Steps toward a dynamic constructivism

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Prologue. The philosophy of mathematics and logic has been an object of my reflections ever since I first undertook serious study of these subjects. Over many years I have slowly developed a global way of thinking about the foundations, not only of mathematics, but of abstract concepts in general (see (Sambin, 1987), (Sambin, 1991), (Sambin, 1998) and the introduction of (Sambin and Valentini, 1998)). Recently, the fertility of such a global approach, at which I have arrived independently of the usual schools, has been confirmed by two specific and novel technical developments in logic and in mathematics. This fact has finally convinced me to address a wider audience, and, moreover, in English.[1]

Since I cannot convey everything in a few pages, this paper is necessarily an overview. I hope in the future to find the energy to write up a complete version, which will include all the details, arguments, connections and references omitted here. Also, if my exposition is somewhat ideological and my language nontechnical, it is because my goal is to take the initial steps towards a new conception.

1 A general claim.

Right from the start, the philosophy of mathematics of Brouwer – the founder of intuitionism – was rejected by most because of its links with mysticism. This seems correct. Certainly it is not a sound practice to base scientific knowledge on any form of religious belief.

Now the real problem is that, if by religion we mean any form of unprovable belief in something beyond our perceptions, then apparently also all other existing philosophies of mathematics are based no less on religion. Logicism, which claims that mathematics can be reduced to logic, gives no explanation of why logic should stand by itself, given *a priori*. Formalism purposely deprives mathematics of all intentionality and thus accepts it, without any discussion, as given in its totality, since its aim is to justify all of it by reducing it to its 'objective' external appearance, that is, to signs. In general, all of the present day approaches seem still to be permeated by some form of platonism: something, be it called 'truth', 'existence', 'objectivity', or the like, is assumed to exist independently of our perceptions and of our minds. Assumptions of this sort may be understandable from a psychological point of view, since they relieve the anxieties of emptiness and uncertainty as well as furnish an aim for our trials and errors, but it must be emphasized that they have no scientific basis.

Of course, we all remain free to believe whatever we wish. This does not mean, however, that all is relative: the important things in the end are the consequences one draws from such beliefs. In my view the problem with religion is that it has often been accepted quite uncritically, in the form of 'blind faith', and this can lead to the making of very bad, even catastrophic mistakes. Among these, a very bad example – but one providing an excellent illustration of my claim – is the massacres by Cortes and Pizzarro of several million people in Central and South America on the supposition that they did not possess souls. Similarly, the supposition, dogma even, that an absolute truth must exist and that classical axiomatic set theories such as ZF must form a part of this truth, has resulted in a number of cultural errors – intellectual atrocities even – of which the Banach-Tarski paradox is just another very bad – or excellent – example. More than just being philosophically shaky, ZF has severely impaired our intuition by splitting it from what ZF would have us believe. (To see this, just try to teach certain statements of ZF to any nonmathematician equipped with common sense; cf. also (Boolos, 1998)).

This is a pathological state which requires healing. It has been present for at least a century, but little has been done about it. I believe the reason for this is the lack of a good general philosophy, capable of providing sound guiding principles. But now such a thing seems possible. We must go beyond our founding fathers Frege, Peano, Russell, Hilbert, Brouwer, Gödel. They did their task; now it is our turn.

What do we know today that was not known sixty years ago, in 1939? We must recognize that the last sixty years 1939-1999 have been much less productive of the philosophy of mathematics and logic than have the previous sixty, 1879-1939. This being the case, where should we now seek new ideas? On what can we base our hope for a freer, healthier attitude? Many novel ideas have come from *biology*, chiefly since the discovery of DNA in the 50s. Of the greatest interest here is the fact that, after fifty years, it is now generally believed by biologists and by neuroscientists that nothing beyond biology and evolution is theoretically necessary to explain human beings, that is, their bodies *and* their minds (it must be stressed that this is to be meant in a theoretical sense, independently of present inability to give a detailed account of phenomena like consciousness and creativity). No metaphysics, no ghosts, no homunculi. This is a big intellectual change, which has led to a new global cultural attitude.[2] My general claim is simply that the same holds for logic and mathematics. That is, a naturalistic, evolutionary attitude is fully sufficient, and, as I hope to show, actually convenient for explaining not only the human body and mind, but also all human intellectual products, including logic and mathematics, which are just the most exact of these. That is, mathematics is a product of our minds and so to explain it we require no more than what is needed by biologists to explain the mind.

I hasten to point out that this does not mean replacing old religions and previous principles by new ones, the principle of evolution, say. In fact, we shall deduce nothing from the assumption of evolutionary theory. Evolution is only a useful guide for finding good explanations and new results. When these are convincing by themselves, they can stand alone, under no assumptions, and evolution can be discarded. We shall see two examples of this in sections 5 and 6.

The change of attitude I propose towards mathematics is similar to the recent change of attitude towards mind among biologists. It is a matter of fact that each of us possesses both a brain, and a mind, whatever *that* is; it has not been proved, and I believe it cannot be proved, that we have a spirit, distinct from mind in the sense that it can live independently of the body. Biology helps us to accept this as a matter of fact. Similarly, it is a matter of fact that human beings have produced some kind of mathematics; it is by no means proved, and I claim it is in fact unprovable, that it exists in itself, independently of our minds, in some metaphysical mathematical universe.

Of course, while accepting this, the debate remains open and many other points should be made. Let me here recall just one experimental fact about our brains, since it may help to remove a common source of resistance to a biological explanation of mind. It has been argued that accepting a biological, i.e., materialistic, explanation of mind means asserting that the human brain is a machine, that is, in the end, a Turing machine, and hence that there is no room for human creativity. Basing their claims on experimental evidence, certain neuroscientists[3] strongly deny that the brain works like a Turing machine. I see no solid grounds for doubting this. However, even assuming that the brain is a Turing machine, its mere quantitative complexity is so high that it totally eludes our own control and certainly contains room for anything, including that which we term human creativity and the elusiveness of mathematics. For consider: according to recent estimates, the human cerebral cortex contains 10^{11} neurons, each possessing perhaps 10^4 synaptical buttons. If 10 pages are needed to describe one neuron with all its synapses, then the whole cortex would require 10^{12} pages, that is $10,000$ libraries, each with $100,000$ books of $1,000$ pages each. And this is still probably a tiny number when compared with that of potential brain states...

2 The familiar questions.

I would like to make clear that the evolutionary-naturalistic frame of mind should not be imported passively from biology, as a *deus ex machina* for solving problems in the foundations of logic and mathematics. In actuality, this frame of mind was built from below, step-by-step as it were, over a very long reflective journey, starting from questions familiar to anybody interested in foundations. It was only after reading several books on evolution that I became aware, *a posteriori*, of the perfect compatibility with my own conclusions. Some of the familiar questions are:

> What is the nature of mathematical entities? Where do they come from? Are they discovered or invented?
>
> What is mathematical and logical truth? Is it absolute? If so, how can we attain it? Why is a proof convincing? What is objectivity? And if objectivity is really there and accessible to everybody, why has all discussion not simply ceased?
>
> What is the meaning of existence in mathematics? Which entities exist? Where, when, and how do they exist? Is it possible to delineate the difference in the modes of existence of Nero Wolfe and Robin Hood? Of black holes and ether? Of 17 and an inaccessible cardinal?

Also intriguing are certain specific questions:

> Which set theory should we accept? Why should we believe in ZF? and in the axiom of choice? and in paradoxes such as Banach-Tarski?
>
> Which logic is 'correct', classical or intuitionistic? Who was right, Brouwer or Hilbert?
>
> etc.

This list has been made precisely to show that they are just familiar questions which anyone can put to himself. Perhaps what characterizes my approach is an aversion to answers 'by authority', an incorrigible aversion to concede anything as true unless I see it with my own eyes. Yes, I admit, I am exactly like a child of 4 or 5 who goes on asking why? why? why? and doesn't stop until he is satisfied. And even when his father, or any father, asserts that one should just believe, and no further questions are tolerated, that child may fall silent, but still doesn't stop thinking. No silence or act of faith has ever convinced me; I have always been seeking something more reliable, and actually so reliable that it can withstand any possible question. Taken in this way, any of the above questions has the potential to lead us to new domains of thought. It takes no more than a couple of defenses of common sense against widespread opinion, apparently, for one to be considered a revolutionary!

It is only relatively recently that I have, after many years, been able to reach a satisfactory understanding, that is, answers to these questions which in my opinion satisfy even the demands of my incorrigible aversion. It is important to emphasize that I mean *all* such and similar questions, because I believe that this is required of any good philosophy of mathematics (otherwise one can easily cheat, for example, by assuming the existence of God, or the like). But, of course, I do not expect to be able to furnish every detail (the mere mention of which seems to me a bit megalomaniac). My answers are of a philosophical, nontechnical nature, and yet genuine answers, since the feelings of anxiety sometimes engendered by the questions have been appreciably lowered! Gradually, I have acquired a whole new way of thinking, one which works seven days a week, and I have also been freed of all traces of guilt for not believing in something I simply don't see, such as classical logic and set theory. I thank life for having given me all this.

The answers which I have found to these questions now seem to me so satisfying, so simple and natural, that I can hardly see the distance I have moved from more widespread opinions. Among the many things I should say, it is now difficult to make the right choices, and in the right order.

It seems to me that the principal change has been the move to seeing everything from a dynamic, rather than a static, viewpoint. This applies in particular to foundations – so maybe the word ‘foundation’ itself should be changed, since it refers to something reliable insofar as it is static. I mean what I say: the price that must be paid is that traditional concepts such as truth, existence, objectivity, usually taken as pillars, must also be regarded as the results of a process. Nothing in mathematics exists in itself, or by divine *fiat*.

This may look at first as too high a price to pay, as a lunatic attack on the sources of any certainties we may happen to possess, an attack explicable only as the product of a self-destructive attitude. But, thank heavens!, just the opposite is the case. It is a false impression, similar to that produced when one contemplates giving up smoking, or freeing oneself of any other form of addiction. At first, the idea of quitting looks like a suicidal jump into emptiness, but after a while you see that you don’t need the addictive substance, and that actually the world (in this case, the world of mathematics) is much nicer, more complex, more interesting and – what matters most – more alive than it appeared to be before. So, in the end, the gain far outweighs the pain of withdrawal! One should not, therefore, expect from me relief from that pain – much less in the form of another, stronger addiction. The only help I can give is to offer myself as an example: it should show that it is possible to free the self of addictions or pathologies, and yet remain a mathematician or logician.

3 The dynamic process.

Any abstract concept, as expressed by a word, is the result of a dynamic process of abstraction. Actually, it is not the result, but is, rather, the dynamic process itself, going on in each of our minds and hence in the community. It would take quite long to justify this claim properly, and I hope I have done it already in my previous paper (Sambin, 1991). So let me here give just some examples with comments.

Even understanding such common words as ‘apple’ requires a lot of mental work. Each of us somehow reconstructs, in his or her mind, the process by which the concept of apple has been formed. So we must abstract not only from time, position, ... but also from defects, colour, taste and so on. The concept of apple, however precise, still contains some element of idealization, that is, some property which does not correspond to reality, but only serves the purpose of convenience. An example might help. A drawing like Fig. 1 shows the axis of symmetry of our concept of apple.

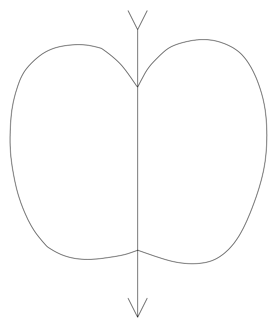

Fig.1

In reality, there is no such symmetry, and all real apples are slightly asymmetric. To test this, put an apple on a table, and place a piece of cardboard on top of it. You will see that the plane of the cardboard is not parallel to that of the table’s surface, which shows that the apple is not perfectly symmetric. So why is it symmetric in our mind? Is it we who are ‘wrong’? Certainly not; it is just a matter of saving mental energy. Keeping in mind a concept of apple which includes being symmetric-but-not-perfectly-so would be costly to the brain. So we add symmetry even if in reality it is not perfect, revealing it to be just an idealization we impose on the concept to make it simpler.

This shows that while of course the concept of apple has a lot to do with apples, in nature only apples but no concepts of apple are available. So there is no identification, and not even an a priori fixed link, between our concept and what we call the ‘object in itself’. The concept is constructed by us, socially and individually, and it changes continually in a dynamic way, in response to changes in our needs or our knowledge. This last statement can be checked experimentally. I bet that I would be able to change, for example, the concept of pipe in the reader’s mind, if I were given just a few minutes of attention. My confidence in being able to achieve this, of course, derives from the fact that I know enough about pipes to be able to increase the reader’s knowledge (by talking about shapes, names, materials, etc. or by explaining why each smoker has so many pipes, etc.), even if he is a pipe smoker (by talking about my pipes and how I took care of them, etc., when I used to smoke). Even such a small experience would suffice to attach new data to the word ‘pipe’ and thus change for ever the concept of pipe in the reader’s mind (and for me to win the bet). This shows just how dynamic concepts are.

The third example shows that an excess of idealization may be not convenient. It comes from my recent experience. Fig. 2 gives the shape of the kitchen in the old flat I am renovating. It was

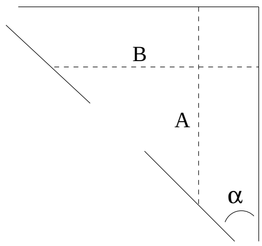

Fig. 2

impossible to measure the length of the walls, since the angles were still under reconstruction. So I measured A, and it came out at 219 cms. Measuring was not an easy matter, because neither plaster nor floor was yet in place, and one had to make complicated estimates. Then I assumed B was the same length as A. But if that were the case, then fridge and dishwasher could not both be put in B, which was the only way I could have both. I was so disappointed that I measured B directly, just to check. This turned out to be 228 cms, which was surely enough! I measured it three times to make sure. What was wrong with my first idealization? Simply that the angle α was not, as I had implicitly assumed, 45 degrees. I think the whole story is typical. I needed to measure my kitchen, that is to make an idealization. The first was too simple, and it did not meet my needs. Further data allowed me to make it more precise, until it was adequate. Note that there is no such thing as the perfect idealization, the platonic idea of my kitchen! I stopped when I had the idealization which just sufficed for my purposes.

The conclusion I suggest is that it is always thus: all concepts, including the most abstract ones like those of logic and of mathematics, are the result of a dynamic process of abstraction. Indeed, why should they be essentially different from the familiar, everyday concepts in the first place?

4 A dynamic constructivism.

I know very well that the attitude I am describing requires a radical change of perspective, a turning upside-down of the customary sources of certainty. It is usually thought that the certainty and reliability of mathematics is guaranteed by its possession of some special ingredients – objectivity, absolute truth, etc. I propose to reverse this, and say that it is *we* who call mathematics whatever is certain and reliable to a fully satisfactory degree (which we usually express by saying that it is objectively true or the like).

Such a change of perspective is a liberating experience which is usually undergone on first learning about the evolutionary explanation of the natural world, or, more generally, in passing from a static to a dynamic view of the world. I can only give an example here. It is usually taken for granted that we all have a strong will to live, and that this is a primitive instinct (whatever that means) which cannot be explained. The change of perspective in this case consists just in realizing that no special assumption is necessary, and that the explanation is quite simple: all those who had a little weaker will to live by natural selection are all already dead! As R. Dawkins reminds us, there are many more ways to be dead than to be alive. Those who found the way to be still alive today, now possess what we call a strong will to live. But there is no need to assume it at the beginning, as an *a priori*.

The situation is quite similar also in the mathematical world. Consider for instance the natural numbers. To be able to rely on certainty and objectivity of facts pertaining them, there is no need to think of them as given *a priori* (however indirectly, through sets, for example). It is enough to recall that they are the result of an extremely long evolution, right up to Dedekind's and Peano's insights on induction. All those conceptions which were even just a tiny bit less precise or less efficient (such as Roman notation, lack of zero, improper induction principles, and so on) have already been selected out by history and are now defunct.

I have spoken of the natural numbers, but it should be clear that I mean to claim that the same applies to all of mathematics. All of it is produced through a dynamic process, and the feeling of certainty, stability, objectivity, etc. however strong it may be, is only something that we attain *a posteriori*. It is neither a matter of pure convention nor of existence *a priori*, but of survival of the fittest: what remains stable is only what works best (also in respect of our own demands on conceptualization), all the rest is sooner or later wiped out.

At this point the next question is naturally: if all is constructed, what meaning should one attach to words already mentioned such as *objectivity*, *truth*, *existence* in mathematics? Let us begin with *objectivity*.

What does objectivity become in this new perspective? If it includes being absolutely sure of what we say, how can we attain it? From our new standpoint, there is no such absolute notion of objectivity and so it is the concept of objectivity itself which must be changed and newly defined. The word 'objectivity' clearly comes from 'object', and the reason also seems to me clear: objectivity is reached when we get the same feeling of certitude and the same independence of the subject that we get with what we call concrete, or external objects. So let us look closely at the source of our certitude when we speak about objects.

We must admit that even what we call a concrete object does not exist in itself, independently of our mind. What exists in itself is only a certain distribution of matter in a portion of space at a certain moment, which causes certain sensory inputs to enter our brains. When we speak of an object as existing in itself, what we are actually referring to is a mental construction, obtained by a process of abstraction from what it is convenient to count as contingent, such as place, time, illumination, our mood, etc. We attribute existence in itself simply for reasons of convenience: it is much quicker, more efficient, and reliable, exactly as in the case of all those mental functions which have become almost automatic, like pieces of 'hardware'. While the strong agreement we can reach concerning objects is ultimately based on the fact that we have the same sensory organs, still it must include much mutual adaptation. What we consider to be 'objectively true', that is part of the external world and hence necessary, is often just one possible way of organizing our perceptions, and hence our knowledge of the world. Witness, for example, the fact that different people may see different colours (typical is the pair blue-green) and that different languages use different concepts for the same reality (Italian has three words – vetro, bicchiere, occhiali – for glass and its plural, while Chinese has no word for wrist).

Similarly, objectivity in mathematics is reached when some entity is considered an object, that is, acted upon by automatic mental functions similar to those used for the (internal) image of an (external) object. As with concrete objects, to reach this stage the process of abstraction must include the interaction with other individuals, and with the outcomes of *their* mental processes. This is the principal, and deepest difference from Brouwer. It is through a democratic, though occasionally turbulent, dynamic process of achieving consensus that we arrive at what *a posteriori* we call objective or objectively true.

This is my understanding of the word 'objectivity'. Since no objects are assumed to exist by themselves, the meaning of objectivity becomes very close to that of inter-subjectivity and also of complete certainty. Of course, one may disagree, but note that at least I am giving an explanation. In the absence of dynamics, it is very difficult to talk about objectivity without assuming it to be universally clear, that is without falling exactly into what one tries to avoid, the – very subjective – claim that objective is what is considered as true in a certain period and by a certain community, usually ours! Here is how unjustified assumptions, such as set theory or logic taken as *a priori*, creep in.

The explanation of what is to be meant by truth and by existence follows the same general idea, namely that none of them is absolute, that our feeling that they are independent of us is due to reasons of convenience and that a new, positive meaning must be found.

The traditional definition of truth as *adaequatio rei et intellectus*, equality of reality and intellect, is essentially correct, once it is understood in a dynamic way, as something to be achieved. So it is not a case of two terms being equal, but rather of two processes, the results of which are to be made to converge (this is confirmed by the fact that the prefix *ad* in *adaequatio* gives the idea of movement) (see Fig. 3). Reality is first taken in (which is the original meaning of 'assuming')

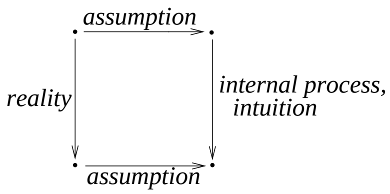

Fig. 3

and elaborated by pure thought, or intuition, or internal mental processes; if the outcome agrees with a second test on reality, then truth is confirmed. Otherwise, something must be changed.

This point of view allows one to see also how agreement on truth is built up. Since reality also includes other individuals – possibly figures of authority such as teachers, authors, etc – when we are presented with something which is put forward as true, we have an alternative: either it agrees with our intuition and we also recognize it as true, thus confirming it as a piece of truth and contributing to its becoming universal, or we reject it and look for something different. This is how the dynamic process, and hence truth, is kept alive.

Such dynamics involving many different subjects is also needed, in my view, to explain how new concepts and new theories are created, and how the meaning of words can change. Moreover, the dynamics involved is immediately visible in some cases, both individually, as in learning and in doing research, and socially, in historical development. In these cases a static view is clearly not sufficient. But unfortunately dynamics is not an ingredient like salt, which can be added at the last minute, after tasting; it is rather like yeast, which must be present right from the start.

And finally, what is existence? It is we who give existence to mathematical entities, just as it is we who create abstract concepts and who attain objectivity. This is more easily grasped in certain cases such as large cardinals, ultrafilters, or infinitesimals: their form of existence is not so different from that of Santa Claus or of Nero Wolfe, in that they have been created by human beings and exist only in the sense that certain human beings (mathematicians, children, readers of novels) keep them alive, both in their minds and in their discussions.

But, if one looks carefully, one sees that all of mathematics is like that. Consider for instance the number $10^{10^{10}}$; this probably exceeds the number of electrons in the known physical universe, and thus I feel safe in saying that it does not exist in nature, in the sense that it cannot be used to count something existing concretely (of course, it is a relatively small number when our abstract constructions, such as permutations, possible states, subsets, etc. are to be counted). So one can see that the natural numbers too are kept alive by our minds; the strong feeling of external existence is due to their very long history, their universal utility and their conceptual simplicity (they are just the simplest never-ending procedure). In other words, the natural numbers are now completely reliable, and almost everybody believes that they are perfectly acceptable as they stand. But if this were considered as a sufficiently good reason to declare them to exist 'in themselves', *a priori*, should we believe that also, say, large inaccessible cardinals exist a priori? and if not, where should we put the border between the natural numbers and large cardinals? Any choice would be artificial and easily criticized.

However psychologically difficult at first, the correct explanation is that existence is never an absolute attribute, independent of our minds; it follows also that different forms of existence are possible. Thus in a certain sense the natural numbers 'exist more strongly' than do large cardinals, in something like the same way that Pinocchio, Lewis Carroll's Alice and Donald Duck 'exist more strongly' than other less well-known characters. One could – paradoxically – even say that, from this point of view, Nero Wolfe 'exists more strongly' than Robin Hood does, since we know him better![4]

It should by now be clear that the familiar dilemma 'discovered or invented?' has in my opinion a definite answer: *invented*. But, as with abstract concepts and with objectivity, it is often much more efficient to look for something in the belief that it has to be discovered rather than invented. (This is quite similar to the well known fact that, given a problem admitting a yes-no answer, it is much easier to find a full proof if the pure answer yes or no is known.)

The feeling that one has truly discovered something can on occasion be quite strong and fully justified. This is particularly the case when that thing is just a logical consequence of previous knowledge. Here 'discovery' is just the bringing to awareness of a detail which was implicit in a whole invented picture. That is, it is again only a local psychological phenomenon, which does not change the global, simple substance: the whole mathematical universe exists only in the minds of human beings. Does the self-deception of believing that it exists in itself make it nobler, or more interesting, or more reliable?

One can begin to see that reflecting on familiar questions has actually produced more than a collection of isolated answers. The answers can now be seen as arising from a unitary and global philosophical attitude towards logic and mathematics. This attitude I call *dynamic constructivism*. *Constructivism*, because at the root of everything I see, as did Brouwer, the mental constructions of an individual; and *dynamic* to emphasize that such constructions involve the interplay between intuition and reality (which includes other individuals) and not just pure introspection, as in Brouwer's intuitionism.

I hope that what I have said so far is sufficient at least to suggest that what I call dynamic constructivism has some kind of internal consistency. Assuming this, another way to clarify what I mean is to characterize it in terms of the changes that should be made to be able to reach it from any one of the common views in the philosophy of mathematics. I do this very briefly, in terms of rough but, I hope, suggestive images.

Brouwer's main claim, namely that no mathematical object or truth is given and that all must be constructed, is, in my view, correct. Actually, I believe that most of Brouwer's insights are correct. In the last analysis, my claim is that to obtain dynamical constructivism it is enough to drop Brouwer's justification for the intersubjectivity of mathematics, which is essentially the direct mystical inspiration of the creative subject, and simply replace it with the existence of other individuals. Contrary to Brouwer, I am happy to recognize the existence of my fellow human beings (I must confess, however, that this is not always achieved without effort, and that there are certain human beings whose existence I am happier to accept than that of some others). Each of us is a creative subject, or could be. But then, instead of hoping the One to give the same inspiration to every subject, it is simpler to explain intersubjectivity as a dialectical process, the communication – debate, agreement, fight... – between many creative subjects. And this may take place also less directly, in the form of adjustment, within the same creative subject, between intuition (content, meaning, what has to be expressed) and its expression in an intersubjective language. These are the reasons why I have added the adjective dynamic; but perhaps one should speak just of present-day constructivism, liberated from useless and troublesome assumptions.

The aim of formalism is apparently to reach the certitude of objectivity by cutting off whatever involves a subject, and so all intentions, intuitions, meanings, etc.; that is why mathematics is reduced to signs and formal systems. This is not only unsatisfactory, since it fails to explain the meaning of mathematics and how it is done, but also illusory: formalism seems to ignore the fact that as soon as a language (or formal system) is considered, a subject reading that language (which involves, at least, recognizing the signs) is automatically on the scene. So the change here would be to put metalanguage back in its proper place, which automatically means restoring the role of subject, with its intentions, intuitions and all such nuisances, unavoidable when human beings are involved. The one safe path to objectivity is to *transcend* the subject; pretending that there is no subject is a false shortcut, which in the end is similar to Brouwer's pretence that there is only one.

In a certain sense, it is easier to obtain dynamic constructivism from platonism, at least in verbal terms, since it is only a matter of subtracting: drop the static view, the fact that concepts, or ideas, are statically given somewhere. Maybe it is easier to grasp this in the converse direction: think of platonism as obtained from dynamic constructivism by postulating that the process of construction of a concept actually has an aim, a télos, so that all our trials and errors are just approximations converging to something which exists already. My suggestion is then to do without the useless, troublesome assumption of the convergence point, and accept that there is *only* the process of trial and error, and that this *is* the concept. In other words, I share the platonists' conviction that we need concepts, but, less optimistically, I expect that we must work hard to get them, and that they are never to be found (*where* would that be?) already prepared by someone (*who* could have prepared them?) for us (*why* would they be so obliging to have done so?).

So far the philosophical setting has been presented mainly in terms of cultural attitude; we will see that it also has a technical continuation into logic and mathematics. Here by 'continuation' I mean not just the drawing of consequences, but also a process of confirmation, since the new cultural attitude has allowed us to find new structures, and these in turn prove that it is possible to develop logic and mathematics in consonance with the general philosophy, thereby confirming it.

A vital difference between dynamic constructivism and other approaches is the stress placed on a *multiplicity* of individuals who must perforce communicate with one another. As a consequence, crucial importance is here attached to the interplay between the self and the others, between what the self has in mind and what he/she is able to communicate, or, more abstractly, between meaning and expression, content and form, metalanguage and language (see Fig. 4). Usually, the

$$\begin{array}{rcl} \textit{I} & & \textit{others} \\ \textit{what I have in mind} & \longrightarrow & \textit{what I communicate} \\ \textit{meaning} & & \textit{expression} \\ \textit{content} & \longleftarrow & \textit{form} \\ \textit{metalanguage} & & \textit{language} \end{array}$$

Fig. 4

discrepancy between meaning and its expression is taken as a misfortune and simply ignored – intuitionism and formalism each escaping this misfortune simply by denying one half of the picture, to wit, language and, respectively, metalanguage. By contrast, in the present framework this discrepancy is accepted as an essential fact of life, thus revealing it as the very source of the dynamics and the underlying engine for the creation of concepts. It has been a deeply moving experience to see how well this philosophical expectation is confirmed by rigorous mathematical development in the case of logic: forcing the convergence of language and metalanguage turns out to be precisely the way to generate all logical connectives, in a fully precise sense which is outlined in section 5.

It is often assumed that adopting a 'weaker' logic and foundation gives a 'smaller' body of truths. This makes sense only if truth is considered to be independent of us and immutable; then of course one can see no real reason for any form of self-limitation, and it must be said that some constructivists have contributed to nurturing this false impression. Qualifying constructivism as dynamic means in particular rejecting the idea that a fixed foundation should be given once and for all (since this would make it static!). So, it is not a matter of 'rescuing' mathematics in terms of such a foundation: there are no magic wands, named ZFC, category theory or even constructive type theory, which are able automatically to transform whatever is expressed in their language into meaningful mathematics. In fact, the complexity of mathematics cannot be reduced to fixed ingredients (logic, formulae, intuition, sets, categories,...). Rather, one by one, each piece of mathematics, at the moment it is developed, should be provided with the simplest and most convincing underlying conceptual structure, by means of which the peculiar nature of that fragment is faithfully explained. In a dynamic approach, a 'weaker' foundation is simply a means of maintaining finer distinctions and achieving better control of the idealization, that is, of those elements of information which have been retained as well as those which have been discarded (see also (Sambin and Valentini, 1998)). A bonus is that sometimes a deep structure can emerge, whose presence was previously hidden by foundational assumptions. A typical example of this is what I call the basic picture, that is a structure underlying topology. This is sketched in section 6 in just enough detail to allow some other general comments.

5 Basic Logic.

Since Brouwer's revolution, which has introduced plurality into the space of logics, any foundation claiming to adequacy is faced with an alternative: *either* a good argument has to be found showing that there can be only one logic, which is 'truer' than all others – and by 'good' here I mean that the argument should not presuppose that same logic in some equivalent form – *or* the existence of a variety of logics must be explicitly acknowledged and explained.

It is difficult to accept a plurality of logics and still believe that logic deals with a static, *a priori*, and hence univocal notion of truth. If instead truth is understood as constructed, and hence also relative to a degree of abstraction, then it is possible to conceive of a logic as a stock of rules allowing one to pass, at that level of abstraction, from a given state of (grasped) knowledge to a new piece of knowledge and making no use of further data. In this more open view, what remains to be done is to find some structure in the space of possible logics, in other words, to explicate just how a logic is created. This we now do.

First comes the metalanguage, then the language, just as first comes the content we want to express and only then the means of its expression (the word 'meaning' in Latin is *significatum*, that is 'what has been made into a sign'). So we assume we have propositions A, B, C, ...; it is not necessary here to know exactly what a proposition is, so long as it is sharply distinguished from a judgement, or assertion, based on it, which I write as $A\ is$, $B\ is$ (the common denominator of assertions like $A\ is\ true$, $A\ is\ available$, $A\ is\ measurable$,...). The proposition A is at the object level, the assertion $A\ is$ at the metalevel. In addition, we need just two metalinguistic links, *and* and *yields*, for building compound assertions.

This setting is sufficient for introducing and explaining all the notation of a calculus of sequents, in the style of (Gentzen, 1935). In fact, a sequent $\Gamma \vdash \Delta$ usually abbreviates $C_1, \ldots, C_m \vdash D_1, \ldots, D_n$, which in turn here is an abbreviation of ($C_1\ is\ and \ldots and\ C_m\ is$) *yields* ($D_1\ is\ and \ldots and\ D_n\ is$). Note that $\Gamma \vdash A$ and $C \vdash A$ abbreviate ($C_1\ is\ and \ldots\ and\ C_m\ is$) $yields\ A\ is$ and $C\ is\ yields\ A\ is$, respectively. A rule like

$$\frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'}$$

is also an abbreviation, of ($\Gamma \vdash \Delta$) *yields* ($\Gamma' \vdash \Delta'$), and similarly in the case of two premises, where the empty space abbreviates *and*.

All the assumptions we need are:

identity $\qquad A \vdash A$

$$\text{composition:}\quad \text{on the left}\quad \frac{\Gamma \vdash A \quad A, \Gamma' \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta} \qquad \text{on the right}\quad \frac{\Gamma \vdash \Delta, A \quad A \vdash \Delta'}{\Gamma \vdash \Delta, \Delta'}$$

for any A, Γ, Γ', Δ, Δ'. Composition is just the formal expression of a very intuitive idea: if there is a product (or conclusion) Δ which results from some ingredients (assumptions) A, Γ', then we can substitute the ingredient A by the ingredients Γ of a previous production of A, and have the same product Δ. The same is assumed on the other side of *yields*, on succedents.

Now I can show, under these assumptions alone, that all connectives are created in accordance with a single, general scheme. First we have what I call a definitional equation, for example

$$\Gamma \vdash A\&B \;\; \textit{if and only if} \;\; \Gamma \vdash A \textit{ and } \Gamma \vdash B$$

The connective we want to create (& in the example) appears on the left side, which I then call *definiendum*, while the *definiens* is on the right. I say that the connective (& in the example) is the reflection of a metalinguistic link (*and* in the example) in a certain configuration. Thus the definitional equation is still a desideratum: it just expresses the wish for a connective producing a new proposition ($A\&B$ in the example) which behaves, when asserted in a given situation (at the right of $\vdash$, in the example), just as the compound assertion ($\Gamma \vdash A$ *and* $\Gamma \vdash B$, in the example) with that link. I say that & reflects *and* at the right of $\vdash$ and that & obeys the principle of reflection, since, as we shall now see, its inference rules are obtained by solving the definitional equation.

The direction from definiens to definiendum gives what I call the formation rule (in the example, it is

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}$$

recalling that *and* is expressed here by the empty space). The other direction, from definiendum to definiens (in the example, $\Gamma \vdash A\&B/\Gamma \vdash A$ and $\Gamma \vdash A\&B/\Gamma \vdash B$), does not yield a good rule, since the connective to be defined appears in the premises. I call it *implicit reflection* since it expresses the required information only implicitly. We can however solve the equation by finding an acceptable rule which is equivalent to implicit reflection. First we trivialize the premiss of implicit reflection to obtain an axiom (in the example, replacing Γ with $A\&B$ we obtain the &-axioms $A\&B \vdash A$ and $A\&B \vdash B$). Then by composition we reach the result that the connective to be defined appears in the conclusion, but on the other side (in the example, if $A \vdash \Delta$ then from the &-axiom by composition also $A\&B \vdash \Delta$). This yields the rule I call *explicit reflection* (in the example, $A \vdash \Delta/A\&B \vdash \Delta$), which is perfectly good. Now we can see that we can also proceed in the converse direction, from explicit reflection to the axiom by trivializing the premiss (in the example, putting $\Delta = A$ one obtains $A\&B \vdash A$) and from the axiom to implicit reflection by composition (in the example, from $A\&B \vdash A$ to $\Gamma \vdash A\&B/\Gamma \vdash A$, and similarly when the succedent is B). So the three rules are all equivalent. Therefore we have found inference rules, those of formation and of explicit reflection, which taken together are equivalent to the definitional equation. So, finally, the corresponding wish is satisfied, and the new connective fully defined.

The principal discovery is that all connectives in the most familiar logics are created according to the principle of reflection and, moreover, the solution of the definitional equation always adheres to the same pattern as above.[5] Looking again at the above example for &, one sees immediately that all arguments continue to apply if we interchange the left-hand and right side of each sequent; in fact, all our assumptions are fully symmetric. So the equation $A \vee B \vdash \Delta$ *if and only if* $A \vdash \Delta$ *and* $B \vdash \Delta$ is also solved; it gives the usual inference rules for disjunction $\vee$ (symmetric to those for &). So, interestingly enough, no metalinguistic link for disjunction is required – fortunately, as it happens, since such a link would not be easy to explain – once we are ready to pay the price of accepting a formation rule for $\vee$ acting on the left.

A less common connective is the solution of a simpler definitional equation, that is $A \otimes B \vdash \Delta$ *if and only if* $A, B \vdash \Delta$; here it is perhaps easier to grasp that $\otimes$ reflects *and* on the left of $\vdash$, since in that position the assertion $A \otimes B$ *is* can equivalently replace A *is and* B *is*. The solution here yields the connective $\otimes$ of Girard's linear logic. Actually this is not exactly the case, since in linear logic the inference rules have contexts on both sides, that is they are obtained by solving the definitional equation $\Gamma, A \otimes B \vdash \Delta$ *if and only if* $\Gamma, A, B \vdash \Delta$, with two free parameters Γ and Δ, one on each side. Definitional equations with only one free parameter (Δ in the example of $\otimes$) give rise to a new logic, strictly below linear logic. I have called it *basic logic*, since it is the least logic satisfying the principle of reflection. This means it is as weak as possible but still possess a clear and simple conceptual structure. The sequent calculus **B** for basic logic is similar to a two-sided sequent calculus for linear logic (without exponentials), except for the absence of all contexts of active formulae. The standard reference for basic logic, containing more motivation and of course all technical details, is (Sambin et al., 1998).

Basic logic furnishes a simple structure in which all familiar logics find a place. In fact, by performing on the sequent calculus **B** all the combinations of actions **L**, of restoring contexts on the left, **R**, of restoring contexts on the right, and **S**, of adding the structural rules of weakening and contraction, one obtains a cube of logics in which the equivalences shown in Fig. 5 hold. (N.B. **BLS** is intuitionistic logic equipped with an extra connective which is symmetric of implication;

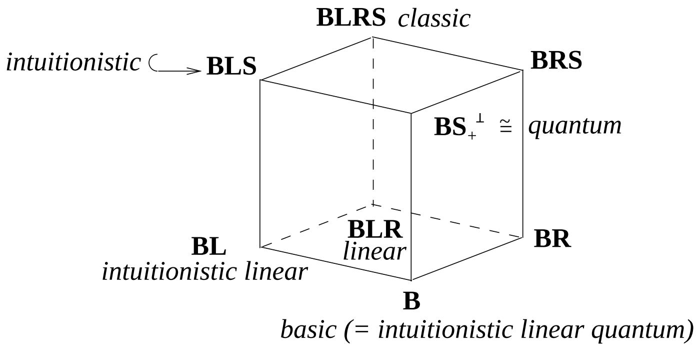

Fig. 5

equipped with a primitive negation, **BS** gives basic orthologic, one of the weak quantum logics).

The fertility of basic logic from a technical standpoint is confirmed by two results, both of which had been sought for a long time. One is an easy theorem of normalization of proofs (cut-elimination) for orthologic, one of the best-known quantum logics, see (Faggian and Sambin, 1998). The other is a single cut-elimination procedure which works simultaneously for all logics in the cube, once they are formulated as subsystems of the same sequent calculus, see (Faggian and Sambin, 1999).

But in my view it is within the conceptual domain that basic logic has consequences of the greatest interest. The rules governing connectives are sharply distinguished from structural rules, including those for handling contexts. Basic logic is the minimal core, with no structural rules, and it brings fully to view the idea that any logic is characterized by a choice of structural rules,[6] that is, rules operative at the metalevel, dealing just with assertions. In this way the important problem of clarifying (and classifying) the diverse notions of truth associated with the variety of logics is reduced simply to the analysis of structural rules.

It seems to me that the theory of meaning for the connectives implicit in the principle of reflection is an improvement on that of Dummett and Prawitz. For here the meaning of each connective is given, not by an introduction rule, but by a definitional equation, which provides a clear description of the role of the connective and specifies exactly how it is created. As a consequence, one can derive not only the elimination rules but also the introduction rules (note that 'introduction' is sometimes what I have called 'formation', sometimes what I have called 'reflection', and this is why the terminology had to be changed). So one sees that all rules of inference are derivable from meaning; moreover, this has been achieved by means of tools which have been introduced explicitly beforehand, and which include just two metalinguistic links.

In every logic of the cube, each connective is defined by a definitional equation, whose solution is obtained by always employing the same method. Thus all logics are born in the same way, which is independent of the choice of structural rules. So we can conclude that the dynamic relationship between metalanguage and language provides a structure which is common to all logics – the search for which is an important philosophical goal – and which is deeper than any specific notion of truth.

These brief remarks should at least suggest how basic logic can be of help in 'putting some order' into the space of logics. One should however be careful not to expect any new absolute to appear. For example, noncommutative basic logic is quite possible, it is strictly below basic logic, and some linguists have already shown interest in it.

6 The basic picture.

A fully constructive (intuitionistic and predicative) theory of subsets is obtained by conceiving of a subset of a given set S as a predicate, or propositional function, over S, that is a proposition $U(a)$ depending on a variable a ranging over S (see (Sambin and Valentini, 1998)). For any element $a \in S$, we write $a \,\epsilon\, U$ to mean that $U(a)$ holds. The next step is to consider two sets, let's call them X and S, and the weakest link between them, namely a binary relation $\Vdash$, that is a propositional function $x \Vdash a$ with two arguments, $x \in X$ and $a \in S$. I call this a *basic pair*.

For any subset $U \subseteq S$ we define the existential anti-image of U along the relation $\Vdash$ to be the subset

$$\mathsf{ext}\, U \equiv \{x \in X : (\exists a)(x \Vdash a \,\&\, a \,\epsilon\, U)\}.$$

Similarly,

$$\mathsf{rest}\, U \equiv \{x \in X : (\forall a)(x \Vdash a \to a \,\epsilon\, U)\}$$

is called the universal anti-image of U along $\Vdash$. Notice the strict logical duality here: $x \,\epsilon\, \mathsf{rest}\, U$ is expressed by a formula which is obtained from that expressing $x \,\epsilon\, \mathsf{ext}\, U$ by formally replacing $\exists$ and & with their duals $\forall$ and $\to$, respectively.

In a fully symmetric way, the existential and the universal image along $\Vdash$ of a subset $Z \subseteq X$ are defined by

$$\Diamond Z \equiv \{a \in S : (\exists x)(x \Vdash a \& x \,\epsilon\, Z)\} \qquad \text{and} \qquad \Box Z \equiv \{a \in S : (\forall x)(x \Vdash a \to x \,\epsilon\, Z)\},$$

respectively. This suffices to express the first discovery. Think of X as a set of points, S as a set of names for formal neighbourhoods, and $x \Vdash a$ as x lies in a, or x belongs to the extension of a. Then the usual definition of the interior of a subset $Z \subseteq X$ is

$$\mathsf{int} Z \equiv \{x \in X : (\exists a)(x \Vdash a \,\&\, (\forall z)(z \Vdash a \to z \,\epsilon\, Z))\};$$

we now see immediately that this is just the composition of ext after $\Box$, that is

$$\mathsf{int} Z = \mathsf{ext}\,(\Box Z).$$

One then quickly sees also that the closure of Z, defined as usual by

$$\mathsf{cl} Z \equiv \{x \in X : (\forall a)(x \Vdash a \to (\exists z)(z \Vdash a \,\&\, z \,\epsilon\, Z))\},$$

is exactly the logical dual of the interior, so that we have also

$$\mathsf{cl} Z = \mathsf{rest}\,(\Diamond Z).$$

By symmetry, one can define another pair of operators, defined on subsets of S, by putting

$$\mathcal{A} U \equiv \Box(\mathsf{ext}\, U) \ \text{ and } \ \mathcal{J} U \equiv \Diamond(\mathsf{rest}\, U)$$

for any $U \subseteq S$. The operator $\mathcal{A}$ also has a topological content, since it is none other than the principal operator of formal topology, yielding the formal open subsets of S. The novelty here is that the operator $\mathcal{J}$, to which we have been led solely by considerations of symmetry and logical duality, also has a topological content, namely that of furnishing the definition of formal closed subsets of S. We say that Z is *concrete open* if $Z = \mathsf{int}Z$ and *concrete closed* if $Z = \mathsf{cl}Z$. Symmetrically, we say that U is *formal open* if $U = \mathcal{A}U$ and *formal closed* if $U = \mathcal{J}U$. Like int, $\mathcal{J}$ is, by symmetry, an interior operator and, like cl, $\mathcal{A}$ is, *ipso facto*, a closure operator. Thus

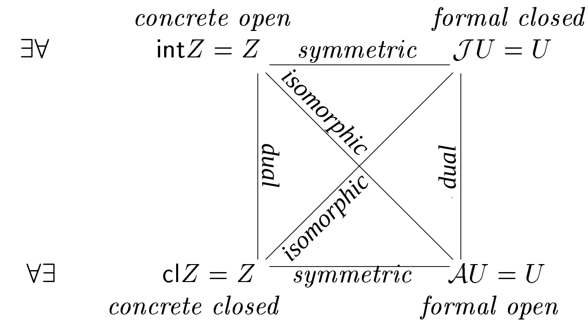

Fig. 6

concrete and formal, open and closed subsets form complete lattices. The whole structure is depicted in Fig. 6. The isomorphism between the lattice of concrete open subsets and that of formal open subsets, and similarly for closed subsets, confirms that the definitions are correct.

To obtain a topology in the usual sense on the set X, it is enough to add the condition that the neighbourhood system at each point be convergent; this is related to distributivity of the lattice of formal open subsets. So the structure illustrated above underlies both usual, or pointwise, and formal, or pointfree, topology. I have called it the *basic picture*. It also includes a similar symmetric structure which underlies the notion of continuity. It shows, among other things, that one can develop the field of nondistributive topology, in which the symmetry between concrete and formal, and that between open and closed, is still unbroken. A first paper on the basic picture is (Sambin and Gebellato, 1999); a full exposition, together with developments, is in preparation.

I hope this brief summary at least makes the reader curious. It should be enough, in any case, to show that topology arises simply in passing from one to two sets (linked by a relation), and that it is deeply connected with logic. One could even maintain that topological notions are just a quick, convenient way of treating combinations of quantifiers.

It is also important to note that a new foundational attitude like the present one, engenders not only new understanding, but also new mathematics. An example is the operator $\mathcal{J}$, which yields the new notion of binary positivity predicate. This is the dual of a formal cover, also in the sense that it is generated by co-induction, rather than by induction.

7 Comments and open problems.

Basic logic and the basic picture have emerged as two hitherto unnoticed natural portions of the landscape of logic and of mathematics. Their principal ingredients are basic and universal concepts: symmetry, logical duality, convergence between language and metalanguage. The logic needed is just intuitionistic predicate logic and any set theory is suffficient, as long as a set can be the domain of quantifications. Morover, they help to explain many facts about logic and topology by inserting them in a general, but quite elementary structure. Actually, these structures – in particular the fact that $\mathsf{int} = \mathsf{ext}\,\Box$ and $\mathsf{cl} = \mathsf{rest}\,\Diamond$ – are so elementary that they could have been discerned at any time since the 20s or 30s. But, incredible at it seems, as far as I know they have remained unnoticed up to now. So it is very natural to ask: why now? On reflection, this fact is less surprising when the rigidity of ideological attitudes is taken into account. Paradoxically, it is precisely the fact that no specific foundation is involved which made their discovery harder rather than easier.

Consider the basic picture. Could an orthodox intuitionist have seen it? Certainly not, because of his idiosyncrasy against formal language: it is virtually impossible to see symmetries and logical dualities without ever writing, for instance, signs like $\forall$ and $\exists$ for quantifiers. A formalist believes wholeheartedly in classical logic, which makes the whole structure collapse and become invisible, because, if one assumes the principle of double negation, the operator cl becomes definable as $-\mathsf{int}-$, and similarly for $\Box$, just as $\forall$ is definable as $\neg\exists\neg$. And finally, if one believes in the powerset, platonists certainly do, the discovery would even never be conceived, since there would be no need to present the base of a topological space as indexed by a second set, like S above. Similar remarks apply to basic logic; just note that technically nothing else is required besides a metalanguage, a fact which has to be accepted since the time of Gödel's theorems, and sequent calculus, which has been known since Gentzen.

So it would seem that ideological blocks have been chiefly responsible for the previous failure to discover these structures. The conclusion is, I believe, that a freer, less dogmatic frame of mind, or foundation, keeps open the possibility of new discoveries. A 'weaker' foundation, with less rigid assumptions concerning the nature of mathematics, can be more consonant with our mental processes, and so may help us to see deep structures which are rendered invisible, and hence destroyed, by a 'stronger' foundation. In this precise sense, it is the 'stronger' foundation which becomes an unreasonable self-limitation, something like making love while wearing boxing gloves.

The two discoveries I have briefly described are just the beginning of the development of logic and of mathematics free of metaphysical constraints. Still, they show that such a development is possible. Thus, in connection with dynamic constructivism, there is a lot of further technical work to be done and many open problems to be solved. This shows how vital the new philosophy is. Among the many open problems, I conclude with three which seem to me particularly meaningful.

We have seen that logic can be justified simply in terms of the principle of reflection, that is of the dynamical interplay between language and metalanguage. I expect that some form of set theory can be explicated in a similar way. So the first open problem is to find the mathematical development of this philosophical expectation. A solution of course would be extremely interesting and relevant for the development of mathematics with no metaphysical assumptions. My impression is that one will arrive at a fragment of constructive type theory.

The second problem is to find a mathematical, complete semantics for basic logic. Since basic logic yields all known logics in a modular way, one would expect to be able to formulate a mathematical semantics which, in a similarly modular way, yields the well-known semantics for the well-known logics. It is natural to expect the basic picture to furnish the right framework for the correct definition of validity, since, like basic logic, it is nondistributive and based on symmetry.

A theorem of Gödel and Kreisel says that there is no possible complete naive semantics for intuitionistic logic. But the extension of basic logic BLS is not exactly the same as intuitionistic logic, since it has an extra connective, symmetric of implication. So by adding this connective, and imposing a new notion of naive semantics (possibly in which refutability and not only validity is defined for formulae), one might hope to obtain a naive complete semantics. This is the third problem, a positive solution to which would be an important extra benefit conferred by dynamic constructivism.

Epilogue. So we reach our conclusion. As you can see, I have chosen a new name, but at the same time it has not been my purpose to come up with a new recipe to replace those already on the market, a new 'final foundation', for the simple reason that there is no such thing. I have sometimes spoken very subjectively, placing myself and my experience (my memories of my father, my present problems with my kitchen,...) in center stage. I am aware of the fact that this is unorthodox in a scientific communication, or perhaps simply inelegant. But it coheres fully with the viewpoint I have tried to put forward. It is only through a subject, with her or his intuitions and subjectivity, that new truths can be attained. Truth is born as subjective truth; only if later accepted by other subjects, can it become universal and finally objectified.

In more specific terms, everything I said in this paper comes from what I have been able to see, things that have caused me suffering, but which I now firmly believe to be true and for which finally I have tried to argue. Now it is your turn: you should evaluate it, choose what to reject and what to make yours, find better answers, and so on. Feedback from you will also modify my views. And precisely in this way the dynamic process has no end.

Notes

[1]I am sincerely grateful to John L. Bell for his invaluable role in tailoring my text into an English adequate for expressing what I had in mind and to Juliette Kennedy for an intense correspondence leading to several improvements of exposition.

[2]Certain philosophers now share this opinion: see e.g. (Dennett, 1995).

[3]E.g. (Edelman, 1992).

[4]I am not suggesting that existence has degrees measured by numbers. Still, it might be an instructive exercise to develop a formal theory with values of existence ranging over for instance the real interval $[0, 1]$, so as to facilitate our understanding some key aspects of what we call existence, in the same way as the various many-valued logics may help to understand the concept of truth.

[5]For lack of space here, I cannot reproduce a visual explanation which is possible in an actual talk and is obtained by first showing a slide with the common, bare structure, and then superposing on it a slide for each of the connectives.

[6]This is also formally precise if the sequent calculus **UB** of (Faggian and Sambin, 1999) is adopted.

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