Pretopologies and a uniform presentation of sup-lattices, quantales and frames

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Abstract

We introduce the notion of infinitary preorder and use it to obtain a predicative presentation of sup-lattices by generators and relations. The method is uniform in that it extends in a modular way to obtain a presentation of quantales, as "sup-lattices on monoids", by using the notion of pretopology.

Our presentation is then applied to frames, the link with Johnstone's presentation of frames is spelled out, and his theorem on freely generated frames becomes a special case of our results on quantales.

The main motivation of this paper is to contribute to the development of formal topology. That is why all our definitions and proofs can be expressed within an intuitionistic and predicative foundation, like constructive type theory.

Introduction

The notion of pretopology, as in [S89], [S95], is a natural generalization of that of formal topology, introduced in [S87]. Formal topology is by now also the name of the field, whose aim is to develop topology within an intuitionistic and predicative foundation, such as Martin-Löf's type theory [ML84] (henceforth, simply type theory). To pursue this aim, one has to reformulate definitions and theorems of more traditional topology so that they can be expressed in type theory. This often leads also to a sharpening of the mathematical content. This is what happened, in our opinion, with the topic of presentation of frames.

In this paper, which is the outcome of an engagement we undertook long ago [BS93], we show how the notion of formal cover, and its generalizations, allow one to obtain a uniform presentation of sup-lattices, quantales and frames. Our treatment is centered on the notion of infinitary preorder, that is a relation between elements and subsets of a set X which satisfies a suitable kind of reflexivity and transitivity. The biunivocal correspondence between infinitary preorders, closure operators on $\mathcal{P}(X)$ (the power of X) and congruences on $\mathcal{P}(X)$, allows one to construct in a simple way the sup-lattice which is freely presented by a set of generators and by some relations, or conditions on them. Following the same pattern, such results are extended to the case of quantales simply by adding suitable conditions to deal with the monoidal operation; in particular, the notion of precover (and hence of pretopology) is obtained by supplying infinitary preorders with an extra stability condition which corresponds to distributivity. Thus the slogan that quantales are just sup-lattices over monoids [JT84] gets further evidence. Finally, frames are treated as particular quantales, simply by adding conditions which force the monoid operation to coincide with the meet, and the precover to become a formal cover. In particular, we obtain a characterization of the frame freely generated by a monoid, which gives Johnstone's well-known frame of C-ideals over a site as a special case ([J82]). Though our formal covers correspond to Johnstone's coverages, in the precise way shown in the last section here, it is the choice of expressing conditions by inequalities (as with formal covers) rather than equalities (as with coverages) which allows one to find a suitably weak form for conditions and hence which makes our modular construction possible.

Our approach is uniform also in the sense that all our results hold independently of the foundational theory, in the following sense. On one hand, unless otherwise stated, all our definitions and proofs are expressible in type theory; this is the main motivation for this paper, whose origin, and hence also notation, is to be found in formal topology. On the other hand, all arguments are compatible with a classical foundation like ZFC and with an intuitionistic but impredicative foundation like topos theory; in particular, we never use in this paper any argument which is valid in type theory, like the choice principle, but which would destroy constructivity of topos theory.

Developing mathematics in type theory means that the logic used is intuitionistic, like in topos theory, but it means also that the set theory is predicative. In particular, the collection of all subsets of a set is not a set, and thus quantification over all subsets of a set is not allowed. More precisely, a universal or existential quantification over subsets does not produce a proposition, and so it cannot be used to construct an object, like a set or subset, while of course free parameters on subsets get a meaning by means of substitutions, and so they can appear in a definition, like that of infinitary preorder.

One advantage of such a discipline lies in the fact that type theory is itself a functional programming language (see [NPS90]), and so all the mathematics developed within type theory is ipso facto expressible and checkable in a computer. We hasten to note that this paper remains a piece of mathematics, written in a language which is not too far from that usual in mathematics. We can leave the details and problems of an actual formalization in type theory, since this is automatic, as long as we use the methods developed in [SV98] (we will use definitions of [SV98] even without mention, beginning with that of subset, since they are equivalent to the traditional ones for a non-predicativist reader).[1] This is, in our opinion, the best way to develop a deep conceptual interaction between mathematics and computing science.

We have put some effort in simplifying proofs and the structure of exposition, and this often allows us to give detailed proofs; besides being a matter of taste, this has the purpose of showing in practice that all our arguments fully preserve constructivity in the strongest sense (an impredicative treatment would bring to a more abstract, and sometimes shorter, exposition).

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1 Infinitary terms and relations

We are going to describe structures equipped with an infinitary operation by means of generators and relations. In the case of finitary operations, one simply defines inductively the set of all terms, or polynomials, over a given set of generators X. This is not possible in the case of an infinitary operation (see [J82] and [V89]), and hence one has to look for a different approach. We here describe our method on the simplest infinitary structure to which it applies, namely that of sup-lattice. The usual definition is (cf. [JT84]):

Definition 1.1 *A sup-lattice $\mathcal{L} = (L, \leq, \bigvee)$ is a partially ordered set $(L, \leq)$ provided with an operation of infinitary join $\bigvee$, that is an operation which applies to every subset of L, and gives the supremum with respect to the order $\leq$.*

A morphism between the sup-lattices $\mathcal{L} = (L, \leq, \bigvee)$ and $\mathcal{L}' = (L', \leq, \bigvee)$ is a map $f : L \to L'$ such that $f(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} f(x_i)$ for every family $x_i \in L$ $(i \in I)$.

The above definition, taken literally, is definitely too restrictive if the notion of set is interpreted as in type theory, where for instance $\mathcal{P}(X)$ is never a set (see [MV99]). Therefore one has to give up the fact that L is a set, and require L to be a collection (or category, see [ML84]). The notion of subset (of which one requires the supremum to exist) is then replaced by that of set-indexed family of elements $x_i \in L$ $(i \in I)$. Then we reach the following definition (see [SV98], section 2.8):

Definition 1.1 (predicative) *A sup-lattice $\mathcal{L}$ is a collection L with a partial order $\leq$ such that for any family of elements of L indexed by a set I, that is $x_i \in L$ $(i \in I)$, the supremum $\bigvee_{i\in I} x_i$ exists in L.*

We must admit that this definition is not very satisfactory, since it contains reference to all subsets and to all elements of a collection (this is implicit in the definition of supremum). Thus by no means it

[1] A reader aquainted with the notation of [SV98] should however be aware that in this paper we don't distinguish typographically between an element of a set $a \in S$ and an element of a subset, written $a \,\epsilon\, U$.

can be used to construct sup-lattices, and rather it should be seen as a requirement to be fulfilled. But it is a fact that it *can* be fulfilled, that is, that there are examples of sup-lattices which are constructed fully within type theory. The main example is of course the power $\mathcal{P}(X)$ of a set X: the ordering is inclusion $\subseteq$ between two subsets and for any set-indexed family of subsets $U_i \subseteq X$ $(i \in I)$, the supremum is simply the union $\bigcup_{i\in I} U_i$ (which is defined through an elementary existential quantification on I, see [SV98]). So we keep the definition, and think of it as a way to abstract some of the properties of the examples, and the examples will be obtained fully constructively.

A convenient characterization of the join of any family $x_i \in L$ $(i \in I)$ of elements of $\mathcal{L}$ is that, for any $y \in L$,

$$\bigvee_{i\in I} x_i \leq y \textit{ iff for every } i \in I,\ \ x_i \leq y.$$

As a consequence one obtains the usual link between $\leq$ and $\bigvee$, that is $x \leq y$ iff $x \vee y = y$, where $x \vee y \equiv \bigvee\{x, y\}$. This is why every morphism of sup-lattices (see definition 1.1) preserves $\leq$.

For our purposes, the following formulation of sup-lattices is more convenient:

Theorem 1.2 *Sup-lattices can be characterized as pairs $(L, \bigvee)$ where $\bigvee$ is an infinitary operation on L satisfying*

(i) $\bigvee\{x\} = x$ *for every* $x \in L$*;*

(ii) $\bigvee_{i\in I}(\bigvee U_i) = \bigvee(\bigcup_{i\in I} U_i)$ *for every family* $(U_i)_{i\in I}$ *of subsets of* L.[2]

Proof. It is immediate to see, by using the above characterization of joins, that a sup-lattice $\mathcal{L} = (L, \leq, \bigvee)$ satisfies conditions (i) and (ii).

Conversely, given a pair $(L, \bigvee)$ satisfying (i) and (ii), one defines $x \leq y$ putting $\bigvee\{x, y\} = y$. Conditions (i) and (ii) are enough to prove that $\leq$ is a partial order (or equivalently, that the binary operation $x \vee y \equiv \bigvee\{x, y\}$ is associative, commutative and idempotent). We now see that $\bigvee U$ gives indeed the join of an arbitrary subset U w.r.t. the order $\leq$ so defined. In fact, for every $a \in U$, it is $a \leq \bigvee U$ because $\bigvee\{a, \bigvee U\}$ is equal to $\bigvee\{\bigvee\{a\}, \bigvee U\}$ by condition (i), and hence also to $\bigvee(\{a\} \cup U)$ by condition (ii), but $\{a\} \cup U = U$ since $a \in U$. Now let b such that $a \leq b$, i.e. $\bigvee\{a, b\} = \bigvee\{b\}$ for every $a \in U$. Then $\bigvee_{a\in U}(\bigvee\{a, b\}) = \bigvee_{a\in U}(\bigvee\{b\})$. By condition (ii), the right member is equal to $\bigvee(\bigcup_{a\in U}\{b\})$ which is the same as $\bigvee\{b\}$, which is equal to b by (i). By condition (ii), the left member is equal to $\bigvee(\bigcup_{a\in U}\{a, b\})$ and hence to $\bigvee(U \cup \{b\})$, that is $\bigvee(\bigvee U, \bigvee\{b\})$, again by condition (ii). So, $\bigvee(\bigvee U, \bigvee\{b\}) = b$, that is $\bigvee U \leq b$. □

After the above characterization, it is easy to prove the following:

Proposition 1.3 *For any set X, the power $\mathcal{P}(X)$ is the free sup-lattice generated by X.*

Proof. $\mathcal{P}(X)$ is a sup-lattice, as we have seen above. Now, for any sup-lattice $\mathcal{L}$ and for any $g : X \to L$, define $g : \mathcal{P}(X) \to L$ by putting $\tilde{g}(U) \equiv \bigvee_{b\in U} g(b)$, for every $U \subseteq X$. The map $\tilde{g}$ extends g, in the sense that $g = \tilde{g} \circ i$ where $i : x \mapsto \{x\}$ is the embedding of X into $\mathcal{P}(X)$. In fact $\tilde{g}(\{x\}) = g(x)$ for every $x \in X$ by (i) of theorem 1.2. Moreover, $\tilde{g}$ is a sup-lattice morphism, since $\tilde{g}(\bigcup_{i\in I} U_i) \equiv \bigvee_{x\in\bigcup_{i\in I} U_i} g(x) = \bigvee_{i\in I}\bigvee_{x\in U_i} g(x) = \bigvee_{i\in I} \tilde{g}U_i$. Then $\tilde{g}$ is the unique sup-lattice morphism extending g to the subsets of X. In fact, for every $U \subseteq X$, it is $U = \bigcup_{b\in U}\{b\}$ and then a morphism f extending g satisfies $f(U) = \bigvee_{b\in U} f(\{b\}) = \bigvee_{b\in U} g(b) \equiv \tilde{g}(U)$. □

We now see that the sup-lattice $\mathcal{P}(X)$ can be considered as the sup-lattice of terms, that is, arbitrary subsets of a given set X of generators take the place, in the case of infinitary join, of the usual (finitary) terms. Since the arguments we give to this aim have the only purpose of motivating intuitively our definitions, we will not be rigorous with expressibility in type theory, up to the end of this section. We recall that the sup-lattice $\mathcal{L}$ is *generated* by a set X if L is the closure of X under the infinitary operation $\bigvee$. By condition (ii) of theorem 1.2 above, X generates $\mathcal{L}$ iff every element of L is the join

[2] We thank Ales Pultr for observing that this proposition amounts to saying that sup-lattices are the (Eilenberg-Moore) P-algebras where P is the monad $(\mathcal{P}, \mu, \eta)$ with $\mathcal{P}$ the powerset functor $Set \to Set$, $\mu_x = (U \mapsto \cup U) : \mathcal{P}\mathcal{P}(X) \to \mathcal{P}(X)$, and $\eta_x = (x \mapsto \{x\}) : X \to \mathcal{P}(X)$. We do not spell out a similar translation into categorical language for other propositions in this paper.

$\bigvee U$ for some subset U of X. In other words, the first level (on subsets of X) of closure under joins is enough to obtain every possible further join. Thus, every element of any sup-lattice generated by X can be labelled by a subset U of X, so that subsets of X can be considered the *infinitary terms* on the set X of generators.

The next question arising is: when are two terms identifiable? The usual extensional equality between subsets, defined by $U = V$ *iff* $\forall a \in X(a \in U \leftrightarrow a \in V)$, is now a sort of *syntactical* equality between the two terms U and V, since it tells that the two terms have the same components (generators). On the other side, two subsets may contain different generators and denote the same element of a given sup-lattice $\mathcal{L}$. We need to conceive a new equality, identifying the terms denoting the same object: in this sense, subsets get a new extension, that is the object they denote as terms. So we put

$$U \,\theta_{\mathcal{L}}\, W \equiv (\bigvee U = \bigvee W)$$

where U, W are subsets of a set X which generates $\mathcal{L}$. We now prove that $\theta_{\mathcal{L}}$ is a *congruence* on the sup-lattice of terms $\mathcal{P}(X)$, as one could expect generalizing from the finitary case.

We say that θ is a congruence on a sup-lattice $\mathcal{L} = (L, \bigvee)$ if it is an equivalence relation on L which moreover respects the infinitary operation $\bigvee$, that is such that $x_i \,\theta\, y_i$ for all $i \in I$ implies $\bigvee_{i\in I} x_i \;\theta\; \bigvee_{i\in I} y_i$. The notion of quotient sup-lattice $\mathcal{L}/\theta$ is then defined, as usual, by considering the quotient L/θ with join defined by $\bigvee_{i\in I}[x_i]_\theta \equiv [\bigvee_{i\in I} x_i]_\theta$. Clearly, $\theta_{\mathcal{L}}$ is an equivalence relation; we see that it respects joins. In fact, $U_i \,\theta_{\mathcal{L}}\, W_i$ for all $i \in I$ means that $\bigvee U_i = \bigvee W_i$ for all $i \in I$, hence $\bigvee_{i\in I}(\bigvee U_i) = \bigvee_{i\in I}(\bigvee W_i)$ from which, by condition (ii) of theorem 1.2, $\bigvee(\bigcup_{i\in I} U_i) = \bigvee(\bigcup_{i\in I} W_i)$, that is $\bigcup_{i\in I} U_i \,\theta_{\mathcal{L}}\, \bigcup_{i\in I} W_i$.

Congruences on the sup-lattice of terms permit to obtain a presentation of sup-lattices; in fact, it is easy to prove that:

Proposition 1.4 *For any sup-lattice $\mathcal{L}$, if L is generated by a set X, then $\mathcal{L}$ is isomorphic to $\mathcal{P}(X)/\theta_{\mathcal{L}}$, where $\theta_{\mathcal{L}}$ is the congruence on the sup-lattice $\mathcal{P}(X)$ defined by:*

$$\textit{for any } U, W \subseteq X,\ U\theta_{\mathcal{L}}W \textit{ iff } \bigvee U = \bigvee W.$$

Proof The map π from the quotient sup-lattice $\mathcal{P}(X)/\theta_{\mathcal{L}}$ to $\mathcal{L}$, defined by $\pi : [U]_{\theta_{\mathcal{L}}} \mapsto \bigvee U$, is an isomorphism: in fact, π is onto because $\mathcal{L}$ is generated by X, and π is one-one because by definition $\pi[U] = \pi[V]$ iff $[U] = [V]$. Finally, π preserves joins because $\pi(\bigvee_{i\in I}[U_i]) \equiv \pi[\bigcup_{i\in I} U_i] \equiv \bigvee(\bigcup_{i\in I} U_i) \equiv \bigvee_{i\in I}(\bigvee U_i) \equiv \bigvee_{i\in I} \pi[U_i]$. □

A congruence on the sup-lattice of terms is an *infinitary relation*; in general, we call *infinitary* any relation on a set in which at least one of the arguments is a subset. So proposition 1.4 above says that any sup-lattice can be impredicatively presented by (infinitary) generators and (infinitary) relations. Unfortunately, the notion of congruence is not very convenient to work with and moreover it is not well suited for an inductive generation, which is necessary in a predicative approach. So we need a different kind of infinitary relations. We dedicate the next three paragraphs to solve this problem; we will then come back to the presentation of sup-lattices, using the most elementary and handy notion we have been able to find, namely that of infinitary preorder.

1.1 Infinitary preorders

We first give a full definition of the notion of congruence on the sup-lattice of terms $\mathcal{P}(X)$:

Definition 1.5 *A congruence θ on a set X is a relation between two subsets of X which is closed under:*

(i) $\quad \dfrac{U = V}{U\theta V}$ *(reflexivity)*

(ii) $\quad \dfrac{U\theta V \quad V\theta W}{U\theta W}$ *(transitivity)*

(iii) $\quad \dfrac{U\theta V}{V\theta U}$ *(symmetry)*

$$(iv) \qquad \frac{U_i \theta V_i \quad \textit{for all } i \in I}{\bigcup_{i\in I} U_i \theta \bigcup_{i\in I} V_i} \qquad \textit{(congruence property)}$$

Note that reflexivity amounts to the requirement that the extensional equality between subsets, that is the syntactical equality between terms, is preserved.

The first step towards a more convenient form is to replace equalities by inequalities, that is to induce a relation $\prec$ between subsets, where the intended meaning of $U \prec W$ is that $\bigvee U \leq \bigvee W$, rather than $\bigvee U = \bigvee W$. The resulting definition is:

Definition 1.6 *For any set X, a relation $\prec$ between subsets of X is called a congruence preorder*[3] *if for all $U, V, W, U_i, W_i \subseteq X$ it satisfies:*

$$(SR_G) \qquad \frac{U \subseteq V}{U \prec V} \quad \textit{(global strong reflexivity)}$$

$$(T_G) \qquad \frac{U \prec V \quad V \prec W}{U \prec W} \quad \textit{(global transitivity)}$$

$$(U) \qquad \frac{U_i \prec V \quad \textit{for all } i \in I}{\bigcup_{i\in I} U_i \prec V} \quad \textit{(union is respected)}$$

Congruences and congruence preorders are linked in the same way as $=$ and $\leq$ are linked, through $\bigvee$, in any lattice:

Proposition 1.7 *Let X be any set. If θ is any congruence on X, the relation $\prec_\theta$ defined by*

$$U \prec_\theta W \equiv (U \cup W)\theta W$$

is a congruence preorder. Viceversa, if $\prec$ is a congruence preorder on X, then the relation $\theta_\prec$ defined by

$$U \theta_\prec W \equiv (U \prec W) \,\&\, (W \prec U)$$

is a congruence on X. The two mappings so defined give a bijection between congruences and congruence preorders.

Proof. Let θ be a congruence. Then (SR_G) holds for $\prec_\theta$, since $U \subseteq V$ means that $U \cup V = V$, from which by reflexivity $(U \cup V)\theta V$, that is $U \prec_\theta V$. The proof of (T_G) is a bit longer: assume $U \prec_\theta V$ and $V \prec_\theta W$, that is $(U \cup V)\theta V$ and $(V \cup W)\theta W$. From $(U \cup V)\theta V$ one derives $(U \cup V \cup W)\theta(V \cup W)$, because $W\theta W$ and θ preserves joins, which, together with $(V \cup W)\theta W$, gives $(U \cup V \cup W)\theta W$ by transitivity. Again from $(V \cup W)\theta W$ one derives $(U \cup V) \cup W \theta (U \cup W)$, and so finally $(U \cup W)\theta W$ i.e. $U \prec_\theta W$. To prove (U), assume $U_i \prec_\theta V$, for all $i \in I$, that is $(U_i \cup V)\theta V$. Taking $V_i = V$ in (iv) of definition 1.5, one obtains $\bigcup_{i\in I}(U_i \cup V)\theta V$; since $\bigcup_{i\in I}(U_i \cup V) = (\bigcup_{i\in I} U_i) \cup V$, one has $\bigcup_{i\in I}(U_i \cup V)\theta(\bigcup_{i\in I} U_i) \cup V$, by reflexivity, and hence by transitivity $((\bigcup_{i\in I} U_i) \cup V)\theta V$, i.e. $\bigcup_{i\in I} U_i \prec_\theta V$.

Let $\prec$ be a congruence preorder. From (SR_G) reflexivity of $\theta_\prec$ follows immediately: $U = V$ means that $U \subseteq V \,\&\, V \subseteq U$, and hence, by (SR_G), $U \prec V \,\&\, V \prec U$, that is $U\theta_\prec V$. From (T_G) it is straightforward to obtain transitivity of $\theta_\prec$. Symmetry of $\theta_\prec$ is obvious. To see that $\theta_\prec$ preserves joins, assume $U_i \theta_\prec V_i$ for all $i \in I$; then $U_i \prec V_i$, hence $U_i \prec \bigcup_{i\in I} V_i$, for all $i \in I$, from which $\bigcup_{i\in I} U_i \prec \bigcup_{i\in I} V_i$ by (U), and similarly $\bigcup_{i\in I} V_i \prec \bigcup_{i\in I} U_i$, so that $\bigcup_{i\in I} U_i \theta_\prec \bigcup_{i\in I} V_i$.

Finally, $\theta_{\prec_\theta}$ is equal to θ, because $U\theta_{\prec_\theta} V \equiv U \prec_\theta V \,\&\, V \prec_\theta U \equiv (U \cup V)\theta V \& (U \cup V)\theta U$ iff $U\theta V$, and $\prec_{\theta_\prec}$ is equal to $\prec$ because $U \prec_{\theta_\prec} V \equiv (U \cup V)\theta_\prec V \equiv (U \cup V) \prec V \& V \prec (U \cup V)$ iff $U \prec V$. $\square$

Our aim is now to reduce $\prec$ to a relation $\lhd$ between elements and subsets. Recalling that $a = \bigvee\{a\}$, the characterizing property of joins, given after 1.1, can be rewritten in terms of $\prec$ as

$$(S) \qquad (\forall a \in U)(\{a\} \prec V) \qquad \textit{iff} \qquad U \prec V$$

Condition (S) can equivalently replace condition (U)[4]:

[3] We borrow this name from [V].

[4] Condition (S) is actually stronger than condition (U), in the sense that, if (S) is assumed, (SR_G) is equivalent to the apparently weaker condition of global reflexivity:

$(R_G) \quad U \prec U$

while (SR_G) cannot be replaced by (R_G) in the original definition of congruence preorder. We leave the details of the proof.

Proposition 1.8 *A relation $\prec$ between two subsets of a set X is a congruence preorder iff it satisfies (SR), (T_G) and (S) above.*

Proof. It is enough to show that, assuming (SR_G) and (T_G), condition (S) is equivalent to (U). So assume (U). From $(\forall a \in U)(\{a\} \prec U)$ since $U = \bigcup_{a \in U}\{a\}$ by (U) it follows that $U \prec V$; viceversa, if $U \prec V$ then from $a \in U$, i.e. $\{a\} \subseteq U$, it follows $\{a\} \prec U$ by (SR_G) and hence $\{a\} \prec V$ by (T_G), and this means that $(\forall a \in U)(\{a\} \prec V)$.

Conversely, assume (S). If $U_i \prec V$ for every $i \in I$, then by the right-to-left direction of (S) it follows $(\forall a \in U_i)(\{a\} \prec V)$ for every $i \in I$, that is $(\forall a \in \bigcup_{i\in I} U_i)(\{a\} \prec V)$ and then the conclusion $\bigcup_{i\in I} U_i \prec V$ follows by the left-to-right direction of (S). □

Now, the point is that we can read (S) as a characterization of congruence preorders in terms of a subrelation using only singletons at the left. So, for any congruence preorder $\prec$, we define a relation $\lhd$ between elements and subsets by putting

$$a \lhd U \equiv \{a\} \prec U.$$

We now see that enough conditions on $\lhd$ can be found, to get a notion equivalent to that of congruence preorder. From (SR_G) it follows that strong reflexivity:

$$(SR) \qquad \frac{a \in U}{a \lhd U}$$

must be valid, because $a \in U$ gives $\{a\} \subseteq U$, and hence $\{a\} \prec U$. Transitivity for $\lhd$ takes the form:

$$(SR) \qquad \frac{a \lhd U \quad \forall b \in U(b \lhd V)}{a \lhd V}$$

In fact, $\forall b \in U(b \lhd V)$, i.e. $\forall b \in U(\{b\} \prec V)$, is equivalent by (S) to $U \prec V$, which together with $a \lhd U$, i.e. $\{a\} \prec U$, gives by transitivity of $\prec$ the conclusion $\{a\} \prec V$, i.e. $a \lhd V$.

We have thus reached the basic definition of our approach:

Definition 1.9 *For any set X, a relation $\lhd$ between elements and subsets of X is called an infinitary preorder if it satisfies (SR) and (T) above.*

A pair $(X, \lhd)$ is called an infinitary preordered set if $\lhd$ is an infinitary preorder on the set X.

The notions of infinitary preorder and congruence preorder are actually interchangeable:

Proposition 1.10 *Every congruence preorder $\prec$ gives rise to an infinitary preorder $\lhd_\prec$ defined by:*

$$a \lhd_\prec U \equiv \{a\} \prec U.$$

Viceversa, every infinitary preorder $\lhd$ gives rise to a congruence preorder $\prec_\lhd$ defined by:

$$U \prec_\lhd V \equiv \forall a \in U(a \lhd V).$$

Such a correspondence gives a bijection between congruence preorders and infinitary preorders.

Proof. The remarks preceding definition 1.9 show that $\lhd_\prec$ is an infinitary preorder whenever $\prec$ is a congruence preorder.

Conversely, assume $\lhd$ is an infinitary preorder. To prove (SR_G) for $\prec_\lhd$, assume $U \subseteq V$; then for any $a \in U$ it is $a \in V$, hence $a \lhd V$ by (SR) and therefore $\forall a \in U(a \lhd V)$, i.e. $U \prec_\lhd V$. (T_G) is easily seen to hold by definition. To prove (U), assume $U_i \prec_\lhd V$ for every $i \in I$; then for any $a \in \bigcup_{i\in I} U_i$ it is $a \in U_i$ for some $i \in I$, and hence $a \lhd V$, which means that $\forall a \in \bigcup_{i\in I} U_i (a \lhd V)$ holds, as wished.

Finally, since $a \lhd_{\prec_\lhd} U \equiv \{a\} \prec_\lhd U \equiv \forall b \in \{a\}(b \lhd U)$, it is $a \lhd_{\prec_\lhd} U$ iff $a \lhd U$ and since $U \prec_{\lhd_\prec} V \equiv \forall b \in U(b \lhd_\prec V) \equiv \forall b \in U(\{b\} \prec V)$, it is $U \prec_{\lhd_\prec} V$ iff $U \prec V$. Hence the correspondence is bijective. □

By such proposition, any congruence preorder is obtained in a unique way by extending an infinitary preorder to subsets on the left, the intended meaning of $U \lhd V$ being $\forall a \in U(a \lhd V)$. So we can from now on leave out the notion of congruence preorder.

Finally, we define the category of infinitary preordered sets. We need the notion of morphism between two objects $\mathcal{C} = (X, \lhd_{\mathcal{C}})$ and $\mathcal{C}' = (X', \lhd_{\mathcal{C}'})$ of the category, that is we need maps which can transform the generators and preserve the relations. It is enough to put:

Definition 1.11 *A morphism between the infinitary preordered sets $\mathcal{C} = (X, \lhd_{\mathcal{C}})$ and $\mathcal{C}' = (X', \lhd_{\mathcal{C}'})$ is a map $f : X \to \mathcal{P}(X')$ such that*

$$a \lhd_{\mathcal{C}} U \Rightarrow f(a) \lhd_{\mathcal{C}'} f(U)$$

for every $a \in X, U \subseteq X$, where we put $f(U) \equiv \bigcup_{b \in U} f(b)$.

It is easy to see that, given $f : (X, \lhd_{\mathcal{C}}) \to (X', \lhd_{\mathcal{C}'})$ and $g : (X', \lhd_{\mathcal{C}'}) \to (X'', \lhd_{\mathcal{C}''})$, their composition, defined by $g \circ f(a) \equiv g(f(a))$, that is $\bigcup_{b \in f(a)} g(b)$ by the above definition, is a morphism. Moreover, it is immediate to see that the morphism which maps every $a \in X$ into the singleton $\{a\} \in \mathcal{P}(X)$ is the identity with respect to such composition. Hence the infinitary preordered sets form a category, called **IP**.

1.2 Infinitary preorders and closure operators

We show that the notion of infinitary preorder is equivalent also to a well known and general notion, namely that of closure operator. This will be used in the next section in the presentation of sup-lattices. To see the equivalence, the first step is to note that infinitary relations on a set X correspond to operators on X. An operator $\mathcal{O}$ on X is a map $\mathcal{O} : \mathcal{P}(X) \to \mathcal{P}(X)$; given a pair $\mathcal{O}, \mathcal{O}' : \mathcal{P}(X) \to \mathcal{P}(X)$ we say that $\mathcal{O}$ is finer than $\mathcal{O}'$ when $\mathcal{O}(U) \subseteq \mathcal{O}'(U)$ for every $U \subseteq S$. This defines a partial order between operators on X, as it is easy to see.

Proposition 1.12 *For any set X, there is a bijection between infinitary relations $R(a, U)$ and operators $\mathcal{O}$ on X, which is given by the maps $R \mapsto \mathcal{O}_R$ and $\mathcal{O} \mapsto R_{\mathcal{O}}$ defined by putting:*

$$\mathcal{O}_R(U) \equiv \{a \in X : R(a, U)\}$$

and

$$R_{\mathcal{O}}(a, U) \equiv a \in \mathcal{O}(U).$$

Moreover, such a bijection preserves order.

Proof. Straightforward, because $R_{\mathcal{O}_R}(a, U) \equiv a \in \mathcal{O}_R(U) \equiv R(a, U)$ and $\mathcal{O}_{R_{\mathcal{O}}}(U) \equiv \{a : R_{\mathcal{O}}(a, U)\}$ $\equiv \{a : a \in \mathcal{O}(U)\} = \mathcal{O}(U)$. Moreover, it is $R \subseteq R'$ if and only if, for every $U \subseteq X$, $a \in \mathcal{O}(U)$ implies $a \in \mathcal{O}'(U)$ for every $a \in X$, that is $\mathcal{O}(U) \subseteq \mathcal{O}'(U)$ for every $U \subseteq X$. $\Box$

The link between relations and operators is very convenient and will often be used; in the sequel, we will jump from one notation to the other simply by saying that "$R(a, U)$" is rewritten as "$a \in \mathcal{O}_R(U)$" and conversely. As we have just seen, the rewriting technique preserves the order. This simple fact will play an important role in the sequel.

We now see that infinitary preorders correspond to closure operators. Recall that a closure operator $\mathcal{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ is any operator satisfying the conditions $U \subseteq \mathcal{C}U$ (reflexivity), $U \subseteq V \Rightarrow \mathcal{C}U \subseteq \mathcal{C}V$ (monotonicity) and $\mathcal{C}\mathcal{C}U \subseteq \mathcal{C}U$ (idempotency), for every $U, V \subseteq X$. Note that from the last and the first property the equality $\mathcal{C}\mathcal{C}U = \mathcal{C}U$ follows. Now we can see that:

Proposition 1.13 *The correspondence given in proposition 1.12 specializes to a bijection between infinitary preorders and closure operators on a set X.*

Proof If $\lhd$ is a relation and $\mathcal{C}$ is the associated operator, as in 1.12, one can see that the conditions for $\lhd$ to be an infinitary preorder are rewritten into properties for $\mathcal{C}$ to be a closure operator. Rule (SR) is rewritten as

$$\frac{a \in U}{a \in \mathcal{C}U}$$

that is $\forall x(x \in U \Rightarrow x \in \mathcal{C}U)$, which is a definition of $U \subseteq \mathcal{C}U$. So $\lhd$ satisfies (SR) iff $\mathcal{C}$ satisfies reflexivity. Rule (T) is rewritten as:

$$\frac{a \in \mathcal{C}U \quad \forall b \in U(b \in \mathcal{C}V)}{a \in \mathcal{C}V}$$

which means that from the right premiss, which by definition is $U \subseteq \mathcal{C}V$, one can conclude $a \in \mathcal{C}U \Rightarrow a \in \mathcal{C}V$ for arbitrary a, that is:

$$\frac{U \subseteq \mathcal{C}V}{\mathcal{C}U \subseteq \mathcal{C}V} \tag{$*$}$$

Now $(*)$ is easily seen to be equivalent to monotonicity together with idempotency for $\mathcal{C}$. In fact, if $(*)$ holds, then from $U \subseteq V$, and hence $U \subseteq \mathcal{C}V$ by reflexivity, it follows that $\mathcal{C}U \subseteq \mathcal{C}V$, so that $\mathcal{C}$ is monotonic; idempotency follows by $(*)$ from $\mathcal{C}U \subseteq \mathcal{C}U$. Conversely, from the premiss $U \subseteq \mathcal{C}V$ one has $\mathcal{C}U \subseteq \mathcal{C}\mathcal{C}V$ if $\mathcal{C}$ is monotonic, and hence $\mathcal{C}U \subseteq \mathcal{C}V$, if $\mathcal{C}$ is idempotent, so that $(*)$ holds. $\square$

An additional characterization says that $\mathcal{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operator if and only if it satisfies the equivalence

$$U \subseteq \mathcal{C}V \qquad \text{iff} \qquad \mathcal{C}U \subseteq \mathcal{C}V \tag{$**$}$$

In fact, one direction of $(**)$ is $(*)$ above, while the other is equivalent to reflexivity.

Putting together propositions 1.7, 1.10, 1.13, we can conclude that congruences, infinitary preorders and closure operators are all just different formulations of the same mathematical content. That is, summing up, for any set X there is a bijection among the following:

(i) Congruences on the sup-lattice $(\mathcal{P}(X), \bigcup)$;

(ii) Infinitary preorders on X;

(iii) Closure operators on X.

In spite of such a biunivocal correspondence, it is quite convenient to keep both the notation (and intuition) of a closure operator $\mathcal{C}$ and that of the corresponding infinitary preorder, which we denote by $\lhd_{\mathcal{C}}$ (sometimes also without the subscript). In fact, the former often allows shorter statements and proofs, while the latter is necessary from the predicative point of view, since it allows to see that $\lhd$ can be generated inductively by some axioms and rules, as we will see in the next section.

Moreover, it is useful to grasp the correspondences just summarized without the intermediate step of congruence preorders. Given $\mathcal{C}$, we have just seen that $a \lhd_{\mathcal{C}} U$ is a rewriting for $a \in \mathcal{C}U$, and hence $U \lhd_{\mathcal{C}} V$, that is a rewriting for $U \subseteq \mathcal{C}V$, is equivalent to the inclusion $\mathcal{C}U \subseteq \mathcal{C}V$, by $(**)$. So, in particular, if we denote by $=_{\mathcal{C}}$ (instead of $\theta_{\lhd_{\mathcal{C}}}$!) the congruence associated with $\lhd_{\mathcal{C}}$, then the congruence relation $U =_{\mathcal{C}} V$, which is by definition $U \lhd_{\mathcal{C}} V \& V \lhd_{\mathcal{C}} U$, is equivalent to the equality $\mathcal{C}U = \mathcal{C}V$. Summing up, we have the equivalences

$$U \lhd_{\mathcal{C}} V \qquad \text{iff} \qquad \mathcal{C}U \subseteq \mathcal{C}V$$

$$U =_{\mathcal{C}} V \qquad \text{iff} \qquad \mathcal{C}U = \mathcal{C}V$$

So $=_{\mathcal{C}}$ is the finest equivalence turning the preorder relation $\lhd_{\mathcal{C}}$ between subsets into a partial order. In fact, if $\sim$ is an equivalence relation between subsets such that $U \lhd_{\mathcal{C}} V \& V \lhd_{\mathcal{C}} U$ implies $U \sim V$, it is by definition that $U =_{\mathcal{C}} V$ implies $U \sim V$. Note in addition that the equivalence $(**)$ can be rewritten also into

$$\forall a(a \in U \to a \lhd_{\mathcal{C}} V) \quad \text{iff} \quad \forall a(a \lhd_{\mathcal{C}} U \to a \lhd_{\mathcal{C}} V) \tag{$***$}$$

So, given a relation $\lhd$ between elements and subsets of a set X and putting $U \lhd V \equiv \forall a \in U(a \lhd V)$, the relation $\lhd$ is an infinitary preorder when $U \lhd V$ holds if and only if $\forall a(a \lhd U \to a \lhd V)$. We stress finally that $U =_{\mathcal{C}} V$ can be written $\forall a(a \lhd_{\mathcal{C}} U \leftrightarrow a \lhd_{\mathcal{C}} V)$, that has the form of an extensional equality, depending on the relation $\lhd_{\mathcal{C}}$ rather than membership. This is the extension of subsets considered as terms for the elements of a sup-lattice, and in this sense the congruence $=_{\mathcal{C}}$ is the equality of the infinitary preordered set $(X, \lhd_{\mathcal{C}})$.

In this setting, it is significant to observe that, for any morphism $f : \mathcal{C} \to \mathcal{C}'$ of the category **IP**, $U =_{\mathcal{C}} V$ implies $f(U) =_{\mathcal{C}'} f(V)$. This means that a morphism respects the equalities of the infinitary preordered sets $\mathcal{C}$ and $\mathcal{C}'$. Accordingly, we characterize a morphism with respect to congruences, so that we consider two morphisms $f, f' : \mathcal{C} \to \mathcal{C}'$ equal when it is $f(U) =_{\mathcal{C}'} f'(U)$ in $\mathcal{C}'$, for every $U \subseteq X$. This amounts again to considering the extension of the terms as the extension of the object they denote, quite independently from how the term is given. The two maps are then identified when "their graphs are equal", quite independently from how the maps themselves are given.

2 Presentation of sup-lattices

A presentation of sup-lattices by means of infinitary preordered sets is now easily obtained, by way of the link with closure operators. The fixed points of the closure operator $\mathcal{C}$, i.e. the subsets U of X such that $U = \mathcal{C}U$, are usually called closed; here we prefer to call them $\mathcal{C}$*-saturated*, or simply *saturated*, subsets. Accordingly, the collection of $\mathcal{C}$-saturated subsets of X is denoted by $Sat(\mathcal{C})$. Since U is saturated iff $U = \mathcal{C}V$ for some $V \in X$, it follows that

$$Sat(\mathcal{C}) = \{\mathcal{C}U : U \subseteq X\}.$$

Since $U \lhd_{\mathcal{C}} V$ is equivalent to $\mathcal{C}U \subseteq \mathcal{C}V$, the order $\lhd_{\mathcal{C}}$ between subsets becomes the inclusion between saturated subsets. Moreover, the characterization $(**)$ allows to prove quite easily the well known result that the partially ordered collection $Sat(\mathcal{C})$ is indeed a sup-lattice.

Theorem 2.1 *For any closure operator $\mathcal{C}$ on the set X, the following hold:*

(i) *$(Sat(\mathcal{C}), \bigvee)$, with join given by $\bigvee_{i \in I} \mathcal{C}U_i \equiv \mathcal{C}(\bigcup_{i \in I} \mathcal{C}U_i)$ is a sup-lattice.*

(ii) *The closure operator $\mathcal{C}$, considered as a mapping from the sup-lattice $(\mathcal{P}(X), \bigcup)$ onto the sup-lattice $(Sat(\mathcal{C}), \bigvee)$, is a sup-lattice morphism, i.e. the equality $\mathcal{C}(\bigcup_{i \in I} U_i) = \mathcal{C}(\bigcup_{i \in I} \mathcal{C}U_i)$ holds.*

(iii) *The family $\{\mathcal{C}\{b\} : b \in S\}$ generates $Sat(\mathcal{C})$, that is $\mathcal{C}U = \bigvee_{b \in U} \mathcal{C}\{b\}$ for every $U \subseteq X$.*

Proof. (i) By its definition, $\bigvee_{i \in I} \mathcal{C}(U_i)$ satisfies the characterization of supremum in $Sat(\mathcal{C})$ given after definition 1.1. In fact, if $\mathcal{C}U_i, i \in I$, is a family in $Sat(\mathcal{C})$, then $\mathcal{C}U_i \subseteq \mathcal{C}V$ for all $i \in I$ if and only if $\bigcup_{i \in I} \mathcal{C}U_i \subseteq \mathcal{C}V$, which holds if and only if $\mathcal{C}(\bigcup_{i \in I} \mathcal{C}U_i) \subseteq \mathcal{C}V$ by $(**)$.

(ii) $\mathcal{C}(\bigcup_{i \in I} U_i)$ is an upper bound of the family $\mathcal{C}U_i$, $i \in I$, because $U_i \subseteq \bigcup_{i \in I} U_i$ implies $\mathcal{C}U_i \subseteq \mathcal{C}(\bigcup_{i \in I} U_i)$ for every $i \in I$, and hence $\mathcal{C}(\bigcup_{i \in I} \mathcal{C}U_i) \subseteq \mathcal{C}(\bigcup_{i \in I} U_i)$; the opposite inclusion is immediate.

(iii) One has $U = \bigcup_{b \in U} \{b\}$, and hence, by (ii) and the definition of join, $\mathcal{C}U = \bigvee_{b \in U} \mathcal{C}\{b\}$. $\square$

The sup-lattice $Sat(\mathcal{C})$ described by means of the closure operator $\mathcal{C}$ is isomorphic to the sup-lattice obtained as a quotient of $\mathcal{P}(X)$ over the congruence $=_{\mathcal{C}}$ corresponding to $\mathcal{C}$. This confirms the equivalence of the two approaches.

Proposition 2.2 *The sup-lattice $\mathcal{P}(X)_{/=_{\mathcal{C}}}$ is isomorphic to $Sat(\mathcal{C})$.*

Proof. We denote by $[U]$ the equivalence class of the subset U modulo $=_{\mathcal{C}}$. The map $\phi : [U] \mapsto \mathcal{C}U$ is the isomorphism. In fact the equivalence $U \lhd_{\mathcal{C}} V$ iff $\mathcal{C}U \subseteq \mathcal{C}V$ tells both that ϕ is well defined and that it preserves the order, and hence that it is injective. Obviously, ϕ is onto. Finally $\phi(\bigvee_{i \in I}[U_i]) \equiv \mathcal{C}(\bigcup_{i \in I} U_i) = \mathcal{C}(\bigcup_{i \in I} \mathcal{C}U_i) \equiv \bigvee_{i \in I} \phi[U_i]$. $\square$

We say that a sup-lattice $\mathcal{L}$ is *based* on a set X if there is a function $g : X \to L$ such that the image $g(X)$ generates $\mathcal{L}$, that is for any $a \in L$,

$$\downarrow_g a \equiv \{g(x) : g(x) \leq a \text{ and } x \in X\}$$

is a set-indexed family of elements and $a = \bigvee \downarrow_g a$. We say that $\mathcal{L}$ is *set-based* if it is based on some set (see [S87] and [A97]).

The typical example is $Sat(\mathcal{C})$, where $\mathcal{C}$ is any closure operator on a set X. In fact, consider the function $i : X \to Sat(\mathcal{C})$ defined by $i(x) \equiv \mathcal{C}\{x\}$ for any $x \in X$. Then clearly $\downarrow_i \mathcal{C}U = \{\mathcal{C}\{x\} : \mathcal{C}\{x\} \subseteq \mathcal{C}U\}$ and so $\downarrow_i \mathcal{C}U$ is the image along the function i of the subset $\mathcal{C}U$; this is enough to conclude that it is a set-indexed family (see [SV98]). Now $\mathcal{C}U = \bigvee\{\mathcal{C}\{x\} : \mathcal{C}\{x\} \subseteq \mathcal{C}U\}$ is immediate.

The next proposition says that this is essentially the only example:

Theorem 2.3 *If $\mathcal{L}$ is any sup-lattice based on a set X via the function $g : X \to L$, then the relation $\lhd_g$ defined by putting*

$$a \lhd_g U \equiv g(a) \leq \bigvee_{b \in U} g(b)$$

is an infinitary preorder and, writing $\mathcal{C}_g$ for the corresponding closure operator, $Sat(\mathcal{C}_g)$ is isomorphic to $\mathcal{L}$.

Proof. The relation $\lhd_g$ is an infinitary preorder. In fact, if $a \in U$, then it is $g(a) \leq \bigvee_{b \in U} g(b)$, and so reflexivity is satisfied. Assume that $a \lhd_g U$ and $U \lhd_g V$; then $g(a) \leq \bigvee_{b \in U} g(b)$ and $\forall b \in U(g(b) \leq \bigvee_{c \in V} g(c))$, which by definition of join is equivalent to $\bigvee_{b \in U} g(b) \leq \bigvee_{c \in V} g(c)$. So $g(a) \leq \bigvee_{c \in V} g(c)$ by transitivity of $\leq$, and hence $\lhd_g$ is transitive. Now $\tilde{g} : Sat(\mathcal{C}_g) \to \mathcal{L}$, defined putting $\tilde{g}(\mathcal{C}_g U) \equiv \bigvee_{b \in U} g(b)$, is an isomorphism. In fact, $\tilde{g}$ is well defined and one-one, since $\mathcal{C}_g U \subseteq \mathcal{C}_g V$ if and only if $\tilde{g}(\mathcal{C}_g U) \subseteq \tilde{g}(\mathcal{C}_g V)$; in fact $\mathcal{C}_g U \subseteq \mathcal{C}_g V$ if and only if $U \lhd_g V$, which is equivalent to $\forall b \in U(g(b) \leq \bigvee_{c \in V} g(c))$ by definition of $\lhd_g$, and hence also to $\bigvee_{b \in U} g(b) \leq \bigvee_{c \in V} g(c)$, which by definition of $\tilde{g}$ is $\tilde{g}(\mathcal{C}_g U) \subseteq \tilde{g}(\mathcal{C}_g V)$.

Moreover, $\tilde{g}^{-1} : \mathcal{L} \to Sat\mathcal{C}_g$ defined putting $\tilde{g}^{-1}(l) \equiv \mathcal{C}_g\{x \in X : g(x) \leq l\}$ is the inverse of $\tilde{g}$. In fact, one has $\tilde{g}(\tilde{g}^{-1}(l)) \equiv \bigvee_{b \in \{x \in X : g(x) = l\}} g(b) = l$ since $\mathcal{L}$ is based on the set X via the function g, and $\tilde{g}^{-1}(\tilde{g}(\mathcal{C}_g U)) \equiv \mathcal{C}_g\{x \in X : g(x) \leq \bigvee_{b \in U} g(b)\} = \mathcal{C}_g U$ by definition of $\lhd_g$. $\square$

When the carrier L of a sup-lattice $\mathcal{L}$ is a set, following the definition contained in the above theorem, one obtains the infinitary preordered set induced by the identical map on L, that is $\mathcal{C}_{id_L} \equiv (L, \lhd_{id_L})$, where $a \lhd_{id_L} U$ is $a \leq \bigvee_{b \in U} b$. We can consider $\mathcal{C}_{id_L}$ a sort of translation of the structure $\mathcal{L}$ into the language of the infinitary preordered sets: the elements of L are translated into the infinitary terms, the order relation into $\lhd$. Let us put $Transl(\mathcal{L}) \equiv \mathcal{C}_{id_L}$. The above theorem amounts to saying that $\mathcal{L}$ is isomorphic to $Sat(Transl(\mathcal{L}))$. Such isomorphism is given by the map $\downarrow_L$, defined by $\downarrow_L (l) \equiv \{x \in X : x \leq l\}$ (it is the map $\tilde{id}_L^{-1}$ in the above notation), whose inverse (namely $\tilde{id}_L$) maps $\mathcal{C}_{id_L} U$ into $\bigvee U$.

The carrier of $\mathcal{L}$ is always a set in an impredicative setting, where, hence, one can state the following corollary:

Corollary 2.4 *Every sup-lattice $\mathcal{L}$ is isomorphic to $Sat(Transl(\mathcal{L})$.*

In the next section we deal with the predicative case.

2.1 Sup-lattices presented by axioms

Fix a set X and an infinitary relation $R(a, U)$, defined for any $a \in X$ and $U \subseteq X$. We think of $R(a, U)$ as giving conditions on an infinitary preorder, and thus we say that an infinitary preorder satisfies R if it includes it, that is if it satisfies

$$R\text{-ax:} \qquad \text{for every } a \in X \text{ and } U \subseteq X, \quad \frac{R(a,U)}{a \lhd U}$$

The name R-ax should recall that R is thought of as giving axioms. When the least infinitary preorder satisfying R exists, we call it $\lhd_R$ and say that the sup-lattice corresponding to $(X, \lhd_R)$ is presented through the set of generators X and the conditions given by the relation R. Thus if $\mathcal{C}_R$ is the closure operator corresponding to $\lhd_R$, the sup-lattice presented by X and R is $Sat(\mathcal{C}_R)$. So note that here the condition $R(a, U)$ requires an inequality to hold, namely $\mathcal{C}_R\{a\} \subseteq \bigvee_{b \in U} \mathcal{C}_R\{b\}$, or equivalently $\mathcal{C}_R\{a\} \subseteq \mathcal{C}_R U$, rather than the equality $\mathcal{C}_R\{a\} = \mathcal{C}_R U = \bigvee_{b \in U} \mathcal{C}_R\{b\}$ as in other approaches (like [J82] and [V89]).

When $\lhd_R$ exists, it must satisfy:

1. $\lhd_R$ is an infinitary preorder;
2. $\lhd_R$ satisfies R;
3. $\lhd_R$ is the least infinitary preorder satisfying R; that is, for any infinitary preorder $\lhd$,

$$\text{if } \frac{R(a,U)}{a \lhd U} \text{ for any } a \in X \text{ and } U \subseteq X, \text{ then also } \frac{a \lhd_R U}{a \lhd U} \text{ for any } a \in X \text{ and } U \subseteq X.$$

To find a solution, that is, to construct an infinitary relation $\lhd_R$ satisfying the requirements 1.-3., it is useful to translate them in terms of the closure operator $\mathcal{C}_R$ corresponding to $\lhd_R$. It is also convenient to introduce the notation

$$RU \equiv \{a : R(a, U)\}$$

for any $U \subseteq X$. Then $\lhd$ satisfies R can be rewritten as: $RU \subseteq \mathcal{C}U$ for any U, where $\mathcal{C}$ is the closure operator corresponding to $\lhd$. And then it is also immediate to see that the following are equivalent:

a. $\lhd$ satisfies R, that is $\lhd$ is closed under R-ax, that is $RU \subseteq \mathcal{C}U$ for any U

b. $\lhd$ is closed under R-trax: $\dfrac{R(a,V) \quad V \lhd U}{a \lhd U}$, that is $\dfrac{V \subseteq \mathcal{C}U}{RV \subseteq \mathcal{C}U}$ for any U, V

(the name R-trax comes from "transitivity on axioms", see [CSSV]). This suggests the following definition (which is the natural generalization of the notion of C-*ideal* of [J82], cf. also definition 5.2.1 of [AV93] and the last section here):

Definition 2.5 *For any set X and infinitary relation R on X, a subset $Z \subseteq X$ is called R-saturated if*

$$\frac{R(a,U) \quad U \subseteq Z}{a \in Z}, \quad \textit{that is} \quad \frac{U \subseteq Z}{RU \subseteq Z}$$

holds for every $a \in X$ and $U \subseteq X$.

The notion of R-saturated subset allows to rewrite easily conditions 1.–3. into a simple equivalent formulation in terms of the closure operator $\mathcal{C}_R$:

1'. $\mathcal{C}_R$ is a closure operator;

2'. for every $U \subseteq X$, $\mathcal{C}_R U$ is R-saturated;

3'. for any closure operator $\mathcal{C}$,

if $\mathcal{C}U$ is R-saturated for any $U \subseteq X$, then also $\mathcal{C}_R U \subseteq \mathcal{C}U$ for any $U \subseteq X$.

These conditions can be further simplified. In fact, suppose that the least R-saturated subset containing U exists, and is denoted by $\mathcal{C}_R U$. That is, assume that

1". $U \subseteq \mathcal{C}_R U$;

2". $\dfrac{V \subseteq \mathcal{C}_R U}{RV \subseteq \mathcal{C}_R U}$;

3". if $U \subseteq Z$ and $V \subseteq Z \Rightarrow RV \subseteq Z$, then $\mathcal{C}_R U \subseteq Z$.

hold for any $U \subseteq X$. Then $\mathcal{C}_R$ satisfies 1.'-3'. In fact, $\mathcal{C}_R$ is a closure operator because it satisfies $(**)$, that is $U \subseteq \mathcal{C}_R W$ iff $\mathcal{C}_R U \subseteq \mathcal{C}_R W$. In fact, one direction holds by 3". applied to $Z \equiv \mathcal{C}_R W$: since $U \subseteq \mathcal{C}_R W$ by assumption and $V \subseteq \mathcal{C}_R W \Rightarrow RV \subseteq \mathcal{C}_R W$ by 2"., then by 3". also $\mathcal{C}_R U \subseteq \mathcal{C}_R W$. The other direction of $(**)$ holds by 1". Moreover, 2'. and 3'. follow immediately from 2". and 3". respectively. Now the point is that, by the minimality property 3'., if a solution of 1'.-3'. exists, it is unique. So it is enough to find $\mathcal{C}_R U$ which satisfies 1".-3".[5]

Since the intersection of R-saturated subsets is clearly R-saturated, the common solution is to define $\mathcal{C}_R U$ simply as the intersection of all R-saturated subsets containing U:

$$\mathcal{C}_R U \equiv \bigcap \{Z : U \subseteq Z \textit{ and } Z \text{ is } R\textit{-saturated}\}$$

So, accepting the definition of $\mathcal{C}_R$, any set of generators and any infinitary relation R on X present a sup-lattice, which is $Sat(\mathcal{C}_R)$.

[5] One can also formally prove that 1'.-3'. imply 1".-3".; we leave the details, except for the remark that a constructive proof of 3". from 3'. is possible because for any choice of an R-saturated subset $Z \subseteq X$, $\mathcal{C}^Z U \equiv \{a \in X : U \subseteq Z \to a \in Z\}$ is a closure operator with $\mathcal{C}^Z Z = Z$ and $\mathcal{C}^Z U$ R-saturated for any U.

The trouble with the above definition of $\mathcal{C}_R$ is that it is not justified in type theory since it involves quantifications over subsets in an essential way; more specifically, the quantification over Z corresponding to intersection is not bounded, in the sense that it is not indexed by a set, and moreover it is nested with the quantification on subsets needed to express R-saturation, which is also unbounded.

The solution is to require R itself to be given more explicitly, that is through families of subsets indexed by sets. Following [CSSV], we say that an infinitary relation R on X *has an axiomset* if there exist a family of sets $I(a)$ *set* $(a \in X)$ and a family of subsets $C(a,i) \subseteq X$ $(a \in X, i \in I(a))$ such that, for any $a \in X$ and any $U \subseteq X$,

$$R(a,U) \text{ if and only if } (\exists i \in I(a))(C(a,i) \subseteq U).$$

It is immediate to check that, when R has an axiomset I, C, then Z is R-saturated if and only if for any $a \in X$, $(\exists i \in I(a))(C(a,i) \subseteq Z) \rightarrow a \in Z$. So $\mathcal{C}_R$, or equivalently $\triangleleft_R$, is defined inductively by the rules:

reflexivity: $\dfrac{a \in U}{a \triangleleft_R U}$

infinity: $\dfrac{i \in I(a) \qquad C(a,i) \triangleleft_R U}{a \triangleleft_R U}$

This is an inductive definition of a kind which is acceptable in type theory (see [Dy94]). This means that proofs by induction on the generation of $\triangleleft_R$ are justified:

if $U \subseteq Z$ and $(\forall i \in I(a))(C(a,i) \subseteq Z \rightarrow a \in Z)$
then $a \triangleleft_R U$ implies $a \in Z$

Note that this is exactly a rewriting of 3"., when R has an axiomset I, C. It is easy to prove by induction (see [CSSV]) that the relation $\triangleleft_R$ satisfies transitivity, and hence that it is an infinitary preorder. As a conclusion, the construction of $Sat(\mathcal{C}_R)$ is possible within type theory whenever R has an axiomset.

We have devised exposition in such a way that from now on we don't need any explicit reference to inductive definitions. What we will need is that $\triangleleft_R$ exists, however it is conceived and defined. So from now on whenever we mention $\triangleleft_R$ we mean that it is the least infinitary preorder satisfying R, and that it exists. It is understood that if a predicative treatment is wished, one must understand also that the relation R is given through an axiomset.

In the following theorem we extend proposition 1.3 to sets equipped with a relation, and, moreover, we extend to sup-lattices a result proved for frames in [J82], p. 58, proposition 2.11. Given a pair (X,R), and a sup-lattice $\mathcal{L}$, we say that a map $f : X \rightarrow L$ *preserves* R if and only if $R(a,U)$ implies $f(a) \leq \bigvee_{b \in U} f(b)$. We need the following lemma:

Lemma 2.6 *Let (X,R) be a set equipped with an infinitary relation R and $\mathcal{L}$ any sup-lattice. Then a map $f : X \rightarrow L$ preserves R if and only if it preserves the preorder $\triangleleft_R$.*

Proof. The "if" direction is obvious, since $\triangleleft_R$ contains R. Let $f : X \rightarrow L$ be a map which preserves R. The infinitary preorder $a \triangleleft_f U \equiv f(a) \leq \bigvee fU$, defined as in theorem 2.3, is exactly the maximum relation which is preserved by f. Then $\triangleleft_f$ is a preorder which includes R by hypothesis, and hence it includes also $\triangleleft_R$, which is the minimum infinitary preorder including R. □

It is now quite easy to prove that $Sat(\mathcal{C}_R)$ is the sup-lattice freely generated by (X,R):

Theorem 2.7 *For any pair (X,R), where X is a set and R an infinitary relation on X, the map $i : X \rightarrow Sat(\mathcal{C}_R)$ defined by $x \mapsto \mathcal{C}_R\{x\}$ is universal among maps $g : X \rightarrow \mathcal{L}$, where $\mathcal{L}$ is any sup-lattice and g is any map preserving R. That is, for any such g there is a unique morphism $\tilde{g} : Sat(\mathcal{C}_R) \rightarrow \mathcal{L}$ such that $g = \tilde{g} \circ i$.*

Proof. Notice that the canonical embedding $i : X \rightarrow Sat(\mathcal{C}_R)$ preserves R; in fact, $R(a,U)$ implies $a \triangleleft_R U$, that is $\mathcal{C}_R\{a\} \subseteq \mathcal{C}_R U$, and $\mathcal{C}_R U = \bigvee_{b \in U} \mathcal{C}_R\{b\}$, by theorem 2.1. Since the diagram must be commutative, it must be $\tilde{g}(\mathcal{C}_R\{a\}) = g(a)$ for any $a \in X$. This defines $\tilde{g}$ on the image of X under $\mathcal{C}_R$. Such image generates the whole $Sat(\mathcal{C}_R)$, that is $\mathcal{C}_R U = \bigvee_{b \in U} \mathcal{C}_R\{b\}$ for any U, by theorem 2.1; so we put $\tilde{g}(\mathcal{C}_R U) \equiv \bigvee_{b \in U} g(b)$, and $\tilde{g}$ is the only possible way to define a morphism making the diagram commute. So it only remains to check that $\tilde{g}$ is indeed a morphism. To see that $\tilde{g}$ is well defined, that is that $\mathcal{C}_R U = \mathcal{C}_R V$ implies $\tilde{g}\mathcal{C}_R U = \tilde{g}\mathcal{C}_R V$, it is enough to see that $\tilde{g}$ preserves inequalities: if $\mathcal{C}_R U \subseteq \mathcal{C}_R V$ then $\forall a \in U(a \lhd_R V)$ and hence, since g preserves R and so also $\lhd_R$ by the lemma, it is $g(a) \leq \bigvee_{b\in V} g(b)$ for every $a \in U$, that is $\bigvee_{a\in U} g(a) \leq \bigvee_{b\in V} g(b)$. We can finally see that $\tilde{g}$ preserves joins. Since it is $\bigvee_{i\in I} \mathcal{C}_R U_i = \mathcal{C}_R(\bigcup_{i\in I} U_i)$ by 2.1, one has the equalities: $\tilde{g}(\bigvee_{i\in I} \mathcal{C}_R U_i) = \tilde{g}(\mathcal{C}_R(\bigcup_{i\in I} U_i)) \equiv \bigvee_{b\in\bigcup U_i} g(b) = \bigvee_{i\in I}(\bigvee_{b\in U_i} g(b)) = \bigvee_{i\in I} \tilde{g}(C_R U_i)$. □

Note that proposition 1.3 can be obtained from theorem 2.7. In fact, any $g : X \to L$ such that $g(X)$ generates L trivially preserves the empty set of axioms (that is $R = \emptyset$) and hence it extends in a unique way to $Sat(\mathcal{C}_\emptyset)$. Now $Sat(\mathcal{C}_\emptyset)$ is just $\mathcal{P}(X)$ since the infinitary preorder generated by $\emptyset$ is membership.

In theorem 2.7, $\tilde{g}$ is onto, in the strong sense that it has a right inverse defined by putting $\tilde{g}^{-1}(a) \equiv \mathcal{C}_R\{x \in X : g(x) \leq a\}$, if and only if $g(X)$, the image of X along g, generates $\mathcal{L}$. Similarly, $\tilde{g}$ is one-one if and only if $\lhd_R$ coincides with the infinitary preorder $\lhd_g$ defined in theorem 2.3. In fact, by definition $\tilde{g}$ is one-one iff $\tilde{g}\mathcal{C}_R U = \tilde{g}\mathcal{C}_R V$ implies $\mathcal{C}_R U = \mathcal{C}_R V$. This amounts to saying that $g(a) \leq \bigvee_{c\in V} g(c)$ implies $a \lhd_R V$ for every $a \in U$, that is $a \lhd_g U$ implies $a \lhd_R U$. The converse implication holds because g preserves R, so $\tilde{g}$ is one-one when $\lhd_R = \lhd_g$. So corollary 2.3 could be obtained as a consequence of theorem 2.7.

Let us say that a sup-lattice $\mathcal{L}$ is predicatively presentable if there is a set X and an infinitary relation R with an axiomset such that $\mathcal{L}$ is isomorphic to $Sat(\mathcal{C}_R)$. A natural question now is: which sup-lattices are predicatively presentable? If $h : Sat(\mathcal{C}_R) \to \mathcal{L}$ is the isomorphism, then clearly $\mathcal{L}$ is based on X via the function $g = h \circ i : x \mapsto h(\mathcal{C}_R\{x\})$; in fact, this is the meaning of thm. 2.1.(iii). So we certainly must restrict to set-based sup-lattices. Then we can define $\lhd_g$ as in theorem 2.3, and obtain that $\mathcal{L}$ is isomorphic to $Sat(\mathcal{C}_g)$. The proof of such theorem is all right, but it is relative to the knowledge of the ordering $\leq$ of $\mathcal{L}$. In other words, the difficulty for a predicativist is only that the definition of $\lhd_g$ relies on the order of $\mathcal{L}$, which in general is not given predicatively. This means that we must add a condition which is satisfied by $Sat(\mathcal{C}_R)$ only when R has an axiomset. By a result of P. Aczel (see theorem 3.2 of [CSSV]), when R has an axiomset, also $\lhd_R$ has an axiomset. Recalling that $\lhd_R$ is just the ordering of $Sat(\mathcal{C}_R)$, we say that a sup-lattice $\mathcal{L}$ which is set-based on X via the function $g : X \to L$ is also *set-presented* (see [A97]) if $\lhd_g$ has an axiomset. Then every predicatively presented sup-lattice is clearly set-based and set-presented. The converse also holds, since $\lhd_g$ coincides with the infinitary preorder it generates, and so $Sat(\mathcal{C}_g)$ is predicatively presented. We thus have:

Theorem 2.8 *A sup-lattice $\mathcal{L}$ can be presented predicatively if and only if it is set-based and set-presented.*

Impredicatively, theorem 2.7 leads to the equivalence between the category **IP** and the category of sup-lattices, here denoted by **SL**.

Proposition 2.9 *The categories* **IP** *and* **SL** *are equivalent.*

Proof. By theorem 2.1 we have a map $Sat : Ob(\mathbf{IP}) \to Ob(\mathbf{SL})$ which maps any $\mathcal{C}$ into $Sat(\mathcal{C})$; by theorem 2.3 and corollary 2.4 Sat has a right inverse $Transl : Ob(\mathbf{SL}) \to Ob(\mathbf{IP})$, where $Transl(\mathcal{L}) = (L, \lhd_{id_L})$ is the infinitary preorder defined on the carrier L of the sup-lattice $\mathcal{L}$ putting $a \lhd_{id_L} U \equiv a \leq \bigvee_{b\in U} b$ (cf. theorem 2.3). We remind also that the isomorphism $\mathcal{L} \to Sat(Transl(\mathcal{L}))$ of corollary 2.4 is obtained by mapping $l \in L$ into $\downarrow_L (l) \equiv \{x \in L : x \leq l\}$.

Now one can define Sat on morphisms as follows: for any $f : \mathcal{C} \to \mathcal{C}'$ one can first define a map $g : X \to Sat(\mathcal{C}')$ preserving $\lhd_\mathcal{C}$, putting $g(a) \equiv \mathcal{C}'(f(a))$ for every $a \in X$. Then, by 2.7, one can extend it to a sup-lattice morphism $\tilde{g} : Sat(\mathcal{C}) \to Sat(\mathcal{C}')$. So, put $Sat(f) \equiv \tilde{g}$. By definition of $\tilde{g}$ one has $Sat(f)(\mathcal{C}U) = \bigvee_{b\in U} g(b)$, that is $\mathcal{C}'(\bigcup_{b\in U} \mathcal{C}'(g(b)))$ in $Sat(\mathcal{C}')$. Easy calculations show that this last is equal to $\mathcal{C}'(fU)$, so one has $Sat(f)(\mathcal{C}U) = \mathcal{C}'(fU)$.

To define $Transl$ on morphisms, put simply $Transl(m) = m$, for any two objects $\mathcal{L}$, $\mathcal{L}'$ and any morphism $m : \mathcal{L} \to \mathcal{L}'$[6]; $Transl(m)$ is then extended to subsets of L as usual. We see that $Sat(Transl(m)) = m$: in fact, it is $Sat(Transl(m))(\downarrow_L (l)) \equiv \bigvee(\downarrow_{L'} m(l)) = m(l)$ for every $l \in L$. Conversely, let us consider $\mathcal{C} = Transl(\mathcal{L})$, $\mathcal{C}' = Transl(\mathcal{L}')$ and $f : \mathcal{C} \to \mathcal{C}'$ any morphism. Then

[6] to be pedantic, $Transl(m(l))$ is the singleton $\{m(l)\}$

$Transl(Sat(f))(\mathcal{C}U) = \mathcal{C}'(f(U)) =_{\mathcal{C}'} f(U) =_{\mathcal{C}'} f(\mathcal{C}U)$, for every $U \subseteq L$; so $Transl(Sat(f)) = f$ as arrows of **IP**. □

3 Pretopologies and the presentation of quantales

3.1 Precovers and stable closure operators

We now extend our method to richer algebraic infinitary structures, namely quantales (cf. [M86], [R90]). The idea is to present quantales as "sup-lattices on monoids" (cf. [JT84], p. 7, see also [AV93]). In this way we can extend the results proved in the previous section, and then apply such extension to frames, as we shall see in the next section. So we reach a good modularity in the treatment of infinitary structures.

Even if most of our results on quantales could easily generalize to the non-commutative case, we will deal with commutative quantales, as ancestors of frames, that are commutative. We remind here the basic definitions.

Definition 3.1 *A (commutative, unital) quantale is a structure $\mathcal{Q} = (Q, \cdot, 1, \bigvee)$ such that:*
i) $(Q, \bigvee)$ is a sup-lattice,
ii) $(Q, \cdot, 1)$ is a commutative monoid,
iii) Infinite distributivity of $\cdot$ with respect to $\bigvee$ holds, that is $p \cdot \bigvee_{i \in I} q_i = \bigvee_{i \in I}(p \cdot q_i)$, for every $p \in Q$ and $q_i \in Q\, (i \in I)$.

Given two quantales $\mathcal{Q}$ and $\mathcal{Q}'$, a map $f : Q \to Q'$ is a quantale morphism if it is a sup-lattice morphism and a monoid morphism, i.e. $f(\bigvee_{i \in I} q_i) = \bigvee_{i \in I} f(q_i)$ for every family $q_i \in Q\, (i \in I)$, $f(p \cdot q) = f(p) \cdot f(q)$ for every $p, q \in Q$ and $f(1) = 1$.

The following normal form lemma is the key which allows to extend notions and results concerning sup-lattices to quantales. To simplify exposition, we say that X is a subset of Q even if actually it is $X = g(X')$ for some set X' and some map $g : X' \to Q$.

Lemma 3.2 *For every quantale $\mathcal{Q}$ and every subset X of its carrier Q, any element q of the sub-quantale $\mathcal{Q}(X)$ generated by X can be expressed by $q = \bigvee U$ for some subset $U \subseteq S$, where S is the monoid generated by X in Q.*

Proof. By induction on the construction of q. If $q \in X$, the claim is trivial since $q = \bigvee\{q\}$. If $q = 1$, then $1 = \bigvee\{1\}$ and $\{1\} \subseteq S$. If $q = p \cdot r$, by the inductive hypothesis $p = \bigvee U$ and $r = \bigvee V$ hold for some $U, V \subseteq S$ and then $q = (\bigvee U) \cdot \bigvee(V) = \bigvee\{u \cdot v : u \in U, v \in V\}$ by distributivity. If $q = \bigvee_{i \in I} p_i$ for some $p_i \in S\, (i \in I)$, by the inductive hypothesis for every $i \in I$ there exists some $U_i \subseteq S$ such that $p_i = \bigvee U_i$ holds, hence $q = \bigvee_{i \in I}(\bigvee U_i) = \bigvee(\bigcup_{i \in I} U_i)$. □

From now on, let S stay for a monoid $(S, \cdot, 1)$ where S is a set. By the above lemma, if two subsets U and V denote two elements q and q' of a quantale $\mathcal{Q}(X)$, then the subset $U \cdot V \equiv \{a \cdot b : a \in U, b \in V\}$ denotes the product $q \cdot q'$. Hence the subsets of a monoid can be seen as the infinitary terms for a quantale.

The power $\mathcal{P}(S)$ is itself a quantale, with the above operation $U \cdot V$ and with unit $\{1\}$; in fact, $\mathcal{P}(S)$ is a sup-lattice and distributivity holds, because by definition $U \cdot \cup_{i \in I} V_i = \cup_{i \in I}(U \cdot V_i)$. We always write $U \cdot b$ for $U \cdot \{b\}$. In particular, $U \cdot 1 = U$ is obvious.

By proposition 1.3, we obtain that $\mathcal{P}(S)$ is actually the free quantale:

Proposition 3.3 *For any monoid S, the power $\mathcal{P}(S)$ is the quantale freely generated by S. Hence for any set X, $\mathcal{P}(CMon(X))$ is the quantale freely generated by X, if $CMon(X)$ is the commutative monoid freely generated by X.*

Proof. For any $g : S \to \mathcal{Q}$ preserving the monoid operation of S, the sup-lattice morphism $\tilde{g}$, defined in 1.3 by $\tilde{g}(U) \equiv \bigvee_{b \in U} g(b)$, preserves the pointwise defined monoid operation of $\mathcal{P}(S)$. In fact it is $\tilde{g}(U \cdot V) \equiv \bigvee_{x \in U \cdot V} g(x) = \bigvee_{b \cdot c \in U \cdot V} g(b \cdot c) = \bigvee_{b \cdot c \in U \cdot V} g(b) \cdot g(c) = \bigvee_{b \in U} g(b) \cdot \bigvee_{c \in V} g(c) \equiv \tilde{g}U \cdot \tilde{g}V$. Note that distributivity is necessary in the proof. As for the last statement, any $f : X \to Q$ extends in a unique way to a monoid morphism $\dot{f} : CMon(X) \to Q$ and then in a unique way to $\mathcal{P}(CMon(X))$. □

So we have seen that the equation "quantales = sup-lattices on monoids" is true for the quantales of terms, in which distributivity holds by definition of product and join. But, in general, generating a sup-lattice from a given monoid under some conditions R does not produce a quantale. So, in order to describe quantales by means of generators and relations, the elements of S are enough as generators, but we need further conditions on infinitary preorders (or closure operators, congruence relations, etc.) on S, to capture the characterizing property of quantales, that is distributivity. We first need a technical lemma:

Lemma 3.4 *Let $(Q, \cdot, 1, \bigvee)$ be a structure with $(Q, \cdot, 1)$ a commutative monoid and $(Q, \bigvee)$ a sup-lattice. Then the distributivity property $(\bigvee_{i\in I} c_i) \cdot b = \bigvee_{i\in I}(c_i \cdot b)$ holds (so that Q is a quantale) if and only if the rule:*

$$\frac{a \leq \bigvee_{i\in I} c_i}{a \cdot b \leq \bigvee_{i\in I}(c_i \cdot b)}$$

is valid in Q.

Proof. If distributivity holds, then from $a \leq \bigvee_{i\in I} c_i$, i.e. $a \vee \bigvee_{i\in I} c_i = \bigvee_{i\in I} c_i$, it follows that $a \cdot b \vee \bigvee_{i\in I}(c_i \cdot b) = \bigvee_{i\in I}(c_i \cdot b)$, that is $a \cdot b \leq \bigvee_{i\in I}(c_i \cdot b)$ as wished.

Conversely, if the above rule is valid, from $\bigvee_{i\in I} c_i \leq \bigvee_{i\in I} c_i$ it follows that $(\bigvee_{i\in I} c_i) \cdot b \leq \bigvee_{i\in I}(c_i \cdot b)$. To prove the other inequality, first note that, since $c = \bigvee\{c\}$, the rule

$$\frac{a \leq c}{a \cdot b \leq c \cdot b}$$

is obtained as a particular case of the rule assumed. So from $c_i \leq \bigvee_{i\in I} c_i$ it follows $c_i \cdot b \leq (\bigvee_{i\in I} c_i) \cdot b$ for all $i \in I$, hence also $\bigvee_{i\in I}(c_i \cdot b) \leq (\bigvee_{i\in I} c_i) \cdot b$. □

By the above lemma, to extend the presentation of sup-lattices to the case of quantales, closure under the rule of localization:

$$(L) \qquad \frac{a \lhd U}{a \cdot b \lhd U \cdot b}$$

must be required, in addition to the rules of infinitary preorder. So the basic notion to study quantales via infinitary terms and relations will be the following:

Definition 3.5 *A precover on a monoid $(S, \cdot, 1)$, is an infinitary preorder satisfying localization, that is a relation $\lhd$ satisfying:*

$$(SR) \quad \frac{a \in U}{a \lhd U} \qquad (T) \quad \frac{a \lhd U \quad U \lhd V}{a \lhd V} \qquad (L) \quad \frac{a \lhd U}{a \cdot b \lhd U \cdot b}$$

A pretopology is a quadruple $\mathcal{F} = (S, \cdot, 1, \lhd_{\mathcal{F}})$, where $(S, \cdot, 1)$ is a monoid, called the base of $\mathcal{F}$, and $\lhd_{\mathcal{F}}$ is a precover on S.

An alternative definition of precovers (as in [S89]) requires closure under the apparently stronger rule of stability

$$(St) \qquad \frac{a \lhd U \quad b \lhd V}{a \cdot b \lhd U \cdot V}$$

but actually an infinitary preorder $\lhd$ is closed under localization (L) iff it is closed under stability (St). In fact, assume $a \lhd U$ and $b \lhd V$. Then by (L) $a \cdot b \lhd U \cdot b$ and similarly $u \cdot b \lhd u \cdot V$ for any $u \in U$; since $u \cdot V \lhd U \cdot V$, by transitivity it follows $u \cdot b \lhd U \cdot V$ for any $u \in U$, that is $U \cdot b \lhd U \cdot V$. So $a \cdot b \lhd U \cdot V$ by transitivity. Conversely, assuming (St) closure under (L) is obtained as a special case, when a premiss is $b \lhd b$. Sometimes the versions with subsets on the left, that is

$$(L_G) \qquad \frac{U \lhd V}{U \cdot Z \lhd V \cdot Z}$$

and

$$(St_G) \qquad \frac{Z \lhd U \quad W \lhd V}{Z \cdot W \lhd U \cdot V}$$

are more convenient. It is easy to see that (L_G) is equivalent to (L), and (St_G) is equivalent to (St), so that also the equivalence of (L_G) with (St_G) follows.

The bijection between infinitary preorders and closure operators can be specialized to precovers once we obtain the condition on closure operators corresponding to stability. Simply by rewriting stability of $\lhd_{\mathcal{F}}$ in terms of the corresponding closure operator $\mathcal{F}$, one obtains

$$\frac{a \in \mathcal{F}U \quad b \in \mathcal{F}V}{a \cdot b \in \mathcal{F}(U \cdot V)}$$

that is

$$\mathcal{F}U \cdot \mathcal{F}V \subseteq \mathcal{F}(U \cdot V)$$

for any $U, V \subseteq S$; we say that a closure operator $\mathcal{F}$ on a monoid S is stable if it satisfies such condition. Note that stability is also equivalent to $\mathcal{F}U \cdot b \subseteq \mathcal{F}(U \cdot b)$, which is just a rewriting of localization. The restriction of the bijection of proposition 1.13 immediately gives:

Proposition 3.6 *There is a bijection between precovers and stable closure operators.*

In the sequel, we often need an equivalent formulation of stability of $\mathcal{F}$ in terms of equality, namely:

$$\mathcal{F}(U \cdot V) = \mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V).$$

To see the equivalence, first note that $\mathcal{F}(U \cdot V) \subseteq \mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V)$ holds for every closure operator; in fact, $U \cdot V \subseteq \mathcal{F}U \cdot \mathcal{F}V$ by reflexivity (and stability of membership, to be pedantic) and then $\mathcal{F}(U \cdot V) \subseteq \mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V)$ by monotonicity. So the equality $\mathcal{F}(U \cdot V) = \mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V)$ is equivalent to $\mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V) \subseteq \mathcal{F}(U \cdot V)$, that is stability of $\mathcal{F}$, because $\mathcal{F}$ is a closure operator.

Congruences on quantales are obtained by adding a condition on congruences on sup-lattices:

Definition 3.7 *For any monoid S, a relation θ is called a quantale congruence on S if it is a congruence on the quantale $(\mathcal{P}(S), \cdot, \{1\}, \bigcup)$ that is if θ is a sup-lattice congruence on S (as in definition 1.5) which moreover respects the monoid operation, i.e. is closed under the rule*

$$(v) \qquad \frac{U \,\theta\, V}{U \cdot Z \,\theta\, V \cdot Z}.$$

Recalling the bijection which associates a congruence $=_{\mathcal{F}}$ with an infinitary preorder $\lhd_{\mathcal{F}}$ (see proposition 1.7), it is now possible to see that quantale congruences are exactly sup-lattice congruences which are induced by a precover:

Proposition 3.8 *For any monoid S, the bijection between infinitary preorders on S and sup-lattice congruences on S restricts to a bijection between precovers and quantale congruences on S.*

Proof. If $=_{\mathcal{F}}$ satisfies (v), the induced preorder $\lhd_{\mathcal{F}}$ satisfies localization: if $U \lhd_{\mathcal{F}} V$, that is $U \cup V =_{\mathcal{F}} V$, then $(U \cup V) \cdot Z = U \cdot Z \cup V \cdot Z =_{\mathcal{F}} V \cdot Z$ which means $U \cdot Z \lhd_{\mathcal{F}} V \cdot Z$.

Conversely, if $\lhd_{\mathcal{F}}$ satisfies localization, then from $U =_{\mathcal{F}} V$, i.e. $U \lhd_{\mathcal{F}} V$ and $V \lhd_{\mathcal{F}} U$, by (L) it follows $U \cdot Z \lhd_{\mathcal{F}} V \cdot Z$ and $V \cdot Z \lhd_{\mathcal{F}} U \cdot Z$, i.e. $U \cdot Z =_{\mathcal{F}} V \cdot Z$. □

3.2 Presentation of quantales

We apply the results already obtained for sup-lattices to quantales, and see that the characterizing properties of quantales are satisfied. The analogue of theorem 2.1 in the case of quantales is:

Theorem 3.9 *For any pretopology $\mathcal{F} = (S, \cdot, 1, \lhd_{\mathcal{F}})$, the structure $(Sat(\mathcal{F}), \cdot_{\mathcal{F}}, \mathcal{F}(1), \bigvee)$, where $\cdot_{\mathcal{F}}$ is defined by:*

$$\mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V \equiv \mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V),$$

is a quantale.

Proof. After theorem 2.1, it is enough to see that $\cdot_{\mathcal{F}}$ is a monoid operation and that it satisfies distributivity with respect to joins. Since by stability $\mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V = \mathcal{F}(U \cdot V)$, the operation $\cdot_{\mathcal{F}}$ is obviously commutative, and $\mathcal{F}1$ is its unit because $\mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}1 = \mathcal{F}(U \cdot 1) = \mathcal{F}U$; moreover for any $U, V, W \subseteq S$ it is $\mathcal{F}U \cdot_{\mathcal{F}} (\mathcal{F}V \cdot_{\mathcal{F}} \mathcal{F}W) = \mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}(V \cdot W) = \mathcal{F}(U \cdot (V \cdot W))$ and similarly $(\mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V) \cdot_{\mathcal{F}} W = \mathcal{F}((U \cdot V) \cdot W)$, so that associativity of $\cdot_{\mathcal{F}}$ follows by stability from associativity of $\cdot$ in S. Finally, since $U \cdot \bigcup_{i \in I} V_i = \bigcup_{i \in I}(U \cdot V_i)$ for any U and V_i, distributivity follows by repeated use of stability: $\mathcal{F}U \cdot_{\mathcal{F}} (\bigvee_{i \in I} \mathcal{F}V_i) \equiv \mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}(\bigcup_{i \in I} \mathcal{F}V_i) = \mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}(\bigcup_{i \in I} V_i) = \mathcal{F}(U \cdot \bigcup_{i \in I} V_i) = \mathcal{F}(\bigcup_{i \in I} U \cdot V_i) = \mathcal{F}(\bigcup_{i \in I} \mathcal{F}(U \cdot V_i)) = \bigvee_{i \in I} \mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V_i$. □

As one can expect, proposition 2.2 becomes:

Proposition 3.10 *Let $=_{\mathcal{F}}$ be a quantale congruence on S. Then the quotient quantale $\mathcal{P}(S)/=_{\mathcal{F}}$, where $[U] \cdot [V] \equiv [U \cdot V], 1 \equiv [1]$, is isomorphic to $Sat(\mathcal{F})$.*

Proof The isomorphism of proposition 2.2 is, in this case, a quantale isomorphism since $\phi[U] \cdot_{\mathcal{F}} \phi[V] \equiv \mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V \equiv \mathcal{F}(U \cdot V) \equiv \phi[U \cdot V]$ and $\phi[1] \equiv \mathcal{F}1$. □

We also have the analogue of theorem 2.3, that is a presentation of quantales by means of pretopologies:

Theorem 3.11 *For any monoid S, any quantale $\mathcal{Q}$ and any monoid morphism $g : S \to Q$ such that the monoid $g(S)$ generates Q, there is a pretopology $(S, \triangleleft_{\mathcal{F}})$ such that $Sat(\mathcal{F})$ is isomorphic to $\mathcal{Q}$.*

Proof. By theorem 2.3, $\triangleleft_g$ is an infinitary preorder and $\tilde{g} : Sat(\mathcal{F}) \to Q$ gives a sup-lattice isomorphism. Actually, $\triangleleft_g$ is a precover, since, if $a \triangleleft_g U$, that is $g(a) \leq \bigvee_{x \in U} g(x)$ then for any b it is $g(a \cdot b) = g(a) \cdot g(b) \leq \bigvee_{x \in U} g(x) \cdot g(b)$ and hence by distributivity $a \cdot b \triangleleft_g U \cdot b$. Moreover, $\tilde{g}$ is a quantale isomorphism; in fact, $\tilde{g}(\mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V) = \tilde{g}\mathcal{F}(U \cdot V) = \bigvee_{z \in U \cdot V} g(z) = \bigvee_{x \in U} g(x) \cdot \bigvee_{y \in V} g(y) = \tilde{g}\mathcal{F}U \cdot \tilde{g}\mathcal{F}V$. □

The functor $Transl$, already defined for sup-lattices in the impredicative case, can be defined for quantales as well, leading to the translation of any quantale into a pretopology. It follows that every quantale $\mathcal{Q}$ is impredicatively isomorphic to $Sat(\mathcal{F})$, where $\mathcal{F} = Transl(\mathcal{Q})$.

Corollary 3.12 *Every quantale $\mathcal{Q}$ is isomorphic to $Sat(Transl(\mathcal{Q}))$.*

3.3 Quantales presented by axioms

It is now a relatively simple task to extend the presentation of sup-lattices by axioms to obtain analogous results for quantales. In fact, we will see that the precover generated by an infinitary relation R is the same as the infinitary preorder generated by the closure of R under localization. Thus, in a certain sense, the equation "quantales = sup-lattices on monoids" is satisfied not only by the generators, but also by the relations.

Let R be any infinitary relation on a monoid S. If it exists, the least precover satisfying R can be characterized by saying that the corresponding operator $\mathcal{F}_R$ satisfies:

1. $\mathcal{F}_R$ is a stable closure operator;
2. $\mathcal{F}_R$ satisfies R;
3. $\mathcal{F}_R$ is the least stable closure operator satisfying R.

We now wish to find a solution to such requirements by reducing to the case of closure operators (or sup-lattices) satisfying a relation, treated in section 2.1. The new task is to obtain that $\mathcal{F}_R$ satisfies localization, that is

$$\mathcal{F}_R U \cdot b \subseteq \mathcal{F}_R(U \cdot b).$$

One idea is to force localization on the relation, that is construct R^{loc} as the least extension of R which satisfies

$$R^{loc} U \cdot b \subseteq R^{loc}(U \cdot b),$$

then generate $\mathcal{C}_{R^{loc}}$ as known from section 2.1, and finally prove that actually $\mathcal{C}_{R^{loc}} = \mathcal{F}_R$. We now prove that it is indeed so.

We first make sure that it is possible to construct R^{loc} as required. Allowing a quantification on susbets, one defines R^{loc} by

$$R^{loc}(c, V) \equiv (\exists a, b \in S)(\exists U \subseteq S)(\ c = a \cdot b \ \&\ V = U \cdot b \ \&\ R(a, U))$$

By a little logic, it is easy to see that R^{loc} satisfies localization, and obviously it is the least such. The same idea is expressed in type theory by saying that R^{loc} is defined by the introduction rule:

$$\frac{R(a, U) \qquad b \in S}{R^{loc}(a \cdot b, U \cdot b)}$$

In both cases, it is clear that $RU \subseteq R^{loc}U$ for any U, because S contains 1.

So the next step is to construct $\mathcal{C}_{R^{loc}}$, that is the least closure operator satisfying R^{loc}. Knowing that the relation is of the form R^{loc}, we can improve a bit on the characterization given in section 2.1. First note that, by minimality of R^{loc}, any stable closure operator $\mathcal{F}$ satisfying R must also satisfy R^{loc}; so $\mathcal{F}_R U \supseteq \mathcal{C}_{R^{loc}}U$ for any U. Now it is not difficult to show that for any closure operator $\mathcal{C}$, the conditions

a. $R^{loc}U \subseteq \mathcal{C}U$ for any U, that is $\mathcal{C}U$ is R^{loc}-saturated for any U

b. $\dfrac{V \cdot b \subseteq \mathcal{C}U}{RV \cdot b \subseteq \mathcal{C}U}$ for any U

are equivalent. In fact, assume a. and suppose $V \cdot b \subseteq \mathcal{C}U$; then $RV \subseteq R^{loc}V$ gives $RV \cdot b \subseteq R^{loc}V \cdot b \subseteq R^{loc}(V \cdot b) \subseteq \mathcal{C}(V \cdot b) \subseteq \mathcal{C}U$. Conversely, suppose $c \in R^{loc}V$; then there exists b such that $V = U \cdot b$ and $c \in RU \cdot b$, so that $U \cdot b = V \subseteq \mathcal{C}V$ by b. gives $RU \cdot b \subseteq \mathcal{C}V$, and hence $c \in \mathcal{C}V$ as wished.

Thus the characterization of $\mathcal{C}_{R^{loc}}U$ as the least R^{loc}-saturated subset containing U now brings to characterize the operator $\mathcal{C}_{R^{loc}}$ as follows. We write $\mathcal{F}_R$ for $\mathcal{C}_{R^{loc}}$, since we will show immediately that it satisfies 1.-3.

1'. $U \subseteq \mathcal{F}_R U$

2'. $\dfrac{V \cdot b \subseteq \mathcal{F}_R U}{RV \cdot b \subseteq \mathcal{F}_R U}$

3'. $\mathcal{F}_R U$ is the least subset satisfying 1'. and 2'., that is: if $U \subseteq P$ and $V \cdot b \subseteq P \Rightarrow RV \cdot b \subseteq P$, then $\mathcal{F}_R U \subseteq P$

We now can see that such $\mathcal{F}_R U$ is a solution to 1.-3.. The proof of transitivity for $\mathcal{F}_R$, that is $V \subseteq \mathcal{F}_R U \Rightarrow \mathcal{F}_R V \subseteq \mathcal{F}_R U$, is exactly as in section 2.1. So to prove 1. we need to show localization $\mathcal{F}_R U \cdot b \subseteq \mathcal{F}_R(U \cdot b)$. To this aim, we must exploit minimality expressed by 3'. We put $P \equiv \{c : c \cdot b \in \mathcal{F}_R(U \cdot b)\}$. Then $U \subseteq P$ because $U \cdot b \subseteq \mathcal{F}_R(U \cdot b)$ by 1'. Also, $V \cdot c \subseteq P \Rightarrow RV \cdot c \subseteq P$ because $V \cdot c \subseteq P$ means that $(V \cdot c) \cdot b \subseteq \mathcal{F}_R(U \cdot b)$, hence by associativity $V \cdot (c \cdot b) \subseteq \mathcal{F}_R(U \cdot b)$, and so $RV \cdot (c \cdot b) \subseteq \mathcal{F}_R(U \cdot b)$ by 2'., and so finally $RV \cdot c \subseteq P$. So 3'. gives $\mathcal{F}_R U \subseteq P$, which means exactly that $\mathcal{F}_R U \cdot b \subseteq \mathcal{F}_R(U \cdot b)$ as wished.[7] Clearly $\mathcal{F}_R$ satisfies R by 2'., and so 2. holds. Finally, assume $\mathcal{F}$ is any precover satisfying R. Then $RU \subseteq \mathcal{F}U$ gives $RU \cdot b \subseteq \mathcal{F}U \cdot b \subseteq \mathcal{F}(U \cdot b)$, which is immediately seen to be equivalent to b. above, alias 2'. So by minimality 3'. it is $\mathcal{F}_R U \subseteq \mathcal{F}U$ for any U, and so 3. is proved.

In this way we have proved that a solution of 1.-3. is given by the operator $\mathcal{F}_R$ associating with any U the least R^{loc}-saturated subset containing U. As in section 2.1, such $\mathcal{F}_R$ is easily defined impredicatively by putting

$$\mathcal{F}_R U \equiv \bigcap \{Z : U \subseteq Z \text{ and } Z \text{ is } R^{loc}\text{-saturated }\}$$

[7] This last step of the proof of localization of $\mathcal{F}_R$ is more perspicuous if one defines

$$U \to_{\mathcal{F}} V \equiv \{a : a \cdot U \subseteq \mathcal{F}V\}$$

from which it is immediate that

$$Z \cdot U \subseteq \mathcal{F}V \text{ iff } Z \subseteq U \to_{\mathcal{F}} V$$

Then $P \equiv \{b\} \to_{\mathcal{F}} U \cdot b$, so that assuming $V \cdot c \subseteq \{b\} \to_{\mathcal{F}} U \cdot b$ gives $V \cdot c \cdot b \subseteq \mathcal{F}(U \cdot b)$, hence $RV \cdot c \cdot b \subseteq \mathcal{F}(U \cdot b)$ by 2'., hence $RV \cdot c \subseteq \{b\} \to_{\mathcal{F}} U \cdot b$. This seems to show that the proof is essentially the same as that showing that any complete lattice with a good implication $\to$ satisfies infinite distributivity.

Predicatively, one must again assume that R has an axiomset I, C; then $\triangleleft_R$ is constructed by an inductive definition with rules:

reflexivity: $$\frac{a \in U}{a \triangleleft_R U}$$

loc-infinity: $$\frac{i \in I(a) \qquad C(a,i) \cdot b \triangleleft_R U}{a \cdot b \triangleleft_R U}$$

which clearly correspond to 1'. and 2'. above. It is then not difficult to prove, by induction on the generation of $\triangleleft_R$, that it satisfies localization, and hence that it is a precover.

It is now easy to prove that $Sat(\mathcal{F}_R)$ is the quantale freely generated by (S, R):

Theorem 3.13 *For any pair (S, R), where S is a monoid and R an infinitary relation on it, let $\mathcal{F}_R$ be the least pretopology on S satisfying R. Then the map $i : S \to Sat(\mathcal{F}_R)$ defined by $x \mapsto \mathcal{F}_R\{x\}$ is universal among maps $g : S \to \mathcal{Q}$, where $\mathcal{Q}$ is any quantale and g is any monoid morphism preserving R. In other terms, $Sat(\mathcal{F}_R)$ is the quantale freely generated by (S, R).*

In particular, if X is any set of generators and R any relation on it, then every map $f : X \to \mathcal{Q}$ preserving R factors through the quantale $Sat(\mathcal{F}_R)$, where $\mathcal{F}_R$ is the pretopology given by the precover generated by R on the free monoid on X.

Proof It is immediate to see that any monoid morphism $g : S \to \mathcal{Q}$ preserving R preserves also its closure under localization R^{loc}. Then, by theorem 2.7, g extends uniquely to the sup-lattice morphism $\tilde{g} : Sat\mathcal{C}_{R^{loc}} \to Q$, where $\mathcal{C}_{R^{loc}}$ is the infinitary preordered set given by the infinitary preorder generated by R^{loc} on S. As seen above, $\mathcal{C}_{R^{loc}}$ coincides with the pretopology $\mathcal{F}_R$. To conclude, it is enough to check that the maps i and $\tilde{g}$ preserve the monoid operations of S and of $Sat\mathcal{F}_R$, respectively. We have: $i(a \cdot b) \equiv \mathcal{F}_R(a \cdot b) = \mathcal{F}_R(a) \cdot_{\mathcal{F}_R} \mathcal{F}_R(b) \equiv i(a) \cdot i(b)$ by stability of the precover $\triangleleft_{R^{loc}}$; and $\tilde{g}(\mathcal{F}_R U \cdot_{\mathcal{F}_R} \mathcal{F}_R V) \equiv \tilde{g}\mathcal{F}_R(U \cdot V) \equiv \bigvee_{u \in U, v \in V} g(u \cdot v) = \bigvee_{u \in U} g(u) \cdot \bigvee_{v \in V} g(v) \equiv \tilde{g}\mathcal{F}_R U \cdot \tilde{g}\mathcal{F}_R V$, by distributivity of quantales.

In particular, if $CMon(X)$ is the monoid freely generated by X, then any map $f : X \to Q$ preserving R factors uniquely through a monoid morphism $f' : CMonX \to Q$ and then also through a quantale morphism $\tilde{f}' : Sat(\mathcal{F}_R) \to Q$ as seen above. □

In particular, proposition 3.3 is a consequence of the above theorem, putting $R = \emptyset$, as is the case of sup-lattices. Presentation of quantales (theorem 3.11) is another consequence of the above theorem, as we have also already discussed for sup-lattices.

As for a predicative presentation, let us say that $\mathcal{Q}$ is predicatively presentable if and only if it is isomorphic to $Sat(\mathcal{F}_R)$, where R has an axiomset. One can see that, if R has an axiomset, then R^{loc} has an axiomset (see [CSSV], p.25). So, since $Sat(\mathcal{F}_R) = Sat(\mathcal{C}_{R^{loc}})$, as we have seen above, a quantale $\mathcal{Q}$ is predicatively presentable if and only if it is predicatively presentable as a sup-lattice (see theorem 2.8).

We consider now **Pretop**, the subcategory of **IP**, whose objects are pretopologies. A morphism $f : \mathcal{F} \to \mathcal{F}'$ is a morphism of **Pretop** if it is a morphism of **IP** (see definition 1.11) preserving the monoid operation, i.e. satisfying the clauses $f(1) =_{\mathcal{F}'} 1$ and $f(a) \cdot f(b) =_{\mathcal{F}'} f(a \cdot b)$ for every a, b in the base of $\mathcal{F}$. We see that the equivalence given in 2.9 restricts to an equivalence between **Pretop** and the category of quantales, **Quant**:

Theorem 3.14 *The categories* **Pretop** *and* **Quant** *are equivalent.*

Proof. The functor $Sat : \mathbf{IP} \to \mathbf{SL}$, when restricted to **Pretop**, maps the objects of **Pretop** into the objects of **Quant**, by theorem 3.9: the functor $Transl$ is its right inverse by corollary 3.12. Moreover, it is easy to see that, if $\mathcal{F}, \mathcal{G}$ are pretopologies and $f : \mathcal{F} \to \mathcal{G}$ is a morphism of pretopologies, then $Sat(f) : Sat(\mathcal{F}) \to Sat(\mathcal{G})$ is a morphism of quantales. Hence the equivalence in proposition 2.3 restricts to **Pretop** and **Quant**. □

4 Presentation of frames

4.1 Formal topologies and the presentation of frames

In the following we shall see that the results obtained for quantales are enough to present frames as a particular case. The usual definition of frame (see e.g. [J82], p. 39) is the following:

Definition 4.1 *A frame $\mathcal{H} = (H, \wedge, \bigvee)$ is a complete lattice in which distributivity of meets with respect to infinitary joins holds, i.e. $a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I}(a \wedge b_i)$, for every $a \in H$, $b_i \in H$ $(i \in I)$.*

After the previous results, it is more convenient for our purposes to adopt the following equivalent characterization:

Proposition 4.2 *Frames can be characterized as those quantales $(Q, \cdot, 1, \bigvee)$ in which $a \cdot b = a \wedge b$ for all $a, b \in Q$, where $\wedge$ is the meet with respect to the order $\leq$ induced by $\bigvee$.*

By this characterization, given two frames $\mathcal{H}$ and $\mathcal{H}'$, a map $f : H \to H'$ is a frame morphism if and only if it is a quantale morphism.

Since frames are particular quantales, to obtain a presentation of frames the first step is to describe those precover relations which generate frames. Let $\mathcal{A} = (S, \cdot, 1, \lhd_{\mathcal{A}})$ be a pretopology presenting a frame $\mathcal{H}$, i.e. $\mathcal{H} \cong Sat(\mathcal{A})$. Then, by proposition 4.2 above, it must be $\mathcal{A}U \cdot_{\mathcal{A}} \mathcal{A}V = \mathcal{A}U \wedge \mathcal{A}V$ for every $U, V \subseteq S$. Actually, as in any sup-lattice of the form $Sat\mathcal{C}$, the meet operation in $Sat(\mathcal{A})$ is just intersection: since the ordering is inclusion, to show that $\mathcal{A}U \wedge \mathcal{A}V = \mathcal{A}U \cap \mathcal{A}V$, it is enough to see that $\mathcal{A}U \cap \mathcal{A}V$ is $\mathcal{A}$-saturated, a fact which is well known to hold (and easy to see) for any closure operator. So, by stability, the condition which characterizes pretopologies giving rise to frames is:

$$\mathcal{A}(U \cdot V) = \mathcal{A}U \cap \mathcal{A}V$$

that is $\mathcal{A}(U \cdot V) \subseteq \mathcal{A}U \cap \mathcal{A}V$ and $\mathcal{A}U \cap \mathcal{A}V \subseteq \mathcal{A}(U \cdot V)$. We give below some useful equivalents of such inclusions, and for both we find an equivalent condition involving only the elements of the base.

Proposition 4.3 *For any infinitary preorder $\lhd_{\mathcal{A}}$, the following are equivalent:*

- (i) $\mathcal{A}(U \cdot V) \subseteq \mathcal{A}U \cap \mathcal{A}V$ *for every* $U, V \subseteq S$
- (ii) $U \cdot V \lhd_{\mathcal{A}} U$ *for every* $U, V \subseteq S$
- (iii) $a \cdot b \lhd_{\mathcal{A}} a$ *for every* $a, b \in S$
- (iv) $\lhd_{\mathcal{A}}$ *is closed under the rule* $(\cdot L)$: $\dfrac{a \lhd_{\mathcal{A}} U}{a \cdot b \lhd_{\mathcal{A}} U}$

Proof. Since $U \cdot V = V \cdot U$, (i) is equivalent to $\mathcal{A}(U \cdot V) \subseteq \mathcal{A}U$ for any $U, V \subseteq S$, and hence equivalence of (i) with (ii) follows from $\mathcal{A}(U \cdot V) \subseteq \mathcal{A}U$ iff $U \cdot V \lhd_{\mathcal{A}} U$. (iii) is a special case of (ii), obtained by taking $U = \{a\}$ and $V = \{b\}$. Assuming (iii), closure under $(\cdot L)$ is immediate by transitivity. Finally, to show that (iv) implies (ii), assume $x \in U \cdot V$; then $x = a \cdot b$ for some $a \in U$ and $b \in V$ and hence $a \lhd_{\mathcal{A}} U$. By $(\cdot L)$ it follows $a \cdot b \lhd_{\mathcal{A}} U$, that is $x \lhd_{\mathcal{A}} U$, so that $U \cdot V \lhd_{\mathcal{A}} U$ as wished. □

Proposition 4.4 *For any pretopology $\mathcal{A} = (S, \cdot, 1, \lhd_{\mathcal{A}})$, the following are equivalent:*

- (i) $\mathcal{A}U \cap \mathcal{A}V \subseteq \mathcal{A}(U \cdot V)$ *for every* $U, V \subseteq S$
- (ii) $\lhd_{\mathcal{A}}$ *is closed under the rule* $(\cdot R)$: $\dfrac{a \lhd_{\mathcal{A}} U \quad a \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} U \cdot V}$
- (iii) $\lhd_{\mathcal{A}}$ *is closed under the rule* $(\cdot R_G)$: $\dfrac{Z \lhd_{\mathcal{A}} U \quad Z \lhd_{\mathcal{A}} V}{Z \lhd_{\mathcal{A}} U \cdot V}$
- (iv) $U \lhd_{\mathcal{A}} U \cdot U$ *for every* $U \subseteq S$.

(*v*) $a \lhd_{\mathcal{A}} a \cdot a$ *for every* $a \in S$.

Proof. (*i*) is equivalent to (*ii*), since (*ii*) is just a rewriting of (*i*) in terms of $\lhd_{\mathcal{A}}$; (*ii*) implies (*iii*), since (*iii*) is just the variant with subsets on the left of (*ii*); (*iii*) implies (*iv*), taking $Z = U = V$, and (*iv*) implies (*v*) taking $U = \{a\}$. Finally, (*v*) implies (*ii*) because stability applied to the premises of (*ii*) gives $a \cdot a \lhd_{\mathcal{A}} U \cdot V$, from which the conclusion $a \lhd_{\mathcal{A}} U \cdot V$ by (*v*) and transitivity. □

So we adopt the following definition, that is the most convenient in order to obtain the presentation of frames as a corollary of the presentation of quantales. We recall that, since $(\cdot L)$ and $(\cdot R)$ together imply stability, as one can easily see, covers can equivalently be defined as infinitary preorders closed under $(\cdot L)$ and $(\cdot R)$.[8]

Definition 4.5 *A precover* $\lhd$ *satisfying* $a \cdot b \lhd a$ *and* $a \lhd a \cdot a$ *is called a cover. A pretopology* $\mathcal{A} = (S, \cdot, 1, \lhd_{\mathcal{A}})$*, where* $\lhd_{\mathcal{A}}$ *is a cover is called a formal topology.*

By the theorem 3.9 and the discussion so far, we have:

Proposition 4.6 *For any formal topology* $\mathcal{A}$*,* $Sat(\mathcal{A})$ *is a frame.*

Conversely, by corollary 3.12, every frame is impredicatively isomorphic to $Sat(\mathcal{A})$, for some $\mathcal{A}$, since frames are particular quantales and morphisms of frames are morphisms of quantales.

Furtherly, if we consider the full subcategory of **Pretop** whose objects are formal topologies, called **FTop**, and the category of frames, **Frm**, we obtain the following immediate consequence of theorem 3.14:

Theorem 4.7 *The categories* **Ftop** *and* **Frm** *are equivalent.*

4.2 Frames presented by axioms

The formal cover generated with conditions, or axioms, given by an infinitary relation R is just the precover which is generated by the relation R' obtained from R by adding all pairs $(a \cdot b, a)$ and $(a, a \cdot a)$ for any a, b. In fact, in such case the generated precover satisfies conditions 4.3.*iii*) and 4.4.*v*). So we consider the relation P, defined by requiring only that $P(a \cdot b, a)$ and $P(a, a \cdot a)$ hold for any a, b, and for any infinitary relation R we consider $R \cup P$, that is the relation obtained by joining R with P. Then $\mathcal{F}_{R \cup P}$ is the least pretopology satisfying $R \cup P$, and then it is the least formal topology satisfying R. Let us term it $\mathcal{A}_R$ and consider the frame $Sat(\mathcal{A}_R)$. We see that theorem 3.13 for quantales specializes to frames as follows:

Theorem 4.8 *For any pair* (S, R)*, where* S *is a monoid and* R *an infinitary relation on it, let* $\mathcal{A}_R$ *be the least formal topology on* S *satisfying* R*. Then the map* $i : S \to Sat(\mathcal{A}_R)$ *defined by* $x \mapsto \mathcal{A}_R\{x\}$ *is universal among maps* $g : S \to \mathcal{H}$*, where* $\mathcal{H}$ *is any frame and* g *is any monoid morphism preserving* R*. In other terms,* $Sat(\mathcal{A}_R)$ *is the frame freely generated by* (S, R)*.*

Proof. Any monoid morphism g preserves P, since the inequalities $g(a \cdot b) \equiv g(a) \wedge g(b) \leq g(a)$ and $g(a) \leq g(a) \wedge g(a) \equiv g(a \cdot a)$ hold in a frame. So, theorem 3.13 can be applied to $R \cup P$. □

Finally, as for sup-lattices and quantales, by taking R to be empty in the statement of theorem 4.8, one finds out what the free frame is: it is the frame $Sat(\mathcal{F}_P)$. This, in turn, coincides with $Sat(\mathcal{C}_{P^{loc}})$ where P^{loc} is the closure of P under localization.

Can such frame be characterized more directly? Yes; indeed, we now see that the cover generated by the empty set of axioms, that is the infinitary preorder generated by P^{loc}, can be described in terms of a natural preorder on the base S. For every $a, b \in S$, we put

$$a \leq b \quad \text{iff, for some } n \in N \text{ and } d \in S, \quad a^n = b \cdot d$$

[8] This is the course taken in [S87]. Note that the full definition of formal topology includes an additional predicate Pos, which is necessary to express constructively that a formal open is inhabited, but is not relevant to present frames.

It is easy to see that $\leq$ is a preorder (sometimes called the natural preorder on the monoid S). In fact, $a \leq a$ because $a^1 = a \cdot 1$ and if $a \leq b$ and $b \leq c$, then $a^n = b \cdot d$ and $b^m = c \cdot e$ for some m, n, d, e, from which $(a^n)^m = (b \cdot d)^m = c \cdot e \cdot d^m$, that is $a^p = c \cdot f$ for some p, f, i.e. $a \leq c$.

We now see that the infinitary preorder generated by P^{loc} is the least infinitary preorder extending the natural preorder on the base. We first need the following general result:

Lemma 4.9 *Let B be a binary relation between elements of a set X. Then the infinitary preorder generated by B on X satisfies*

i) $x \lhd_B y$ if and only if $x \leq_B y$, where $\leq_B$ is the preorder generated by B on X, that is the reflexive and transitive closure of B.

ii) $\mathcal{C}_B(U) = \bigcup_{b \in U} \mathcal{C}_B\{b\}$, and hence $a \lhd_B U$ if and only if there is $b \in U$ such that $a \leq_B b$.

Proof. *i)* Since $\lhd_B$ is reflexive and transitive, $x \leq_B y$ implies $x \lhd_B y$. Conversely, assume $x \lhd_B y$, that is $x \in \mathcal{C}_B y$. It is immediate to verify that $\downarrow_B y \equiv \{z : z \leq_B y\}$ is B-saturated, and hence $\mathcal{C}_B(y) \subseteq \downarrow_B y$. So $x \leq_B y$, that is $x \in \downarrow_B y$ as wished.

ii) For every closure operator $\mathcal{C}$ it is $\mathcal{C}(U) \supseteq \bigcup_{b \in U} \mathcal{C}\{b\}$. Here the equality holds because $\bigcup_{b \in U} \mathcal{C}_B\{b\}$ is B-saturated: if $B(x, y)$ and $y \in \bigcup_{b \in U} \mathcal{C}_B\{b\}$, then $y \in \mathcal{C}_B\{b\}$ for some $b \in U$, and hence $x \in \mathcal{C}_B\{b\} \subseteq \bigcup_{b \in U} \mathcal{C}_B\{b\}$, because $\mathcal{C}_B\{b\}$ is B-saturated. □

Note that, for every binary relation B on X, $x \leq_B y$ if and only if there exist $d_1, \ldots d_n \in X$ such that $d_1 = x$, $d_n = y$ and $B(d_i, d_{i+1})$ for every $1 \leq i < n$. Applying this to P^{loc}, which is a binary relation, we see that P^{loc} generates the natural preorder on S. In fact, the preorder $\leq_{P^{loc}}$ generated by P^{loc} is contained in the natural preorder, since for every a, b one gets $a \cdot b \leq a$ by taking $n = 1$ and $d = b$ in the definition of $\leq$ above, and for every a, c one gets $a \cdot c \leq a \cdot a \cdot c$ by taking $n = 2$ and $d = c$ and by commutativity of S. Conversely, assume $x \leq y$, that is $x^n = y \cdot d$ for some natural number n and $d \in S$. Then from $P^{loc}(y \cdot d, y)$ we have $P^{loc}(x^n, y)$, and moreover clearly $P^{loc}(x^i, x^{i+1})$ for $1 \leq i < n$, so that $x \leq_{P^{loc}} y$.

Then, by the above lemma 4.9, one has $a \in \mathcal{C}_{P^{loc}} U$ if and only if there is $b \in U$ such that $a \leq b$ in the natural preorder of S. So, by theorem 4.8, we have:

Theorem 4.10 *The free frame generated by a monoid is the frame of downward closed subsets with respect to the natural preorder.*

4.3 Formal covers on semilattices and their connection with coverages

The above theorem 4.10 is usually stated when the base is a semilattice, let us say a $\wedge$-semilattice $(T, \wedge, 1)$. In such case, the natural preorder coincides with the partial order induced by the infimum $\wedge$. In fact, since $a^n = a$ for any $n \in N$ and any $a \in T$, we have $a \leq b$ if and only if $a = b \wedge c$ for some c, if and only if $a = b \wedge (b \wedge c) = b \wedge a$. Then theorem 4.10 gives also the well-known (see [J82]):

Theorem 4.11 *The free frame generated by a semilattice is the frame of its downward closed subsets.*

Precovers defined on a semilattice satisfy the conditions of proposition 4.4 and hence, in such case, any of the conditions of proposition 4.3 characterize covers among precovers.

Proposition 4.12 *Let $(T, \wedge, 1)$ be a semilattice. A relation $\lhd_{\mathcal{A}}$ defined on T is a formal cover if and only if it is closed under the rules (SR), (T), and $(\wedge L)$.*

The cover generated by an infinitary relation R on a semilattice $(T, \wedge, 1)$ is the precover generated by R joined with all pairs $(a \wedge b, a)$ (since $a \wedge a = a$, pairs $(a \wedge a, a)$ are not necessary), that is with the semilattice ordering. Hence, by the results in section 3.3 and since obviously $\leq$ is closed under localization, the cover generated by R is the same thing as the infinitary preorder generated by R^{loc} joined with $\leq$.

Then the frame freely generated by T with conditions given by R is formed by all subsets of T which are R^{loc}-saturated and downward closed (that is, all $U \subseteq T$ s.t. $R^{loc}(a, V)$ & $V \subseteq U \Rightarrow a \in U$ and $a \in U \Rightarrow \downarrow a \subseteq U$, where $\downarrow a \equiv \{b : b \leq a\}$).

The well-known C-ideals of a coverage C (cf. [J82], p. 58) are exactly the C-saturated and downward closed subsets. We remind that a coverage is just an infinitary relation C on a semilattice T which satisfies:

(i) if $C(a,U)$ then $U \subseteq \downarrow a$

(ii) meet-stability: $\dfrac{C(a,U) \quad b \leq a}{C(b, U \wedge b)}$

and it is easy to see that (see [V89]) in presence of (i) meet-stability is equivalent to:

$$\frac{C(a,U)}{C(a \wedge b, U \wedge b)}$$

namely localization. So a frame can be presented as C-ideals of a coverage C if and only if it can be presented as here, as R-saturated subsets of some relation R.

A direct link between coverages and covers can be obtained by noting that coverages, apart from condition (i), are just relations, that is axioms, closed under localization. So to be able to compare them with our covers one must first close the axioms under deductions. Because of the presence of (i), this is not as natural as with covers. However, one can do it, and say that a coverage is *closed* if it satisfies the following additional conditions:

(iii) reflexivity: $\dfrac{a \in U}{C(a, U \wedge a)}$

(iv) transitivity: $\dfrac{C(a,U) \qquad (\forall b \in U) C(b, W \wedge b)}{C(a, W \wedge a)}$

As the next proposition shows in detail, the correspondence between covers and closed coverages is indeed a bijection:

Proposition 4.13 *Let T be any semilattice. For any closed coverage C on T, we put:*

$$a \lhd^C U \equiv C(a, U \wedge a).$$

Then $\lhd^C$ is a cover on T. Conversely, for any cover $\lhd$ on T, we put:

$$C^{\lhd}(a,U) \equiv a \lhd U \ \& \ U \subseteq \downarrow a.$$

Then $C^{\lhd}$ is a closed coverage. This gives a bijective correspondence between covers and closed coverages.

Proof. Let C be a closed coverage. Then (SR) for $\lhd^C$ is exactly reflexivity for C. To prove closure of $\lhd^C$ under (T), assume $a \lhd^C U$ and $(\forall b \in U)(b \lhd^C W)$, that is $C(a, U \wedge a)$ and $(\forall b \in U) C(b, W \wedge b)$ respectively. From the latter by localization we obtain $(\forall b \wedge a \in U \wedge a) C(b \wedge a, W \wedge b \wedge a)$. Putting $U' \equiv U \wedge a$ one has $C(a, U')$ and $(\forall b' \in U') C(b', W \wedge b')$, so finally transitivity of C allows to conclude $C(a, W \wedge a)$, that is $a \lhd^C W$. (L) and $(\wedge L)$ for $\lhd^C$ are both obtained by localization of C; in fact, if $a \lhd^C U$, that is $C(a, U \wedge a)$, one gets $C(a \wedge b, U \wedge a \wedge b)$, that is $a \wedge b \lhd^C U$; but also $a \wedge b \lhd^C U \wedge b$ because of the equality $U \wedge a \wedge b = (U \wedge b) \wedge (a \wedge b)$.

Viceversa, let $\lhd$ be a cover relation on T. Condition (i) is forced by the definition. Localization follows from localization of $\lhd$: if $C^{\lhd}(a,U)$, from $a \lhd U$ one gets $a \wedge b \lhd U \wedge b$, while from $U \subseteq \downarrow a$ one has $U \wedge b \subseteq \downarrow (a \wedge b)$, so that $C^{\lhd}(a \wedge b, U \wedge b)$ holds. Reflexivity for $C^{\lhd}$ follows by (SR) and because $U \wedge a \subseteq \downarrow a$. As for transitivity, if $C^{\lhd}(a,U)$ and $(\forall b \in U) C^{\lhd}(b, W \wedge b)$, then $a \lhd U$ and $U \lhd W \wedge U$, so from $W \wedge U \lhd W$ one gets $a \lhd W$ by (T), and from this $a \lhd W \wedge a$ by (L), so that $C^{\lhd}(a, W \wedge a)$ follows.

The correspondence is bijective: if $C(a,U)$, then $C(a, U \wedge a)$, that is $a \lhd^C U$, so that $C^{\lhd^C}(a,U)$; viceversa $C^{\lhd^C}(a,U)$ means $C(a, U \wedge a)$ and $U \subseteq \downarrow a$, that is $U \wedge a = U$, so that $C(a,U)$. If $a \lhd U$, then $a \lhd U \wedge a$, so $(a \lhd U \wedge a) \& (U \wedge a \subseteq \downarrow a) \equiv C^{\lhd}(a, U \wedge a) \equiv a \lhd^{C^{\lhd}} U$; viceversa from $a \lhd^{C^{\lhd}} U$ by the same equivalences one gets $a \lhd U \wedge a$, but $U \wedge a \lhd U$, so $a \lhd U$. $\square$

The above results show that the two methods to present frames are "quantitatively" equivalent. There are mainly two reasons why it has been chosen (in [S87]) to change Johnstone's definitions. The first reason is that in a predicative treatment it is necessary to generate covers inductively, and thus one must keep both notions, that of axioms given by a relation R and that of cover $\lhd$ (which is closed under deductions). That is why one is free to consider arbitrary relations R, with no conditions like (i) or (ii) to be satisfied. The second reason is that the presence of condition (i), and hence the interpretation of $C(a, U)$ as an equality,[9] makes it difficult to express weaker infinitary relations, corresponding to sup-lattices or quantales. In fact, as the proof of proposition 4.13 shows, $(\wedge L)$ is implicit in the definition of coverage $(\wedge L)$ for $\lhd^C$ is obtained by localization of C, and conversely $(\wedge L)$ of $\lhd$ is not used to prove that $C^{\lhd}$ is a coverage. So it is not possible to express a "pre-coverage" relation, analogous to precovers. Also note that, in the presence of (i), localization is necessary to be able to express transitivity (see the proof above of the fact that $\lhd^C$ is closed under transitivity).

So an advantage of our infinitary relations is that they can express several conditions as independent, which would be linked in the approach of coverages, and that is why they can produce a uniform presentation of sup-lattices, quantales and frames.

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[9] Which is perhaps due to the origin of the definition of coverage, namely Grothendieck's topologies; see the comments in [J82], pag. 77, and see [MLM92] for a definition.

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