A new and elementary method to represent every complete boolean algebra

GIOVANNI SAMBIN
Dipartimento di Matematica Pura ed Applicata
Via Belzoni 7 – 35131 Padova, Italy

Abstract. *For any semilattice* $(S, \wedge, 1)$ *and any* $X, Z \subseteq S$, *define* X^Z *as* $\{a \in S : (\forall b \in X)(a \wedge b \in Z)\}$ *and say that* X *is* Z*-stable if* $X = X^{ZZ}$. *We prove that for any* S *and any downward closed* $Z \subseteq S$, Z*-stable subsets form a complete boolean algebra, and that all cBa's can isomorphically be represented in this way. The proof is obtained by specializing a similar representation theorem for boolean quantales, already known in the literature. Contrary to Stone's representation theorems, the proof is carried over in a fully constructive set theory (that is, no prime filter principle or similar is necessary, and no argument by reductio ad absurdum or proof by cases is used).*

Foreword.

The aim of this paper is to present in all details[1] a new method to represent every complete boolean algebra. Such a method is substantially different from the classic representation by Stone; contrary to Stone's, where the prime filter principle is necessary, it does not need any set-theoretical principle.

This is not pursued just as a debatable curiosity; more positively, all definitions and results can be carried over in a specific highly constructive set theory, namely Martin-Löf's intuitionistic type theory (see e.g. [ML] or [TvD], chap. 11). However, little attention is here payed to details of formalization, with the aim of showing, by an example, that such theory can be put into practice also by the working mathematician.

For the purpose of the proof of the theorem, it is convenient to look at cBa's as quantales satisfying some additional conditions. A unital commutative quantale (cf. e.g. [R]), here simply a *quantale*, is a structure $\mathcal{Q} = (Q, \cdot, 1, \vee)$ where $(Q, \cdot, 1)$ is a commutative monoid, $(Q, \vee)$ is a complete lattice and infinite distributivity $(\vee_{i\in I} a_i) \cdot b = \vee_{i\in I}(a_i \cdot b)$ holds for any family of elements $(b_i)_{i\in I}$ and any element a of Q. We denote the top and bottom element of $\mathcal{Q}$ by $\top$ and 0 respectively. For example, given any monoid $(M, \cdot, 1)$ (here and in the whole paper commutativity is understood), the structure $(\mathcal{P}M, \cdot, \{1\}, \cup)$ is a quantale, where $\mathcal{P}M$ is the powerset of M, $\cdot$ is defined on subsets by $A \cdot B \equiv \{a \cdot b : a \in A, b \in B\}$ and $\cup$ is set-theoretic union. A boolean quantale is a quantale equipped with an operation $-$ such that $a = --a$ for any a. In section 1 we review basic properties of quantales and of boolean quantales up to the point where it can be proved that cBa's are the same thing as boolean quantales satisfying two simple additional conditions.

A phase space $\mathcal{P}M_Z$ (cf. [R], p. 142) is a structure formed by those subsets of M which are stable under double Z-complementation (defined below). In section 2 we repeat a theorem in [R] showing that a quantale is boolean iff it is isomorphic to a phase space. Putting such representation theorem together with the characterization of cBa's, in section 3 the main theorem is easily obtained: phase spaces of the form $\mathcal{P}S_Z$, where S is a semilattice and Z is any downward closed subset of S, exhaust, up to isomorphism, all complete boolean algebras.

Contrary to Stone's, the method presented here unfortunately does not seem to be applicable to complete distributive lattices without modifications.[2] Also, the question remains to obtain Stone's

[1] A first proof was obtained as a by-product of the development of a uniform method to prove completeness of several different logics w.r.t. the many valued semantics given by the notion of pretopology (cf. [S], thm. 14); I here give a simpler proof, and pruned of any reference to logic and pretopologies. I am deeply grateful to Silvio Valentini who has played an important role in the writing of this paper, through conversation and encouragement.

[2] However, I thank John L. Bell for pointing out a connection with the method of polarities, cf. [B], p. 122.

representation of a cBa $\mathcal{B}$ directly from ours, thus detecting precisely the role of the prime filter principle. To make the paper selfcontained and readable also by the non-specialist, we review all properties of quantales which are needed (even if most of them are in [R]) and assume only a little acquaintance with lattices.

1. Preliminaries on quantales and boolean quantales.

The next four lemmas contain the standard facts we need about quantales.

LEMMA 1.1. *In any quantale $\mathcal{Q}$, the following hold for any a, b, c, $d \in Q$:*

(1) *localization: if $a \leq b$ then $a \cdot c \leq b \cdot c$*

(2) *stability: if $a \leq b$ and $c \leq d$ then $a \cdot c \leq b \cdot d$*

PROOF: Localization: $a \leq b$ means that $a \vee b = b$, hence also $(a \vee b) \cdot c = b \cdot c$; but by distributivity $(a \vee b) \cdot c = a \cdot c \vee b \cdot c$, hence $a \cdot c \vee b \cdot c = b \cdot c$, that is $a \cdot c \leq b \cdot c$. Stability: by localization, from $a \leq b$ we have $a \cdot c \leq b \cdot c$ and similarly $b \cdot c \leq b \cdot d$ from $c \leq d$, so by transitivity $a \cdot c \leq b \cdot d$.

In any quantale, the operation $\rightarrow$ is defined putting, for any $a, b \in Q$,

$$a \rightarrow b \equiv \vee\{c : a \cdot c \leq b\}$$

LEMMA 1.2. (BASIC PROPERTIES OF $\rightarrow$). *In any quantale, the operation $\rightarrow$ satisfies the following properties:*

(1) $a \cdot (a \rightarrow b) \leq b$

(2) $c \cdot a \leq b$ *iff* $c \leq a \rightarrow b$

(3) $1 \rightarrow a = a$

(4) $a \rightarrow (b \rightarrow c) = a \cdot b \rightarrow c$

PROOF: (1) By definition, $a \cdot (a \rightarrow b) \equiv a \cdot \vee\{c : a \cdot c \leq b\}$; by distributivity, $a \cdot \vee\{c : a \cdot c \leq b\} \leq \vee\{a \cdot c : a \cdot c \leq b\}$, which by definition of sup is less than b. (2) Assume $a \cdot c \leq b$; then, since $c \in \{c : a \cdot c \leq b\}$, it follows that $c \leq \vee\{c : a \cdot c \leq b\} \equiv a \rightarrow b$. Conversely, if $c \leq a \rightarrow b$ then by localization $a \cdot c \leq a \cdot (a \rightarrow b)$, so that $a \cdot c \leq b$ by (1). (3) $a = a \cdot 1 \leq a$ hence $a \leq 1 \rightarrow a$; conversely, $1 \rightarrow a = 1 \cdot (1 \rightarrow a) \leq a$ (4) By (1), $(a \cdot b \rightarrow c) \cdot a \cdot b \leq c$, hence by (2) $(a \cdot b \rightarrow c) \cdot a \leq b \rightarrow c$ and $a \cdot b \rightarrow c \leq a \rightarrow (b \rightarrow c)$; similarly, $(a \rightarrow (b \rightarrow c)) \cdot a \leq b \rightarrow c$ gives $(a \rightarrow (b \rightarrow c)) \cdot a \cdot b \leq c$ hence $a \rightarrow (b \rightarrow c) \leq a \cdot b \rightarrow c$.

In any quantale, for any element d the operation of *d-complementation* is defined putting

$$-_d a \equiv a \rightarrow d.$$

If d is such that $-_d -_d a = a$ for any a, then d is called a *dualizer* for Q. A quantale is called *boolean*[3] if it has a dualizer d; in any case the d-complement of a is abbreviated by $-a$.

LEMMA 1.3. *In any quantale, for any d, d- complementation satisfies:*

(1) $a \leq b$ *implies* $-b \leq -a$

(2) $-1 = d$

(3) $-0 = \top$

(4) $a \leq --a$

(5) $-a = ---a$

(6) $a \cdot --b \leq --(a \cdot b)$

(7) $--(a \cdot --b) = --(a \cdot b)$

[3]Following a suggestion in [R], where however it is called 'Girard'.

PROOF: (1) holds because $a \le b$ implies $a \cdot (b \to d) \le b \cdot (b \to d) \le d$ and hence $b \to d \le a \to d$. (2) since $-1 \equiv 1 \to d$ and $1 \to d = d$, $-1 = d$ holds. (3) To show $-0 = \top$ it is enough to see that $\top \le -0$: from $0 \le -\top$ one has $0 \cdot \top \le d$ hence $\top \le -0$. (4) follows immediately from $a \cdot -a \le d$. (5) holds since the inequality $-a \le ---a$ is an application of (5), while $---a \le -a$ comes from $a \le --a$ by (1). (6) By (1), $(a \to (b \to d)) \cdot a \le b \to d$, that is, by (4) of lemma 2, $-(a \cdot b) \cdot a \le -b$; by localization, $-(a \cdot b) \cdot a \cdot --b \le -b \cdot --b \le d$, hence by (2) of lemma 2 the claim $a \cdot --b \le --(a \cdot b)$. (7) is an easy consequence of (4) and (6).

LEMMA 1.4. *In any boolean quantale, d- complementation satisfies:*

(1) $a \le b$ *iff* $-b \le -a$

(2) $-1 = d$, $-d = 1$

(3) $-\top = 0$, $-0 = \top$

(4) $-(a \wedge b) = -a \vee -b$ *(de Morgan law)*

PROOF: (1) By properties of $\to$, from $a \le b$ we have $-b \le -a$; conversely, by the same reason, $-b \le -a$ gives $--a \le --b$, i.e. $a \le b$. (3) From $-0 = \top$, which holds in any quantale, one has $0 = --0 = -\top$. (2) Similarly, from $-1 = d$ one has $1 = --1 = -d$ (4) $a \wedge b \le a$ gives $-a \le -(a \wedge b)$ and similarly $-b \le -(a \wedge b)$, so $-a \vee -b \le -(a \wedge b)$. Conversely, $-a \le -a \vee -b$ gives $-(-a \vee -b) \le a$ and similarly for b, hence $-(-a \vee -b) \le a \wedge b$, from which $-(a \wedge b) \le -a \vee -b$.[4]

LEMMA 1.5. *In any boolean quantale, the following are equivalent:*

(1) $a \cdot b \le a$ *for any* a, b

(2) $a \cdot b \le a \wedge b$ *for any* a, b

(3) $a \le 1$ *for any* a

(4) $\top \le 1$

(5) $d \le 0$

PROOF: From (1) it follows that $a \cdot b \le a$ and $a \cdot b \le b$, hence $a \cdot b \le a \wedge b$, i.e. (2); the converse is trivial. By taking $a = 1$ in (1) one has $b = b \cdot 1 \le 1$ for any b, i.e. (3); the converse holds by localization. Finally, (3) iff (4) because $a \le \top$ for any a and (4) iff (5) because by lemma 4 $d \le 0$ iff $-0 \le -d$ iff $\top \le 1$. Note that the equivalences (1)-(4) hold in any quantale.

LEMMA 1.6. *In any quantale, the following are equivalent:*

(1) $a \wedge b \le a \cdot b$ *for any* a, b

(2) $a \le a \cdot a$ *for any* a

PROOF: From (1) by taking $b = a$ it follows that $a = a \wedge a \le a \cdot a$. Conversely, $a \wedge b \le a$ and $a \wedge b \le b$ by stability give $(a \wedge b) \cdot (a \wedge b) \le a \cdot b$, from which (1) since $a \wedge b \le (a \wedge b) \cdot (a \wedge b)$.

COROLLARY 1.7. *In a boolean quantale, if* 0 *is a dualizer, then it is the unique one.*

PROOF: By lemma 5, if $d = 0$ then $\top = 1$ and hence $-d' = 1 = \top$ for any dualizer d', so that $d' = --d' = -\top = 0$.

We are now ready to characterize complete boolean algebras as a special class of boolean quantales. *Boolean algebras* are here defined as usual (apart from notation), namely as distributive lattices with bottom 0 and top $\top$ and with an operation of complementation ν s.t. $a \wedge \nu a = 0$ and $a \vee \nu a = 1$. We say that a quantale $\mathcal{Q}$ is *idempotent* if all its elements are idempotent, that is $a \cdot a = a$ holds for any $a \in Q$. Then we have:

[4] Note that also the second de Morgan law $-(a \vee b) = -a \wedge -b$ holds, but we don't need it in this paper.

THEOREM 1.8. *Complete boolean algebras can be characterized as idempotent boolean quantales in which* 0 *is a dualizer.*

PROOF: It is an exercise on boolean algebras to show that any cBa $\mathcal{B} = (B, \wedge, \vee, 0, \top, \nu)$ is a quantale with the desired properties; however, here is a hint. Trivially, $(B, \wedge, \top)$ is a semilattice, that is a monoid in which $a = a \wedge a$ holds, and $(B, \vee, 0, \top)$ is a complete lattice. Infinite distributivity is easily derivable from the fact that $\rightarrow$ is definable, putting

$$a \rightarrow b \equiv \nu a \vee b$$

It is easy to check that $\rightarrow$ so defined satisfies $a \wedge c \leq b$ iff $c \leq a \rightarrow b$; in fact, if $a \wedge c \leq b$ then $c \leq (\nu a \vee a) \wedge (\nu a \vee c) = \nu a \vee (a \wedge c) \leq \nu a \vee b$, and conversely if $c \leq \nu a \vee b$ then $c \wedge a \leq a \wedge (\nu a \vee b) = (a \wedge \nu a) \vee (a \wedge b) \leq b$. Therefore $(B, \wedge, \top, \vee)$ is an idempotent quantale. Finally, since $\nu a = \nu a \vee 0 = a \rightarrow 0$, the equation $\nu\nu a = a$, which holds in any boolean algebra, means that $a = (a \rightarrow 0) \rightarrow 0$, i.e. 0 is a dualizer.

Conversely, assume $\mathcal{Q} = (Q, \cdot, \vee, 1)$ is a boolean quantale. The assumptions $a \cdot a = a$ and $0 = d$ by lemmas 1.5 and 1.6 mean that $a \cdot b = a \wedge b$; moreover $1 = \top$ by lemma 1.5, so that $(Q, \cdot, \vee, \top)$ is a complete distributive lattice. Finally $a \wedge -a = 0$ because $a \wedge -a = a \cdot -a \leq d = 0$ and $a \vee -a = \top$ because $a \vee -a = - - a \vee -a =$ (by de Morgan law) $-(-a \wedge a) = -0 = \top$, and so $-a$ is the (boolean) complement of a.

2. Phase spaces and the representation of boolean quantales.

As the last result of the preceding section suggests, the representation of cBa's is obtained by specializing the representation of boolean quantales (see [R], thm. 6.1.1); we here repeat it shortly. We need a lemma which shows how any quantale can be transformed into a boolean quantale:

LEMMA 2.1. *For any quantale* $\mathcal{Q}$ *and any* $d \in Q$*, define* d*-complementation* $-a \equiv a \rightarrow d$ *as usual, and say that an element* $a \in Q$ *is* d*-stable if* $a = - - a$*. Then the set of* d*-stable elements of* Q *forms a boolean quantale* $\mathcal{Q}_d$ *with dualizer* d *and with operations defined by:*

$$\vee^d_{i \in I} a_i \equiv - - (\vee_{i \in I} a_i) \qquad a \cdot^d b \equiv - - (a \cdot b) \qquad 1^d \equiv - - 1$$

PROOF: The proof is obtained by standard applications of the properties of d-complementation (cf. lemma 1.3). In fact, $\vee^d_{i \in I} a_i$ is the supremum, since $a_i \leq \vee_{i \in I} a_i \leq - - (\vee_{i \in I} a_i)$ and if $a_i \leq b$ for all $i \in I$, then $\vee_{i \in I} a_i \leq b$ from which $- - (\vee_{i \in I} a_i) \leq - - b = b$. The operation $\cdot^d$ is associative, because $(a \cdot^d b) \cdot^d c = a \cdot^d (b \cdot^d c)$ is by definition $- - (- - (a \cdot b) \cdot c) = - - (a \cdot - - (b \cdot c))$, which is obtained from $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ by repeated use of (7) of lemma 1.3. The element $- - 1$ is the unit element for $\cdot^d$, since $- - 1 \cdot^d a = - - (- - 1 \cdot a) = - - (1 \cdot a) = - - a = a$. Distributivity follows from distributivity of $\mathcal{Q}$, with an argument quite similar to that for associativity above.

Finally, d is d-stable, because $d \cdot (d \rightarrow d) \leq d$ gives $d \leq (d \rightarrow d) \rightarrow d$ and $1 \cdot d \leq d$ gives $1 \leq d \rightarrow d$, hence $(d \rightarrow d) \rightarrow d \leq 1 \rightarrow d = d$; trivially $- - a = a$ for any d-stable a. So d is a dualizer in $\mathcal{Q}_d$ and $\mathcal{Q}_d$ is a boolean quantale.

For any monoid M and any $Z \subseteq M$, the quantale $\mathcal{P}M_Z$, obtained from the quantale $\mathcal{P}M$ as in the preceding lemma, is called the *phase space* on M with dualizer Z. We denote by F, G,..., the Z-stable subsets, sometimes called facts; the explicit definition of operations is:

$$\vee^Z_{i \in I} F_i \equiv - - (\cup_{i \in I} F_i) \qquad F \cdot^Z G \equiv - - (F \cdot G) \qquad 1^Z \equiv - - 1$$

Note that by the definition of $\rightarrow$ in a quantale, it is $A \rightarrow B \equiv \cup\{C : C \cdot A \subseteq B\}$ and it is easily seen that $A \rightarrow B = \{a \in M : a \cdot A \subseteq B\}$. So in particular Z-complementation is

$$-_Z A \equiv A^Z = \{a \in M : a \cdot A \subseteq Z\}$$

We say that a submonoid M of a quantale $\mathcal{Q}$ is a *base* for $\mathcal{Q}$ if for any $q \in Q$, $q = \vee\{a \in M : a \leq q\}$. Note that any quantale has a base, at worst $\mathcal{Q}$ itself[5].

[5] Actually all results below could be stated without mentioning bases. Here some care is taken towards keeping the highest possible degree of constructivity, and this is why bases are essential: in the terminolgy of [ML], a base M for $\mathcal{Q}$ could be a set, even if $\mathcal{Q}$ is a proper category.

Theorem 2.2. *Any quantale is boolean iff it is isomorphic to a phase space. More specifically, any phase space $\mathcal{P}M_Z$ is a boolean quantale, and for any boolean quantale $\mathcal{Q}$, for any submonoid M of $\mathcal{Q}$ which is a base for $\mathcal{Q}$, there exists $Z \subseteq M$ s.t. $\mathcal{Q} \cong \mathcal{P}M_Z$.*

Proof: $\mathcal{P}M_Z$ is a boolean quantale by lemma 2.1. Conversely, let $\mathcal{Q}$ be a boolean quantale and let M be a base for $\mathcal{Q}$. For any $A \subseteq M$, we put $jA \equiv \{a \in M : a \leq \vee A\}$. We now see that $jA = -_Z -_Z A$, where we have put $Z \equiv \{a \in M : a \leq d\}$ for a dualizer d of $\mathcal{Q}$. In fact, for any $a \in M$, by definition $a \in - - A$ iff $a \cdot (A \to Z) \subseteq Z$ iff $(\forall b)(b \in A \to Z \Rightarrow a \cdot b \in Z)$. Now $a \cdot b \in Z$ iff $a \cdot b \leq d$ iff $a \leq -b$ and similarly $b \in A \to Z$ iff $b \cdot A \subseteq Z$ iff $(\forall c \in A)(b \cdot c \in Z)$ iff $(\forall c \in A)(c \leq -d)$ iff $\vee A \leq -d$. So $a \in - - A$ iff $(\forall b)(\vee A \leq -b \Rightarrow a \leq -b)$, which is the same as $a \leq \vee A$, i.e. $a \in jA$.

So the Z-stable subsets of $\mathcal{P}M_Z$ are all of the form jA for some $A \subseteq M$. Note that for any $A \subseteq M$, it is $\vee(- - A) = \vee A$, since $- - A = jA$ and $\vee(jA) = \vee A$.

Then the isomorphism $k : \mathcal{P}M_Z \to \mathcal{Q}$ is given by $k : F \mapsto \vee F$. In fact, since obviously $F \subseteq G$ iff $\vee F \leq \vee G$, k is one-one and preserves inclusion. Since M is a base, any $q \in Q$ is equal to $\vee A$ where $A \equiv \{a \in M : a \leq q\}$ and hence $k(jA) = \vee A = q$, that is k is onto. A few equalities show that k preserves suprema:

$$k(\vee_{i \in I} F_i) \equiv k(- - (\cup_{i \in I} F_i)) = \vee(\cup_{i \in I} F_i) = \vee_{i \in I}(\vee F_i) \equiv \vee_{i \in I} kF_i$$

and that k preserves the monoid operation:

$$k(F \cdot_Z G) \equiv k(- - (F \cdot G)) = \vee(F \cdot G) = \vee F \cdot \vee G \equiv kF \cdot kG$$

3. The representation of complete boolean algebras.

We begin with a corollary to theorem 2.2 above:

Corollary 3.1. *$\mathcal{Q}$ is a boolean quantale in which 0 is a dualizer iff it is isomorphic to $\mathcal{P}M_Z$ for some monoid M and some $Z \subseteq M$ s.t. $Z = - - \emptyset$.*

Proof: By the theorem, it is enough to show that 0 is a dualizer of $\mathcal{P}M_Z$ iff $Z = - - \emptyset$. Recall that $- - \emptyset$ is the bottom of $\mathcal{P}M_Z$ and that Z is always a dualizer of $\mathcal{P}M_Z$. Then the claim follows from corollary 1.7.

The condition $Z = - - \emptyset$ does not give a direct description of how Z should be. However, since obviously $-\emptyset = M$, we can bring $Z = - - \emptyset$ to the form

$$Z = M \to Z$$

which is a sort of fixed point equation to be solved. The solutions are all of the form $M \to A$ for any $A \subseteq M$. In fact, if $Z = M \to Z$ then Z is of such form, and conversely for any A it is $M \to (M \to A) = M \cdot M \to A = M \to A$. It also follows that $Z = \emptyset^{ZZ}$ iff $Z = \emptyset^{AA}$ for some A. Note also that, since the inclusion $M \to Z \subseteq Z$ holds for any Z (because $a \in M \to Z \equiv a \cdot M \subseteq Z$ implies $a = a \cdot 1 \in Z$), the equation $Z = M \to A$ is equivalent to $Z \subseteq M \to Z$, and hence to $Z \cdot M \subseteq Z$; that is, Z is an ideal in the monoid M. So, in the case M is in fact a semilattice $(S, \wedge, 1)$, the condition becomes $S \wedge Z \subseteq Z$, which is equivalent to $\downarrow Z \subseteq Z$, where $\downarrow Z \equiv \{a \in S : (\exists z \in Z)(a \leq z)\}$ is the downward closure of Z. Summing up, we have:

Lemma 3.2. *For any monoid M, the following are equivalent for any $Z \subseteq M$:*

(1) *Z is the bottom in $\mathcal{P}M_Z$, i.e. $Z = - - \emptyset$*
(2) *$Z = \emptyset^{AA}$ for some $A \subseteq M$*
(3) *$Z = M \to Z$*
(4) *$Z = M \to A$ for some $A \subseteq M$*
(5) *Z is an ideal, i.e. $M \cdot Z \subseteq Z$*

In a semilattice S, any of the above is equivalent to:

(6) *Z is downward closed, i.e. $\downarrow Z \subseteq Z$*

We are finally ready to prove the main theorem:

Theorem 3.3. *A structure $\mathcal{B}$ is a complete boolean algebra iff it is isomorphic to a phase space on a semilattice with a downward closed dualizer. More specifically, for any semilattice S and any downward closed $Z \subseteq S$, $\mathcal{P}S_Z$ is a cBa. Conversely, for any cBa $\mathcal{B}$, there exists a semilattice S and a downward closed $Z \subseteq S$ s.t. $\mathcal{B} \cong \mathcal{P}S_Z$.*

Proof: First we prove that, for any semilattice S and any downward closed $Z \subseteq S$, $\mathcal{P}S_Z$ is a complete boolean algebra. By theorem 2.2, $\mathcal{P}S_Z$ is a boolean quantale. Since Z is downward closed, 0 is a dualizer by lemmas 1 and 2. So, by theorem 1.8, it only remains to prove that $\mathcal{P}S_Z$ is idempotent. Since 0 is a dualizer, by lemma 1.5 $U \cdot^Z U \subseteq U$ holds for any Z-stable subset $U \subseteq S$; for the other inclusion $U \subseteq U \cdot^Z U$ it is enough to show that $U \subseteq U \cdot U$, which is immediate since for any $a \in S$, $a \in U$ implies $a = a \cdot a \in U \cdot U$.

Now let $\mathcal{B}$ be a cBa. By theorem 1.8 $\mathcal{B}$ is a boolean quantale, and hence by theorem 2.2 it is $\mathcal{B} \cong \mathcal{P}S_Z$ for some base S and $Z \subseteq S$; note that S, like any base for $\mathcal{B}$, is a semilattice. By the isomorphism $\mathcal{B} \cong \mathcal{P}S_Z$, since 0 is a dualizer in $\mathcal{B}$, it follows that Z is equal to bottom in $\mathcal{P}S_Z$, that is $Z = --\emptyset$, which is equivalent to Z downward closed by lemma 2.

It may be worthwhile to notice that, given a cBa $\mathcal{B}$, it may well happen that $Z \equiv \{a \in M : a \leq 0\}$ is empty, which happens if $0 \notin S$. In this case, it is easy to check that $\mathcal{P}S_Z$ is (isomorphic to) the two-element boolean algebra $\{0, \top\}$; in fact, for any $A \subseteq S$, it is $A^\emptyset = \emptyset$ if $A \neq \emptyset$, and $A^\emptyset = S$ if $A = \emptyset$, so that $A^{\emptyset\emptyset}$ is either $\emptyset$ or S. This does not affect the theorem, however, nor its uniformity; in fact, one finds out (as suggested by Silvio Valentini) that when $\mathcal{B}$ is different from $\{0, \top\}$, then any base S must contain 0:

Proposition 3.4. *If a cBa $\mathcal{B}$ contains some element different from bottom and top, then any base S for $\mathcal{B}$ must contain* 0.

Proof: If $\mathcal{B}$ is different from $\{0, \top\}$, then any base S must be different from the singleton $\{\top\}$; in fact, $\{\top\}$ can generate only $\top$ and 0. So let $b \in S$ with $b \neq \top$; then $0 = b \wedge -b = b \wedge \bigvee\{c \in S : c \leq -b\}$, and since $b \neq \top$, it must be $-b \neq 0$, hence $\{c \in S : c \leq -b\} \neq \emptyset$. Let c be an element of S s.t. $c \leq -b$; then $0 = b \wedge -b \geq b \wedge c$, hence $0 \in S$ because $b, c \in S$.

Concluding general remark.

As a concluding remark, note that the fact that the proof of the representation theorem does not need any set- theoretic principle is not in contrast with the well known situation for representation theorems in the style of Stone; the constructivity of the present proof is a trade off of the fact that our presentation is not a field of sets, that is we have given up with the idea that $\emptyset$ and $\bigcup$ should be the zero and sup, respectively[6]. So, while it is true that $\mathcal{P}S_Z$ is a family of subsets of S, it is *not* true that it is a field of sets, i.e. a subalgebra of the powerset $\mathcal{P}S$ with usual set-theoretic operations.

This is perfectly in line with the spirit of pointfree (or pointless) topology, where for example Tychonoff's theorem on products of compact topological spaces does not need the axiom of choice, at the price of giving up points (for a discussion on why a mathematician should be interested in pointless topology, see [J]). Actually, also our example can be interpreted in this way: a constructive representation is obtained by giving up the requirement that an element of the boolean algebra is to be represented as a set of prime filters, i.e. points. The intriguing problem remains to explain why such a representation seems to work only for the "classical" case of boolean algebras, rather than for an arbitrary lattice of opens.

References

[B] G. Birkhoff, "Lattice theory," 3rd edition, Amer. Math. Soc. Colloquium Publications XXV, 1967.

[G] J-Y. Girard, *Linear logic*, Theor. Computer Sc. **50** (1987), 1–102.

[G] P. Johnstone, *The point of pointless topology*, Bull. Amer. Math. Soc. **8** (1983), 41–53.

[ML] P. Martin-Löf, "Intuitionistic type theory," Notes by G. Sambin of a series of lectures given in Padua, June 1980, Bibliopolis, 1984.

[6]For the logician: this should be confronted with the fact that, to obtain a constructive proof of completeness of Kripke models for intuitionistic logic, one has to give up the idea that absurdity is forced in the empty set of points.

[R] K. Rosenthal, “Quantales and their applications,” Longman, 1990.
[S] G. Sambin, *Pretopologies and completeness proofs*, J. Symbolic Logic (to appear).
[T] A. S. Troelstra – D. Van Dalen, “Constructivism in mathematics. An introduction,” North Holland, 1988.