) - F(t,X_{t-})\right] dN_t \end{aligned} \tag{19}$$

and where, again, $N_t = N_t(E) = p\left((0,t], E\right)$ and $(\cdot)$ stands for (t, X_t); the pedices in F denote partial derivatives. Notice that, if (19) is written in integral form, for the last term on the right we have the two equivalent representations

$$\int_0^t \left[F\left(s, X_{s-}(1+\gamma(s,Y_s))\right) - F(s,X_{s-})\right] dN_s = \sum_{n=1}^{N_t} \left[F(T_n, X_{T_n}) - F(T_n, X_{T_n^-})\right]$$

where the right hand side remains the same also in the more general case of (18).

We shall now use the generalized Ito formula (19) to obtain the solution (17) of equation (14). Choosing $F(t,X) = \log X$, from (19) and (14) we have

$$dF = \alpha_t dt - \frac{1}{2}\beta_t^2 dt + \beta_t dw_t + \log(1+\gamma(t,Y_t))\, dN_t$$

from which

$$\log X_t = \log X_0 + \int_0^t \left(\alpha_s - \frac{1}{2}\beta_s^2\right) ds + \int_0^t \beta_s dw_s + \int_0^t \log(1+\gamma(s,Y_{s-}))\, dN_s \tag{20}$$

i.e. we obtain (17) by taking the exponential on both sides in (20).

2.5 Absolutely continuous transformation of measures

We recall that in the classical case of Wiener driven diffusion processes, the Girsanov-type measure transformation concerns a translation of the Wiener process that in turn induces a change in the drift of the diffusion equation. In view of its generalization below, we recall here the basic result of Girsanov's transformation, conveniently reformulated for a finite time horizon $t \in [0, T]$.

Theorem 2.4 *[Girsanov's measure transformation] Given a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *with* $\mathcal{F} = \cup_t \mathcal{F}_t$*, let* $t \in [0, T]$ *with* T *given and* θ_t *be a square integrable predictable process. Define* $L = (L_t)$ *by*

$$dL_t = L_t \theta_t dw_t \quad , \quad L_0 = 1 \tag{21}$$

and suppose that, for all t, $E^P\{L_t\} = 1$*. Then there exists a probability measure* Q *on* $\mathcal{F}$*, equivalent to* P*, with* $dQ = L_T\, dP$ *such that*

$$dw_t = \theta_t dt + dw_t^Q \tag{22}$$

where w_t^Q *is a* $Q-$*Wiener process.*

Conversely, if $\mathcal{F}_t = \mathcal{F}_t^w$*, then every probability measure* Q*, equivalent to* P*, has the above structure.*

Notice that the second statement relies on martingale representation and requires thus the filtration $\mathcal{F}_t$ to be the one generated by the Wiener process.

As mentioned, Girsanov's measure transformation allows to change the drift in a diffusion equation. In fact, suppose that under P we have

$$dX_t = a_t X_t dt + \sigma_t X_t dw_t$$

and that we would like to change to a measure $Q \sim P$ ($\sim$ meaning *equivalent to*), under which the same X_t satisfies

$$dX_t = r_t X_t dt + \sigma_t X_t dw_t^Q$$

In this case just take $\theta_t = \sigma_t^{-1}(r_t - a_t)$.

If, besides a Wiener w_t, we now have also a marked point process represented by a counting measure $p(dt, dy)$, a Girsanov-type measure transformation allows, in addition to the translation of the Wiener, to perform also a change in the intensity process of the point process part. We have (see Thm. VIII,T10 in [12], see also [10])

Theorem 2.5 *On the finite time interval* $[0, T]$ *let* $p(dt, dy)$ *be an* $E-$*marked point process with* $(P, \mathcal{F}_t)-$*local characteristics* $(\lambda_t, m_t(dy))$*. Let* $\psi_t \geq 0$ *be* $\mathcal{F}_t-$ *predictable and* $h_t(y) \geq 0$ *an* $\mathcal{F}_t-$*predictable* $E-$*indexed process such that,* $P-$*a.s. and for all* $t \in [0, T]$*,*

$$\int_0^t \psi_s \lambda_s ds < \infty \quad ; \quad \int_E h_t(y) m_t(dy) = 1$$

Define $L_t = L_t^{(1)} \cdot L_t^{(2)}$ where $L_t^{(1)}$ satisfies (21) and $L_t^{(2)}$ satisfies

$$dL_t^{(2)} = \int_E (\psi_t h_t(y) - 1)\ L_{t-}^{(2)} q(dt, dy) \tag{23}$$

with $q(dt, dy) = p(dt, dy) - \lambda_t m_t(dy)dt$ the martingale measure associated with $p(dt, dy)$. If $E^P\{L_t^{(2)}\} = 1$ for all t, then all the statements of Theorem 2.4 hold true in addition to the fact that $p(dt, dy)$ has the $(Q, \mathcal{F}_t)-$local characteristics $(\psi_t \lambda_t, h_t(y)m_t(dy))$.

Notice that, using (21) and (23), we have for the Radon-Nikodym derivative L_t

$$\begin{aligned} dL_t &= d\left(L_t^{(1)} \cdot L_t^{(2)}\right) = L_{t-}^{(1)} dL_t^{(2)} + L_t^{(2)} dL_t^{(1)} \\ &= L_t \theta_t dw_t + L_{t-} \int_E (\psi_t h_t(y) - 1)\ q(dt, dy)\,, \quad L_0 = 1 \end{aligned} \tag{24}$$

Using the exponential formula (17), we have that a solution of (24) is given by

$$\begin{aligned} L_t &= \exp\left\{-\tfrac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s dw_s\right\} \\ &\cdot \exp\left\{\int_0^t \int_E (1 - \psi_s h_s(y))\lambda_s m_s(dy) ds\right\} \prod_{n=1}^{N_t} (\psi_{T_n} h_{T_n}(Y_n)) \end{aligned} \tag{25}$$

In the case of a multivariate (in particular univariate) point process $(N_t(1), \cdots, N_t(K))$ with $(P, \mathcal{F}_t)-$intensities $(\lambda_t(1), \cdots, \lambda_t(K))$, consider an $\mathcal{F}_t-$predictable process $(\psi_t(1), \cdots, \psi_t(K))$ such that, $P-$a.s. and for $t \in [0, T]$, $\sum_{k=1}^K \int_0^t \psi_s(k)\lambda_s(k) ds < \infty$. Define then $L_t^{(2)}$ by

$$dL_t^{(2)} = \sum_{k=1}^K (\psi_t(k) - 1) L_{t-}^{(2)}\ (dN_t(k) - \lambda_t(k)dt) \tag{26}$$

instead of by (23) making also corresponding changes in (24) and (25) for the Radon-Nikodym derivative L_t, namely

$$L_t = \exp\left\{-\frac{1}{2}\int_0^t \theta_s^2 ds + \int_0^t \theta_s dw_s\right\} \prod_{k=1}^K \left[\exp\left\{\int_0^t (1 - \psi_s(k))\lambda_s(k) ds\right\} \prod_{n=1}^{N_t(k)} \psi_{T_n}(k)\right] \tag{27}$$

Then, under Q, the intensities become $(\psi_t(1)\lambda_t(1), \cdots, \psi_t(K)\lambda_t(K))$.

Notice, finally, that a condition to have $E^P\{L_t^{(2)}\} = 1$ can be found in Thm. VIII,T11 of [12].

3 Market models with jump-diffusions

In this section we introduce various jump-diffusion type models that were studied in the literature and that we shall be dealing with in the sequel. In the first two subsections we discuss, for asset price and term structure models respectively, the canonical jump-diffusion models in which there are two additive terms : a diffusion term and a jump term. In the last two subsections we then discuss diffusion/jump-diffusion models with stochastic volatility, where the latter is also described in terms of a jumping process. In addition, in the last subsection we model asset price behaviour on very small time scales where actual prices do not change continuously in time but rather at discrete random time points in reaction to trades and significant information. This then leads to a rather peculiar combination of diffusion and jump processes.

3.1 Asset-price and term structure models with additive jumps

As mentioned in the Introduction, the asset price evolution can perhaps be adequately described by a GBM for most of the time, but from time to time a large jump may occur and this cannot be adequately captured by a GBM. It appears thus natural to introduce models, where a jump process can be superimposed on a GBM, e.g. by adding to the diffusion term also a jump term. In a first subsection we discuss this modeling issue in the context of asset prices, while in the second subsection we concentrate on interest rate modeling.

3.1.1 Asset price models with jumps

In this section we adapt the outline of section 7.2 in [36]. Let the price S_t of a risky asset jump at the random times $T_1, \cdots, T_n, \cdots$ and suppose that the relative/proportional change in its value at a jump time is given by $Y_1, \cdots, Y_n, \cdots$ respectively. We may then assume that, between two jump times, the price S_t follows a Black and Scholes model for a Wiener process w_t, that T_n are the jump times of a Poisson process N_t with intensity λ_t and that Y_n is a sequence of random variables with values in $(-1, \infty)$. This description can be formalized by letting, on the intervals $[T_n, T_{n+1})$,

$$dS_t = S_t \left(\mu_t dt + \sigma_t dw_t\right) \tag{28}$$

while, at $t = T_n$, the jump is given by $\Delta S_n = S_{T_n} - S_{T_n^-} = S_{T_n^-} Y_n$ so that

$$S_{T_n} = S_{T_n^-}(1 + Y_n) \tag{29}$$

which, by the assumption that $Y_n > -1$, leads always to positive values of the prices. Using the standard Ito formula to obtain the solution to (28) as well as a recursive argument based on (29), it is easily seen that, at the generic time t, S_t can be given the following equivalent representations

$$\begin{aligned} S_t &= S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dw_s\right] \left[\prod_{n=1}^{N_t} (1 + Y_n)\right] \\ &= S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dw_s + \sum_{n=1}^{N_t} \log(1 + Y_n)\right] \\ &= S_0 \exp\left[\int_0^t \left(\mu_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dw_s + \int_0^t \log(1 + Y_s)\, dN_s\right] \end{aligned} \tag{30}$$

where, as before, Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation. By the generalized Ito formula (19), the process S_t in (30) is easily seen to be a solution of

$$dS_t = S_{t-}\left[\mu_t dt + \sigma_t dw_t + Y_t dN_t\right] \tag{31}$$

This equation corresponds to (28) with the addition of a jump term and is a particular case of the general jump-diffusion model (14) ((15)) when $\gamma(t,y) = y$. In what follows we shall thus consider the more general version of (31) given by

$$dS_t = S_{t-}\left[\mu_t dt + \sigma_t dw_t + \gamma(t,Y_t)\, dN_t\right] \tag{32}$$

that corresponds to (14) in the version of (15) and can thus equivalently be represented as

$$dS_t = S_{t-}\left[\mu_t dt + \sigma_t dw_t + \int_E \gamma(t,y)\, p(dt,dy)\right] \tag{33}$$

If the marked point process is in particular a multivariate (or univariate) point process $(N_t(1),\cdots,N_t(K))$, then (32) ((33)) takes the form (see also (16))

$$dS_t = S_{t-}\left[\mu_t dt + \sigma_t dw_t + \sum_{k=1}^{K} \gamma_t(k)\, dN_t(k)\right] \tag{34}$$

We finally point out that the marked point process in (32) ((33)) may be doubly stochastic in the sense specified in sections 2.1 and 2.2 and this allows for further flexibility when it comes to modeling.

Remark 3.1 *Occcasionally, in the financial literature one finds model (32) ((33)) written in the form*

$$dS_t = S_{t-}\left[\mu_t dt + \sigma_t dw_t + dJ_t\right]$$

where, in the specific case when (32) reduces to (31), $J_t := \sum_{n=1}^{N_t} Y_n$*, while in the general case* $J_t := \sum_{n=1}^{N_t} \gamma(T_n, Y_n)$*. Furthermore, in models of the form (31) one may find the last term* $Y_t dN_t$ *written as* $(Y_t - 1)dN_t$*; in this latter case, instead of (29), we would then have* $S_{T_n} = S_{T_n^-} Y_n = S_{T_n^-} Y_{T_n}$*.*

3.1.2 Term structure models with jumps

Among the basic objects in term structure models we have the zero-coupon bonds with prices $p(t,T)$ (the price, at t, of a bond maturing at T), forward rates $f(t,T)$ (the rate, contracted at t, for instantaneous borrowing at T), and the short rate $r(t)$. There exist some well-known relationships among these quantities, in particular $f(t,T) = -\partial \log p(t,T)/\partial T$; $r(t) = f(t,t)$. Since interest rates, and therefore also bond prices may indeed jump, one may consider the following jump-diffusion models for the above three quantities

$$dr(t) = a_t dt + b_t dw_t + \int_E c(t,y)\, p(dt,dy) \tag{35}$$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dw_t + \int_E \delta(t,T;y)\, p(dt,dy) \tag{36}$$

$$dp(t,T) = p(t-,T)\left\{m(t,T)dt + v(t,T)dw_t + \int_E n(t,T;y)\, p(dt,dy)\right\} \tag{37}$$

where the differential is with respect to the time argument t, not the maturity T. Notice that only (37) has the factor $p(t-,T)$ also in the right hand side. This guarantees (see the exponential formula (17)) positivity of $p(t,T)$ as it should be since $p(t,T)$ is the price of an asset; the interest rates $r(t)$, $f(t,T)$ need not necessarily be positive. Given the well-known relationships between the three quantities in (35)-(37), there obviously has to exist a relationship also between the coefficients in these models. This relationship can be found in Prop. 2.2. of [10].

So far we have mentioned only continuously compounded interest rates. In financial markets also discretely compounded or simple rates such as LIBOR rates play an important role. Given a fixed accrual period δ, denote by $L(t,T)$ the forward rate, contracted at $t < T$, for the interval from T to $T+\delta$. Jump-diffusion models for $L(t,T)$ are studied in [26] under the form

$$dL(t,T) = L(t-,T)\left[\mu(t,T)dt + \sigma(t,T)dw_t + dJ(t,T)\right] \tag{38}$$

where (see Remark 3.1) $J(t,T) = \sum_{n=1}^{N_t} \gamma(T_n, Y_n)$ for a given marked point process represented by the double sequence (T_n, Y_n) (for a more general setup beyond jump-diffusions see [30]). Notice that the relationship

$$L(t,T) = \frac{1}{\delta}\left[\exp\left\{\int_T^{T+\delta} f(t,s)ds\right\} - 1\right] \tag{39}$$

between discretely and continuously compounded forward rates induces a relationship between the coefficients of the corresponding dynamic equations (36) and (39).

3.2 Jump-diffusion models driven by hidden jump processes

As mentioned in the Introduction, empirical studies have led to consider also combinations of jumps and stochastic volatility, where the volatility presents a jump-type behaviour and is possibly also correlated with the jumps in the prices. As pointed out in [41], it is in fact natural to expect that, when the volatility jumps, also the price should jump. One can capture these aspects by a jump-diffusion model, where the coefficients depend on a hidden/latent jump process Z_t that affects also the intensity of the marked point process in the jump term (doubly stochastic marked point process). Formally, and limiting ourselves to asset price models of the form of (33) (that are equivalent to (32) and include (34)), we then have

$$dS_t = S_{t-}\left[\mu_t(Z_t)dt + \sigma_t(Z_t)dw_t + \int_E \gamma(t,y;Z_{t-})\,p(dt,dy)\right] \tag{40}$$

where Z_t is any jump process with non-predictable jumps (could also be a Markov jump process) and $p(dt,dy)$ is the counting measure of a doubly stochastic marked point process with intensity $\lambda_t(Z_{t-},dy)$. Notice that Z_t affects the jump part both through the intensity as well as through the proportional jump sizes and it affects them in a predictable way.

3.3 Asset prices as diffusions sampled at the jump times of a jump process

As was mentioned in the Introduction, on very small time scales the real asset prices do not change continuously over time, but rather only at discrete random points in time in reaction to trades and/or significant new information. This makes jump processes attractive also for modeling high frequency data and here we give a description of such a modeling approach according to [23], [21]. Marked point processes as models for high frequency data were also studied independently by various authors in the recent literature (see e.g. [25], [46], [48]). The models in [23] and [21] are more in the spirit of jump-diffusions in that they consider a combination, although not an additive one, of a diffusion and a jump process as follows : given is a *background price process* of the diffusion type and this process is then sampled according to the random jump times of a jump process. This setup allows also to incorporate a possible correlation between (stochastic) volatility and price jumps in the way mentioned in the previous section, by letting again Z_t be a hidden process that drives the volatility of the background diffusion process and at the same time also the intensity of the (doubly stochastic) jump process that determines the random sampling times. In more formal terms, the logarithm Λ_t of the *background price process* is supposed to satisfy

$$d\Lambda_t = \sqrt{v_t(Z_t)}\, dw_t \tag{41}$$

with w_t a Wiener process independent of Z_t. The process Z_t is the hidden or latent state variable process that can be interpreted as modeling the rate at which new information is absorbed by the market. It may be given as a diffusion or as a finite state Markov process. Next consider a univariate doubly stochastic Poisson process (a Cox process) N_t with intensity $\lambda_t = \lambda_t(Z_{t-})$. The time dependence of this λ as well as of v in (41) is introduced to incorporate systematic patterns in trading activity. The actual price process is now such that its logarithm L_t satisfies

$$L_t = \Lambda_{T_{n-1}} \quad \text{for} \quad t \in [T_{n-1}, T_n) \tag{42}$$

with T_n the jump times of N_t. The given model can thus be interpreted as a stochastic volatility model, evaluated at random times T_n. It is easily seen that the process L_t in (42) satisfies

$$dL_t = \left(\Lambda_t - \Lambda_{T_{N_t-}}\right)\, dN_t \tag{43}$$

where $T_{N_{t-}}$ is the time of the last jump strictly prior to t and it is thus a marked point process with local characteristics $\left(\lambda_t(Z_t), \mathcal{N}\left(0, \int_{T_{N_t-}}^t v_s ds\right)\right)$ where $\mathcal{N}(m,\sigma^2)$ denotes a Gaussian r.v. with mean m and varaince σ^2.

Notice that we may choose an intensity of the form

$$\lambda_t(Z_t) = \lambda_t^{(1)} + \lambda_t^{(2)}\, Z_t \tag{44}$$

so that N_t can be seen as the sum $N_t = N_t^{(1)} + N_t^{(2)}$ of two independent jump processes : $N_t^{(1)}$ with deterministic intensity $\lambda_t^{(1)}$ corresponding to *noise trading* and $N_t^{(2)}$ corresponding to *informed trading*.

One interesting aspect of the above model is that it makes it clear how sample path properties matter when it comes to volatility estimation : the volatility in a diffusion model, i.e. its quadratic variation, can be approximated arbitrarily well by the sum of the observed squared increments. For the given piecewise constant processes the empirical quadratic variation is useless for volatility estimation, even if computed over very small time intervals.

We finally point out that the definition, that was given in section 2 concerning a doubly stochastic Poisson process, in particular that λ_t is $\mathcal{F}_0-$measurable, has as consequence the fact that N_t and Z_t cannot have common jumps and that the actual trading activity, namely the realization of the point process N_t, does not affect the law of Z_t. In economic terms this means that, in the given model, trading is caused purely by exogenous factors such as fundamental information, and not by the observed past trading activity.

4 Martingale measures: existence and uniqueness. (Market price of risk and market completion)

In each of the models discussed in section 3, individual asset prices are driven by at least two independent sources of randomness so that the corresponding market models are incomplete. Based on the extended Girsanov-type measure transformation recalled in section 2.5, in this section we shall discuss existence and, where applicable, uniqueness of martingale measures, thereby exhibiting also the market price of (jump-diffusion) risk. Uniqueness of the martingale measure will be mainly related to *completion of the market*. We want to point out that, as will be shown in more detail in the next section 5 on hedging, it is not necessarily true that, if a market is *completed* to yield a unique martingale measure, then it is also genuinely complete in the sense that every contingent claim can be duplicated by a self financing portfolio. In fact, for marked point process with an infinite mark space, i.e. with an infinite number of sources of randomness, it will be shown in section 5.1.2 that uniqueness of the martingale measure implies only some form of *approximate completeness*. In this section 4 we shall limit ourselves to the jump-diffusion asset price and term structure models of section 3.1. In subsection 4.1 below we treat the case of jump-diffusion models for asset prices and show that the market can relatively easily be completed to yield a unique martingale measure if the jump part corresponds to a marked point process with a finite number of marks (multivariate point processes). For an infinite number of marks the situation is studied in more detail in subsection 4.2 below in the context of term structure models.

4.1 The case of jump-diffusion asset price models

We start with a jump-diffusion model, where the jump part corresponds to a univariate Poisson point process with $P-$intensity λ_t, namely (see (34) for $K=1$)

$$\begin{aligned} dS_t &= S_{t-}\left[\mu_t dt + \sigma_t dw_t + \gamma_t dN_t\right] \\ &= S_{t-}\left[(\mu_t + \gamma_t\lambda_t)\, dt + \sigma_t dw_t + \gamma_t dM_t\right] \end{aligned} \tag{45}$$

with (see (2)) $M_t = N_t - \int_0^t \lambda_s ds$ the $P-$martingale corresponding to N_t. The Radon-Nikodym derivative for an absolutely continuous change of measure from P to Q, that implies a translation of the Wiener by θ_t and a change of the Poisson intensity from λ_t to $\psi_t\lambda_t$, is (see (27) for $K=1$)

$$L_t = \exp\left\{\int_0^t \left[(1-\psi_s)\lambda_s - \frac{1}{2}\theta_s^2\right] ds + \int_0^t \theta_s dw_s + \int_0^t \log\psi_s dN_s\right\} \tag{46}$$

Definig the Wiener and Poisson martingales w_t^Q and M_t^Q by (see (22) and (2))

$$\left\{\begin{array}{lcl} dw_t^Q &=& dw_t - \theta_t dt \\ dM_t^Q &=& dN_t - \psi_t\lambda_t dt \end{array}\right. \tag{47}$$

the dynamics of S_t under Q become

$$dS_t = S_{t-}\left[(\mu_t + \sigma_t\theta_t + \gamma_t\psi_t\lambda_t)\, dt + \sigma_t dw_t^Q + \gamma_t dM_t^Q\right] \tag{48}$$

Taking as numeraire the usual money market account B_t, where $dB_t = r_t B_t dt$, we immediately see that Q is a martingale measure, i.e. a measure under which $\tilde{S}_t := B_t^{-1} S_t$ is a martingale, if θ_t and $\psi_t \geq 0$ are chosen such that $\mu_t + \sigma_t\theta_t + \gamma_t\psi_t\lambda_t = r_t$. From here we see that, for each pair (θ_t, ψ_t) with $\psi_t \geq 0$ arbitrary and

$$\theta_t = \sigma_t^{-1}\left(r_t - \mu_t - \gamma_t\psi_t\lambda_t\right), \tag{49}$$

we obtain a martingale measure, i.e. we can obtain infinitly many martingale measures, one for each choice of ψ_t.

Concerning the market price of risk ρ_t, from (45) and (49) we have

$$\rho_t := \mu_t + \gamma_t\lambda_t - r_t = \gamma_t\lambda_t - \sigma_t\theta_t - \gamma_t\psi_t\lambda_t = -\sigma_t\theta_t - \gamma_t\lambda_t(\psi_t - 1) \tag{50}$$

from where we see that $(-\theta_t)$ can be interpreted as *risk premium per unit of diffusion volatility*, whereas $(-\lambda_t(\psi_t - 1))$ can be interpreted as *risk premium per unit of jump volatility*. On an arbitrage-free market all assets have, at a given time t, the same diffusion- and jump-risk premia and they determine, via the Girsanov transformation i.e. via (46), the equivalent martingale measure Q.

We obtained infinitely many martingale measures because, for a single risky asset, we had two independent sources of randomness. One may thus expect that, by adding a further asset, one can *complete* the market to obtain a unique martingale measure. Consider then, in addition to S_t in (45), an asset with price $\bar{S}_t$ satisfying

$$d\bar{S}_t = \bar{S}_{t-}\left[\bar{\mu}_t dt + \bar{\sigma}_t dw_t + \bar{\gamma}_t dN_t\right] \tag{51}$$

Notice that $\bar{S}_t$ could correspond to the price of a derivative asset with underlying S_t. In fact, if one is given the explicit expression of this derivative price in terms of S_t, i.e. $\bar{S}_t = F(t, S_t)$, then (51) is straightforwardly obtained from (45) by use of the generalized Ito formula (19). Since the two risk premia θ_t and $\lambda_t(\psi_t - 1)$ have to be the same for all assets, we may impose (49) on both assets with prices S_t and $\bar{S}_t$ respectively, namely

$$\theta_t = \sigma_t^{-1}\left(r_t - \mu_t - \gamma_t\psi_t\lambda_t\right) = \bar{\sigma}_t^{-1}\left(r_t - \bar{\mu}_t - \bar{\gamma}_t\psi_t\lambda_t\right) \tag{52}$$

from where one immediately gets

$$\psi_t\lambda_t = \frac{r_t(\sigma_t - \bar{\sigma}_t) + (\mu_t\bar{\sigma}_t - \sigma_t\bar{\mu}_t)}{\sigma_t\bar{\gamma}_t - \gamma_t\bar{\sigma}_t} \tag{53}$$

Inserting this expression in (49) it follows that

$$\theta_t = \frac{\gamma_t(\bar{\mu}_t - r_t) - \bar{\gamma}_t(\mu_t - r_t)}{\sigma_t\bar{\gamma}_t - \gamma_t\bar{\sigma}_t} \tag{54}$$

We have thus obtained unique risk premia and, consequently, a unique martingale measure provided the coefficients in (45) and (51) are such that $\sigma_t\bar{\gamma}_t - \gamma_t\bar{\sigma}_t \neq 0$ and that $\psi_t\lambda_t$ in (53) is positive.

With the unique martingale measure we may expect to have also obtained a *complete* market in the sense that, by investing in a self financing way in the two assets with prices S_t and $\bar{S}_t$, one can duplicate any claim. In subsection 5.1.1 we shall show that, for the given market model, this is indeed the case.

It is easily seen that, if the jump part in the jump-diffusion model corresponds to a multivariate Poisson process, i.e. if instead of (45) we have (see (34))

$$\begin{aligned} dS_t &= S_{t-}\left[\mu_t dt + \sigma_t dw_t + \sum_{k=1}^{K}\gamma_t(k)dN_t(k)\right] \\ &= S_{t-}\left[\left(\mu_t + \sum_{k=1}^{K}\gamma_t(k)\lambda_t(k)\right)dt + \sum_{k=1}^{K}\gamma_t(k)dM_t(k)\right] \end{aligned} \tag{55}$$

with $M_t(k) = N_t(k) - \int_0^t \lambda_s(k)ds$, then the previous results admit a straightforward extension. In particular, (49) becomes

$$\theta_t = \sigma_t^{-1}\left(r_t - \mu_t - \sum_{k=1}^{K}\gamma_t(k)\psi_t(k)\lambda_t(k)\right) \tag{56}$$

and the market price of risk is

$$\rho_t = -\sigma_t\theta_t - \sum_{k=1}^{K}\gamma_t(k)\lambda_t(k)(\psi_t(k) - 1) \tag{57}$$

This time the generic $k-$th term $\gamma_t(k)\lambda_t(k)(\psi_t(k) - 1)$ on the right can be interpreted as *risk premium per unit of jump volatility of type k*.

Again we obtain infinitely many martingale measures by choosing freely $\psi_t(k) \geq 0$, $(k = 1, \cdots, K)$, and θ_t according to (56). Having now $K + 1$ independent sources of randomness, we may expect that one can *complete* the market by adding K further assets to obtain a unique equivalent martingale measure. This can be done along the lines of (52)-(54) although this time the calculations are more complicated and the conditions on the coefficients more cumbersome.

Finally, we consider the more general model (33) (or, equivalently, (32)) with a possibly infinite number of marks. Using the $P-$martingale measure $q(\cdot)$ in (8), by analogy to (45) and (55) we may rewrite (33) as

$$\begin{aligned} dS_t &= S_{t-}\left[\mu_t dt + \sigma_t dw_t + \int_E \gamma(t,y)\,p(dt,dy)\right] \\ &= S_{t-}\left[\left(\mu_t + \int_E \gamma(t,y)\lambda_t(dy)\right)dt + \sigma_t dw_t + \int_E \gamma(t,y)\,q(dt,dy)\right] \end{aligned} \tag{58}$$

Using the particular form of the intensity given in (9), we also have

$$\int_E \gamma(t,y)\lambda_t(dy) = \bar{\gamma}_t\lambda_t \quad \text{with} \quad \bar{\gamma}_t = \int_E \gamma(t,y)\,m_t(dy) \tag{59}$$

and so (58) becomes, quite analogously to (45),

$$dS_t = S_{t-}\left[(\mu_t + \bar{\gamma}_t\lambda_t)\,dt + \sigma_t dw_t + \int_E \gamma(t,y)\,q(dt,dy)\right] \tag{60}$$

Consider then, instead of (46), the more general Radon-Nikodym derivative (25) that we rewrite here in the form analogous to (46) as

$$L_t = \exp\left\{\int_0^t \left[(1-\psi_s\bar{h}_s)\lambda_s - \frac{1}{2}\theta_s^2\right]ds + \int_0^t \theta_s dw_s + \int_0^t \log\left(\psi_s h_s(Y_s)\right)dN_s\right\} \tag{61}$$

where $\bar{h}_s = \int_E h_s(y)m_s(dy)$. Define next the Wiener and jump martingales w_t^Q and $q^Q(dt,dy)$ by (see (22) and (8)- (9) as well as (47))

$$\begin{cases} dw_t^Q = dw_t - \theta_t dt \\ q^Q(dt,dy) = p(dt,dy) - \psi_t\lambda_t h_t(y)m_t(dy)dt \end{cases} \tag{62}$$

The dynamics of S_t under Q then become

$$dS_t = S_{t-}\left[\left(\mu_t + \sigma_t\theta_t + \bar{\Gamma}_t\psi_t\lambda_t\right)dt + \sigma_t dw_t^Q + \int_E \gamma(t,y)\,q^Q(dt,dy)\right] \tag{63}$$

where $\bar{\Gamma}_t = \int_E \gamma(t,y)h_t(y)m_t(dy)$. The measure Q is now a martingale measure if θ_t and $\psi_t \geq 0$ as well as $h_t(y) \geq 0$ are chosen so that $\mu_t + \sigma_t\theta_t + \bar{\Gamma}_t\psi_t\lambda_t = r_t$, which leads to the following relation corresponding to (49)

$$\theta_t = \sigma_t^{-1}\left(r_t - \mu_t - \bar{\Gamma}_t\psi_t\lambda_t\right) \tag{64}$$

Again, this leads to infinitely many martingale measures but, unless the mark space is finite, to *complete* the market in order to obtain a unique equivalent martingale measure one needs infinitely many assets. We shall discuss this situation in more detail in the context of bond markets in the next subsection.

To complete the analogy with the previous cases, notice that this time the market price of risk becomes (by (60) and (64))

$$\begin{aligned} \rho_t &:= \mu_t + \bar{\gamma}_t \lambda_t - r_t = \bar{\gamma}_t \lambda_t - \sigma_t \theta_t - \bar{\Gamma}_t \psi_t \lambda_t = -\sigma_t \theta_t - \lambda_t \left(\bar{\Gamma}_t \psi_t - \bar{\gamma}_t \right) \\ &= -\sigma_t \theta_t - \lambda_t \int_E \gamma(t, y) \left[\psi_t h_t(y) - 1 \right] m_t(dy) \end{aligned} \tag{65}$$

This time one may interpret $[\psi_t h_t(y) - 1] \ m_t(dy)$ as *risk premium per unit of jump volatility of type* y.

In this latter context of a more general model of type (45) we want to point out that a methodology to obtain all equivalent martingale measures has also been worked out in [45].

We close this subsection by mentioning that, depending on the purpose, one can single out some specific martingale measures among the various possible ones in a jump-diffusion model, where the market has not been completed. As an example, the construction of the so-called *minimal martingale measure* in a univariate Poisson jump diffusion model can be found in [47]. From a more practical point of view, an obvious possibility is always that of calibrating the model to market data.

4.2 The case of jump-diffusion term structure models

Consider first a term structure model where, under a given measure P, the (continuously compounded) forward rates $f(t,T)$ and the (zero coupon) bond prices $p(t,T)$ satisfy (36) and (37) respectively, namely

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dw_t + \int_E \delta(t,T;y)\, p(dt,dy) \tag{66}$$

$$dp(t,T) = p(t-,T) \left\{ m(t,T)dt + v(t,T)dw_t + \int_E n(t,T;y)\, p(dt,dy) \right\} \tag{67}$$

We shall also make the ad hoc assumptions that all objects are specified in a way to guarantee the validity of the various operations that will have to be performed, such as differentiation under the integral sign and interchange of the order of integration.

For later use we recall from [10] the relationship between the coefficients in (66) and (67) : if $f(t,T)$ satisfies (66), then $p(t,T)$ satisfies (67) with

$$\left\{ \begin{array}{lcl} m(t,T) &=& r(t) + A(t,T) + \frac{1}{2} \|S(t,T)\|^2 \\ v(t,T) &=& S(t,T) \\ n(t,T;y) &=& e^{D(t,T;y)} - 1 \end{array} \right. \tag{68}$$

where $r(t) = f(t,t)$ is the short rate and

$$\left\{ \begin{array}{lcl} A(t,T) &=& -\int_t^T \alpha(t,s)ds \\ S(t,T) &=& -\int_t^T \sigma(t,s)ds \\ D(t,T;y) &=& -\int_t^T \delta(t,s;y)ds \end{array} \right. \tag{69}$$

In the given bond market there are, at least theoretically, infinitely many assets, namely the bonds for all possible maturities $T > t$. A martingale measure Q is now a measure under which all these bond prices, discounted with respect to the money market account, are (local) martingales. We are therefore not even sure whether in such a given market model there exists a martingale measure and so our first purpose is to investigate the existence of such a measure.

Following essentially [10] and considering general marked point processes, we also take the general form of the Radon-Nikodym derivative L_t, namely (see (24) where, for simplicity, we put $\psi_t \equiv 1$)

$$dL_t = L_t \theta_t dw_t + L_{t-} \int_E (h_s(y) - 1)\; q(ds, dy) \tag{70}$$

where (see (8) and (9))

$$q(ds, dy) = p(ds, dy) - \lambda_s m_s(dy) \tag{71}$$

i.e. we assume that, under P, the local characteristics of the marked point process $p(ds, dy)$ are $(\lambda_t, m_t(dy))$. By Theorem 2.5 we know that, under the measure Q that corresponds to L_t in (70), the local characteristics become $(\lambda_t, h_t(y)m_t(dy))$ so that, defining (see also (62))

$$\begin{cases} dw_t^Q = dw_t - \theta_t dt \\ q^Q(dt, dy) = p(dt, dy) - \lambda_t h_t(y) m_t(dy) ds \end{cases} \tag{72}$$

the bond prices $p(t, T)$ satisfy, under Q, the dynamics

$$\begin{aligned} dp(t,T) = p(t-,T) \Big\{ &\Big[m(t,T) + v(t,T)\theta_t + \lambda_t \int_E n(t,T;y) h_t(y) m_t(dy) \Big]\, dt \\ &+ v(t,T) dw_t^Q + \int_E n(t,T;y)\, q^Q(dt, dy) \Big\} \end{aligned} \tag{73}$$

A necessary condition for the existence of martingale measure Q is then that there exist a predictable process θ_t and a predictable $E-$indexed process $h_t(y) \geq 0$ such that the conditions of Theorem 2.5 hold and

$$m(t,T) + v(t,T)\theta_t + \lambda_t \int_E n(t,T;y) h_t(y) m_t(dy) = r(t) \tag{74}$$

Notice that this implies for the market price of risk a relation analogous to (65), namely

$$\begin{aligned} \rho_t \quad &:= \quad m(t,T) + \lambda_t \int_E n(t,T;y) m_t(dy) - r(t) \\ &= \quad -v(t,T)\theta_t - \lambda_t \int_E n(t,T;y)\, [h_t(y) - 1]\; m_t(dy) \end{aligned} \tag{75}$$

We shall now translate condition (74), involving the coefficients of (67), into a condition involving the coefficients of (66), namely of the forward rates. Using (68), condition (74) becomes

$$A(t,T) + \frac{1}{2}\|S(t,T)\|^2 + S(t,T)\theta_t + \int_E h_s(y) \nu(t,T;dy) = 0 \tag{76}$$

with $\nu(t,T;dy) := \left(e^{D(t,T;y)} - 1\right) \lambda_t m_t(dy)$ and with A, S and D as in (69).

When building a term structure model it is often convenient to specify all objects directly under a martingale measure Q and this obviously imposes some restrictions on the coefficients in the models. Concentrating on forward rates, assume that we want model (66) to be valid under a martingale measure Q, i.e. we are postulating that $P = Q$ and so we have to choose $\theta_t \equiv 0,\ h_t(y) \equiv 1$. Notice now that (76) has to hold for all maturities so that, inserting the above choices of θ_t and $h_t(y)$ and differentiating with respect to T, we obtain (using also (69)) the following necessary condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \int_E \delta(t,T;y)e^{D(t,T;y)}\lambda_t h_t(y)m_t(dy) \tag{77}$$

which is a clear extension of the classical *Heath-Jarrow-Morton drift condition* for the pure diffusion case.

Having investigated the existence of a martingale measure, we may next look for conditions implying its uniqueness. Concentrating again on forward rates, a necessary condition for the existence of a martingale measure has been seen to be the existence of a predictable θ_t and a predictable $E-$indexed $h_t(y) \geq 0$ such that relation (76) holds. Quite obviously then, if (76) admits a unique solution in θ_t and $h_t(y)$, the martingale measure is unique. To formalize this fact, consider the following linear operator (for technical details, that for simplicity we neglect here, we refer to [10])

$$\mathcal{K}_t \;:\; (\theta, h(y)) \;\longrightarrow\; S(t,\cdot)\theta + \int_E h(y)\left(e^{D(t,\cdot;y)} - 1\right)\lambda_t m_t(dy) \tag{78}$$

The operator $\mathcal{K}_t$ is an integral operator of the first kind and we refer to it as *martingale operator*. The martingale measure is then unique if and only if, $dP, dt - a.e$, we have

$$Ker\,\mathcal{K}_t = 0 \tag{79}$$

We may now wonder whether, in the present context of infinitely many sources of randomness, the uniqueness of the martingale measure implies completeness in the sense that every contingent claim can be replicated by a self financing portfolio. The answer is no; in fact, as we shall mention in subsection 5.1.2 below, we obtain only a form of *approximate completeness*.

We finally remark that the relationship (39) between discretely and continuously compounded forward rates has allowed the authors in [26] to carry over the just mentioned results for continuously compounded forward rates also to the case when one has simple forwards instead. In fact, a model of the term structure of simple forwards $L(t,T)$ (see (38)) is defined in [26] to be arbitrage-free, if it can be embedded in an arbitrage-free model of instantaneous forwards $f(t,T)$ via (39).

5 Hedging in jump-diffusion market models

In the previous section we have seen that, as a consequence of its incompleteness, in a jump-diffusion market model we have in general infinitely many martingale measures.

We have then investigated the method of *market completion* as a tool to obtain a unique martingale measure. On the other hand, from the *second fundamental theorem of asset pricing* one has that, in general, if a market admits a unique equivalent martingale measure, then it is also complete in the sense that every contingent claim can be hedged by a self financing portfolio.

We shall investigate the hedging problem in a jump-diffusion market model having in mind two goals : for the first goal, in the context of asset price models, we shall show in subsection 5.1.1 that completed market models with a unique martingale measure are complete also in the sense of hedging if there are only a finite number of marks for the jumping component (there is a finite number of sources of randomness). If however there are an infinite number of marks (an infinite number of sources of randomness) then, in the context of bond markets, in subsection 5.1.2 we shall show that the completed market models with a unique martingale measure are only *approximately complete* in the sense of hedging.

In the context of the first goal we also want to add here that, in the paper [34], the author approximates a given jump-diffusion market model, having an infinite number of marks, by a sequence of jump-diffusion models with a finite number of marks that are therefore complete also in the sense of hedging.

For the second goal, in section 5.2 we shall consider the case when one cannot have a complete market or when it is not appropriate to complete it. In such a case one has to determine the hedging strategy according to some specific hedging criterion. We shall consider the (local) risk minimization and the related minimum variance criteria and show that they lead to hedging strategies that are quite natural extensions of those in complete markets. While so far only the models of section 3.1 have been further investigated, the discussion in section 5.2 will center mainly around the model of section 3.3.

In part, this section can also be seen as preliminary to the next section 6 on pricing. In fact, if a market is complete in the sense of hedging, then by the criterion of absence of arbitrage the initial value of the self financing and hedging strategy has to correspond to the arbitrage-free price of the contingent claim. If the market cannot be completed, the criterion of absence of arbitrage alone is not sufficient to define a price and the preference structure of the investors has to come into play. Since, typically, the initial value of a hedging portfolio satisfying a specific hedging criterion can be expressed as expectation of the discounted claim under a specific martingale measure, the choice of a hedging criterion implies also the choice of a martingale measure and thus of a pricing kernel. We shall discuss these issues in more detail in section 6.1 below.

5.1 Hedging when the market is completed

5.1.1 Asset-price models

In this subsection we consider the univariate jump-diffusion model of section 4.1. We had seen that, considering in addition to the asset with price S_t satisfying (45), also the asset with price $\bar{S}_t$ satisfying (51) with coefficients such that $\psi_t \lambda_t$ in (53) is positive and $\sigma_t \bar{\gamma}_t - \gamma_t \bar{\sigma}_t \neq 0$, then there exists a unique martingale measure Q corresponding to the choice of ψ_t and θ_t according to (53) and (54). Basing ourselves on [33], we show now that in this situation any claim can be hedged with a self financing portfolio.

Given a maturity T, consider as *claim* a (square-integrable) random variable H_T, measurable with respect to $\mathcal{F}_T$, where $\mathcal{F}_t := \sigma \left\{ S_0, \bar{S}_0, w_s, N_s,\ s \leq t \right\}$, completed with the null sets. In addition to the two risky assets with prices S_t and $\bar{S}_t$, we suppose given also a nonrisky asset, whose price we take for simplicity identically equal to 1 (equivalent to assuming all prices discounted with respect to the nonrisky asset). An investment strategy is then a triple $\Phi_t = \left[\phi_t, \bar{\phi}_t, \eta_t \right]$, where η_t denotes the number of units of the nonrisky asset held in the portfolio at time t and $\phi_t, \bar{\phi}_t$ are the number of shares of the two risky assets respectively. Let $\phi_t, \bar{\phi}_t$ be predictable and η_t be adapted. The value, at time t, of a portfolio corresponding to the strategy Φ is then

$$V_\Phi(t) = \phi_t S_t + \bar{\phi}_t \bar{S}_t + \eta_t \tag{80}$$

We want Φ to be such that the corresponding portfolio is self financing and duplicates the claim, i.e. that it satisfies

$$\left\{ \begin{array}{rcl} dV_\Phi(t) & = & \phi_t dS_t + \bar{\phi}_t d\bar{S}_t \\ V_\Phi(T) & = & H_T \end{array} \right. \tag{81}$$

It follows from section 4.1 that, under the unique martingale measure Q, the discounted prices of the two risky assets, that for simplicity we continue denoting by S_t and $\bar{S}_t$, are the martingales satisfying

$$\left\{ \begin{array}{rcl} dS_t & = & S_{t-} \left[\sigma_t dw_t^Q + \gamma_t dM_t^Q \right] \\ d\bar{S}_t & = & \bar{S}_{t-} \left[\bar{\sigma}_t dw_t^Q + \bar{\gamma}_t dM_t^Q \right] \end{array} \right. \tag{82}$$

where w_t^Q and M_t^Q are as in (47) with $\psi_t \lambda_t$ and θ_t according to (53) and (54). Replacing dS_t and $d\bar{S}_t$ from (82) in (81), it follows that also $V_\Phi(t)$ is a $(Q, \mathcal{F}_t)-$martingale satisfying

$$V_\Phi(t) = V_\Phi(0) + \int_0^t \left[\phi_s S_s \sigma_s + \bar{\phi}_s \bar{S}_s \bar{\sigma}_s \right] dw_s^Q + \int_0^t \left[\phi_s S_{s-} \gamma_s + \bar{\phi}_s \bar{S}_{s-} \bar{\gamma}_s \right] dM_t^Q \tag{83}$$

Consider next the $(Q, \mathcal{F}_t)-$martingale

$$M(t) := E^Q \left\{ H_T \mid \mathcal{F}_t \right\} \tag{84}$$

By the martingale representation theorem (see Theorem 2.3 applied here to the particular case of a univariate Poisson point process) there exist two $\mathcal{F}_t-$predictable processes $\xi_t^{(1)}$

and $\xi_t^{(2)}$ such that

$$M(t) = M(0) + \int_0^t \xi_s^{(1)} dw_s^Q + \int_0^t \xi_s^{(2)} dM_s^Q \tag{85}$$

Comparing (83) and (85), one sees immediately that, by putting

$$V_\Phi(0) = M(0) = E^Q \{H_T \mid \mathcal{F}_0\} \tag{86}$$

and choosing $\phi_t, \bar{\phi}_t$ such that (integrating with respect to a Wiener process one may change S_t into S_{t-})

$$\left\{ \begin{array}{lcl} \phi_t S_{t-}\sigma_t + \bar{\phi}_t \bar{S}_{t-}\bar{\sigma}_t & = & \xi_t^{(1)} \\ \phi_t S_{t-}\gamma_t + \bar{\phi}_t \bar{S}_{t-}\bar{\gamma}_t & = & \xi_t^{(2)} \end{array} \right. \tag{87}$$

we have $V_\Phi(t) = M(t)$. Since $M(T) = H_T$ by definition, with the choices (86) and (87) we obtain a self financing and hedging strategy (the value of η_t follows from (80)). Notice that, in order to obtain a unique solution of (87), we have to require that $\sigma_t\bar{\gamma}_t - \gamma_t\bar{\sigma}_t \neq 0$, which is exactly one of the conditions required after (53) and (54) to obtain a unique equivalent martingale measure.

What we have just shown is an existence result leading to the completeness (in the sense of hedging) of the given market when the martingale measure is unique. To actually determine the hedging strategy, we need an explicit expression for the processes $\xi_t^{(1)}$ and $\xi_t^{(2)}$ that, in the case of a simple claim of the form $H_T = H(S_T, \bar{S}_T)$, can be obtained by analogy to the pure diffusion case using the generalized Ito formula (19). Due to the Markov property of $(S_t, \bar{S}_t)$, we may in fact put

$$M(t) = M(t; S_t, \bar{S}_t) = E^Q \left\{ H(S_T, \bar{S}_T) \mid \mathcal{F}_t \right\} \tag{88}$$

Formula (19) then leads to

$$\begin{array}{l} dM(t) = \big[M_t(\cdot) + \frac{1}{2} M_{SS}(\cdot) S_{t-}^2 \sigma_t^2 + \frac{1}{2} M_{\bar{S}\bar{S}}(\cdot) \bar{S}_{t-}^2 \bar{\sigma}_t^2 + M_{S\bar{S}}(\cdot) S_{t-}\bar{S}_{t-}\sigma_t\bar{\sigma}_t \\ + \big(M\left(t; S_{t-}(1+\gamma_t), \bar{S}_{t-}(1+\bar{\gamma}_t)\right) - M\left(t; S_{t-}, \bar{S}_{t-}\right) - M_S(\cdot)\gamma_t - M_{\bar{S}}(\cdot)\bar{\gamma}_t \big) \; \psi_t \lambda_t \big] \; dt \\ + \left[M_S(\cdot) S_t \sigma_t + M_{\bar{S}}(\cdot) \bar{S}_t \bar{\sigma}_t \right] \; dw_t^Q \\ + \left[M\left(t; S_{t-}(1+\gamma_t), \bar{S}_{t-}(1+\bar{\gamma}_t)\right) - M\left(t; S_{t-}, \bar{S}_{t-}\right) \right] \; dM_t^Q \end{array} \tag{89}$$

Since $M(t)$ is a $Q-$martingale, the drift (finite variation) term in (89) has to vanish and so it follows from (89) and (85) that

$$\left\{ \begin{array}{lcl} \xi_t^{(1)} & = & M_S(t; S_t, \bar{S}_t)\, S_t \sigma_t + M_{\bar{S}}(t; S_t, \bar{S}_t)\, \bar{S}_t \bar{\sigma}_t \\ \xi_t^{(2)} & = & M\left(t; S_{t-}(1+\gamma_t), \bar{S}_{t-}(1+\bar{\gamma}_t)\right) - M\left(t; S_{t-}, \bar{S}_{t-}\right) \end{array} \right. \tag{90}$$

For a related result see also [50]. We conclude this subsection by pointing out that, analogously to section 4.1, the procedure that we have described here for the case of a univariate point process can quite naturally be extended to the case of multivariate point processes, provided the market is completed with the addition of an appropriate number of further assets.

5.1.2 Term structure models

We consider the term structure model discussed in section 4.2 assuming that the condition for uniqueness of the martingale measure given by the injectivity (see (79)) of the integral operator $\mathcal{K}_t$ in (78) is satisfied. This subsection is mainly based on [10] (see also [32] for a related approach).

In this market, where the basic assets are zero-coupon bonds with prices $p(t,T)$ for any maturity $T > t$ in addition to a nonrisky asset (money market account B_t), we have first to define a portfolio.

Definition 5.1 *On the given bond market a portfolio is a pair* $(\eta_t, \xi_t(dT))$ *where*

i) η_t *is predictable;*

ii) $\forall t$, $\xi_t(\cdot)$ *is a signed finite measure on* $[t,\infty)$.

Intuitively, η_t is the number of units of the riskfree asset held in the portfolio at time t, $\xi_t(dT)$ is the "number" of bonds, with maturities in $[T, T+dT)$, held at time t. Some integrability assumptions are also required, but we leave them here as implicit. The *value process* of the portfolio (η,ξ), discounted with respect to B_t, is

$$V_t(\eta,\xi) = \eta_t + \int_t^\infty p(t,T)\xi_t(dT) \tag{91}$$

where, with some abuse of notation, we denote by $p(t,T)$ also the discounted value of a $T-$bond.

Definition 5.2 *The portfolio* (η,ξ) *is self-financing if*

$$dV_t(\eta,\xi) = \int_t^\infty \xi_t(dT)dp(t,T) \tag{92}$$

The integral in the right hand side in (92) needs an appropriate definition. Justified by the development in [9], we shall simply replace here $dp(t,T)$ in (92) by its expression under the (unique) martingale measure. To obtain this expression, recall the condition (77) (or, equivalently, (76) with $\theta_t = 0$, $h_t(y) = 1$) on the coefficients of the forward rate dynamics in order that these dynamics hold under a martingale measure. Translating, via (68), these conditions back to the bond price dynamics and taking also into account the definition of $q^Q(dt,dy)$ in (72), one has

$$dp(t,T) = p(t-,T)\left[S(t,T)dw_t^Q + \int_E \left(e^{D(t,T;y)} - 1\right) q^Q(dt,dy)\right] \tag{93}$$

(recall that we take here for $p(t,T)$ the discounted values). Given a contingent claim $H_T \in \mathcal{F}_T$, that we assume here to be bounded, the conditions for self financing and perfect hedging can be expresses as (combining (92) with (93))

$$\left\{\begin{array}{rcl} V_t(\eta,\xi) & = & V_0(\eta,\xi) + \int_0^t \int_s^\infty \xi_s(dT)p(s,T)S(s,T)dw_s^Q \\ & + & \int_0^t \int_E \int_s^\infty \xi_s(dT)p(s-,T)\left(e^{D(s,T;y)} - 1\right) q^Q(ds,dy) \\ V_T(\eta,\xi) & = & H_T \end{array}\right. \tag{94}$$

where the inner integral is with respect to T and the outer with respect to s.

Paralleling the development in the previous subsection 5.1.1, consider next the $(Q, \mathcal{F}_t)-$martingale

$$M(t) := E^Q \{H_T \mid \mathcal{F}_t\} \tag{95}$$

which, by the martingale representation theorem 2.3, admits the representation (see (12) under the measure Q)

$$M(t) = M(0) + \int_0^t \phi_s dw_s^Q + \int_0^t \int_E H(s,y)\, q^Q(ds, dy) \tag{96}$$

for predictable (and appropriately integrable) ϕ and H. Comparing (94) with (96) one sees that, by putting

$$V_0(\eta, \phi) = M(0) = E^Q \{H_T \mid \mathcal{F}_0\} \tag{97}$$

and choosing $\xi_t(dT)$ such that

$$\begin{cases} \int_t^\infty \xi_t(dT) p(t,T) S(t,T) = \phi_t \\ \int_t^\infty \xi_t(dT) p(t-,T) \left(e^{D(t,T;y)} - 1\right) = H(t,y) \end{cases} \tag{98}$$

we have $V_t(\eta, \xi) = M(t)$ and, in particular, $V_T(\eta, \xi) = H_T$, i.e. we have obtained a self financing and hedging strategy (the value of η_t follows from (91)). Everything now hinges upon the (unique) solvability of (98). To this effect consider the integral operator $\mathcal{K}_t^*$ implicit in the left hand side of (98), namely

$$\mathcal{K}_t^* \;:\; \xi \longrightarrow \begin{bmatrix} \int_t^\infty p(t,T) S(t,T) \xi(dT) \\ \int_t^\infty p(t-,T) \left(e^{D(t,T;\cdot)} - 1\right) \xi(dT) \end{bmatrix} \tag{99}$$

so that the conditions (98) become

$$\mathcal{K}_t^* \xi = \begin{bmatrix} \phi_t \\ H(t,\cdot) \end{bmatrix} \tag{100}$$

The integral operator $\mathcal{K}_t^*$ will be called *hedging operator* and the market is complete if $\mathcal{K}_t^*$ is surjective. Combining this result with that of section 4.2 on the uniqueness of the martingale measure, namely (79), we may synthesize them into

Proposition 5.3 *For the given term structure model (66), (67) we have that*

i) *the martingale measure is unique , if the martingale operators $\mathcal{K}_t$ in (78) are injective;*

ii) *the market is complete if the hedging operators $\mathcal{K}_t^*$ in (99) are surjective.*

It turns out that the operators $\mathcal{K}_t^*$ are adjoint to $\mathcal{K}_t$. If the spaces, on which they act, are finite-dimensional, then the injectivity of $\mathcal{K}_t$ implies surjectivity of $\mathcal{K}_t^*$ and thus that uniqueness of the martingale measure implies completeness. Unfortunately, our spaces here are infinite-dimensional and so, due to the duality relationship $(Ker\mathcal{K})^\perp =$

$cl\,(Im\,\mathcal{K}^*)$ between bounded linear operators, the injectivity of $\mathcal{K}_t$ implies denseness of $\mathcal{K}_t^*$. In other words, the uniqueness of the martingale measure implies only an *approximate completeness*. For details we refer to [10].

For the case when the mark space E is infinite, the authors in [10] also give a characterization of the hedgeable claims, based on a Laplace-transform technique and under assumptions that hold e.g. in the case of an affine term structure. When the mark space E is finite, in [10] it is furthermore shown that, under appropriate assumptions, any claim can be hedged with a finite number of bonds, whose maturities can be chosen in an essentially arbitrary way and such that they remain fixed as the running time t varies.

5.2 Hedging when the market is not complete

If one cannot have a complete market or market completion is not appropriate, one has to accept some residual risk, due either to non self financing or non perfect hedging, and choose an investment strategy that minimizes the unhedgeable risk. For this purpose various criteria have been proposed and here we describe one such criterion for the case of a slight variant of the market model described in section 3.3.

We assume here that the actual price S_t of the risky asset satisfies a model of the form of (41), namely

$$dS_t = S_t \sqrt{v_t(Z_t)}\, dw_t \tag{101}$$

where Z_t is supposed to be a diffusion-type process of the form

$$dZ_t = \alpha_t(Z_t)dt + \beta_t(Z_t)\, d\bar{w}_t \tag{102}$$

for a Wiener $\bar{w}_t$, independent of w_t. Given a univariate, doubly stochastic Poisson process N_t with intensity $\lambda_t = \lambda_t(Z_t)$, suppose that the prices of the risky asset can only be observed at the jump times T_n of N_t, i.e. the observation process Y_t is given by (see (43))

$$dY_t = \left(S_t - S_{T_{N_{t-}}}\right) dN_t \tag{103}$$

so that the information of the hedger can be modeled by the filtration

$$\mathcal{F}_t^Y = \sigma\left\{N_s, Y_s\,;\ s \le t\right\} \subset \mathcal{F}_t = \sigma\left\{S_0, Z_0, w_s, \bar{w}_s, N_s\,;\ s \le t\right\}$$

Notice that the only difference with respect to the model described in section 3.3 is that here the actual price process S_t varies continuously in time according to (101), but is observed only at the discrete time points T_n; there, the process according to (101) is only a *background process* and the actual price process is given by the values of the background process, sampled at the time points T_n according to (103). Notice also that, according to (101), the process S_t is implicitly assumed to be a $(P, \mathcal{F}_t)-$martingale. On one hand, this will make our hedging procedure below applicable; on the other hand it can be justified by assuming that (see e.g. [7]) S_t is discounted with respect to a $P-$numeraire portfolio, which is a tradable numeraire such that the discounted assets become martingales with respect to the original measure P.

Our hedging criterion will be that of (local) risk minimization according to [18],[19], that keeps the requirement of perfect hedging and relaxes the self financing requirement into mean self financing. More precisely, considering as strategy a pair (η_t, ξ_t) of $\mathcal{F}_t^Y-$predictable processes with η_t and ξ_t denoting the number of units of the numeraire and the given asset respectively, that are held in the portfolio at time t, we give the following

Definition 5.4 *Assuming prices are discounted with respect to the numeraire, define*

$$\begin{aligned} V_t = V_t(\eta,\xi) &:= \xi_t S_t + \eta_t && \textit{as value process} \\ C_t = C_t(\eta,\xi) &:= V_t - \int_0^t \xi_s dS_s && \textit{as cost process} \end{aligned}$$

Notice that, if $C_t(\eta,\xi) = const.$, the strategy (η,ξ) is self financing. We shall now relax this assumption by allowing $C_t(\eta,\xi)$ to be a $(P,\mathcal{F}_t^Y)-$martingale and, given a (square-integrable) claim $H(S_T)$ (already discounted with respect to the numerarire), determine a hedging strategy (η^*,ξ^*) that, for all $t = T_n \ (n = 1, 2, \cdots)$, minimizes

$$R_t^Y(\eta,\xi) := E\left\{ (C_T(\eta,\xi) - C_t(\eta,\xi))^2 \mid \mathcal{F}_t^Y \right\} \tag{104}$$

with respect to the hedging strategies (η,ξ) for which $C_t(\eta,\xi)$ is a $(P,\mathcal{F}_t^Y)-$martingale. The strategy (η^*,ξ^*) will be called an $\mathcal{F}_t^Y-$*risk minimizing strategy*.

Notice that there is a close relationship between risk minimizing strategies in the just specified sense and variance-minimizing strategies that are self financing and minimize the variance of the residual hedging error.

To compute an $\mathcal{F}_t^Y-$risk minimizing strategy we shall proceed in two steps following [21] (see also [17] and [22]). In the first step we determine an $\mathcal{F}_t-$risk minimizing strategy, namely a risk minimizing strategy where the (hypothetical) information of the hedger corresponds to the full filtration $\mathcal{F}_t$, instead of the subfiltration $\mathcal{F}_t^Y$. For this purpose define the $P-$martingale

$$g(t, S_t, Z_t) := E\left\{ H(S_T) \mid \mathcal{F}_t \right\} \tag{105}$$

where the notation is justified by the Markov property of (S_t, Z_t). Assuming sufficient regularity of $g(\cdot)$, we proceed analogously to the last part of section 5.1.1 applying Ito's formula to $g(t, S_t, Z_t)$ thereby obtaining

$$\begin{aligned} H(S_T) = g(0, S_0, Z_0) &+ \int_0^T \left[g_t(\cdot) + g_Z(\cdot)\alpha_t(\cdot) \right] dt \\ &+ \int_0^T \left[\frac{1}{2} g_{SS}(\cdot) v_t(\cdot) S_t^2 + \frac{1}{2} g_{ZZ}(\cdot) \beta_t^2(\cdot) \right] dt + \int_0^T g_S(\cdot) dS_t + \int_0^T g_Z(\cdot)\beta_t(\cdot) d\bar{w}_t \end{aligned} \tag{106}$$

Since $g(t, S_t, Z_t)$ is a $P-$martingale, the finite variation terms in (106) vanish, leading to

$$H(S_T) = g(0, S_0, Z_0) + \int_0^T g_S(t, S_t, Z_t) dS_t + M_T^H \tag{107}$$

which is of the form of a *Kunita-Watanabe decomposition* of $H(S_T)$, namely a decomposition of the form

$$H(S_T) = H_0 + \int_0^T \xi_t^H dS_t + M_T^H \tag{108}$$

where M^H is a $P-$martingale that, due to the independence of $\bar{w}_t$ and w_t, is orthogonal to the $P-$martingale S. It then follows from [18] and [19] that the $\mathcal{F}_t-$risk minimizing strategy is given by

$$\left\{ \begin{array}{lcl} \xi_t^{\mathcal{F}} & = & \xi_t^H = g_S(t, S_t, Z_t) \\ \eta_t^{\mathcal{F}} & = & g(t, S_t, Z_t) - \xi_t^{\mathcal{F}} S_t \end{array} \right. \tag{109}$$

so that $V_t(\eta^{\mathcal{F}}, \xi^{\mathcal{F}}) = g(t, S_t, Z_t)$. This strategy appears as a very natural extension of the classical Black Scholes strategy in the pure diffusion case. Notice that, to actually determine $(\eta_t^{\mathcal{F}}, \xi_t^{\mathcal{F}})$ and its value, one needs to compute $g(t, S_t, Z_t)$, which can be achieved either by computing the expectation in (105) (numerical simulations may be used) or by solving the PDE that results from (106) by setting equal to zero the finite variation terms. Details can be found in [21].

Coming to the second step, it follows from a general result in [49] (see also [15]) that the $\mathcal{F}_t^Y-$risk minimizing strategy is obtained by projecting the $\mathcal{F}_t-$risk minimizing strategy onto the subfiltration $\mathcal{F}_t^Y$. This *projection property*, which is due to the quadratic nature of the risk minimizing criterion, makes this latter criterion very attractive every time one has to deal with partial information. More precisely, the $\mathcal{F}_t^Y-$risk minimizing strategy (η^*, ξ^*) is given by

$$\left\{ \begin{array}{lcl} \xi_t^* & = & E\left\{ v_t(Z_t) S_t^2 \xi_t^{\mathcal{F}}(S_t, Z_t) \mid \mathcal{F}_{t-}^Y \right\} / E\left\{ v_t(Z_t) S_t^2 \mid \mathcal{F}_{t-}^Y \right\} \\ \eta_t^* & = & E\left\{ H(S_T) - \xi_t^* S_t \mid \mathcal{F}_t^Y \right\} \end{array} \right. \tag{110}$$

Notice that, according to the model, the hedger will compute the strategy (η^*, ξ^*) only at the jump times T_n of N_t, when he receives new information (for details and a stochastic filtering-type algorithm to compute the projection in (110) see again [21]).

We close ths section mentioning that, for a standard jump-diffusion model of the type of section 3.1.1 with a marked point process, a self financing strategy that minimizes the variance of the residual hedging error can be found in chapter 7 of [36].

6 Pricing in jump-diffusion models

6.1 General aspects

With the introduction of jumps and/or stochastic volatility the market becomes incomplete. Consequently, the principle of absence of arbitrage does not lead to a uniquely defined price. One obtains actually an entire range of prices (see [16],[8]) and the preference structure of the investors has to come into play to determine the *pricing measure*. From the point of view of pure pricing, the problem then reduces to determining a specific martingale measure or, equivalently, the market price of risk. To this effect there are various possibilities and in this section we mention some of them, the last two of which will be discussed in more detail.

i) Historically it appears that a first approach to pricing in markets that are incomplete due to jumps in the prices and to a jumping volatility has been based on *general equilibrium* with a representative agent (see e.g. [3],[42],[1]).

ii) A somewhat related and rather recent approach is that of *pricing by utility maximization*, in which the density of the martingale measure (the *pricing kernel*) is related to the marginal utility of terminal wealth (see e.g. [24] and the references therein; for a specific jump-diffusion setting see [39]).

iii) An alternative possibility is given by more econometric-type approaches based on *estimating/filtering the market price of risk* on the basis of market data. Related to such an approach is the approach described in [27] for a diffusion model with a volatility that may jump at a random time and where the price of a European call turns out to be a monotone function of a parameter λ characterizing the martingale measures. There exists then a unique λ^* consistent with the option price thus allowing to price all the other derivatives consistently with this option. This corresponds basically to completeing the market with the given option.

iv) Approaches based on *market completion*. In the previous secion 4.1 we have discussed various ways to complete both stock as well as bond markets of the jump-diffusion type. As we have seen, this completion leads always to a unique martingale measure, but it does not necessarily imply also completeness in the sense that every claim can be hedged with a self financing portfolio. On the other hand, the uniqueness alone of an equivalent martingale measure is already sufficient to obtain a unique arbitrage-free price of a claim as the expectation of its discounted value under this measure. In all cases where one achieves also completeness in the sense of hedging (essentially all cases except when there are an infinite number of sources of randomness) then, always by absence of arbitrage, the (unique) initial value of the self financing and hedging portfolio has to coincide with the price computed as expectation under the unique martingale measure. The approach based on market completion has been widely used an implemented in various economic setups and here we mention just [50],[51],[33],[41], [37], [31],[32]. It has the advantage to lead to a unique price on the basis of the principle of absence of arbitrage alone, without having to make assumptions on a non-priced jump risk and without the need to introduce a general equilibrium model. On the other hand it requires that the stochastic evolution of more than just the underlying asset has to be specified and, without specific criteria, the completion may occasionally be rather arbitrary.

v) In the previous section 5, in the context of hedging it was mentioned that, if the market cannot be completed, then one has to accept some residual risk and it becomes natural to determine the hedging strategy on the basis of a risk minimization criterion. On the other hand, in the previous point iv) we recalled the fact that, in a complete/completed market the initial value of a self financing and hedging portfolio has to coincide with the arbitrage-free price of the claim. By analogy, it appears then natural to define as price of a claim in a non complete market the *initial value of a portfolio minimizing a given hedging criterion*. Quite typically, the initial value of such a portfolio turns out to be the expectation of the discounted value of the given claim under a specific martingale measure. In other words, there is a correspondence between hedging criteria and martingale measures and the choice of a specific pricing measure can be based on the choice of a specific hedging criterion. An approach along these lines appears thus related to the pricing approach by utility maximization, mentioned in point ii) above. As an example, let us point out that the criterion of risk minimization discussed in section 5.2 leads to the so-called *minimal martingale measure* that was already mentioned at the end of section 4.1. It has been further shown in [47] that, if in a jump-diffusion model claims are priced according to the minimal martingale measure, then convergence of asset prices implies convergence of option prices. This stability result for prices computed according to the minimal martingale measure makes the risk minimization criterion discussed in section 5.2 an attractive criterion for hedging. (For further extensions of this stability property see [44], [28]).

6.2 Computational aspects

Assume that for a jump-diffusion model we have selected a specific martingale measure according to one of the approaches mentioned in the previous section 6.1. We have then to compute the expectation of the (discounted value of the) claim under this martingale measure. In this section we shall mention some of the possible methods to accomplish this.

We consider first the univariate jump-diffusion model (45) under a generic martingale measure Q with intensity of the Poisson process N_t given by $\psi_t \lambda_t$. If Q corresponds to the unique martingale measure obtained from a market completion as in section 4.1, then $\psi_t \lambda_t$ has to be taken according to (53). For simplicity we assume that all the prices are already discounted and so we can put $r_t \equiv 0$. The dynamics of S_t under Q are given by (see (48), (49), (47))

$$dS_t = S_{t-} \left[-\gamma_t \psi_t \lambda_t dt + \sigma_t dw_t^Q + \gamma_t dN_t \right] \tag{111}$$

We want to compute the value of a European call option, namely $E^Q \{(S_T - K)^+\}$. For this purpose we adapt an approach from [37], assuming first that in (111) we have $\gamma_t \equiv \gamma$, i.e. the jump coefficient is constant (for this case see also [2]). We have

$$E^Q \left\{ (S_T - K)^+ \right\} = E^Q \left\{ E^Q \left\{ (S_T - K)^+ \mid N_T \right\} \right\} \tag{112}$$

For a fixed k, i.e. when $N_T = k$, $(k = 0, 1, \cdots)$, using the exponential formula (17) for the specific case when (14) is given by (111), we have

$$S_T^{(k)} = S_0 e^{k\log(1+\gamma)} \exp\left[-\int_0^T \left(\gamma\psi_s\lambda_s + \frac{1}{2}\sigma_s^2 \right) ds + \int_0^T \sigma_s dw_s^Q \right] \tag{113}$$

namely

$$\log S_T^{(k)} \sim \mathcal{N}(\cdot; m_T, \sigma_T^2) \tag{114}$$

with

$$\begin{cases} m_T & = & \log S_0 + k\log(1+\gamma) - \int_0^T \left(\gamma\psi_s\lambda_s + \frac{1}{2}\sigma_s^2 \right) ds \\ \sigma_T^2 & = & \int_0^T \sigma_s^2 ds \end{cases} \tag{115}$$

i.e. $S_T^{(k)}$ is lognormal with mean and variance given by m_T and σ_T respectively. Next compute (with $\Phi(\cdot)$ the cumulative standard Gaussian distribution function)

$$\begin{array}{rcl} V_0^{(k)} & := & E_{S_0}^Q \left\{ (S_T^{(k)} - K)^+ \right\} = \int_{\log k}^{+\infty} (e^x - K) d\mathcal{N}(x; m_T, \sigma_T^2) dx \\ & = & \frac{1}{\sqrt{2\pi\sigma_T^2}} \int_{\log k}^{+\infty} e^x e^{-\frac{1}{2\sigma_T^2}(x-m_T)^2} dx - \frac{K}{2\sigma_T^2} \int_{\log k}^{+\infty} e^{-\frac{1}{2\sigma_T^2}(x-m_T)^2} dx \\ & = & e^{m_T - \frac{1}{2}\sigma_T^2} \Phi\left(\frac{m_T + \sigma_T^2 - \log K}{\sigma_T} \right) - K\Phi\left(\frac{m_T - \log K}{\sigma_T} \right) \\ & := & (1+\gamma)^k G(k, S_0) \end{array} \tag{116}$$

with

$$G(k, S_0) = S_0 \exp\left(-\int_0^T \gamma\psi_s\lambda_s ds \right) \Phi(x) - \frac{K}{(1+\gamma)^k} \Phi(y) \tag{117}$$

where

$$\begin{cases} x & = & \dfrac{\log\left(\frac{S_0(1+\gamma)^k}{K} \right) \int_0^T \left(-\gamma\psi_s\lambda_s + \frac{1}{2}\sigma_s^2 \right) ds}{\sqrt{\int_0^T \sigma_s^2 ds}} \\ y & = & x - \sqrt{\int_0^T \sigma_s^2 ds} \end{cases} \tag{118}$$

Coming back to (112) we then have

$$E^Q\left\{ (S_T - K)^+ \right\} = E^Q\left\{ V_0^{(N_T)} \right\} = \sum_{k=0}^{\infty} (1+\gamma)^k G(k, S_0) \left(\frac{H^k}{k!} e^{-H} \right) \tag{119}$$

with $H = \int_0^T \psi_s\lambda_s ds$. Notice that, for actual computations, the infinite sum in the right in (119) has to be truncated at a sufficiently large positive integer.

The result for $\gamma_t \equiv \gamma$ can be easily extended to the case when γ_t is a piecewise constant deterministic time function. To this effect, given a positive integer m and a subdivision $0 = t_0^m < t_1^m < \cdots < t_m^m = T$, let

$$\gamma_t^{(m)} = \gamma_0 1_{\{0\}}(t) + \sum_{j=1}^{m} \gamma_j 1_{(t_{j-1}^m, t_j^m]}(t) \ ; \quad \gamma_j > -1 \tag{120}$$

Furthermore, let $\mathcal{P}_j$, $(j = 1, \cdots, m)$, be independent Poisson random variables with parameters $H_j = \int_{t_{j-1}^m}^{t_j^m} \psi_s \lambda_s ds$. The generalization of formula (119) is then

$$\begin{aligned} & E^Q\left\{(S_T - K)^+\right\} \\ & = \sum_{k_1,\cdots,k_m=0}^{\infty} \exp\left[\sum_{j=1}^m k_j \log(1+\gamma_j)\right] G(k_1,\cdots,k_m,S_0) \prod_{j=1}^m \left[\frac{(H_j)^{k_j}}{(k_j)!} e^{-H_j}\right] \end{aligned} \tag{121}$$

with

$$G(k_1,\cdots,k_m,S_0) = S_0 \exp\left(-\int_0^T \gamma_s^{(m)} \psi_s \lambda_s ds\right) \Phi(x) - \frac{K}{\prod_{j=1}^m (1+\gamma_j)^{k_j}} \Phi(y) \tag{122}$$

where

$$\left\{ \begin{array}{rcl} x & = & \dfrac{\log\left(\frac{S_0 \prod_{j=1}^m (1+\gamma_j)^{k_j}}{K}\right) \int_0^T \left(-\gamma_t^{(m)} \psi_s \lambda_s + \frac{1}{2}\sigma_s^2\right) ds}{\sqrt{\int_0^T \sigma_s^2 ds}} \\ y & = & x - \sqrt{\int_0^T \sigma_s^2 ds} \end{array} \right. \tag{123}$$

Comig finally to the case of a more general deterministic time function γ_t for the jump coefficient, we assume that there exist piecewise constant deterministic time functions $\gamma_t^{(m)}$ and $\sigma_t^{(m)}$ such that

$$\gamma_t^{(m)} \uparrow \gamma_t \quad , \quad \sigma_t^{(m)} \uparrow \sigma_t \quad \text{as} \quad m \to \infty \tag{124}$$

Consider then a sequence of fictitious risky assets, whose (discounted) values $S_t^{(m)}$ are martingales with respect to the same martingale measure Q as is S_t in (111), namely they satisfy

$$dS_t^{(m)} = S_{t-}^{(m)} \left[-\gamma_t^{(m)} \psi_t \lambda_t dt + \sigma_t^{(m)} dw_t^Q + \gamma_t^{(m)} dN_t\right] \tag{125}$$

For each of the processes $S_t^{(m)}$ we can compute

$$v_0^{(m)} = E^Q\left\{(S_T^{(m)} - K)^+\right\} \tag{126}$$

according to (121), (122), (123). In [37] it is now shown that

$$\lim_{m\to\infty} v_0^{(m)} = v_0 = E^Q\left\{(S_T - K)^+\right\} \tag{127}$$

i.e. if γ_t is a generic time function, that can be approximated from below by a sequence of piecewise constant time functions, then the corresponding option value can be approximated arbitrarily closely by computable expressions. In [37] it is also shown that, for given m, $v_0^{(m)}$ can be interpreted as initial value of a mean self financing and risk minimizing portfolio in the sense of section 5.2 when the asset price evolves in discrete time according to the process $S_t^{(m)}$ of (125), evaluated at the discrete time points t_j. In line with the last part of point v) of the previous subsection 6.1, we may thus consider the approximating values $v_0^{(m)}$ as option values themselves, computed according to the minimal martingale measure.

After having discussed the univariate jump-diffusion model (45), we turn now to the general jump-diffusion model with a marked point process and which can equivalently be represented either by (32) or (33). We opt here for the representation (32). i.e.

$$dS_t = S_{t-}\left[\mu_t dt + \sigma_t dw_t + \gamma(t, Y_t)\, dN_t\right] \tag{128}$$

In what follows we shall make the further

Assumption 6.1 **i)** *$\gamma(t, Y_t) \equiv \gamma(Y_t)$, i.e. γ is independent of the current time;*

ii) *considering the representation of the marked point process as double sequence (T_n, Y_n), assume that T_n is independent of Y_n and the Y_n form a sequence of independent random variables, the generic one Y_n having law $m(dy)$.*

The driving marked point process has thus local $P-$characteristics $(\lambda_t, m(dy))$.

Suppose that we have chosen a specific martingale measure Q and that we want to compute $v_0 = E^Q\{H(S_T)\}$ where, typically, we may have $H(S) = (S-K)^+$. For this purpose, in what follows we adapt a procedure from chapter 7 in [36].

Recall first from Theorem 2.5 that a general absolutely continuous measure transformation from P to Q transforms the $P-$local characteristics into $Q-$local characteristics of the form $(\psi_t\lambda_t, h_t(y)m(dy))$. Recalling furthermore (63) with (62) and (64), it is easily seen that, under the measure Q corresponding to the above local characteristics, the discounted value of S_t satisfies

$$dS_t = S_{t-}\left[-\bar{\Gamma}_t\bar{\lambda}_t dt + \sigma_t dw_t^Q + \gamma(Y_t)dN_t\right] \tag{129}$$

where we have put $\bar{\Gamma}_t = \int_E \gamma(y)h_t(y)m(dy)$ and $\bar{\lambda}_t = \psi_t\lambda_t$. Using the exponential formula (17) to integrate (129), that is of the form of (14) with the representation (15), one immediately finds that, for a given initial asset price S_0, the value $v_0(S_0)$ of the claim $H(S_T)$ is given by

$$v_0(S_0) = E^Q\left\{H\left(S_0 \exp\left[-\int_0^T \left(\bar{\Gamma}_t\bar{\lambda}_t + \frac{\sigma_t^2}{2}\right)dt + \int_0^T \sigma_t dw_t^Q\right]\prod_{n=1}^{N_T}(1+\gamma(Y_n))\right)\right\} \tag{130}$$

Next let

$$V(S_0) := E^Q\left\{H\left(S_0\exp\left[-\int_0^T \frac{\sigma_t^2}{2}dt + \int_0^T \sigma_t dw_t^Q\right]\right)\right\} \tag{131}$$

so that, for $H(S) = (S-K)^+$, the $V(S_0)$ is given by the Black Scholes formula, i.e. $V(S_0) = BS(S_0)$. With the use of $V(S_0)$ we can now write

$$\begin{aligned} v_0(S_0) &= E^Q\left\{V\left(S_0\exp\left[-\int_0^T \bar{\Gamma}_t\bar{\lambda}_t dt\right]\prod_{n=1}^{N_T}(1+\gamma(Y_n))\right)\right\} \\ &= \sum_{k=0}^{\infty} E^Q\left\{V\left(S_0\exp\left[-\int_0^T \bar{\Gamma}_t\bar{\lambda}_t dt\right]\prod_{n=1}^{k}(1+\gamma(Y_n))\right)\right\}\left(\frac{H^k}{k!}e^{-H}\right) \end{aligned} \tag{132}$$

where, due to the local characteristics under Q, we have $H = \int_0^T \psi_s\lambda_s ds$ and where the expectation is with respect to the joint distribution of the Y_n that in Assumption 6.1

were supposed to be independent. This latter expectation can be explicitly computed in special cases, in more complicated cases one has to use simulations. Again, for the actual computations, the infinite sum has to be truncated at a sufficiently large positive integer.

We close this section by mentioning that in [26], for the term structure models of simple forwards in the jump-diffusion setup described therein, the authors study the pricing of some derivative securities after having characterized arbitrage-free dynamics. The derivative prices are also used to investigate what types of patterns in implied volatilities are produced through jumps.

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