

THE BASIC PICTURE
a structural basis
for constructive topology

Giovanni Sambin

including two papers
with Per Martin-Löf and with Venanzio Capretta

to the memory of
my mother Elena Danieli, open heart and acute mind,
who showed me how one can be brave and creative
and of
my father Paolo, tireless worker and rigorous historian,
who taught me how to be tolerant and faithful to my principles

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Giovanni Sambin,
Dipartimento di Matematica Pura ed Applicata, Università di Padova, Italy.
sambin@math.unipd.it

Per Martin-Löf,
Department of Mathematics, University of Stockholm, Sweden.
pml@math.su.se

Venanzio Capretta,
Department of Mathematics and Statistics, University of Ottawa, Canada.
Venanzio.Capretta@mathstat.uottawa.ca

This research has been supported by the Italian Ministero dell'Istruzione, Università e Ricerca, project "Metodi costruttivi in topologia, algebra e fondamenti dell'informatica".

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