Formal topology and domains

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The theme of this paper is the relation between formal topology and the theory of domains. On one hand, domain theory can be seen as a branch of formal topology. On the other hand, the influence of domain theory on formal topology is twofold. Historically, the presence of the subset *Con* in Scott's information systems has been the starting point for the introduction of the positivity predicate Pos in formal topology; also, the notion of approximable mapping has influenced the definition of continuous relations between formal topologies. Conceptually, since domain theory can be seen as a particular case, any notion and result in domain theory becomes a challenge for formal topology: how much of domain theory can be generalized to formal topology?

My impression is that some open problems in one of the two fields could already have a solution in the other, and that is why an intensification of contact should be rewarding.[1]

1. Formal topology

What is formal topology? A good approximation to the correct answer is: formal topology is topology as developed in (Martin-Löf's) type theory [3]. This means that it is intuitionistic *and* predicative. Actually, it is fully formalizable in an implementation of type theory, via what we have called the toolbox for subsets (cf. [7]); as a result, notation is quite standard, except for the use of ϵ, which is different from $\in$, for elements of a subset U (which is a propositional function, and hence *not* a set: when S is a set and $U \subseteq S$, $a \ \epsilon \ U$ means that $a \in S$ and $U(a)$ is true). The adjective "formal" is due to the stress on the pointfree approach to topology, to which one is naturally lead adopting type theory. The original main definition (cf. [4]) was:

Definition 1.1 (1984-1987) *A structure* $\mathcal{A} = (S, \cdot, 1, \lhd, \mathsf{Pos})$ *is a formal topology when:*

[1] I am very grateful to the organizers of the workshop Domains IV, in particular to Dieter Spreen, for inviting me.

S is a set and $(S, \cdot, 1)$ is a semilattice (or commutative monoid), called the base;

$a \lhd U$ prop $(a \in S, U \subseteq S)$ is a cover, that is it satisfies

$$\textit{reflexivity} \quad \frac{a \,\epsilon\, U}{a \lhd U}$$

$$\textit{transitivity} \quad \frac{a \lhd V \quad V \lhd U}{a \lhd U} \quad \textit{where } V \lhd U \equiv (\forall b \,\epsilon\, V)(b \lhd U)$$

$$\cdot\ \textit{- Left} \quad \frac{a \lhd U}{a \cdot b \lhd U}$$

$$\cdot\ \textit{- Right} \quad \frac{a \lhd U \quad a \lhd V}{a \lhd U \cdot V} \quad \textit{where } U \cdot V \equiv \{b \cdot c : b \,\epsilon\, U, c \,\epsilon\, V\}$$

$\mathsf{Pos}(a)$ prop $(a \in S)$ is a positivity predicate, that is it satisfies

$$\textit{monotonicity} \quad \frac{\mathsf{Pos}(a) \quad a \lhd U}{(\exists b \,\epsilon\, U)\mathsf{Pos}(b)}$$

$$\textit{positivity} \quad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$$

With any (infinitary) relation $\lhd$ one can associate an operator on subsets $\mathcal{A}U \equiv \{a \in S : a \lhd U\}$ and conversely, given an operator $\mathcal{A}$, one defines a relation $a \lhd U \equiv a \,\epsilon\, \mathcal{A}U$. One can show that $\lhd$ is a cover iff the associated operator $\mathcal{A}$ is a closure operator satisfying $\mathcal{A}(U \cdot V) = \mathcal{A}U \cap \mathcal{A}V$. A subset U is called $\mathcal{A}$-saturated if $U = \mathcal{A}U$; on the collection $Sat(\mathcal{A})$ of $\mathcal{A}$-saturated subsets one can define as usual meets as intersections and joins by $\bigvee_{i \in I} \mathcal{A}U_i \equiv \mathcal{A}(\cup_{i \in I} U_i)$. Then one can prove:

Theorem 1.2 *For any formal topology $\mathcal{A}$, $Sat(\mathcal{A})$ is a frame (or locale, or complete Heyting algebra).*

$Sat(\mathcal{A})$ is called the frame of formal opens. The intuition is that two subsets $U, V \subseteq S$ are equal in $\mathcal{A}$ if they cover each other, that is $(U =_{\mathcal{A}} V) \equiv U \lhd V\ \&\ V \lhd U$, and clearly $U =_{\mathcal{A}} V$ iff $\mathcal{A}U = \mathcal{A}V$. From an impredicative point of view (for instance, in topos theory), all frames can be represented as formal opens of a suitable formal topology. In this sense, formal topologies are the predicative version of frames, or locales (see [2]).

The second main definition is:

Definition 1.3 *In a formal topology $\mathcal{A}$, a subset $\alpha \subseteq S$ is called a formal point if:*

$$\alpha \textit{ is inhabited} \quad 1 \,\epsilon\, \alpha$$

$$\alpha \textit{ is convergent} \quad \frac{a \,\epsilon\, \alpha \quad b \,\epsilon\, \alpha}{a \cdot b \,\epsilon\, \alpha}$$

$$\alpha \textit{ splits } \lhd \quad \frac{a \,\epsilon\, \alpha \quad a \lhd U}{(\exists b \,\epsilon\, U)(b \,\epsilon\, \alpha)}$$

α *is consistent* $\qquad a \,\epsilon\, \alpha \to \mathsf{Pos}(a)$

One can prove that:

Theorem 1.4 *There is a bijection between formal points of $\mathcal{A}$ and completely prime filters of $Sat(\mathcal{A})$.*

So, again, formal points are the predicative version of points in a locale; note that while a formal point is a subset of S, a completely prime filter of $Sat(\mathcal{A})$ is a collection of subsets of S.

Definition 1.5 *For any formal topology $\mathcal{A}$, the collection $Pt(\mathcal{A})$ of all formal points of $\mathcal{A}$ is called a formal space.*

Intuitively, $Pt(\mathcal{A})$ is provided with the topology generated by the base $\mathsf{ext}\,(a) \equiv \{\alpha : a \,\epsilon\, \alpha\}$ for $a \in S$; but note that in type theory $Pt(\mathcal{A})$ is not necessarily a set, and this is why the adjective "formal" again.

Definition 1.6 *Let $\mathcal{A} = (S, \cdot, \lhd_{\mathcal{A}}, \mathsf{Pos}_{\mathcal{A}})$ and $\mathcal{B} = (T, \cdot, 1, \lhd_{\mathcal{B}}, \mathsf{Pos}_{\mathcal{B}})$ be any formal topologies. A relation aFb prop $(a \in S, b \in T)$ is called a continuous relation from $\mathcal{A}$ into $\mathcal{B}$ if*

$$aF1 \qquad\qquad \frac{aFb \quad aFc}{aFb \cdot c}$$

$$\frac{aFb \quad b \lhd_{\mathcal{B}} V}{a \lhd_{\mathcal{A}} F^{-1}V} \qquad\qquad \frac{Pos_{\mathcal{A}}(a) \quad aFb}{Pos_{\mathcal{B}}(b)}$$

One can read intuitively this definition thinking of F as a constructive approximation of a continuous function $f : Pt(\mathcal{A}) \to Pt(\mathcal{B})$, the idea being that aFb holds iff for all formal points α, $a \,\epsilon\, \alpha \to b \,\epsilon\, f\alpha$.

Formal topologies and continuous relations form a category, called FTop. It is the predicative version of locales, since (reasoning impredicatively) FTop is equivalent to the category of locales (and dual to the category of frames).

2. Scott domains

After this telegraphic resumé of the beginning of formal topology, we can see the connection with Scott domains, again telegraphically (the paper [8], submitted in '92, contains all details).

Definition 1.7 *A formal topology $\mathcal{A}$ is called unary, or scott, if its cover is unary, that is it satisfies*

$$a \lhd U \text{ iff } Pos(a) \to (\exists b \,\epsilon\, U)(a \lhd \{b\})$$

The name is due to the fact that $Pos(a) \to (\exists b \,\epsilon\, U)(a \lhd \{b\})$ iff there exists a subset $u \subseteq U$ with at most one element such that $a \lhd u$; intuitively, no two elements can cooperate to cover a positive element. The beginning of the connection is given by:

Proposition 1.8 *If $\mathcal{A}$ is a unary formal topology, then $Pt(\mathcal{A})$ is a Scott domain.*

But if $\mathcal{A}$ is unary, then $\lhd$ is determined by its trace on elements $a \prec b \equiv a \lhd \{b\}$; the converse also holds. So we put:

Definition 1.9 *An information base is a structure $\mathcal{S} = (S, \cdot, 1, \prec, Pos)$ where*

$$a \prec 1 \qquad a \prec a \qquad \frac{a \prec b \quad b \prec c}{a \prec c}$$

$$\frac{a \prec b}{a \cdot c \prec b}, \qquad \frac{a \prec b}{c \cdot a \prec b} \qquad \frac{a \prec b \quad a \prec c}{a \prec b \cdot c}$$

$$Pos(1) \qquad \frac{Pos(a) \quad a \prec b}{Pos(c)} \qquad \frac{Pos(a) \rightarrow a \prec b}{a \prec b}$$

In other terms, an information base is just a way to present a semilattice (the first two lines) with Pos. Formal points on, and continuous relations between, information bases are defined as expected. One can prove (impredicatively) that:

Theorem 1.10 *The category of Scott domains is equivalent to the category of information bases.*

So information bases give a predicative approach to Scott domains, and, since information bases are just an alternative presentation of unary formal topologies, the theory of Scott domains becomes a subtheory of formal topology. This is the content of [8], where the reader can find details, as well as an interpretation of formal points as concepts and of continuous relations as translations.

The aim here is to begin to see how such results should be modified in the new approach to formal topology.

3. The basic picture

The main definitions of formal topology had to be modified for two reasons: to treat also formal closed subsets predicatively, and to get rid of the operation $\cdot$ so that $\mathcal{P}S$, preorders, trees, etc... could be included more naturally. This has brought to a structure which is deeper than previous formal topology, and which I have called the basic picture.

To see it, one has to analyse carefully the usual definition of topological space $(X, \Omega X)$ so that it can be brought to type-theoretic terms. The main change is that one must add a second set S, of observables or formal basic neighbourhoods, which plays the role of an index-set for the family of open subsets ΩX. We cannot expect, however, to obtain all open subsets with an index in S, otherwise we should obtain also $\mathcal{P}X$ as set-indexed. So we start from X, S sets and a function $\mathsf{ext} : S \rightarrow \mathcal{P}X$, and we want to obtain ΩX as the family of subsets $\mathsf{ext}(U) \equiv \cup_{a \epsilon U} \mathsf{ext}(a)$ for all $U \subseteq S$. Then $\emptyset = \mathsf{ext}(\emptyset)$ and closure under unions are automatic; so, to obtain that ΩX is a topology, only

$X = \mathsf{ext}\,(S)$ and closure under finite intersections must be required. The latter is $\mathsf{ext}\,U \cap \mathsf{ext}\,V = \mathsf{ext}\,\{c : \mathsf{ext}\,(c) \subseteq \mathsf{ext}\,U \cap \mathsf{ext}\,V\}$ which by distributivity is equivalent to

$$\mathsf{ext}\,(a) \cap \mathsf{ext}\,(b) = \mathsf{ext}\,(a{\downarrow}b)$$

where $a{\downarrow}b \equiv \{c : \mathsf{ext}\,(c) \subseteq \mathsf{ext}\,(a) \cap \mathsf{ext}\,(b)\}$. Finally, note that a function $\mathsf{ext} : S \to \mathcal{P}X$ can be identified with a binary relation $x \Vdash a \equiv x \,\epsilon\, \mathsf{ext}\,(a)$. Thus the definition we reach is:

Definition 1.11 *A concrete topological space is a structure $\mathcal{X} = (X, \Vdash, S)$ where X, S are sets and $\Vdash$ is a binary relation from X to S satisfying:*

B1: $\quad \forall x \exists a (x \Vdash a)$

B2: $$\frac{x \Vdash a \qquad x \Vdash b}{x \Vdash a{\downarrow}b}$$

The new discovery is that B1 and B2 are not necessary to define usual topological notions such as open, closed, continuous etc. What remains is just what I call a *basic pair* $\mathcal{X} = (X, \Vdash, S)$, that is two sets X, S linked by a relation $\Vdash$:

$$\begin{array}{ccc} X & \Vdash & S \\ \textit{concrete points } x, y, \cdots & & \textit{observables } a, b, \cdots \end{array}$$

It is easy to check that the usual definition of interior becomes: for any $A \subseteq X$,

$$\mathsf{int}\,A \equiv \{x : (\exists a \in S)(x \Vdash a \;\&\; \mathsf{ext}\,(a) \subseteq A)\}.$$

Similarly for closure of A; adopting the notation

$$A \between B \equiv (\exists x \in X)(x \,\epsilon\, A \;\&\; x \,\epsilon\, B)$$

(which is different from $A \cap B \neq \emptyset$), the duality with interior is more visible:

$$\mathsf{cl}\,A \equiv \{x : (\forall a \in S)(x \Vdash a \to \mathsf{ext}\,(a) \between A)\}$$

We now can see that int and cl are the combination of two very natural operators on subsets. We have already defined the operator $\mathsf{ext} : \mathcal{P}S \to \mathcal{P}X$ by putting $\mathsf{ext}\,(U) \equiv \cup_{a \epsilon U}\, \mathsf{ext}\,(a) \equiv \{x : (\exists a)(x \Vdash a \;\&\; a \,\epsilon\, U\}$ for any $U \subseteq S$. Since a basic pair is fully symmetric, we can define also an operator $\Diamond : \mathcal{P}X \to \mathcal{P}S$, which is symmetric to ext, by putting $\Diamond A \equiv \{a : (\exists x)(x \Vdash a \;\&\; x \,\epsilon\, A)\}$ for any $A \subseteq X$. We write $\Diamond x$ for $\Diamond\{x\}$ so that $a \,\epsilon\, \Diamond x$, just as $x \,\epsilon\, \mathsf{ext}\,(a)$, is a synonym for $x \Vdash a$. Note that $x \,\epsilon\, \mathsf{ext}\,(U) \equiv \Diamond x \between U$ and $a \,\epsilon\, \Diamond A \equiv \mathsf{ext}\,(a) \between A$.

While ext is defined by an existential quantification, the second operator rest from $\mathcal{P}S$ into $\mathcal{P}X$ is defined by a universal quantification: for all $U \subseteq S$, $\mathsf{rest}\,(U) \equiv \{x : (\forall a)(x \Vdash a \rightarrow a \,\epsilon\, U\}$. Then the symmetric of rest is $\Box : \mathcal{P}X \rightarrow \mathcal{P}S$ which is defined by $\Box A \equiv \{a : (\forall x)(x \Vdash a \rightarrow x \,\epsilon\, A)\}$ for all $A \subseteq X$. That is, $x \,\epsilon\, \mathsf{rest}\, U \equiv \Diamond x \subseteq U$ and $a \,\epsilon\, \Box A \equiv \mathsf{ext}\,(a) \subseteq A$.

It is now immediate to see that, for any $A \subseteq X$,

$$\mathsf{int}\, A \equiv \mathsf{ext}\, \Box A \qquad\qquad x \,\epsilon\, \mathsf{int}\, A \equiv \Diamond x \between \Box A$$

$$\mathsf{cl}\, A \equiv \mathsf{rest}\, \Diamond A \qquad\qquad x \,\epsilon\, \mathsf{cl}\, A \equiv \Diamond x \subseteq \Diamond A$$

which shows both the logical nature of int and cl, and the logical duality between them. As usual, we say that A is (concrete) open if $A = \mathsf{int}\, A$ and (concrete) closed if $A = \mathsf{cl}\, A$. By symmetry, we can define two operators on $\mathcal{P}S$ by putting, for any $U \subseteq S$:

$$\mathcal{C}U \equiv \Diamond\, \mathsf{rest}\, U \qquad\qquad x \,\epsilon\, \mathcal{C}U \equiv \mathsf{ext}\,(a) \between \mathsf{rest}\, U$$

$$\mathcal{A}U \equiv \Box\, \mathsf{ext}\, U \qquad\qquad x \,\epsilon\, \mathcal{A}U \equiv \mathsf{ext}\,(a) \subseteq \mathsf{ext}\, U$$

and then say that U is formal open if $\mathcal{A}U = U$ and that U is formal closed if $\mathcal{C}U = U$. The choice of names is due to the fact that in this way it is the lattice of formal open (formal closed) subsets which comes out as isomorphic to that of concrete open (concrete closed) subsets. A summary is expressed in a picture:

concrete closed *formal open*

cl $\forall\exists$, *symmetric* $\mathcal{A}$

int $\exists\forall$, *symmetric* $\mathcal{C}$

concrete open *formal closed*

A few more details can be found in [6], while a full treatment, including also morphisms, will be in [5].

4. The new formulation of formal topology

The new definition of formal topology is obtained by first describing only in terms of the operators $\mathcal{A}$ and $\mathcal{C}$ the properties of the formal side of a concrete topological space (which is nothing but a basic pair in which B1 and B2 hold) and then taking the result as an axiomatic definition. If $\lhd$ is the relation corresponding to the operator $\mathcal{A}$, then $a \lhd U$ (that is $a \,\epsilon\, \mathcal{A}U$) is defined by $\mathsf{ext}\, a \subseteq \mathsf{ext}\, U$, which is exactly the intuitive, pointwise content of covers. The discovery of symmetry, and hence of the new operator $\mathcal{C}$, brings to the introduction of a binary positivity predicate $\mathsf{Pos}(a, F) \equiv a \,\epsilon\, \mathcal{C}F$. Also, the pointfree expression of B2 given by the condition ↓-Right below, allows to get rid of the operation $\cdot$ (and of 1).

Definition 1.12 (1995) *A formal topology is a structure* $\mathcal{S} = (S, \lhd, \mathsf{Pos})$ *where:*

a. S *is a set;*

b. $\lhd$ *is a relation between elements* a *and subsets* U *of* S *which satisfies*

$$\textit{reflexivity} \quad \frac{a \,\epsilon\, U}{a \lhd U}$$

$$\textit{transitivity} \quad \frac{a \lhd V \quad V \lhd U}{a \lhd U}$$

$$\downarrow\textit{-Right} \quad \frac{a \lhd U \qquad a \lhd V}{a \lhd U{\downarrow}V}$$

where $a{\downarrow}b \equiv \{c : c \lhd a \;\&\; c \lhd b\}$ *and* $U{\downarrow}V \equiv \cup_{a\epsilon U, b\epsilon V}\, a{\downarrow}b$*;*

c. Pos *is a relation between elements* a *and subsets* F *of* S *which satisfies*

$$\textit{antireflexivity} \quad \frac{\mathsf{Pos}(a, F)}{a \,\epsilon\, F}$$

$$\textit{transitivity} \quad \frac{\mathsf{Pos}(a, F) \qquad (\forall b)(\mathsf{Pos}(b, F) \to b \,\epsilon\, G)}{\mathsf{Pos}(a, G)}$$

d. $\lhd$ *and* Pos *are linked by*

$$\textit{compatibility} \quad \frac{\mathsf{Pos}(a, F) \qquad a \lhd U}{(\exists b \,\epsilon\, U)\mathsf{Pos}(a, F)}$$

The notation with operators $\mathcal{A}U \equiv \{a : a \lhd U\}$ and $\mathcal{C}F \equiv \{a : \mathsf{Pos}(a, F)\}$ allows a much shorter characterization: $(S, \mathcal{A}, \mathcal{C})$ is a formal topology if $\mathcal{A}$ is a closure operator satisfying $\mathcal{A}(U{\downarrow}V) = \mathcal{A}U \cap \mathcal{A}V$, $\mathcal{C}$ is an interior operator, and they are linked by $\mathcal{C}F \between \mathcal{A}U \to \mathcal{C}F \between U$.

As it was meant to be, if $\mathcal{X} = (X, \Vdash, S)$ is a concrete topological space, then $\mathcal{S}_{\mathcal{X}} \equiv (S, \lhd, \mathsf{Pos})$, where $a \lhd U \equiv \mathsf{ext}\,(a) \subseteq \mathsf{ext}\,U$ and $\mathsf{Pos}(a, F) \equiv \mathsf{ext}\,(a) \between \mathsf{rest}\,F$, is a formal topology; any such formal topology is said to be (pointwise) representable.

Formal topology with the new definition is still under development (see [5]). One can already see, however, that both locale theory and previous formal topology can be thought of as particular cases. In fact, locale theory is obtained when Pos is *improper*, that is $\mathsf{Pos}(a, F)$ is always false, and previous formal topology when Pos is *trivial*, that is $\mathcal{C}S$ is the only nonempty formal closed subset, so that the old Pos is defined by $Pos(a) \equiv \mathsf{Pos}(a, S)$. Note that, when Pos is trivial, the condition of compatibility boils down to monotonicity. Also note that nothing corresponding to positivity is now required in the definition; the idea is that one can add that assumption when convenient.

The definition of formal point has to be changed slightly, to take care of the new positivity predicate:

Definition 1.13 *In any formal topology $(S, \lhd, \mathsf{Pos})$, a subset α of S is called a formal point if, writing $\alpha \Vdash a$ for $a \epsilon \alpha$, for any $a, b \in S$ and $U, F \subseteq S$ it holds that:*

$$\alpha \text{ is inhabited:} \qquad (\exists a)(\alpha \Vdash a)$$

$$\alpha \text{ is convergent:} \qquad \frac{\alpha \Vdash a \qquad \alpha \Vdash b}{\alpha \Vdash a{\downarrow}b}$$

$$\alpha \text{ splits } \lhd: \qquad \frac{\alpha \Vdash a \qquad a \lhd U}{\alpha \Vdash U} \qquad \text{where } \alpha \Vdash U \equiv (\exists b \,\epsilon\, U)(\alpha \Vdash b)$$

$$\alpha \text{ enters } \mathsf{Pos}: \qquad \frac{\alpha \Vdash a \qquad \alpha \subseteq F}{\mathsf{Pos}(a, F)}$$

Again, the notation with $\mathcal{A}$, $\mathcal{C}$ and $\between$ is shorter: α is a formal point if $\alpha \between S$, $\alpha \between U \;\&\; \alpha \between V \to \alpha \between U{\downarrow}V$, $\alpha \between \mathcal{A}U \to \alpha \between U$ and $\alpha \subseteq F \to \alpha \subseteq \mathcal{C}F$. Note that α enters Pos if and only if $\alpha = \mathcal{C}\alpha$, that is, α is formal closed; hence α splits $\lhd$ is derivable by compatibility.

Formal spaces are defined as before.

5. Unary formal topologies and domains

The question now is whether also in the new formulation unary formal topologies are just a reformulation of Scott domains. The answer is negative, but interesting. A relevant result is:

Proposition 1.14 *For any formal topology $\mathcal{S}$ with a trivial Pos satisfying positivity, there exists a formal topology $\mathcal{A}$, according to the old definition and hence with $\cdot$ and 1, such that $\mathcal{A}$ is isomorphic to $\mathcal{S}$.*

(The idea for the proof is to simulate in pointfree terms the standard fact that the closure under finite intersections of a base is still a base, for the same topology.) But then one has a little surprise: keeping the definition of unary cover given before in definition 1.7, it may well happen that the cover of $\mathcal{S}$ is unary while that of $\mathcal{A}$ is not. Given that the class of unary formal topologies, with the old definition, corresponds to the class of Scott domains, to which kind of domains do unary formal topologies, with the new definition, correspond? First of all, we introduce a definition of unary formal topology in which also the new (binary) positivity predicate plays a role. Thus we keep the definition of unary cover (except that $Pos(a)$ is now replaced by $\mathsf{Pos}(a, S)$) and characterize unary positivity predicates by a dual condition:

Definition 1.15 *A formal topology $\mathcal{S} = (S, \lhd, \mathsf{Pos})$ is said to be unary if both $\lhd$ and Pos are unary, in the sense that they satisfy*

$$a \lhd U \;\leftrightarrow\; \mathsf{Pos}(a, S) \to (\exists b \,\epsilon\, U)(a \lhd \{b\})$$

$$\mathsf{Pos}(a, S) \;\leftrightarrow\; \mathsf{Pos}(a, S) \;\&\; (\forall b)((a \lhd \{b\} \to b \,\epsilon\, F)$$

respectively.

The following little result says that, in the case of unary covers, one should not worry about the condition $\downarrow$-Right:

Lemma 1.16 *If $\lhd$ is reflexive and unary, then it also satisfies $\downarrow$-Right.*

Proof: Assume $a \lhd U$ and $a \lhd V$, and let $\mathsf{Pos}(a, S)$. Since $\lhd$ is unary, one obtains $(\exists b \,\epsilon\, U)(a \lhd \{b\})$ and $(\exists c \,\epsilon\, V)(a \lhd \{c\})$, which by definition says that $a \,\epsilon\, U{\downarrow}V$, from which $(\exists d \,\epsilon\, U{\downarrow}V)(a \lhd \{d\})$. So $\mathsf{Pos}(a, S) \to (\exists d \,\epsilon\, U{\downarrow}V)(a \lhd \{d\})$, which means that $a \lhd U{\downarrow}V$ since $\lhd$ is unary. q.e.d.

The idea underlying the condition on Pos is that $\mathsf{Pos}(a, F)$ should hold just when $\mathsf{ext}\,(a)$ is inhabited by a point, no matter where inside $\mathsf{ext}\,(a)$ it is. The peculiarity of unary formal topologies is that they can be characterized in elementary terms, and the framework of basic pairs allows to express this in a simple and precise way.

Any preordered set $\mathcal{P} = (P, \prec)$, that is a set P with a reflexive and transitive binary relation $\prec$, can be looked at as a basic pair, namely $(P, \prec, P)$. Then the formal topology represented by $\mathcal{P}$ is unary. More generally, assume that H is any monotone, or upward closed, subset of P (that is, $a \,\epsilon\, H \;\&\; a \prec b \to b \,\epsilon\, H$). Define $\Vdash$ to be the restriction of the preorder $\prec$ to H, that is

$$a \Vdash b \;\equiv\; a \,\epsilon\, H \;\&\; a \prec b$$

Then the topology $\mathcal{S}_{\mathcal{P}}$ pointwise represented by $\mathcal{P} = (P, \prec, P)$ is unary, as we now prove. The cover and positivity predicate of $\mathcal{S}_{\mathcal{P}}$ are defined by

$$a \lhd_{\mathcal{P}} U \equiv \mathsf{ext}\,(a) \subseteq \mathsf{ext}\, U \equiv (\forall c)(c \Vdash a \to (\exists b \,\epsilon\, U)(c \Vdash b))$$

$$Pos_{\mathcal{P}}(a, F) \equiv \mathsf{ext}\,(a) \between \mathsf{rest}\, F \equiv (\exists c)(c \Vdash a \;\&\; (\forall b)(c \Vdash b \to b \,\epsilon\, F))$$

respectively. As for any basic pair, it can be proved with no difficulty that $\mathcal{S}_{\mathcal{P}}$ is a formal topology. To prove that it is unary, note first of all that $\mathsf{ext}\,(a) \between \mathsf{rest}\, S$ iff $(\exists c)(c \Vdash a)$ iff $a \,\epsilon\, H$. So $Pos_{\mathcal{P}}(a, S)$ iff $a \,\epsilon\, H$. By definition, $a \lhd_{\mathcal{P}} U$ is $(\forall c)(c \Vdash a \to (\exists b \,\epsilon\, U)(c \Vdash b))$, which by definition of $\Vdash$ is equivalent to $(\forall c)(c \,\epsilon\, H \;\&\; c \prec a \to (\exists b \,\epsilon\, U)(c \prec b))$ and hence, since H is monotone, also to $a \,\epsilon\, H \to (\exists b \,\epsilon\, U)(a \prec b)$. We can conclude that $\lhd_{\mathcal{P}}$ is unary because

$$\text{for any } a \,\epsilon\, H,\; a \prec b \text{ iff } a \lhd_{\mathcal{P}} \{b\}$$

In fact, $a \lhd_{\mathcal{P}} \{b\} \equiv \mathsf{ext}\,(a) \subseteq \mathsf{ext}\,(b) \equiv (\forall c)(c \,\epsilon\, H \;\&\; c \prec a \to c \prec b)$ iff $a \,\epsilon\, H \to a \prec b$ and hence the claim since $a \,\epsilon\, H$.

It is convenient now to adopt the notation ${\uparrow}a \equiv \{b : a \prec b\}$; by the last remark above, ${\uparrow}a = \{b : a \lhd_{\mathcal{P}} \{b\}\}$ for any $a \,\epsilon\, H$. So $\mathsf{Pos}_{\mathcal{P}}(a, F)$ can be rewritten as $(\exists c)(c \,\epsilon\, H \;\&\; c \prec a \;\&\; {\uparrow}c \subseteq F)$, which is equivalent to $a \,\epsilon\, H \;\&\; {\uparrow}a \subseteq F$. So also $\mathsf{Pos}_{\mathcal{P}}$ is unary, and the proof is complete.

It is now easy to see that actually all unary formal topologies are of this kind. In fact, if $\mathcal{S} = (S, \lhd, \mathsf{Pos})$ is unary, then $\lhd$ and Pos are uniquely determined by the subset $S^+ \equiv \{a : \mathsf{Pos}(a, S)\}$ and by the restriction of $\lhd$ to elements. By compatibility, if $\mathsf{Pos}(a, S)$ and $a \lhd \{b\}$, then $(\exists c \,\epsilon\, \{b\})Pos(c, S)$, that is

$\mathsf{Pos}(b, S)$; so S^+ is monotone. It is then only a matter of checking that $\lhd$ and Pos coincide with the cover and positivity predicate pointwise defined in the basic pair $(S, \Vdash, S)$, where $a \Vdash b \equiv a \epsilon S^+ \ \&\ a \lhd \{b\}$. This is immediate, because $(\forall c)(c \epsilon S^+ \ \&\ c \lhd \{a\} \rightarrow (\exists b \epsilon U)(c \lhd \{b\}))$ iff $a \epsilon S^+ \rightarrow (\exists b \epsilon U)(a \lhd \{b\}))$ iff $a \lhd U$ since $\lhd$ is unary; quite similarly for Pos. So we have proved that:

Theorem 1.17 *Unary formal topologies are exactly those formal topologies which are represented by basic pairs of the form $(S, \Vdash, S)$, where $\Vdash$ is the restriction of a preorder on S to a monotone subset $H \subseteq S$.*

The above theorem says that the essence of a unary formal topology is just a preordered set with a distinguished monotone subset H. To actually obtain a bijective correspondence, one has to add an extra condition saying that the preorder is determined by its restriction to H:

Definition 1.18 *A structure $\mathcal{P} = (P, \prec, H)$ is called a preorder with positivity if P is a set, $\prec$ is a preorder on P and H is a subset of P satisfying*

$$\textit{monotonicity} \quad \frac{a \epsilon H \qquad a \prec b}{b \epsilon H}$$

$$\textit{positivity} \quad \frac{a \epsilon H \rightarrow a \prec b}{a \prec b}$$

Of course, given any preorder $(P, \prec)$ and any monotone $H \subseteq P$, one can impose positivity by defining $a \prec' b \equiv a \epsilon H \rightarrow a \prec b$.

Proposition 1.19 *There is a bijective correspondence between unary formal topologies and preorders with positivity.*

Proof: With any unary formal topology $\mathcal{S} = (S, \lhd, \mathsf{Pos})$, we associate $L\mathcal{S} \equiv (S, \prec_{\lhd}\ H_{\mathsf{Pos}})$ where

$$a \prec_{\lhd} b \ \equiv\ a \lhd \{b\} \qquad\qquad H_{\mathsf{Pos}} \equiv S^+ \equiv \{a : \mathsf{Pos}(a, S)\}$$

Clearly $\prec_{\lhd}$ is a preorder. We have already seen that S^+ is monotone. To see that it satisfies positivity, note that any unary cover satisfies

$$\text{positivity} \quad \frac{a \epsilon S^+ \rightarrow a \lhd U}{a \lhd U}$$

because $a \epsilon S^+ \rightarrow a \lhd U$ iff $a \epsilon S^+ \rightarrow (a \epsilon S^+ \rightarrow (\exists b \epsilon U)(a \lhd \{b\}))$ iff $a \epsilon S^+ \rightarrow (\exists b \epsilon U)(a \lhd \{b\})$ iff $a \lhd U$. So in particular $a \epsilon S^+ \rightarrow (a \lhd \{b\})$ iff $a \lhd \{b\}$, as wished. So $L\mathcal{S}$ is a preorder with positivity.

Conversely, given any preorder with positivity $\mathcal{P} = (P, \prec, H)$, we define $W\mathcal{P}$ to be the formal topology represented by $(S, \Vdash, S)$, where $a \Vdash b \equiv a \epsilon H \ \&\ a \prec b$. By the same arguments as in the proof of theorem 1.17, this means that $W\mathcal{P}$ is equal to $(P, \lhd_{\prec}, \mathsf{Pos}_H)$ where

$$a \lhd_{\prec} U \ \equiv\ a \epsilon H \rightarrow (\exists b \epsilon U)(a \prec b)$$

$$\mathsf{Pos}_H(a,F) \;\equiv\; a \,\epsilon\, H \;\&\; \uparrow a \subseteq F)$$

It only remains to be proved that the correspondence given by L and W is bijective. To show that $LW\mathcal{P} = \mathcal{P}$, it is enough to note that $a \prec_{\lhd_\prec} b \equiv a \lhd_\prec \{b\}$ iff $a \prec b$ and that $a \,\epsilon\, H_{\mathsf{Pos}_H} \equiv \mathsf{Pos}_H(a,S)$ iff $a \,\epsilon\, H$, as we have already seen above. On the other hand, $WL\mathcal{S} = \mathcal{S}$ because $a \lhd_{\prec_\lhd} U \equiv a \,\epsilon\, H_{\mathsf{Pos}} \to (\exists b \,\epsilon\, U)(a \prec_\lhd b)$ iff $a \,\epsilon\, S^+ \to (\exists b \,\epsilon\, U)(a \lhd b)$ iff $a \lhd U$ because $\lhd$ is unary, and similarly $\mathsf{Pos}_{H_{\mathsf{Pos}}}(a,F) \equiv a \,\epsilon\, H_{\mathsf{Pos}} \;\&\; \uparrow a \subseteq F$ iff $a \,\epsilon\, S^+ \;\&\; \uparrow a \subseteq F$ iff $\mathsf{Pos}(a,F)$ because Pos is unary. q.e.d.

It should be possible to extend to morphisms the definition of L and W given above on objects, and thus obtain an equivalence of categories. However, the general notion of morphism between formal topologies (in the new formulation) is not yet safely stabilized, and this is why I prefer to leave this topic to a future occasion. Let us turn to domains.

We are going to see that for any unary formal topology $\mathcal{S}$, the formal space $Pt(\mathcal{S})$ is an algebraic cpo. The proof is simpler if we exploit the preceding proposition. In any preorder with positivity $\mathcal{P}$, a subset α of P is called a proper filter if it is inhabited, contained in H, upward closed and convergent; writing $\alpha \Vdash a$ for $a \,\epsilon\, \alpha$ as in definition 1.13 and $a{\downarrow}b$ for $\{c : c \prec a \;\&\; c \prec b\}$, the second two conditions are expressed by

$$\frac{\alpha \Vdash a \quad a \prec b}{\alpha \Vdash b} \qquad\qquad \frac{\alpha \Vdash a \quad \alpha \Vdash b}{\alpha \Vdash a{\downarrow}b}$$

respectively. The link with formal topologies is given by:

Lemma 1.20 *For any unary formal topology $\mathcal{S}$, formal points of $\mathcal{S}$ coincide with proper filters of the corresponding preorder $L\mathcal{S}$.*

Proof: Since $c \prec_\lhd a$ iff $c \lhd \{a\}$, the subsets $\{c : c \prec_\lhd a \;\&\; c \prec_\lhd b\}$ and $\{c : c \lhd a \;\&\; c \lhd b\}$ coincide (and this is why the same notation $a{\downarrow}b$ may be used). So it is enough to show that α enters Pos iff α is upward closed and $\alpha \subseteq H$. Assume that α enters Pos. Then $\alpha \Vdash a$ (with $\alpha \subseteq \alpha$) gives $\mathsf{Pos}(a,S)$, that is $a \,\epsilon\, H$; so $\alpha \subseteq H$. From $\alpha \Vdash a$ (and $\alpha \subseteq \alpha$) it follows that $\mathsf{Pos}(a,\alpha)$ and hence, if $a \prec_\lhd b$ that is $a \lhd \{b\}$, by compatibility also $\mathsf{Pos}(b,\alpha)$, from which $\alpha \Vdash b$; so α is upward closed. Conversely, assume that α is upward closed and that $\alpha \subseteq H$. Then $\alpha \Vdash a$ gives $a \,\epsilon\, H$ and $\alpha \subseteq F$ gives $\uparrow a \subseteq F$, so that $\mathsf{Pos}(a,F)$ because Pos is unary. q.e.d.

Proposition 1.21 *For any unary formal topology $\mathcal{S}$, the formal space $Pt(\mathcal{S})$ is an algebraic cpo.*

Proof: By the lemma and by proposition 1.19, it is enough to show that proper filters of any preorder with positivity form an algebraic cpo. This is a standard result (cf. e.g. [1] for a quite similar result). A variant is due to the presence of the subset H, but it gives no problems. The algebraic elements are exactly the principal filters $\uparrow a$ for any $a \,\epsilon\, H$, and for any proper filter α, $\alpha = \cup_{a \epsilon \alpha} \uparrow a$ holds. q.e.d.

Also the converse holds, that is, any algebraic cpo $\mathcal{D} = (D, \leq)$ can be presented as the formal space of a unary formal topology. Equivalently, $\mathcal{D}$ will be presented as the cpo of proper filters of a preorder with positivity. This also is familiar; to obtain a predicative proof, one must assume that $\mathcal{D}$ is set-based, that is, that the collection $K(D)$ of compact elements is given as a set-indexed family, say $a_i \in D(i \in I)$, where I is a set (see [8]). Then we say that $\mathcal{D}$ is algebraic if, for any $x \in D$, the collection $\downarrow_K x$ of compact lower bounds of x is a set-indexed family of elements which is directed (and hence inhabited) and such that $x = \vee \downarrow_K x$.

Then the idea is simple: we take $K(D)$ itself (formally, the set I on which it is indexed) as the preordered set, and associate an arbitrary element $x \in D$ with (the indexes of) compact elements below x. In particular, a compact element a must correspond to (indexes of) the formal point $\uparrow a$. So we provide I with the preorder

$$i \prec j \equiv a_j \leq a_i$$

for any $i, j \in I$. For the positivity predicate, we have no other choice than declaring any element in I to be positive. So $(I, \prec, I)$ is a preorder with (a trivial) positivity. The isomorphism from $\mathcal{D}$ into proper filters of $(I, \prec)$ is given by

$$f : x \mapsto \downarrow_I x \equiv \{i \in I : a_i \leq x\}$$

Firstly, $\downarrow_I x$ is a proper filter, for any $x \in D$. In fact, $\downarrow_I x$ is inhabited, because $i \mathrel{\epsilon} \downarrow_I x \equiv a_i \leq x$ and $\downarrow_K x$ is inhabited. It is convergent because $\downarrow_K x$ is directed. And it is upward closed because $i \mathrel{\epsilon} \downarrow_I x$ and $i \prec j$ mean that $a_i \leq x$ and $a_j \leq a_i$, so that $a_j \leq x \equiv j \mathrel{\epsilon} \downarrow_I x$. Trivially, $\downarrow_I x$ is proper because all elements of I are positive.

It is easy to check that

$$x \leq y \text{ iff } \downarrow_I x \subseteq \downarrow_I y$$

In fact, if $x \leq y$ and $i \mathrel{\epsilon} \downarrow_I x$, then $a_i \leq x$, hence $a_i \leq y$, that is $i \mathrel{\epsilon} \downarrow_I y$. Conversely, $\downarrow_I x \subseteq \downarrow_I y$ gives $\downarrow_K x \subseteq \downarrow_K y$, and hence $x \leq y$ because $x = \vee \downarrow_K x$ and $y = \vee \downarrow_K y$. So f preserves the order and is one-one. Finally, to show that f is onto, assume that α is any proper filter of $(I, \prec)$. Then $\{a_j : \alpha \Vdash j\}$ is directed in $\mathcal{D}$, and hence it has a supremum x; the claim is that $\alpha = \downarrow_I x$. In fact, if $\alpha \Vdash i$ then obviously $a_i \leq x$, that is $i \mathrel{\epsilon} \downarrow_I x$. Conversely, if $i \mathrel{\epsilon} \downarrow_I x$, that is $a_i \leq x \equiv \vee\{a_j : \alpha \Vdash j\}$, then since a_i is compact, there exists j such that $\alpha \Vdash j$ and $a_i \leq a_j \equiv j \prec i$, and hence $\alpha \Vdash i$ because α is upward closed.

We have thus proved:

Proposition 1.22 *Any set-based algebraic cpo is isomorphic to a formal space $Pt(\mathcal{S})$, where $\mathcal{S}$ is some unary formal topology.*

Technically speaking, propositions 1.21 and 1.22 are hardly new (for similar results, see for instance [1], in particular theorem 2.2.29); the novelties here are mainly conceptual, as I now point out. First, all the proofs given here are predicative; more precisely, they can be formalized in type theory, just as all of formal topology. It is nice to know, I believe, that the theory of domains, that is denotational semantics, can be expressed in a safe computer language. Secondly, the expectation is confirmed that a class of domains corresponds to unary formal topologies, also in the new formulation. The fact that such class is that of algebraic cpos, which is wider than that of Scott domains, speaks in favour of the new definition of formal topology (and of course Scott domains can be obtained by adding the condition that for all a and b there is c such that $a{\downarrow}b = {\downarrow}c$, but we leave the details). The interest of the embedding into formal topology is that all the results acquire meaning as part of a more general framework; in particular, the link between preorders and domains is given by the fundamental topological functor mapping a formal topology $\mathcal{S}$ into the formal space $Pt(\mathcal{S})$.

Finally, I admit, the extra generality given by the presence of the subset H of positive elements is not yet fully justified. Further study is necessary; in particular, I would not drop it until a thorough study of morphisms is accomplished. Note however that from a predicative point of view the class of preorders with positivity is strictly wider than that of preorders, because predicatively a subset is not necessarily a set.

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