Formal Topologies on the Set of First-Order Formulae

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1   Introduction

The completeness proof for first-order logic by Rasiowa and Sikorski [13] is a simplification of Henkin's proof [7] in that it avoids the addition of infinitely many new individual constants. Instead they show that each consistent set of formulae can be extended to a maximally consistent set, satisfying the following existence property: if it contains $(\exists x)\phi$ it also contains some substitution $\phi(y/x)$ of a variable y for x. In Feferman's review [5] of [13], an improvement, due to Tarski, is given by which the proof gets a simple algebraic form.

Sambin [15] used the same method in the setting of formal topology [16], thereby obtaining a constructive completeness proof. This proof is elementary and can be seen as a constructive and predicative version of the one in Feferman's review. It is a typical, and simple, example where the use of formal topology gives constructive sense to the existence of a generic object, satisfying some forcing conditions; in this case an ultrafilter satisfying the existence property.

In order to get a formal topology on the set of first-order formulae, Sambin used the Dedekind-MacNeille completion to define a covering relation $\lhd_{DM}$. This method, by which an arbitrary poset can be extended to a complete poset, was introduced by MacNeille [9] and is a generalization of the construction of real numbers from rationals by Dedekind cuts. It is also possible to define an inductive cover, $\lhd_I$, on the set of formulae, which can also be used to give canonical models, see Coquand and Smith [3]. Proof-theoretically, one can notice that $\lhd_I$ is given by a generalized inductive definition, while the definition of $\lhd_{DM}$ is elementary.

Given that Sambin's completeness proof can be seen as a constructive version of the Henkin-Rasiowa-Sikorski proof, it was natural to conjecture that the points of this topology correspond to Henkin sets; this conjecture appears in [15]. For the inductive topology, it is easy to see that the points correspond to Henkin sets. Hence, the natural question: do these two topologies coincide? We show in this paper that the question has a simple negative answer. This raised further natural questions on what can be said about the points of these two topologies; we give some answers.

The observation that topological models for first-order theories can expressed in the framework of locales appears, for instance, in Fourman and

Grayson [6], where the analogy between points of a locale and models of a theory is emphasised; the identification of formal points with Henkin sets, gives a precise form to this analogy. We replace the use of locales by formal topology, which can be expressed in a predicative framework such as Martin-Löf's type theory. Proof-theoretic issues are also considered by Dragalin [4], who presents a topological completeness proof using only finitary inductive definitions. Palmgren and Moerdijk [10] is also concerned with constructions of models: using sheaf semantics, they obtain a stronger conservativity result than the one in [3].

We will first investigate the difference between the Dedekind-MacNeille cover and the inductive cover. It easy to see that $\lhd_{DM}$ is stronger than $\lhd_I$, that is, $\phi \lhd_I U$ implies $\phi \lhd_{DM} U$, but the converse does not hold in general.

The notion of point is not primitive in formal topology and therefore it is natural to require that a formal topology has some notion of positivity defined on the basic neighbourhoods; that a neighbourhood is positive then corresponds to, in ordinary point based topology, that it is inhabited by some point. We will show several negative results on positivity, both for the inductive topology and the Dedekind-MacNeille topology. The points of an inductive topology correspond to Henkin sets, but the Dedekind-MacNeille topology has, in general, no points.

Our reasoning is constructive and, in the same way as Bishop's [1], neutral in the sense that no principles that contradicts classical mathematics are used. The meta-theory is weak: all arguments can be carried out in Martin-Löf's type theory without universes [17, 11]. However, we will be informal and the paper can be read without any knowledge of type theory. We only want to point out that our reasoning is predicative, hence we make a distinction between sets and types. Sets are inductively defined and form the objects of the type Set. Subsets of a set S are propositional functions, that is, they are objects of the function type $S \to$ Set. If U is a subset of S and $a \in S$, we usually write $a \, \epsilon \, U$ for the judgement $U(a)$ true.

Formal topology has been developed in computer systems for type theory [2]; in particular, the completeness proof in [15] has been checked in the ALF system [12].

Some of the results in this paper first appeared in Sara Sadocco's *tesi di laurea* [14].

2   Definition of formal space

A *formal topology*, as defined by Sambin [16], is a commutative idempotent monoid $\langle S, \cdot, 1 \rangle$ with a covering relation $\lhd$, that is, a relation between elements of the set S and subsets of S which satisfies the following rules.

Reflexivity
$$\frac{a \, \epsilon \, U}{a \lhd U}$$

Transitivity
$$\frac{a \lhd U \quad U \lhd V}{a \lhd V}$$

· -left
$$\frac{a \lhd U}{a \cdot b \lhd U}$$

Stability
$$\frac{a \lhd U \qquad b \lhd V}{a \cdot b \ \lhd \ \{c \cdot d : c \ \epsilon \ U \text{ and } d \ \epsilon \ V\}}$$

where $U \lhd V$ means that every element of U is covered by V. Intuitively, the elements of S are the basic open sets and the multiplication · corresponds to intersection. An open set is represented by a subset of basic open sets; in traditional topology with points, this corresponds to that an open set can be represented as the union of a set of basic open sets.

A set U is *saturated* if $a\epsilon U$ if and only if $a \lhd U$. The type of saturated sets form a predicative version of a complete Heyting algebra in the sense that only set indexed families of saturated sets, in general, have least upper bounds.

For the details of formal topology in a type theoretic setting, we refer to Sambin [16].

In this paper, S will always be the set of formulae of some arbitrary first-order theory T, with classical or intuitionistic logic, and $\vdash_T$ the derivability relation of T. We let ϕ, ψ and σ denote arbitrary formulae of T. The monoid operation · is the conjunction, that is, $\phi \cdot \psi \ = \ \phi \& \psi$, and the unit 1 is the true proposition $\top$. The equality of the monoid is provable equivalence, that is, $\phi = \psi$ if and only if $\vdash_T \phi \leftrightarrow \psi$. We will use the quantifiers $\forall$ and $\exists$ both informally and in first-order formulae, but the meaning will always be clear from the context.

3 Covers on the set of first-order formulae

For the first defintion of a cover, we use MacNeille's method [9], by which an arbitrary poset can be extended to a complete poset. In our case, the partial ordering is induced by the derivability relation of a first-order theory T. Let ϕ be a an arbitrary formula of T and U an arbitrary subset of formulae of T. The *Dedekind-MacNeille covering*, $\lhd_{DM}$, is defined by

$$\phi \lhd_{DM} U \ = \ (\forall \psi)((\forall \sigma \ \epsilon \ U)(\sigma \vdash_T \psi) \Rightarrow \phi \vdash_T \psi).$$

So ϕ is covered by a subset U if and only if every formula ψ that can be proved from each of the formulae in U can also be proved from ϕ.

The *inductive cover* is defined by an infinitary inductive definition:

Reflexivity
$$\frac{\phi \ \epsilon \ U}{\phi \lhd_I U}$$

Absurdity
$$\frac{\phi \vdash_T \bot}{\phi \lhd_I U}$$

Provability

$$\frac{\phi \vdash_T \psi \qquad \psi \lhd_I U}{\phi \lhd_I U}$$

Disjunction

$$\frac{\phi \lhd_I U \qquad \psi \lhd_I U}{\phi \lor \psi \lhd_I U}$$

Existence

$$\frac{\phi(t) \lhd_I U \text{ for all terms } t}{\exists x \phi(x) \lhd_I U}$$

It is straightforwardly proved that this defines a formal topology. We call this topology the *Henkin* or *Inductive* topology associated to the first-order theory T.

4 The relation between $\lhd_{DM}$ and $\lhd_I$

The following proposition is easy to prove, by induction on the derivation of $\phi \lhd_I U$,

Proposition 1 *For any theory, $\phi \lhd_I U$ implies $\phi \lhd_{DM} U$.*

As we will show in this section, the converse does not hold.

The set of formulae, Frm_T, of a theory T forms a poset with derivability as the order relation and equality as defined above. By the following proposition, the poset $\langle Frm_T, \vdash_T \rangle$ can be embedded in the type of subsets of Frm_T by identifying a formula ϕ with the singleton $\{\phi\}$ and by interpreting $\vdash_T$ by either $\lhd_{DM}$ or $\lhd_I$.

Proposition 2 *Let $\lhd$ be either $\lhd_{DM}$ or $\lhd_I$. Then $\phi \vdash_T \psi$ if and only if $\phi \lhd \{\psi\}$.*

Proof. The case of the Dedekind-MacNeille covering follows easily from the definition of $\lhd_{DM}$. The implication from left to right in the case of the inductive cover is directly obtained by the definition of $\lhd_I$; the converse implication is proved by induction on the derivation of $\phi \lhd \{\psi\}$. □

For both the Dedekind-MacNeille covering and the inductive covering it is easy to see that $\phi \lhd \{\psi_1, \ldots \psi_n\}$ if and only if $\phi \lhd \{\psi_1 \lor \cdots \lor \psi_n\}$. Hence, we obtain from proposition 2

Corollary 1 *Let U be finite. Then $\phi \lhd_{DM} U$ if and only if $\phi \lhd_I U$.*

A covering relation $\lhd$ induces a partial order on the type of subsets of the set S of basic neighbourhoods by letting $U \le V$ mean that every element of U is covered by V, i.e. $U \lhd V$. If U_I is a family of subsets over the set I, then an immediate consequence of the rules for a covering relation is that the supremum of U_I exists and is equal to the union of the family.

If the subset U has a supremum with respect to a partial order $\le$, we let $sup_{\le} U$ denote the supremum. The Dedekind-MacNeille completion has the property that suprema that exist in $\langle Frm_T, \vdash_T \rangle$ are preserved:

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Proposition 3 *Let $\overline{U} = \{\{u\} : u \,\epsilon\, U\}$. Then $sup_{\vdash_T} U = \phi$ implies $sup_{\lhd_{DM}} \overline{U} = \{\phi\}$.*

Proof. Immediate consequence of the definition of $\lhd_{DM}$. $\square$

In the proof of the next theorem, we will use the notion of inductive subset. A subset U on the set of formulae of the theory T is *inductive* if it satisfies

- If $\phi \vdash_T \psi$ and $\psi \,\epsilon\, U$, then $\phi \,\epsilon\, U$.

- $\perp \,\epsilon\, U$.

- If $\phi \,\epsilon\, U$ and $\psi \,\epsilon\, U$, then $\phi \vee \psi \,\epsilon\, U$.

- If, for all terms t, $\psi(t) \,\epsilon\, U$, then $(\exists x)\psi(x) \,\epsilon\, U$.

Lemma 1 *$\phi \lhd_I U$ if and only if ϕ belongs to every inductive subset that contains U.*

Proof. The implication from left to right is by induction; the other direction is immediate since $\{\phi : \phi \lhd_I U\}$ is inductive. $\square$

Theorem 1 *In general, $\phi \lhd_{DM} U$ does not imply $\phi \lhd_I U$.*

Proof. Let T_P be a theory with only one predicate symbol P and no non-logical axioms. Let $X + Y$, with X infinite and Y nonempty, be a partition of the set of variables and let $U_X = \{P(y) : y \,\epsilon\, X\}$. We will show that $\exists x P(x) \lhd_{DM} U_X$ but not $\exists x P(x) \lhd_I U_X$.

To prove that $\exists x P(x) \lhd_{DM} U_X$ we must show that if $P(y) \vdash_{T_P} \psi$ for all $y \,\epsilon\, X$, then $\exists x P(x) \vdash_{T_P} \psi$. Since X is infinite, there exist a $z \,\epsilon\, X$ which does not occur in ψ; hence $P(z) \vdash_{T_P} \psi$ gives $\exists x P(x) \vdash_{T_P} \psi$.

Define the subset V on the formulae of T_P by

$$\phi \,\epsilon\, V \quad \Leftrightarrow \quad (\exists y_1 \cdots \exists y_k \,\epsilon\, X)(\phi \vdash_{T_P} P(y_1) \vee \cdots \vee P(y_k))$$

Using that Y is nonempty, it is easy to see that V is an inductive subset and that $U_X \subseteq V$. Hence, by the lemma,

$$\exists x P(x) \lhd_I U_X \quad \Rightarrow \quad \exists x P(x) \,\epsilon\, V.$$

But, clearly, $\exists x P(x) \,\epsilon\, V$ does not hold. $\square$

From the proof of theorem 1 we see that $sup_{\vdash_{T_P}} U_X = \exists x P(x)$ and $sup_{\lhd_I} \overline{U_X} \neq \exists x P(x)$. Hence proposition 3 does not hold if we replace $\lhd_{DM}$ by $\lhd_I$:

Corollary 2 *In general, the inductive cover does not preserve suprema from the poset $\langle \mathrm{Frm}_T, \vdash_T \rangle$.*

A cover $\lhd$ is a *Stone cover* if $a \lhd U$ implies that there is a finite subset U_0 of U such that $a \lhd U_0$.

Proposition 4 *Neither $\lhd_{DM}$ nor $\lhd_I$ are in general Stone covers.*

Proof. Assume that the inductive cover is Stone. By the definition of the inductive cover, $\exists x \psi(x) \lhd_I \{\psi(t) : \ t \text{ arbitrary term}\}$. For any theory T we would then have, by proposition 2, $\exists x \psi(x) \vdash_T \psi(t_1) \vee \cdots \vee \psi(t_n)$ which clearly does not hold.

Assume that the Dedekind-MacNeille cover is Stone. By theorem 1, $\phi \lhd_I U$ implies $\phi \lhd_{DM} U$; hence, by corollary 1, also the inductive would be Stone. $\square$

The *Stone compactification* $\lhd_\omega$ of a cover $\lhd$ is defined by

$$a \lhd_\omega U \text{ if there exists a finite subset } U_0 \text{ of } U \text{ such that } a \lhd U_0.$$

From corollary 1 we see that the covers $\lhd_I$ and $\lhd_{DM}$ have the same Stone compactification.

5   Positivity

A *positivity predicate* Pos is a predicate defined on the base of a formal topology and satisfying

$$\frac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists b \ \epsilon \ U)\mathsf{Pos}(b)}$$

$$\frac{\mathsf{Pos}(a) \ \Rightarrow \ a \lhd U}{a \lhd U}$$

The original definition in [16] of formal topology included a positivity predicate. This notion corresponds to the notion of *open* locale in the theory of locales [8]. A formal space does not necessarily have a positivity predicate. However, the following notion of positivity can always be defined:

$$\mathsf{POS}(a) \ = \ (\forall U)(a \lhd U \ \Rightarrow \ U \text{ inhabited})$$

Since the definition of POS involves quantification over subsets, $\mathsf{POS}(a)$ is a type but not a set. We first prove that if the topology has a positivity predicate, then it coincides with POS; this was pointed out to us by Peter Aczel.

Proposition 5 *If a topology has a positivity predicate* Pos, *then for all a in the base,* $\mathsf{Pos}(a)$ *if and only if* $\mathsf{POS}(a)$.

Proof. The implication from left to right is trivial. By the second rule of Pos we have

$$a \ \lhd \ \{x : \mathsf{Pos}(x) \text{ and } x \ \epsilon \ \{a\}\}$$

If we assume $\mathsf{POS}(a)$, then $\{x : \mathsf{Pos}(x) \text{ and } x \ \epsilon \ \{a\}\}$ is inhabited; hence $\mathsf{Pos}(a)$ holds. $\square$

For the inductive topology, $\mathsf{POS}(\phi)$ holds precisely when ϕ is consistent:

Proposition 6 *For the inductive topology on the formulae of a theory T,* $\mathsf{POS}(\phi)$ *holds if and only if* $\neg(\phi \vdash_T \bot)$.

Proof. Assume $\mathsf{POS}(\phi)$. Then $\phi \lhd_I \emptyset$ implies that $\emptyset$ is inhabited. Since $(\phi \vdash_T \bot)$ gives that $\phi \lhd_I \emptyset$, we obtain $\neg(\phi \vdash_T \bot)$.

For the implication in the other direction, we must prove that $\neg(\phi \vdash_T \bot)$ and $\phi \lhd_I U$ implies U inhabited; we do that by proving the stronger proposition

$$\phi \lhd_I U \ \Rightarrow \ (\phi \vdash_T \bot \ \lor \ U \text{ inhabited})$$

by a straightforward induction on the derivation of $\phi \lhd_I U$. Note that this proposition implies that if U is empty then $\phi \lhd_I U$ implies $\phi \vdash_T \bot$; hence, $\phi \lhd_I \emptyset \ \Leftrightarrow \ (\phi \vdash_T \bot)$ since the implication in the other direction is an immediate consequence of the definition of inductive cover; we will use this equivalence in the proof of the corollary below. $\quad \square$

By propositions 5 and 6, if the inductive topology has a positivity predicate then $\mathsf{Pos}(\phi) \Leftrightarrow \neg(\phi \vdash_T \bot)$. By the proof of proposition 6, we have $\phi \lhd_I \emptyset \ \Leftrightarrow \ (\phi \vdash_T \bot)$. Hence, by the second rule for Pos, $(\neg(\phi \vdash_T \bot) \Rightarrow (\phi \vdash_T \bot)) \Rightarrow (\phi \vdash_T \bot)$ which gives $\neg\neg(\phi \vdash_T \bot) \Rightarrow (\phi \vdash_T \bot)$. Let R be an arbitrary decidable predicate on the natural numbers and let T_R be the theory which has no non-logical symbols and the axioms Ax_n defined by

$$Ax_n = \begin{cases} \bot & \text{if } R(n) \\ \top & \text{otherwise} \end{cases}$$

Clearly, T_R is inconsistent if and only if $\exists x R(x)$; so, $\neg\neg(\vdash_{T_R} \bot) \Rightarrow (\vdash_{T_R} \bot)$ implies $\neg\neg(\exists x R(x)) \Rightarrow (\exists x R(x))$. Hence, we obtain

Corollary 3 *If every inductive topology admits a positivity predicate* Pos, *then Markov's principle holds.*

We say that a space is *positive* if, for every subset U, 1 is covered by U implies that U is inhabited. Note that a space is positive if and only if $\mathsf{POS}(1)$ holds. Every space with a positivity predicate Pos such that $\mathsf{Pos}(1)$ holds is positive; this follows immediately from the axioms for Pos and the fact that $a \lhd 1$ for any $a \in S$.

Let $\{1 : A\}$ be the subset $\{x : x = 1 \text{ and } A\}$.

Lemma 2 *A space is positive if and only if, for all propositions A, $1 \lhd \{1 : A\}$ implies A.*

Proof. Let the space with the covering relation $\lhd$ be positive. The definition of positivity gives $1 \lhd \{x : A\} \ \Rightarrow \ A$. Since $\{1 : A\} \lhd \{x : A\}$, transitivity of the covering relation implies $1 \lhd \{1 : A\} \ \Rightarrow \ A$.

Let the space be such that, for all propositions A, $1 \lhd \{1 : A\}$ implies A. Because $\{x : B(x)\} \lhd \{1 : (\exists x)B(x)\}$ we get positivity from $1 \lhd \{1 : (\exists x)B(x)\} \ \Rightarrow \ (\exists x)B(x)$, again using transitivity of the covering relation. $\quad \square$

Proposition 7 *The inductive topology for a consistent theory is positive.*

Proof. By lemma 2, it is enough to prove

$$1 \ \lhd_I \ \{1 : A\} \ \Rightarrow \ A \tag{1}$$

for all A. It is straightforward to see that V defined by $\psi \ \epsilon \ V$ if and only if $A \lor (\psi \vdash \perp)$ is inductive. Lemma 1 then gives

$$\psi \lhd_I U \ \land \ U \subseteq V \ \Rightarrow \ \psi \ \epsilon \ V. \tag{2}$$

By putting ψ equal to 1 and U equal to $\{1 : A\}$ in (2) we obtain (1), provided the theory is consistent. $\square$

The following theorem shows that there is no hope to prove constructively that any Dedekind-MacNeille topology is positive.

Theorem 2 *If the Dedekind-MacNeille topology is positive for a theory, then Markov's principle implies the full law of the excluded middle.*

Proof. By lemma 2 and the definition of $\lhd_{DM}$, positivity of the Dedekind-MacNeille topology is expressed by that

$$(\forall \psi)(((\forall \sigma \ \epsilon \ \{1 : A\})\sigma \vdash_T \psi) \ \Rightarrow \ 1 \vdash_T \psi) \ \Rightarrow \ A \tag{3}$$

holds for all propositions A. (3) is equivalent to

$$(\forall \psi)((A \ \Rightarrow \vdash_T \psi) \ \Rightarrow \vdash_T \psi) \ \Rightarrow \ A. \tag{4}$$

$\vdash_T \phi$ is equivalent to $(\exists n) Proof(\phi, n)$ where $Proof$ is a decidable predicate over the natural number, expressing that n codes a proof of ϕ in T. Assume, for each formula ϕ,

$$\neg\neg(\exists n) Proof(\phi, n) \ \Rightarrow \ (\exists n) Proof(\phi, n) \tag{5}$$

Since, for all propositions B and C, $\neg\neg C \Rightarrow C$ implies $(\neg\neg B \Rightarrow ((B \Rightarrow C) \Rightarrow C)$, we obtain from (5)

$$\neg\neg A \ \Rightarrow \ (\forall \psi)((A \ \Rightarrow \vdash_T \psi) \ \Rightarrow \vdash_T \psi). \tag{6}$$

(4) and (6) give that $\neg\neg A \ \Rightarrow \ A$. Hence, since A is an arbitrary proposition, Markov's principle implies the full law of the excluded middle. $\square$

Note that the proof shows that if a theory is decidable, then positivity of the Dedekind-MacNeille topology implies the full law of the excluded middle.

6 Points

We say that $\alpha \subseteq S$ is a *point* of a topology $\langle S, \cdot, 1, \lhd \rangle$ if it satisfies the following rules:

$$\alpha(1)$$

$$\frac{\alpha(a) \qquad \alpha(b)}{\alpha(a \cdot b)}$$

$$\frac{\alpha(a) \qquad a \lhd U}{(\exists b \; \epsilon \; U)(\alpha(b))}$$

A subset H of Frm_T is a *Henkin set* if it satisfies

$$\bot \;\; \notin H$$

$$\phi \vee \psi \; \epsilon \; H \;\; \Rightarrow \;\; \phi \; \epsilon \; H \text{ or } \psi \; \epsilon \; H$$

$$\exists x \psi(x) \; \epsilon \; H \;\; \Rightarrow \;\; \text{there exists a term } t \text{ such that } \psi(t) \; \epsilon \; H$$

$$H \vdash_T \phi \;\; \Rightarrow \;\; \phi \; \epsilon \; H$$

Note that, for a theory T with classical logic, a Henkin set is an ultrafilter with the existence property.

Proposition 8 *A subset of* Frm_T *is a point in the inductive topology if and only if it is a Henkin set.*

Proof. Both implications are straightforward consequences of the definitions. Notice that the last clause of the definition of a point can be used to show that a point cannot contain $\bot$. $\square$

A Dedekind-MacNeille topology may not have any points at all:

Theorem 3 *In general, the Dedekind-MacNeille topology for a first-order theory has no points.*

Proof. Let T be a theory with classical logic, an infinite number of atomic formulae and no non-logical axioms. We will show that the Dedekind-MacNeille topology for T has no points.

Assume that α is a point. Define the subset V by

$$V \;=\; \{\phi : \alpha(\neg\phi)\}$$

and let ψ be an arbitrary formula and At be an atomic formula which does not occur in ψ. Then $At \vdash_T \psi$ or $\neg At \vdash_T \psi$ implies $\vdash_T \psi$; hence $(\forall \sigma \in V)\sigma \vdash_T \psi \; \Rightarrow \; \vdash_T \psi$ which gives that $1 \lhd_{DM} V$. Since α is a point we then get that there must exist a formula $\phi \in V$ such that $\alpha(\phi)$, which is impossible since then $\alpha(\phi \& \neg\phi)$. $\square$

This theorem was pointed out to one of us by John Bell in a classical meta-theory. It is then well-known that a complete atomless boolean algebra has no points, and it is easy to check that the Dedekind-MacNeille completion of an atomless boolean algebra is atomless.

It is a direct consequence of the definitions that if a topology has a point, then the topology is positive. Hence, theorem 2 gives

Corollary 4 *If the Dedekind-MacNeille topology for a theory has a point, then Markov's principle implies the full law of the excluded middle.*

We say that a formal topology $\langle S, \cdot, 1, \lhd \rangle$ is *pointwise definable* if there exists a set X and a relation $\Vdash$ from X to S such that

$$ a \lhd U \quad \Leftrightarrow \quad (\forall x \in X)(x \Vdash a \Rightarrow x \Vdash U) $$

where $x \Vdash U$ means that $(\exists b \, \epsilon \, U)(x \Vdash b)$.

Corollary 5 *If every inductive topology is pointwise definable, then Markov's principle holds.*

Proof. If the topology is pointwise definable, then a positivity predicate can be defined by $\mathsf{Pos}(\phi) = (\exists x \in X)(x \Vdash \phi)$. Hence, the result follows from corollary 3. $\square$

7   Set based topologies

A disadvantage of the definition of formal topology that we have used is that, in general, it is not clear how to form the Cartesian product of two spaces. A natural way to change the definition so that the Cartesian product of two spaces always can be formed is to require that it is set based. A formal topology is *set based* if there exists a set I and a family of subsets U_i over I such that, for any subset U,

$$ a \lhd U \quad \Leftrightarrow \quad (\exists i \in I)(a \lhd U_i \text{ and } U_i \subseteq U) $$

Peter Aczel has shown that a topology is set based if and only if its cover is inductively defined; hence Henkin topologies are set based. For Dedekind-MacNeille topologies, we conjecture that they are, in general, not set based.

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