INTUITIONISTIC FORMAL SPACES[1]

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The notion of formal space was introduced by Fourman and Grayson [FG] only a few years ago, but it is only a recent though important step of a long story whose roots involve such names as Brouwer and Stone and whose development is due to mathematicians from different fields, mainly algebraic geometry, category theory and logic.

I am not going to tell this story (but see for instance [J] and [G]). For our purposes here, it is enough to say that the main idea is to reverse the traditional conceptual order of definitions in topology and define points as particular filters of neighbourhoods, rather than opens as particular sets of points (I adjust the language to this point of view, by considering open also as a noun). This explains why our topic is sometimes called pointless topology; formal or abstract topology, or just topology tout court would be preferable.

The basic notion is that of locale which, roughly speaking, is a lattice satisfying all those properties of opens in a topological space which are expressible without mention of points. Thus locales are complete lattices, where $\wedge$ and $\bigvee$ correspond to finite intersection and arbitrary union, satisfying the law of infinite distributivity $a \wedge \bigvee\{b_i : i \in I\} = \bigvee\{a \wedge b_i : i \in I\}$. One could also look at locales as the solution of x : topological spaces = boolean algebras : Stone spaces.

In [FG] a method is given to construct a locale from an entailment relation and the result is called a formal space. We will show here that, with minor modifications, their method is general enough to yield all locales. Our basic notion is that of covering relation, which, besides entailment, is strictly related to what is known in the literature under a wide variety of names: coverage, J-operator, congruence, ...

This is not a mere technical device. In fact, from the intuitionistic point of view, which is here taken seriously and thus is not reduced to putting an asterisk where the axiom of choice is used, the notion of covering permits a study of topology which avoids such problematic notions as the powerset of a given set or quantification over subsets. I here try to show how this is possible, that is

[1]This paper is exactly the same as *Intuitionistic formal spaces - a first communication*, in: Mathematical Logic and its Applications, D. Skordev ed., Plenum 1987, pp. 187-204 by the same author, except for: (*i.*) the conditions on the positivity predicate (part 3. of definition 1.1 and end of section 1) and the treatment of Scott domains (section 8), which have been modified as explained in the addendum *Intuitionistic formal spaces vs. Scott domains*, in: Atti del Congresso Temi e prospettive della logica e della filosofia della scienza contemporanee, vol. 1, CLUEB, Bologna 1988, pp. 159-163; (*ii.*) the correction of some of the misprints; (*iii.*) one change in notation (now $\lhd$ is used for covering relations, rather than $\leq$) and one in terminology (now 'weak transitivity' replaces 'weakening'). For an update on the development of formal topology, see the survey *Formal topology - twelve years of development*, in preparation, by the same author.

begin to develop formal topology in the framework of Martin-Löf's intuitionistic type theory (or constructive set theory).

One of the fundamental aspects of Martin-Löf's type theory is the distinction between sets (or (data) types) and categories (or logical types). However, while a formal treatment of sets has by now been developed and reached maturity (see for instance [ITT]), a similar work for categories is still in progress. The lack of texts with rules to handle categories is particularly felt here, where we deal with the category of opens, among others (but see [SI] for the basic logical types of propositions and truths). I thus have to ask the reader to rely on a pragmatic principle, which is based on my understanding of Martin-Löf's views by direct talking: all what we are going to do informally, can also be done formally in Martin-Löf's foundational theory, once it will appear in complete form. Only a few additions to [ITT] are necessary here, and we give them in the preliminaries below.

During my visit to Stockholm in Spring 1984, following Martin-Löf's proposal I began to work with him on the topics of this paper; one of his aims then was to give a general form to the connection he had discovered between denotational and operational semantics for programming languages (cf. [SG] and the course given in Udine at the Course on Computation Theory in September 1984). Since then, we have gradually changed basic definitions, also under the influence of the literature, mainly [FG]. The first public outcome was the 'short course' I gave at the VIII Incontro di Logica Matematica in Siena in January 1985. The present formulation is due to Martin-Löf and was presented by him in a few seminars in Stockholm in May 1985.

This paper contains little beyond definitions and a few meaningful examples, including Scott's domains. Much work remains to be done, which, together with time, will certainly cause modifications on what is presented here. It should be clear however that even the present fragment would not exist without the work by Per Martin-Löf; most ideas, definitions and examples have been suggested by him and discussed together (but of course responsibility here is only mine). I am glad to thank him for teaching me so much. I also thank Isa Bossi and Silvio Valentini.

Preliminaries

The following simple though radical additions to [ITT] will be sufficient here. A subset U of a set S is a unary propositional function with argument ranging over S, shortly $U(a)$ *prop* $(a \in S)$; we write as usual $U \subseteq S$, and $a \in U$ for $U(a)$ (note that, contrary to [ITT], $a \in U$ is a proposition here). We also often write $\{a : U(a)\}$ for U, and $\{f(a) : U(a)\}$ for $(\exists a \in S)(I(S, x, f(a))\ \&\ U(a))$. A relation between A and B, where A, B are either sets or categories, is just a propositional function with two arguments, one in A and one in B. Recall that propositions, and hence a fortiori subsets of any set, form a category which is not a set. For A, B propositions, we here write $A \leq B$ for $A\ true \rightarrow B\ true$ and thus $A \leq B$ iff $A \rightarrow B = \top$, where $\top \equiv$ any true proposition and $(A = B) \equiv (A \leq B\ \&\ B \leq A)$. Finally, recall that any set is given together with an equality relation between elements; such equality is denoted simply by = here since it should always be clear to which set it refers.

1 Formal topologies

The classical notion of topological space is not suitable, as it stands, for an intuitionistic treatment, mainly because opens generally form a proper category and coverings are always defined pointwise. To bring opens into our framework, we build them up from basic neighbourhoods, which are supposed to form a set, by means of an abstract covering relation; and we define points in terms of the algebra of opens ('pointless topology'), and thus ultimately as particular filters of neighbourhoods.

Thus the usual notion of topological space corresponds here to two notions, formal topology and formal space. We must begin with the former. The hints above should justify the following definition (but see also section 2 below for some intuitive motivations):

Definition 1.1 *A formal topology $\mathcal{A}$ consists of:*

1. *a formal base, namely a set $S_{\mathcal{A}}$ with a binary operation $\wedge_{\mathcal{A}}$ and a distinguished element $\Delta_{\mathcal{A}}$, such that $S_{\mathcal{A}}, \wedge_{\mathcal{A}}, \Delta_{\mathcal{A}}$ form a semilattice with one;*

2. *a covering relation, that is a relation $a \lhd_{\mathcal{A}} U$ prop $(a \in S, U \subseteq S)$ which for arbitrary $a, b \in S_{\mathcal{A}}$, $U, V \subseteq S_{\mathcal{A}}$ satisfies*

   reflexivity $$\frac{a \in U}{a \lhd_{\mathcal{A}} U}$$

   transitivity $$\frac{a \lhd_{\mathcal{A}} U \qquad U \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} V} \qquad \textit{where } U \lhd_{\mathcal{A}} V \equiv (\forall a \in U)(a \lhd_{\mathcal{A}} V)$$

   $\wedge$*-left* $$\frac{a \lhd_{\mathcal{A}} U}{a \wedge_{\mathcal{A}} b \lhd_{\mathcal{A}} U} \qquad \frac{b \lhd_{\mathcal{A}} U}{a \wedge_{\mathcal{A}} b \lhd_{\mathcal{A}} U}$$

   $\wedge$*-right* $$\frac{a \lhd_{\mathcal{A}} U \qquad a \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} U \wedge_{\mathcal{A}} V} \qquad \textit{where } U \wedge_{\mathcal{A}} V \equiv \{b \wedge_{\mathcal{A}} c : b \in U, c \in V\}$$

3. *a consistency predicate, that is a property $\mathsf{Pos}_{\mathcal{A}}(a)$ prop $(a \in S)$ which for arbitrary $a \in S_{\mathcal{A}}$, $U \subseteq S_{\mathcal{A}}$ satisfies*

   monotonicity $$\frac{\mathsf{Pos}_{\mathcal{A}}(a) \qquad a \lhd_{\mathcal{A}} U}{\mathsf{Pos}_{\mathcal{A}}(U)} \qquad \textit{where } \mathsf{Pos}_{\mathcal{A}}(U) \equiv (\exists b \in U)\mathsf{Pos}_{\mathcal{A}}(b)$$

   positivity $$\frac{\mathsf{Pos}_{\mathcal{A}}(a) \to a \lhd_{\mathcal{A}} U}{a \lhd_{\mathcal{A}} U}$$

Elements $a, b, c, \cdots$ of $S_{\mathcal{A}}$ are called formal basic neighbourhoods, or simply neighbourhoods; $a \lhd_{\mathcal{A}} U$ is read U covers a, or U is a cover of a, and $\mathsf{Pos}_{\mathcal{A}}(a)$ is read a is positive, or consistent. We read also $U \lhd_{\mathcal{A}} V$ as V covers U; it is then natural to say that two subsets are equal if they cover each other:

$$(U =_{\mathcal{A}} V) \equiv (U \lhd_{\mathcal{A}} V \ \& \ V \lhd_{\mathcal{A}} U)$$

It is immediate to see that also for subsets $\lhd_{\mathcal{A}}$ is reflexive and transitive, and hence that $=_{\mathcal{A}}$ is an equivalence relation.

For convenience, we will omit the subscript $\mathcal{A}$, except when it is necessary to avoid confusion; in particular, it is needed to distinguish $=_{\mathcal{A}}$ above from extensional equality of subsets, denoted by $=$ as usual.

Definition 1.2 *For any formal topology $\mathcal{A}$, $Open(\mathcal{A})$ is the category $\mathcal{P}(S)$ of subsets of S, with equality $=_{\mathcal{A}}$. The objects of $Open(\mathcal{A})$ are called formal opens of $\mathcal{A}$.*

In other words, an open in $\mathcal{A}$ is the equivalence class under $=_{\mathcal{A}}$ of a subset of S.

It is well known that opens of a topological space form a complete Heyting algebra (see for instance [J], p. 39). Our next aim is to see that this is true also for formal opens. However, it is easier if we first have a little stock of derived rules at our disposal.

From reflexivity we have $U \subseteq V \rightarrow U \lhd V$, and hence

weak transitivity $\qquad \dfrac{a \lhd U \qquad U \subseteq V}{a \lhd V}$

Applying weak transitivity and substitution rules of [ITT], we have

substitution $\qquad \dfrac{a = b \quad b \lhd V}{a \lhd V} \qquad \dfrac{a \lhd U \quad U = V}{a \lhd V}$

A fortiori, we obtain that $=_{\mathcal{A}}$ respects $\lhd$; that is, writing $a \lhd b$ for $a \lhd \{b\}$, and hence $a =_{\mathcal{A}} b$ for $\{a\} =_{\mathcal{A}} \{b\}$

$$\frac{a =_{\mathcal{A}} b \quad b \lhd U \quad U =_{\mathcal{A}} V}{a \lhd V}$$

Since $a \wedge \Delta = a$, by $\wedge$-left applied to $\Delta \lhd \Delta$ we have $a \lhd \Delta$ and hence

$$U \lhd \Delta$$

Applying first $\wedge$-left and then $\wedge$-right, we obtain

stability $\qquad \dfrac{a \lhd U \quad b \lhd V}{a \wedge b \lhd U \wedge V}$

which, since $b \lhd b$, gives in particular

localisation $\qquad \dfrac{a \lhd U}{a \wedge b \lhd U \wedge b} \qquad$ where $U \wedge b \equiv \{a \wedge b : a \in U\}$

This completes the list of derived rules we need.

We have already seen that $\lhd$ is a preorder on $\mathcal{P}(S)$. So, to see that it is a partial order on $Open(\mathcal{A})$, it is enough to check that $\lhd$ is respected by $=_{\mathcal{A}}$, namely

$$(U \lhd V') \ \& \ (U =_{\mathcal{A}} U') \ \& \ (V =_{\mathcal{A}} V') \rightarrow U' \lhd V'$$

which is obvious by transitivity.

Next we show that $\wedge$ gives to $Open(\mathcal{A})$ the structure of a semilattice. To begin with, we must check that $=_{\mathcal{A}}$ respects $\wedge$, namely

$$(U =_{\mathcal{A}} U') \ \& \ (V =_{\mathcal{A}} V') \rightarrow U \wedge V =_{\mathcal{A}} U' \wedge V'$$

(Proof. If $a \in U \wedge V$, then there exist $b \in U$, $c \in V$ such that $a = b \wedge c$. Since $U \lhd U'$, $V \lhd V'$, we have $b \lhd U'$, $c \lhd V'$ and hence $b \wedge c \lhd U' \wedge V'$ by stability. This proves $U \wedge V \lhd U' \wedge V'$, and the converse holds by symmetry.) Then it is enough to show that $\wedge$ gives the infimum, or meet, with respect to $\lhd$, namely

$$U \wedge V \lhd U, \ U \wedge V \lhd V$$

$$W \lhd U \ \& \ W \lhd V \rightarrow W \lhd U \wedge V$$

which are obvious by $\wedge$-left and $\wedge$-right respectively.

Since $U \lhd \Delta$ and obviously $\emptyset \lhd U$ for any U, $\emptyset$ and $\{\Delta\}$ are zero and one of $Open(\mathcal{A})$, respectively.

The last step is to define arbitrary suprema, or joins. For any family of subsets $(U_i)_{i \in I}$ we put

$$\bigvee_{i \in I} U_i \equiv \cup_{i \in I} U_i$$

(in section 3 we try to give a reason for this definition). Again, we first have to check that $=_{\mathcal{A}}$ respects $\bigvee$, that is

$$(\forall i \in I)(U_i =_{\mathcal{A}} U_i') \rightarrow \bigvee_{i \in I} U_i =_{\mathcal{A}} \bigvee_{i \in I} U_i'$$

This is not difficult, by using weak transitivity. Now we easily see (the first by weak transitivity, the second by intuitionistic logic) that indeed $\bigvee$ gives joins:

$$(\forall i \in I)(U_i \lhd \bigvee_{i \in I} U_i)$$

$$(\forall i \in I)(U_i \lhd V) \rightarrow \bigvee_{i \in I} U_i \lhd V$$

Finally, the definition of $\wedge$ tells us that $V \wedge \cup_{i \in I} U_i = \cup_{i \in I}(V \wedge U_i)$, and hence a fortiori infinite distributivity holds:

$$V \wedge \bigvee_{i \in I} U_i =_{\mathcal{A}} \bigvee_{i \in I} (V \wedge U_i)$$

We follow [FG] here and use the word frame for complete lattices with infinite distributivity; so we have shown above that $Open(\mathcal{A})$ is a frame. It is well known that in any frame the operation of implication is definable, that is we can make a complete Heyting algebra out of any frame. In our case, such definition reduces to:

$$U \to_{\mathcal{A}} V \equiv \{a \in S : U \wedge a \lhd V\}$$

We leave it to the reader to show, first of all, that $\to_{\mathcal{A}}$ is well defined (i.e. it is respected by $=_{\mathcal{A}}$), and that it satisfies the usual characterization of implication, namely

$$W \lhd U \to_{\mathcal{A}} V \text{ iff } W \wedge U \lhd V$$

We thus have completed the proof of

Proposition 1.3 *$Open(\mathcal{A}), \wedge_{\mathcal{A}}, \bigvee_{\mathcal{A}}, \emptyset, \Delta_{\mathcal{A}}, \to_{\mathcal{A}}$ form a complete Heyting algebra*

Proving that any complete Heyting algebra, or better any frame, is obtainable as above, is now an easy task. We say that a frame H is based on a set D_0 if H is the closure of D_0 under finite meets and arbitrary joins. The closure of D_0 under finite meets, including the top element of H, forms a set D, which is a semilattice with one. We define a covering on D putting

$$a \lhd_K U \equiv a \leq_H \bigvee U$$

Then the assignment $U \to \bigvee U$ is an isomorphism of $Open(\mathcal{A})$ onto H, because by infinite distributivity $\wedge$ is preserved and any element of H can be written as join of elements in D.

From the classical point of view, any frame is trivially set-based and hence representable as above, which means that our approach is not restrictive. On the other hand, it is our claim that the notion of formal topology itself is the intuitionistic counterpart of classical frames, and what follows should justify it.

We now turn to the consistency predicate. First of all, note that the rule

ex falso quodlibet $$\frac{\neg \mathsf{Pos}_{\mathcal{A}}(a)}{a \lhd_{\mathcal{A}} \emptyset}$$

is immediately derivable from positivity by intuitionistic logic. Recalling that $\mathsf{Pos}(U) \equiv (\exists b \in U)\ \mathsf{Pos}(b)$, we can easily extend monotonicity to subsets

$$\frac{\mathsf{Pos}(U) \qquad U \lhd V}{\mathsf{Pos}(V)}$$

In particular $=_{\mathcal{A}}$ respects Pos, that is

$$\mathsf{Pos}(U)\ \&\ (U =_{\mathcal{A}} V) \to \mathsf{Pos}(V)$$

so that we may speak of consistency for opens. The empty open is not consistent,

$$\neg \mathsf{Pos}(\emptyset)$$

since trivially $\neg(\exists a \in \emptyset)\mathsf{Pos}(a)$. Therefore also

$$\frac{U \lhd \emptyset}{\neg\mathsf{Pos}(U)}$$

since $\mathsf{Pos}(U)$ together with $U \lhd \emptyset$ gives $\mathsf{Pos}(\emptyset)$ by monotonicity. But by ex falso quodlibet also the converse holds

$$\frac{\neg\mathsf{Pos}(U)}{U \lhd \emptyset}$$

and hence

$$\neg\mathsf{Pos}(U) \leftrightarrow U =_{\mathcal{A}} \emptyset$$

that is, $\emptyset$ is the only inconsistent open.

We intend $\mathsf{Pos}(U)$ to be a positive way to assert that U is not empty (classically, from what above we would have $\mathsf{Pos}(U) \leftrightarrow U \neq_{\mathcal{A}} \emptyset$). A similar predicate is also introduced in [FG], p. 113, but its definition requires a quantification over subsets, which is not accepted here.

The meaning of our definition is better understood after the following:

Proposition 1.4 *The following conditions are all equivalent:*

1. $\dfrac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$ *positivity*
2. $\dfrac{a^+ \lhd U}{a \lhd U}$ *where* $U^+ \equiv \{b \in U : \mathsf{Pos}_{\mathcal{A}}(b)\}$ *and* $a^+ \equiv \{a\}^+$
3. $U \lhd U^+$
4. $\dfrac{a \lhd U}{a \lhd U^+}$ *openness*
5. $a \lhd a^+$

Proof. The first equivalence is just the observation that $\mathsf{Pos}(a) \to a \lhd U$ is logically equivalent to $a^+ \lhd U$. Now assume positivity holds, and let $a \in U$. From $\mathsf{Pos}(a)$ we then have $a \in U^+$, hence $a \lhd U^+$ by reflexivity, that is $\mathsf{Pos}(a) \to a \lhd U^+$ holds, so that by positivity $a \lhd U^+$; this shows $U \lhd U^+$. The other implications needed to show equivalence follow by very simple deductions in the order suggested.

Openness is necessary to express the fact that only consistent opens contribute to covers. From the classical point of view, openness or any of its equivalents has little meaning, since one can easily prove that:

Proposition 1.5 *Any decidable property satisfying monotonicity and ex falso quodlibet, also satisfies positivity, and hence is a consistency predicate.*

Finally note that ex falso quodlibet can be seen as a special case of positivity, obtained by considering $U = \emptyset$ in 2. of proposition 1.4.

2 Example. Concrete spaces

Consider the following naive definition of topological spaces:

Definition 2.1 *A concrete space $\mathcal{M}$ consists of:*

1. *a set M of concrete points $m, n, \ldots$*
2. *a set S of indices $a, b, \ldots$, with a binary operation $\wedge$ and a distinguished element Δ*
3. *a neighbourhood relation, that is a relation $N(m, a)$ prop ($m \in M, a \in S$) which for arbitrary $m \in M$, $a, b \in S$ satisfies:*

$$N(m, \Delta) = \top, \qquad N(m, a \wedge b) = N(m, a) \;\&\; N(m, b)$$

Of course, a is meant to be an index for the subset $N_a \equiv \{m \in M : N(m, a)\}$, called a concrete neighbourhood. The family $(N_a)_{a \in S}$ is thus a concrete base for a concrete topology on M, since, by the assumptions in 3., $N_\Delta = M$ and $N_{a \wedge b} = N_a \cap N_b$. Concrete opens are then defined, as usual, to be unions of concrete neighbourhoods: for every $U \subseteq S$, $N_U \equiv \{m \in M : (\exists a \in U) N(m, a)\}$ is the concrete open which is the union of the family $(N_a)_{a \in S}$. We might equivalently define concrete opens to be those subsets P of M for which the usual condition $(\forall m \in P)(\exists a \in S)(N_a \subseteq P \;\&\; m \in N_a)$ holds.

The trouble is that many interesting spaces can not be presented in this way, because the points we want to consider do not form a set or the covering relation is not defined pointwise. This is why we introduce the notion of formal topology, which now can be seen as the abstract result of the following concrete actions on concrete spaces:
1. add to $S, \wedge, \Delta$ a relation $a \lhd U$, which holds iff $(N_b)_{b \in U}$ covers N_a and a predicate $\mathsf{Pos}(a)$ which holds iff $(\exists m \in M)(m \in N_a)$;
2. write down all the properties of $\wedge, \Delta, \lhd$ and Pos which can be expressed without mentioning concrete points;
3. get rid of concrete points, and hence also of the neighbourhood relation.

The content of this intuitive explanation is formally expressed as follows: if, for any concrete space $\mathcal{M}$, we put

$$S_{\mathcal{A}(M)} \equiv S, \qquad \wedge_{\mathcal{A}(M)} \equiv \wedge, \qquad \Delta_{\mathcal{A}(M)} \equiv \Delta$$
$$a \lhd_{\mathcal{A}(M)} U \equiv (\forall m \in M)(N(m, a) \to (\exists b \in U) N(m, b))$$
$$\mathsf{Pos}_{\mathcal{A}(M)}(a) \equiv (\exists m \in M)(N(m, a)$$

we obtain a formal topology $\mathcal{A}(\mathcal{M})$ with base S.

Let us look at a concrete example of concrete space. Contrary to propositions, the formulae of a fixed formal language $\mathcal{L}$, say that of predicate logic, form a set. In particular, let S be the set of formulae $A(x)$ with at most x free. Given a structure for $\mathcal{L}$ based on the set M, we put $N(m, A) \equiv A(m)$ is true. Then M, S, N form a concrete space, whose concrete neighbourhoods are the $\mathcal{L}$-definable subsets of M, and of course two formal neighbourhoods are extensionally equal iff they define the same subset of M.

3 Coverings as closure operators

Some reader may be annoyed by the fact that we give formal opens of a formal topology $\mathcal{A}$ only up to $=_{\mathcal{A}}$-equivalence classes. If so, the following alternative approach can be taken.

With any formal topology $\mathcal{A}$, we associate an operator $\mathcal{C}l_{\mathcal{A}}$ acting on subsets of S, by putting

$$\mathcal{C}l_{\mathcal{A}}(U) \equiv \{a \in S : a \lhd_{\mathcal{A}} U\}$$

To minimize subscripts, we often write $\mathcal{A}(U)$ for $\mathcal{C}l_{\mathcal{A}}(U)$. Note that by definition $a \in \mathcal{A}(U)$ iff $a \lhd U$. Therefore

$$U \lhd V \text{ iff } U \subseteq \mathcal{A}(V)$$

Using this equivalence, one easily shows that:

Proposition 3.1 *For any formal topology $\mathcal{A}$, the following hold:*

1. $U \subseteq \mathcal{A}(U)$
2. $U \subseteq V \to \mathcal{A}(U) \subseteq \mathcal{A}(V)$
3. $\mathcal{A}(\mathcal{A}(U)) = \mathcal{A}(U)$

That is, $\mathcal{C}l_{\mathcal{A}}$ is a closure operator on $\mathcal{P}(S)$. Again by the above equivalence, we have

$$U \lhd V \text{ iff } \mathcal{A}(U) \subseteq \mathcal{A}(V)$$

and hence

$$U = V \text{ iff } \mathcal{A}(U) = \mathcal{A}(V)$$

In particular, also

$$U =_{\mathcal{A}} \mathcal{A}(U)$$

Let us say that U is $\mathcal{A}$-closed, or saturated, when $U = \mathcal{A}(U)$; then each $=_{\mathcal{A}}$-equivalence class is represented by one and only one $\mathcal{A}$-closed subset, which is the greatest in the class. That is, $\mathcal{A} : Open(\mathcal{A}) \to Sat(\mathcal{A})$, where $Sat(\mathcal{A})$ is the category of saturated subsets, is a bijection.

It is well known that, when $\mathcal{A}$ is a closure operator, $\mathcal{A}$-closed subsets form a complete lattice, with meets given by intersection and joins defined by $\bigvee_{i \in I} \mathcal{A}(U_i) \equiv \mathcal{A}(\cup_{i \in I} \mathcal{A}(U_i))$. With standard calculations, we see that $\mathcal{A}(\cup_{i \in I} U_i) = \bigvee_{i \in I} \mathcal{A}(U_i)$, which at the same time means that the function $\mathcal{A}$ preserves joins and justifies the very definition of joins as unions which we gave for opens.

Since $\mathcal{A}(\Delta) = S$, $\mathcal{A}$ preserves one, and since $\mathcal{A}(\emptyset)$ is the least saturated subset, $\mathcal{A}$ also preserves zero. Thus to show that $\mathcal{A}$ is a frame homomorphism, we only have to prove that $\mathcal{A}$ preserves meets, i.e.

4. $\mathcal{A}(U \wedge V) = \mathcal{A}(U) \cap \mathcal{A}(V)$

The inclusion from left to right follows immediately from $U \wedge V \lhd U$, $U \wedge V \lhd V$, while the converse inclusion holds by $\wedge$-right.

Finally, note that $U \rightarrow_{\mathcal{A}} V$ is saturated for any $U, V \subseteq S$ (we leave this as an exercise). This means both that $\rightarrow_{\mathcal{A}}$ is a good implication also in $Sat(\mathcal{A})$, and that $\mathcal{A}(U \rightarrow_{\mathcal{A}} V) = \mathcal{A}(U) \rightarrow_{\mathcal{A}} \mathcal{A}(V)$, that is $\mathcal{A}$ preserves $\rightarrow_{\mathcal{A}}$. This completes the proof of:

Proposition 3.2 *$Sat(\mathcal{A})$ is a complete Heyting algebra isomorphic to $Open(\mathcal{A})$.*

We will in the sequel confuse the two algebras; as hinted above, in $Sat(\mathcal{A})$ equality is extensional.

We have shown that for any formal topology $\mathcal{A}$, $\mathcal{C}l_{\mathcal{A}}$ is a closure operator which preserves meets, i.e. is an operator on $\mathcal{P}(S)$ satisfying 1.– 4. An interesting fact is that the converse also holds, in the following sense. For every operator $\mathcal{C} : P(S) \rightarrow P(S)$, we define a relation $\lhd_{\mathcal{C}}$ by putting

$$a \lhd_{\mathcal{C}} U \equiv a \in \mathcal{C}(U)$$

Then it is not difficult to reverse what we did, and show that when $\mathcal{C}$ is a closure operator $\lhd_{\mathcal{C}}$ is reflexive and transitive, and moreover that, when $\mathcal{C}$ satisfies 4., $\wedge$-rules hold for $\lhd_{\mathcal{C}}$. Since the correspondence between $\mathcal{C}$ and $\lhd_{\mathcal{C}}$ is obviously biunivocal, we have

Proposition 3.3 *There is an isomorphism between covering relations and meet-preserving closure operators.*

The notion of J-operator (see e.g. [J], p. 48) is strictly related to that of meet-preserving closure operator. In fact, one can show that J-operators on $Open(\mathcal{A})$ coincide with those meet-preserving closure operators $\mathcal{B}$ which satisfy $\mathcal{A}(U) \subseteq \mathcal{B}(U)$ for any $U \subseteq S$ (cf. definition 4.4 below).

4 Points and spaces

We have introduced bases, opens and coverings up to now, but still miss points. To grasp the following definition, it may help to look at it as the result of describing properties of concrete points without mentioning them.

Definition 4.1 *A formal point on the formal topology $\mathcal{A}$ is a subset $\alpha(a)\,prop$ $(a \in S)$ of S which for arbitrary $a, b \in S$, $U \subseteq S$ satisfies:*

1. $\alpha(a \wedge b) = \alpha(a)\ \&\ \alpha(b), \qquad \alpha(\Delta) = \top$
2. $a \lhd_{\mathcal{A}} U \rightarrow \alpha(a) \leq \alpha(U) \qquad$ *where* $\alpha(U) \equiv (\exists b \in U)\alpha(b)$
3. $\alpha(a) \rightarrow \mathsf{Pos}(a)$

The definition above is simply the translation into our approach of the usual definition of points as completely prime filters over a frame (see e.g. [J], p. 41). To see this, let us first say that a propositional function $F(U)$ *prop* $(U \subseteq S)$ is a filter on $Open(A)$ if $F(U \wedge V) = F(U)$ & $F(V)$, $F(\Delta) = \top$ and $U \lhd V \rightarrow F(U) \leq F(V)$. We then say that the filter F is completely prime if $F(\cup_{i \in I} U_i) = (\exists i \in I) F(U_i)$ and consistent if $F(U) \rightarrow \mathsf{Pos}(U)$ (note that classically consistency is expressed by requiring $\neg F(\emptyset)$). Now, given a point α we put as above $\alpha(U) \equiv (\exists a \in U)\alpha(a)$ and then can easily check that $\alpha(U)$ *prop* $(U \subseteq S)$ is a completely prime consistent filter on $Open(\mathcal{A})$. Conversely, given a completely prime consistent filter F we put $\alpha_F(a) \equiv F(\{a\})$ and check (exercise) that α_F is a point. Since the correspondence is obviously biunivocal, we have

Proposition 4.2 *There is an isomorphism between points on $\mathcal{A}$ and completely prime consistent filters on $Open(\mathcal{A})$.*

We denote by $Pt(\mathcal{A})$ the category of points on $\mathcal{A}$. Writing $\phi(\alpha, a)$ for $\alpha(a)$, we immediately see that $\phi(\alpha, a)$ *prop* $(\alpha \in Pt(\mathcal{A}), a \in S)$ is a neighbourhood relation, since by the definition of point

$$\phi(\alpha, \Delta) = \top, \qquad \phi(\alpha, a \wedge b) = \phi(\alpha, a) \;\&\; \phi(\alpha, b)$$

We don't have a concrete space, however, for the simple reason that $Pt(\mathcal{A})$ is in general not a set. Still we say that a is a formal neighbourhood of α if $\phi(\alpha, a)$, that is $\alpha(a)$, is true. More generally, we can define formal topological notions on $Pt(\mathcal{A})$ in terms of $\mathcal{A}$ (for instance, see the definition of subspaces below and of continuos functions in section 6). This justifies the following

Definition 4.3 *For any formal topology $\mathcal{A}$, we call $Pt(\mathcal{A})$ the formal space induced by $\mathcal{A}$.*

Of course, a reader who doesn't distinguish sets from categories will consider $Pt(\mathcal{A})$ as an ordinary topological space, with the base given by the family $\phi(a) \equiv \{\alpha \in Pt(\mathcal{A}) : \phi(\alpha, a)\}$ for $a \in S$ and where of course the covering relation is just inclusion. Then the assertion that $\phi(U) \equiv \cup\{\phi a : a \in U\}$ covers ϕa in $Pt(\mathcal{A})$ iff U covers a in $\mathcal{A}$, or equivalently $(\forall \alpha \in Pt(a))(\alpha(a) \rightarrow \alpha(U)) \rightarrow a \lhd U$, is far from being trivial, and actually is often a strong existence principle (see for instance end of sections 7 and 9). Here we take the other way round, and define $\phi a \subseteq \phi(U)$ to mean that $a \lhd U$ (which is possible only because in constructive set theory no other meaning is given to quantification over the category of points). For similar reasons, we say that $Pt(\mathcal{A})$ is proper when $\mathsf{Pos}(\Delta)$ is true.

Given any two formal topologies $\mathcal{A}$, $\mathcal{B}$ on the same base S, we say that $\mathcal{A}$ is finer than $\mathcal{B}$ (and $\mathcal{B}$ coarser than $\mathcal{A}$) if every open in $\mathcal{B}$ is also an open in $\mathcal{A}$, and every neighbourhood which is consistent in $\mathcal{B}$ is also consistent in $\mathcal{A}$. To put it in symbols, $\mathcal{A}(U) \subseteq \mathcal{B}(U)$ and $\mathsf{Pos}_{\mathcal{B}}(U) \rightarrow \mathsf{Pos}_{\mathcal{A}}(U)$ for any $U \subseteq S$. It is immediate to see that when $\mathcal{A}$ is finer than $\mathcal{B}$ any point on $\mathcal{B}$ is also a point on $\mathcal{A}$. This justifies

Definition 4.4 *If $\mathcal{A}$ is finer than $\mathcal{B}$, then $Pt(\mathcal{B})$ is called a subspace of $Pt(\mathcal{A})$.*

Two kinds of spaces deserve specific attention, and hence specific names.

Definition 4.5 *If the covering relation $\lhd$ of a formal topology $\mathcal{A}$ satisfies 1. (or 2.) below, then $\mathcal{A}$ is called a Stone (or Scott, resp.) topology, and $Pt(\mathcal{A})$ a Stone (or Scott, resp.) space:*

1. $a \lhd U \to a \lhd U_0$ *for some finite* $U_0 \subseteq U$
2. $\mathsf{Pos}(a)$ & $a \lhd U \to a \lhd b$ *for some* $b \in U$

The names I have chosen are justified below (after my lecture in Druzhba, Y. Ershov has kindly brought to my attention f-spaces, which he introduced independently of Scott and which should correspond exactly to what I here call Scott spaces; thus it seems that the name Scott here is justified not only by the results in section 8, but also by gaps in my knowledge, which I am not able to fill in now).

A perspicuous characterization of Stone and Scott topologies is easily obtained through closure operators. Recall that a closure operator $\mathcal{C}$ is called algebraic if $\mathcal{C}(U) = \cup\{\mathcal{C}(U_0) : U_0$ is a finite subset of $U\}$ holds for any $U \subseteq S$. Here we also say that $\mathcal{C}$ is irreducible if for any consistent $U \subseteq S$, $\mathcal{C}(U) = \cup\{\mathcal{C}(a) : a \in U\}$ (where of course $\mathcal{C}(a) \equiv \mathcal{C}(\{a\})$). Then, working out definitions and using the equality $\mathcal{A}(U) = \mathcal{A}(U^+)$, we obtain the following characterizations:

Proposition 4.6 *For any formal topology $\mathcal{A}$,*

1. *$\mathcal{A}$ is Stone iff $\mathcal{C}l_{\mathcal{A}}$ is algebraic;*
2. *$\mathcal{A}$ is Scott iff $\mathcal{C}l_{\mathcal{A}}$ is irreducible.*

Points on Stone and Scott topologies also have a neat characterization. As usual, let us say that a subset $\alpha(a)$ *prop* $(a \in S)$ is a consistent filter on $S, \wedge, \lhd_{\mathcal{A}}$ if it satisfies 1.1, 1.3 above and $a \lhd_{\mathcal{A}} b \to \alpha(a) \leq \alpha(b)$; we also say that a filter α is prime if it satisfies $a \lhd_{\mathcal{A}} \{b_1, \cdots, b_n\} \to \alpha(a) \leq \alpha(b_1) \vee \cdots \vee \alpha(b_n)$. Then, by a little more than working out definitions, we obtain

Proposition 4.7 *For any formal topology $\mathcal{A}$,*

1. *if $\mathcal{A}$ is Stone, $Pt(\mathcal{A})$ is the category of consistent prime filters;*
2. *if $\mathcal{A}$ is Scott, $Pt(\mathcal{A})$ is the category of all consistent filters.*

It is clear that any Stone topology is compact, in the sense that from $\Delta \lhd U$ we have $\Delta \lhd U_0$ for some finite U_0. Given an arbitrary topology $\mathcal{A}$, we can get its Stone compactification $\mathcal{C}(\mathcal{A})$ quite easily, namely by declaring $a \lhd_{\mathcal{C}(\mathcal{A})} U$ to hold iff $a \lhd_{\mathcal{A}} U_0$ holds for some finite $U_0 \subseteq U$, while keeping the same consistency predicate. It is routine to check that we indeed obtain a topology, which is Stone by definition, and that any other Stone topology finer than $\mathcal{A}$ is also finer than $\mathcal{C}(\mathcal{A})$. We can similarly obtain a Scott topology $\mathcal{S}(\mathcal{A})$ out of $\mathcal{A}$ by requiring, if $\mathsf{Pos}(a)$ is true, $a \lhd_{\mathcal{S}(\mathcal{A})} U$ to hold iff $(\exists b \in U)(a \lhd_{\mathcal{A}} b)$. We thus have

Proposition 4.8 *For any formal topology $\mathcal{A}$, the topology $\mathcal{C}(\mathcal{A})$ (or $\mathcal{S}(\mathcal{A})$) defined above is the coarsest Stone (or Scott, resp.) topology finer than $\mathcal{A}$.*

5 Some examples

We present here some, hopefully instructive, examples of formal topologies and formal spaces. In some of them we have used notions defined in successive sections.

a. The real numbers Let Q be the set of rational numbers, S the set of pairs (p, q) where p, q are either rational or $\pm\infty$. We define Δ to be $(-\infty, +\infty)$ and $\wedge$ by $(p, q) \wedge (p', q') \equiv (max\{p, p'\}, min\{q, q'\})$; then $S, \wedge, \Delta$ form a semilattice. We can define on S a covering relation essentially as in [J], p. 123, and a positivity predicate by putting $\mathsf{Pos}((p, q)) \equiv (p < q)$, where $<$ is the order of Q with top and least elements $\pm\infty$. The result will be a formal topology, whose corresponding formal space is the space of real numbers (see [J], pp. 123 - 125 for details).

b. Compactifications of N The set N of natural numbers is constructed as usual by means of the rules

$$0 \in N \qquad\qquad \frac{n \in N}{s(n) \in N}$$

Basing on them, we introduce the set S of formal neighbourhoods by the production rules

$$\underline{0} \in S \qquad\qquad \Delta \in S$$

$$\frac{a \in S}{s(a) \in S} \qquad\qquad \frac{a \in S \qquad b \in S}{a \wedge b \in S}$$

$+S, \wedge, \Delta$ form a semilattice with one

The idea is that a formal neighbourhood gives a piece of information: Δ gives no information, $s(a)$ is met by successors of those objects which meet a, $\underline{0}$ is met only by 0, etc. We introduce a relation $Approx(n, a)$ $prop$ $(n \in N, a \in S)$ (read: the information a applies to n, or a approximates n) by the rules

$$Approx(0, \underline{0}) \qquad\qquad Approx(n, \Delta)$$

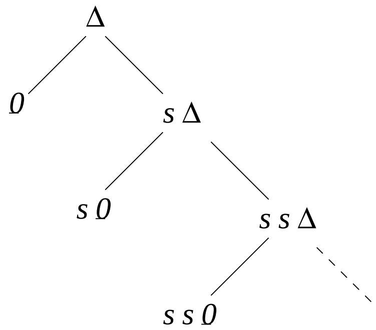
$$\frac{Approx(n, a)}{Approx(s(n), s(a))} \qquad\qquad \frac{Approx(n, a) \qquad Approx(n, b)}{Approx(n, a \wedge b)}$$

Since $Approx$ is clearly a neighbourhood relation, $N, S, Approx$ give a concrete space $\mathcal{N}$. Then we have also a formal topology $\mathcal{A}(\mathcal{N})$, whose covering relation and consistency predicate are defined, as in section 2, by

$$a \lhd U \equiv (\forall n \in N)(Approx(n, a) \rightarrow (\exists b \in U)(Approx(n, b))$$

$$\mathsf{Pos}(a) \equiv (\exists n \in N) Approx(n, a)$$

The consistent formal neighbourhoods and their ordering in $\mathcal{A}(\mathcal{N})$ are indicated in the picture:



However, $\mathcal{A}(\mathcal{N})$ doesn't say much: since $\Delta \lhd \{\underline{0}, s\underline{0}, ss\underline{0}, \cdots\}$, points on $\mathcal{A}(\mathcal{N})$ are just finite branches in the picture, and thus we get nothing but a copy of N.

The situation is quite different if we compactify $\mathcal{A}(\mathcal{N})$. In the Stone compactification, Δ is not covered by $\{\underline{0}, s\underline{0}, ss\underline{0}, \cdots\}$, and therefore, beside all finite branches, the infinite branch $\omega \equiv \{\Delta, s\Delta, ss\Delta, \cdots\}$ is also a point. The idea behind ω is that we can never exclude that it is a natural number; we thus may call it a non standard natural number. In the Scott compactification, a path from any node up to Δ in the picture is also a point; we may call it a partial number.

It is possible to obtain the three formal spaces above without any reference to N; we here only give a hint, and leave the details as an exercise. Think of formal neighbourhoods as pieces of information, as suggested above. Then the rules for S can be integrated with $s(a \wedge b) = sa \wedge sb$; also, we declare $\underline{0}$, Δ to be possible, a to be equipossible with $s(a)$, $\underline{0} \wedge s\Delta$ to be impossible. Defining $a \leq b$ as a is more informative than b, that is $a \leq b \equiv (a \wedge b = a)$, we obtain for possible pieces of information the same picture as above. Now, according to how we handle disjunction of information, we will have three possible coverings. If we accept the infinite disjunction $\Delta \lhd \{\underline{0}, s\underline{0}, ss\underline{0}, \cdots\}$, we obtain the topology $\mathcal{A}(\mathcal{N})$ above, and hence a copy of N. If we only accept finite disjunctions, we obtain the Stone topology above (more precisely, we assume $\Delta \lhd \{\underline{0}, s\Delta\}$ and close under the rule $a \lhd U \rightarrow s(a) \lhd \{s(b) : b \in U\}$, beside the rules for topologies; of course, $\Delta \lhd \{\underline{0}, s\Delta\}$ is interpreted as saying that being 0 or a successor amounts to no information). Finally, in the Scott topology no disjunction is allowed, i.e. $a \lhd U$ means that U already contains one piece of information covering a.

c. The topology of ideals Assume that L is a set with operations $+$, $\cdot$ and constants $0, 1$ such that $L, +, \cdot, 0, 1$ form a distributive lattice with zero and one. We write $\leq_L$ for the partial order defined as usual by $a \leq_L b \equiv (a \cdot b = a)$. For any $U \subseteq L$, we define $I(U)$ to be the ideal of L generated by U, that is we put

$$I(U) \equiv \{a \in L : (\exists b_1, \cdots, b_n \in U)(a \leq_L b_1 + \cdots + b_n)\}$$

It is obvious, and well known, that $I : \mathcal{P}(L) \to \mathcal{P}(L)$ is an algebraic closure operator. So to show that I induces a covering $\lhd_I$ it is enough to show that $I(U \cdot V) = I(U) \cdot I(V)$, which is easily done using distributivity.

Now assume we also have a consistency predicate Pos (the easiest case is when equality in L is decidable, so that we can put $\mathsf{Pos}(a) \equiv (a \neq 0)$). Then $L, \cdot, 1, \lhd_I, \mathsf{Pos}$ give a formal topology $I(L)$, called the topology of ideals of L, which is Stone since I is algebraic. Note that $Sat(I(L))$ is just the well known complete Heyting algebra of ideals of L. We will show in section 7 that all Stone topologies are obtained in this way (up to isomorphism).

d. The subsets of the one-element set Let 1 be a set with just one element, say 0 (N_1 of [ITT] is such a set). 1 is trivially a semilattice with one, and hence it can be taken as a formal base for a topology. We define a covering relation $\lhd$ by putting

$$0 \lhd U \equiv U(0) \equiv 0 \in U$$

for any subset U of 1, and we declare 0 to be consistent. We then trivially have a topology, whose algebra of opens is just $\mathcal{P}(1)$ with extensional equality, because $U \lhd V \equiv U \subseteq V$. Since 0 is the only consistent neighbourhood, $\{0\}$ is the only point of $\mathcal{P}(1)$.

Since subsets of 1 are propositional functions with an argument which can only be 0, $\mathcal{P}(1)$ is isomorphic to the category of propositions (using a tiny bit of Martin-Löf's theory of expressions, we can see that the isomorphism maps a subset U of 1 into $U(0)$, which is a proposition, and conversely a proposition A into its abstraction $(x)A$, which is a unary propositional function; the claim then follows because $U = (x)(U(0))$ in $\mathcal{P}(1)$, while $A \equiv ((x)A)(0)$ in the category of propositions). As a corollary, we obtain that for any formal topology $\mathcal{A}$, $Pt(\mathcal{A})$ is isomorphic to the category of morphisms from $\mathcal{A}$ into $\mathcal{P}(1)$ (cf. [FG], p. 122).

e. Free topologies Let S be any set, $Fin(S)$ the set of finite subsets of S. For any $d, e \in Fin(S)$, we put $d \wedge e \equiv d \cup e$ (and hence $d \leq e \equiv d \supseteq e$) and $\Delta \equiv \emptyset$. Then $Fin(S)$ becomes a semilattice with one, and actually the semilattice freely generated by S (see for instance [J], p. 27). We now define Pos so that $\mathsf{Pos}(d)$ holds for any $d \in Fin(S)$ and put

$$d \lhd U \equiv (\exists e \in U)(d \leq e)$$

Then $\lhd$ is the least covering relation on $Fin(S)$, and the resulting topology $F(S)$ is called the free topology on S. In fact, given any formal topology $\mathcal{A}$ and a surjective map $f_0 : S \to S_0$, where S_0 is a subset of $S_{\mathcal{A}}$ generating $S_{\mathcal{A}}$ as a semilattice, there is a unique full morphism from $F(S)$ to $\mathcal{A}$ which extends f.

If we define a continuous function $f^* : Pt(\mathcal{A}) \to Pt(\mathcal{B})$ to be injective precisely when f is full, then $Pt(F(S))$ can be called a universal space, since all formal spaces which are small enough can be embedded in it. Since $F(S)$ is Scott by definition, $Pt(F(S))$ then plays the role of a universal Scott domain.

6 Morphisms of topologies and continuos functions

The usual definition says that a function between two topological spaces is continuous if its inverse is a homomorphism between the frames of opens. Since here opens are more basic than points, reversing the order we will obtain a continuous function as the adjoint (that is, a sort of generalized inverse) of a homomorphism between the frames of opens, which in its turn can be described in terms of the underlying formal topologies. We are thus led to:

Definition 6.1 *Let $\mathcal{A}$, $\mathcal{B}$ be arbitrary formal topologies. A function $f : S_{\mathcal{A}} \to \mathcal{P}(S_{\mathcal{B}})$ is called a morphism of $\mathcal{A}$ into $\mathcal{B}$, written $f : \mathcal{A} \to \mathcal{B}$, if for any $a, b \in S$, $U \subseteq S$:*

1. $f(a \wedge b) =_{\mathcal{B}} fa \wedge fb$, $f(\Delta_{\mathcal{A}}) = \Delta_{\mathcal{B}}$
2. $a \lhd_{\mathcal{A}} U \to fa \lhd_{\mathcal{B}} f(U)$ *where* $f(U) \equiv \cup\{fb : b \in U\}$
3. $\mathsf{Pos}_{\mathcal{B}}(fa) \to \mathsf{Pos}_{\mathcal{A}}(a)$

In other words, f is a function which preserves all the structure of $\mathcal{A}$. Two little facts, however, should be pointed out: first, we can not require the base $S_{\mathcal{A}}$ always to be mapped into the base $S_{\mathcal{B}}$, which explains why values of f are subsets of $S_{\mathcal{B}}$, i.e. opens; second, consistency is preserved backwards, which is essential when defining f^* below.

It is easy to see that for a morphism $f : \mathcal{A} \to \mathcal{B}$ properties 1. and 2. can be extended to opens, that is $f(U \wedge V) =_{\mathcal{B}} f(U) \wedge f(V)$ and $U \lhd_{\mathcal{A}} V \to f(U) \lhd_{\mathcal{B}} f(V)$ hold. This means that the function $f^\circ : Open(\mathcal{A}) \to Open(\mathcal{B})$, where $f^\circ U \equiv f(U)$, is well defined and that it preserves the order and finite meets. By its definition, f° also preserves arbitrary joins. So f° is a homomorphism between frames.

Conversely, given an arbitrary homomorphism $h : Open(\mathcal{A}) \to Open(\mathcal{B})$, consider its trace h_0 on the base $S_{\mathcal{A}}$, which is defined by $h_0(a) \equiv h(\{a\})$ for $a \in S_{\mathcal{A}}$. Then h_0 is obviously a morphism of topologies and $h_0^\circ = h$ because h preserves infinite joins. So we have:

Proposition 6.2 *Any morphism of topologies $f : \mathcal{A} \to \mathcal{B}$ induces, as described above, a frame homomorphism $f^\circ : Open(\mathcal{A}) \to Open(\mathcal{B})$, and all frame homomorphisms are obtained in this way.*

A morphism $f : \mathcal{A} \to \mathcal{B}$ is called faithful if $a \lhd_{\mathcal{A}} U \leftrightarrow fa \lhd_{\mathcal{B}} f(U)$ and full if for any $V \subseteq S_{\mathcal{B}}$ there exists $U \subseteq S_{\mathcal{A}}$ such that $f(U) =_{\mathcal{B}} V$. A morphism which is both faithful and full is called an equivalence of topologies, since it is not difficult to prove that

Proposition 6.3 *A morphism $f : \mathcal{A} \to \mathcal{B}$ is faithful iff $f^\circ : Open(\mathcal{A}) \to Open(\mathcal{B})$ is injective and full iff f° is surjective.*

Given a morphism $f : \mathcal{A} \to \mathcal{B}$ and a point β of $\mathcal{B}$, we define a propositional function $f^*\beta$ by putting

$$(f^*\beta)(a) \equiv \beta(fa) \text{ for } a \in S_{\mathcal{A}}$$

It is easy to check that $f^*\beta$ is a point on $\mathcal{A}$ (here the fact that f preserves consistency backwards is essential to show that $f^*\beta$ is consistent). Now assume that a is a formal neighbourhood of $f^*\beta$, i.e. $(f^*\beta)(a)$ true. Then $\beta(fa)$ is true, and hence $(\exists b \in fa)\beta(b)$, i.e. there is a neighbourhood b of β such that $b \lhd_{\mathcal{B}} fa$ (which is our way of expressing the classical statement $(\forall \alpha \in Pt(\mathcal{B}))(\alpha(b) \to (f^*\alpha)(a))$, that is $f^*(\phi b) \subseteq \phi a$). This justifies the following:

Definition 6.4 *If $f : \mathcal{A} \to \mathcal{B}$ is a morphism, then $f^* : Pt(\mathcal{B}) \to Pt(\mathcal{A})$ is called a continuous function.*

It is not difficult to check that formal topologies and morphisms form a category in MacLane's sense, and that $Pt(-) : \mathcal{A} \to Pt(\mathcal{A})$, $f \to f^*$ is a contravariant functor. Hence our category of formal spaces corresponds to the category of locales (see e.g. [J], p. 39). A natural question is then how much of the theory of locales can be transferred into our framework; this indeed is a good project, but yet for another paper.

Some reader may not like the definition of morphisms above, since they are functions with subsets as values. Then, instead of f, (s)he may consider the relation $F(a,b)$ *prop* $(a \in S_{\mathcal{A}}, b \in S_{\mathcal{B}})$ defined by $F(a,b) \equiv b \in fa$, and find those properties of F which correspond to 1.-3. above. Then (s)he will find that the job is easier if f is assumed to be saturated, that is if fa is $\mathcal{B}$-closed for each $a \in S_{\mathcal{A}}$ (note that f and its saturation $f_{\mathcal{B}}(a) \equiv \mathcal{B}(fa)$ induce the same frame homomorphism). The result is that f is a saturated morphism iff F satisfies

1. $F(a \wedge b, c) = F(a,c) \,\&\, F(b,c)$, $F(\Delta_{\mathcal{A}}, \Delta_{\mathcal{B}})$
2. $a \lhd_{\mathcal{A}} U \,\&\, F(a,b) \to b \lhd_{\mathcal{B}} \{c \in S_{\mathcal{B}} : (\exists a \in U) F(a,c)\}$
3. $\mathsf{Pos}_{\mathcal{B}}(b) \,\&\, F(a,b) \to \mathsf{Pos}_{\mathcal{A}}(a)$
4. $(\forall b \in V) F(a,b) \,\&\, (b \lhd_{\mathcal{B}} V) \to F(a,b)$

So, since clearly any relation $F(a,b)$ *prop* $(a \in S_{\mathcal{A}}, b \in S_B)$ induces a function $fa \equiv \{b \in S_B : F(a,b)\}$ and the correspondence is biunivocal, there is an isomorphism between saturated morphisms and relations satisfying 1.-4.

Conditions 2. and 4. are not very perspicuous. When $\mathcal{A}$, $\mathcal{B}$ are Scott, however, they acquire a much simpler form, namely

2'. $a \lhd_{\mathcal{A}} a' \,\&\, F(a,b) \to F(a',b)$

4'. $F(a,b) \,\&\, b' \lhd_{\mathcal{B}} b \to F(a,b')$

Apart from minor notational details, 1.,2'.,3.,4'. form the very definition of approximable mappings in [S2].

7 Compact opens and Stone representation

The usual notion of compactness is easily expressed in pointless words:

Definition 7.1 *An open U in the formal topology $\mathcal{A}$ is called compact if, for arbitrary $V \subseteq S$, $U \lhd V \to U \lhd V_0$ for some finite $V_0 \subseteq V$.*

First of all, note that this is a good definition; hence compact opens of $\mathcal{A}$ are just those elements of $Open(\mathcal{A})$ which are compact (or algebraic, or finite) according to the usual definition in complete lattices (see e.g. [J], p. 63). Note that finite subsets of S need not be compact (the standard example here is that of trees, in section 9). On the other hand we do have: if U is compact, then $U =_{\mathcal{A}} U_0$ for some finite $U_0 \subseteq U$. Using this, it is easy to see that compact opens are closed under finite unions, that is form a join-subsemilattice of $Open(\mathcal{A})$.

By the definitions, $\mathcal{A}$ is Stone iff every basic neighbourhood is compact. Thus, because of closure under finite unions, we have:

Proposition 7.2 *A formal topology $\mathcal{A}$ is Stone iff every finite subset of S is compact.*

The following results are our version of Stone's representation theorems (see e.g. [J], pp. 64-66).

Proposition 7.3 *Any Stone topology is equivalent to the topology of ideals on a distributive lattice.*

Proof. The idea is to induce on $Fin(S)$, the set of finite subsets of S, the structure of $\mathcal{A}$. That is, for $d, e \in Fin(S)$, $U \subseteq Fin(S)$, define $d \lhd_I U \equiv d \lhd_{\mathcal{A}} \cup U$, $d \wedge e \equiv \{a \wedge b : a \in d, b \in c\}$, $d \bigvee e \equiv d \cup e$. In particular, $Fin(S)$ with equality $=_I$ is a distributive lattice and the name is well chosen, that is $d \lhd_I U$ iff d is in the ideal of $Fin(S)$ generated by U, because d is compact. It is now obvious that the identity mapping $i : Fin(S) \to P(S_{\mathcal{A}})$ is an equivalence of topologies.

Proposition 7.4 *Every distributive lattice L is isomorphic to the sublattice of compact opens of the Stone topology $I(L)$.*

Proof. Since $I(L)$ is Stone, U is compact in $I(L)$ iff $U =_I \{a_1, \cdots, a_n\}$ for some $a_1, \cdots, a_n \in L$, and hence $U =_I a_0 + \cdots + a_n$. So the identity mapping is the isomorphism we want.

It may be instructive to see how, admitting classical principles, Stone's representation of a distributive lattice L would be derived from the above result. Recall that classically $Pt(I(L))$ is a topological space with base $(\phi a)_{a \in L}$. Then ϕ is a function from $Open(I(L))$ onto the opens of $Pt(I(L))$, since any open is of the form $\phi(I) \equiv \cup\{\phi a : a \in I\}$ for some ideal I of L. To show that ϕ is injective, it is enough and necessary to have

$$\phi a \subseteq \phi(I) \to a \in I$$

for any ideal I. By definition, $\phi a \subseteq \phi(I)$ means, classically, that $(\forall \alpha \in Pt(I(L)))$ $(a \in \alpha \rightarrow (\exists b \in I)(b \in \alpha))$; so by classical logic the condition above is equivalent to

$$a \notin I \rightarrow (\exists \alpha \in Pt(I(L)))(a \in \alpha \;\&\; I \cap \alpha = \emptyset)$$

which, by proposition 4.6, is exactly a formulation of the prime filter theorem. So the prime filter theorem is equivalent to ϕ being an isomorphism. Since obviously U is a compact subset in $I(L)$ iff $\phi(U)$ is compact in the traditional sense, proposition 4 above gives Stone's theorem: assuming the prime filter theorem, L is isomorphic to the lattice of compact opens in $Pt(I(L))$.

8 Scott spaces and Scott domains

Various presentations of the so called Scott domains have been given by D. Scott himself. In the so called axiomatic presentation, a Scott domain is a structure $D, \perp, \leq$ which is a complete partial order (i.e. $\leq$ is a partial order with bottom $\perp$, in which every directed subset $E \subseteq D$ has a supremum $\bigvee E$) which is algebraic (i.e. for every element d of D, algebraic elements below d form a directed subset whose supremum is d) and any two algebraic elements d, e which are majorized in D have a supremum $d \vee e$. The connection with our approach is immediate:

Proposition 8.1 *Any Scott space $Pt(\mathcal{A})$ is a Scott domain.*

In fact, by proposition 4.7, points of $\mathcal{A}$ are just filters on S. So the order $\alpha \leq \beta \equiv (\forall a \in S)(\alpha(a) \rightarrow \beta(a)) \equiv \alpha \subseteq \beta$ is a complete partial order on $Pt(\mathcal{A})$, with the filter generated by Δ as bottom, which is algebraic since algebraic elements are exactly principal filters. The last condition is then taken care of by the consistency predicate: if the filters generated by $a, b \in S$ are majorized by α in $Pt(\mathcal{A})$, then $\alpha(a \wedge b)$ holds, hence $\mathsf{Pos}(a \wedge b)$ and the filter generated by $a \wedge b$ is consistent.

On the other hand, let D be a Scott domain and S the set of its algebraic elements. Then any element of D may be identified with the directed set $E_d \equiv \{a \in S : a \leq d\}$, since $d = \bigvee E_d$. But then we may think of E_d as a filter on the dual of S, that is the structure with order $\geq$ and meet $\bigvee$. In more recent presentations, Scott has introduced neighbourhood systems [S1] and information systems [S2], which can be seen as a way to axiomatize the structure of S above. In particular, information systems are provided with a consistency predicate (which intuitively asserts a to be consistent with b when the meet of a and b exists), and Scott domains are then given as the collection of consistent filters over them. Our aim is to show that Scott formal topologies play the same role, and that they can be identified with much simpler structures, as we now see.

For an arbitrary formal topology $\mathcal{A}$, the *trace* $\mathcal{SA}$ of $\mathcal{A}$ on its base $\mathcal{S}$ is the structure induced by $\mathcal{A}$ on the elements of S; more formally, we put $\mathcal{SA} \equiv (S_{\mathcal{SA}}, =_{\mathcal{SA}}, \wedge_{\mathcal{SA}}, \Delta_{\mathcal{SA}}, \mathsf{Pos}_{\mathcal{SA}})$, where

$$S_{\mathcal{SA}} \equiv S_{\mathcal{A}}, \quad \wedge_{\mathcal{SA}} \equiv \wedge_{\mathcal{A}}, \quad \Delta_{\mathcal{SA}} \equiv \Delta_{\mathcal{A}}$$
$$(a \leq_{\mathcal{SA}} b) \equiv (a \lhd_{\mathcal{A}} \{b\}) \qquad \text{and hence} \quad (a =_{\mathcal{SA}} b) \equiv (\{a\} =_{\mathcal{A}} \{b\})$$
$$\mathsf{Pos}_{\mathcal{SA}}(a) \equiv \mathsf{Pos}_{\mathcal{A}}(a)$$

It is easy to see that $\mathcal{SA}$ is a semilattice with one; in fact, it is the quotient of $S_{\mathcal{A}}$ over $=_{\mathcal{A}}$, which is a congruence, and $\leq_{\mathcal{SA}}$ is then the partial order induced by $\wedge_{\mathcal{SA}}$. Moreover, $\mathcal{SA}$ is provided with a predicate Pos satisfying

$$\frac{\mathsf{Pos}(a) \quad a \leq b}{\mathsf{Pos}(b)} \qquad\qquad \frac{\mathsf{Pos}(a) \to a \leq b}{a \leq b}$$

We put all this into a definition, and hence also a little result:

Definition 8.2 *Any semilattice with one and with a predicate* Pos *satisfying the above two conditions is called a Scott semilattice.*

Proposition 8.3 *For any formal topology $\mathcal{A}$, the trace $\mathcal{SA}$ is a Scott semilattice.*

In the opposite direction, from any Scott semilattice $\mathcal{S} \equiv (S, \wedge, \Delta, \mathsf{Pos}_{\mathcal{S}})$ we construct the least formal topology based on $\mathcal{S}$. We need a lemma.

Lemma 8.4 *For any semilattice with one $\mathcal{S} \equiv (S, \wedge, \Delta)$, putting*

$$a \lhd_{\mathcal{L}} U \equiv (\exists b \in U)(a \leq_{\mathcal{S}} b)$$

defines a covering relation $\lhd_{\mathcal{L}}$ which moreover is the coarsest possible (that is, $a \lhd_{\mathcal{L}} U \to a \lhd_{\mathcal{A}} U$ for any covering $\lhd_{\mathcal{A}}$ on $\mathcal{S}$).

Proof. For any covering $\lhd_{\mathcal{A}}$ on $\mathcal{S}$, by the rule of $\wedge$-left we have

$$\frac{a \leq_{\mathcal{S}} b}{a \lhd_{\mathcal{A}} b}$$

and hence $a \lhd_{\mathcal{L}} U \to a \lhd_{\mathcal{A}} U$. It is now easy to verify that $\mathcal{L}$ is a closure operator on $\mathcal{P}(S)$, for which $\mathcal{L}(U) \cap \mathcal{L}(V) = \mathcal{L}(U \wedge V)$ holds, so that $\lhd_{\mathcal{L}}$ is a covering relation by proposition 3.3.

Proposition 8.5 *For any Scott semilattice $\mathcal{S} \equiv (S, \wedge, \Delta, \mathsf{Pos}_{\mathcal{S}})$, putting*

$$a \lhd_{\mathcal{SA}} U \equiv \mathsf{Pos}_{\mathcal{S}}(a) \to a \lhd_{\mathcal{L}} U$$

gives a covering relation. Then $\mathcal{AS} \equiv (S, \wedge, \Delta, \lhd_{\mathcal{AS}}, \mathsf{Pos}_{\mathcal{S}})$ is a Scott formal topology, and it is coarser than any formal topology $\mathcal{B}$ based on $\mathcal{S}$ and with $\mathsf{Pos}_{\mathcal{B}}(a) \leftrightarrow \mathsf{Pos}_{\mathcal{S}}(a)$.

Proof. From the assumption $\mathsf{Pos}_{\mathcal{S}}(a) \& a \leq_{\mathcal{S}} b \to \mathsf{Pos}_{\mathcal{S}}(b)$, it follows that $\mathsf{Pos}_{\mathcal{S}}$ satisfies monotonicity with respect to $\lhd_{\mathcal{L}}$, and hence also with respect to $\lhd_{\mathcal{SA}}$. Positivity is immediate by definition. The proof that $\lhd_{\mathcal{SA}}$ is a covering relation is a direct verification of the conditions, using the lemma. So $\mathcal{AS}$ is a formal topology. By the definition of $\lhd_{\mathcal{L}}$ and the fact that $a \leq_{\mathcal{S}} b$ implies $a \lhd_{\mathcal{SA}} b$,

$\mathcal{AS}$ is a Scott topology. Finally, by the assumption on $\mathsf{Pos}_{\mathcal{B}}$, from $a \lhd_{\mathcal{AS}} U$ it follows that $\mathsf{Pos}_{\mathcal{B}}(a) \to a \lhd_{\mathcal{L}} U$ and hence, by the lemma, $\mathsf{Pos}_{\mathcal{B}}(a) \to a \lhd_{\mathcal{B}} U$, from which $a \lhd_{\mathcal{B}} U$ by positivity of $\mathcal{B}$.

We now can easily prove that Scott semilattices can be identified with Scott topologies:

Proposition 8.6 *For any Scott semilattice $\mathcal{S}$, the trace of $\mathcal{AS}$ on $\mathcal{S}$ is $\mathcal{S}$ itself, that is $\mathcal{SAS} = \mathcal{S}$.*

Proof. Since the base set S and Pos are the same, it is enough to show that $a \leq_{\mathcal{S}} b$ iff $a \leq_{\mathcal{SAS}} b$. Now $a \leq_{\mathcal{SAS}} b \equiv a \lhd_{\mathcal{AS}} \{b\} \equiv \mathsf{Pos}_{\mathcal{S}}(a) \to (\exists c \in \{b\})(a \leq_{\mathcal{S}} c)$ iff $\mathsf{Pos}_{\mathcal{S}}(a) \to a \leq_{\mathcal{S}} b$ and hence, by the assumptions on $\mathsf{Pos}_{\mathcal{S}}$, if and only if $a \leq_{\mathcal{S}} b$.

Proposition 8.7 *For any formal topology $\mathcal{A}$, $\mathcal{ASA} = \mathcal{A}$ iff $\mathcal{A}$ is Scott.*

Proof. $\mathcal{SA}$ is a Scott semilattice by proposition 8.3, hence $\mathcal{ASA}$ is Scott by proposition 8.5, hence also $\mathcal{A}$ is Scott if we assume that $\mathcal{A} = \mathcal{ASA}$.

Conversely, it is enough to show that $\lhd_{\mathcal{ASA}}$ coincides with $\lhd_{\mathcal{A}}$, when $\mathcal{A}$ is Scott. By definitions,

$$a \lhd_{\mathcal{ASA}} U \equiv \mathsf{Pos}_{\mathcal{A}}(a) \to (\exists b \in U)(a \lhd_{\mathcal{A}} b)$$

$$\text{iff } \mathsf{Pos}_{\mathcal{A}}(a) \to (\mathsf{Pos}_{\mathcal{A}}(a) \to (\exists b \in U)(a \lhd_{\mathcal{A}} b))$$

$$\text{iff } \mathsf{Pos}_{\mathcal{A}}(a) \to a \lhd_{\mathcal{A}} U \quad \text{because } \mathcal{A} \text{ is Scott}$$

$$a \lhd_{\mathcal{A}} U \quad \text{by positivity of } \mathcal{A}.$$

By propositions 8.7 and 8.5, any Scott topology $\mathcal{A}$ is the coarsest topology over its own trace $\mathcal{SA}$, which by proposition 8.3 is a Scott semilattice. Moreover, when $\mathcal{A}$ is Scott, by proposition 4.7 points over $\mathcal{A}$ are just consistent filters, or actually filters (in the traditional sense) over the semilattice $\mathcal{SA}$ which are consistent, i.e. contained in Pos. We thus can forget about Scott topologies, in favour of Scott semilattices, which have a simpler structure.

Then, given any Scott semilattice $\mathcal{S}$, since $\mathcal{S} = \mathcal{SAS}$ by proposition 8.6, the space of consistent filters of $\mathcal{S}$ is the same as the formal space $Pt(\mathcal{AS})$, and hence it is a Scott domain by proposition 8.3. Conversely, any Scott domain can be presented in this way (the idea is that algebraic elements of a Scott domain give rise to a Scott semilattice; we leave details to the reader). Thus Scott semilattices play exactly the same rôle as Scott's neighbourhood systems in [S1] and information systems in [S2].

Therefore our proposal is to take Scott semilattices (may be with a shorter name) as the basic structures on which the theory of Scott domains can be built up. Besides a substantial technical simplification, the improvement here is that of embedding such concepts in the more general theory of intuitionistic formal spaces.

9 Trees and bars

One of the motivations to the study of formal spaces is to give an interpretation, in the foundational framework adopted, of Brouwer's notion of choice sequence. The main idea is that choice sequences on a given tree should be points of a suitable formal topology based on that tree. However, this is easier said than done. In fact, I was able to materialize that idea only at the cost of modifying the definition of formal topology itself, namely by requiring $a \lhd U^+$ to be derivable from $a \lhd U$ only when $\mathsf{Pos}(a)$ holds (weak openness). Whether this is a good reason to adopt this modification from the start, I still do not know. In any case, I believe the topic to be interesting enough to present here, though briefly and informally, the present state of work.

A tree is here given by two families of sets

$$A(n) set(n \in N)$$

$$B(n, x) set(n \in N, x \in A(n))$$

which satisfy

$$A(n+1) = (\Sigma x \in A(n)) B(n, x)$$

Intuitively, we begin with $A(0)$, which usually contains only one element Δ, called the root. Then we have a set $B(0, \Delta)$ of choices on how to proceed. Once $b \in B(0, \Delta)$ is chosen, we form $a_1 \equiv (\Delta, b) \in A(1)$. Then we have a set $B(1, a_1)$ of choices, etc. So $A(n)$ is the set of elements which are obtained with n successive choices and $S \equiv (\Sigma x \in N) A(n)$ is the set of all nodes. May be a better definition can be found, but at least the one above has the advantage of lying entirely within the framework of [ITT]. Of course, common examples of trees (like the complete tree over a given set T, which is often $\{0, 1\}$ or N) fall under it (take $B(n, x) = \mathrm{T}$, $A(0) = \{\Delta\}$ and put $A(n+1) \equiv A(n) \times \mathrm{T}$).

We define the ordering on S in which Δ is the top element by putting $(n, a) \leq (m, b)$ when $n \geq m$ and b is obtained from a by right projection applied $n - m$ times. We often write a for (n, a), and say that a is a node of length n.

We now want to put a topology on a given tree S. To this aim, we first close S under an operation $\wedge$ of formal meet which respects the order of nodes; that is, we impose $\wedge$ to satisfy $a \leq b \leftrightarrow a \wedge b = a$, beside idempotency, commutativity and associativity. What about the 'formal' nodes $a \wedge b$ obtained when a, b are incomparable? The consistency predicate solves this problem: we declare them to be inconsistent (and hence they will not appear in a drawing).

For every node a, we define immediate successors of a to be those nodes which are reached from a with just one choice. More formally, the immediate cover of $(n, a) \in S$ is the set $C(a) \equiv \{(n+1, (a, b)) : b \in B(n, a)\}$ and any element of $C(a)$ is called an immediate successor of a. Now assume we have a predicate Pos satisfying

$$\mathsf{Pos}(a) \ \& \ a \leq b \rightarrow \mathsf{Pos}(b)$$
$$\mathsf{Pos}(a) \rightarrow \mathsf{Pos}(C(a))$$
$$\mathsf{Pos}(a \wedge b) \rightarrow a \leq b \vee b \leq a$$

(this means of course that even 'standard' nodes may be inconsistent, i.e. sterilized in Brouwer's terminology). We then define a covering relation $\lhd_{\mathcal{C}}$ by assuming

$$a \lhd_{\mathcal{C}} C(a)^+ \text{ for any } a \in S$$

and closing under the rules for coverings. We want to show that this indeed defines a formal topology $\mathcal{C}$, called the inductive topology (a name borrowed from [FG]) on the tree S with consistency predicate Pos.

First of all, note that when $a \leq b$ holds, then $a \wedge b = a$ and hence $a \lhd_{\mathcal{C}} b$ follows from $b \lhd_{\mathcal{C}} b$ by $\wedge$-left; so, even if the converse $a \lhd_{\mathcal{C}} b \rightarrow a \leq b$ is in general false (for instance when b is the only immediate successor of a), we will drop the subscript $\mathcal{C}$.

The rule of ex falso quodlibet is easily seen to be derivable: if $\neg\mathsf{Pos}(a)$, then also $\neg\mathsf{Pos}(C(a))$ by the assumptions on Pos, and hence $C(a)^+ = \emptyset$, so that $a \lhd \emptyset$ is simply an axiom.

Any derivation of $a \lhd U$, where $\mathsf{Pos}(a)$ is true, can be reduced to a sort of canonical form as follows. First, it is easy to show that the rule of $\wedge$-right is equivalent to the rule of localisation introduced in section 1; thus we eliminate $\wedge$-right in favour of localisation. The advantage is that applications of localisation can be lifted over all other rules (for example,

$$\frac{\dfrac{a \lhd U}{a \wedge b \lhd U}}{a \wedge b \wedge c \lhd U \wedge c} \quad \text{reduces to} \quad \frac{\dfrac{a \lhd U}{a \wedge c \lhd U \wedge c}}{a \wedge c \wedge b \lhd U \wedge c}$$

and other rules are treated similarly). Thus we can reduce the derivation to one in which localisation is applied only to axioms, that is to obtain covers of the form $b \wedge c \lhd C(b)^+ \wedge c$ (note that any number of consecutive applications of localisation can be reduced to one).

Now the derivation of $a \lhd U$ can be reduced to a derivation of $a \lhd U^+$ in which only consistent nodes occur. To see how this is possible, imagine to apply weak openness to the conclusion of the derivation, and then lift it as much as possible (an example of reduction is

$$\frac{\dfrac{b \lhd V \quad V \lhd W}{b \lhd W}}{b \lhd W^+} \quad \text{reduces to} \quad \frac{\dfrac{b \lhd V}{b \lhd V^+} \quad \dfrac{V^+ \lhd W}{V^+ \lhd W^+}}{b \lhd W^+}$$

where it is better to think of $V \lhd W$ as a free-variable derivation of $c \lhd W$ from the assumption $c \in V$, so that $V^+ \lhd W$ is obtained simply by restricting to the assumption $c \in V^+$; the cases for reflexivity and $\wedge$-left are even simpler). The result is that only consistent nodes appear, at least all the way up to any conclusion $b \wedge c \lhd C(b)^+ \wedge c$ of localisation. Then in particular $\mathsf{Pos}(b \wedge c)$ is true, and hence $b \leq c \vee c \leq b$. So we can substitute localisation either by nothing (when $b \leq c, b \wedge c = b$ and $C(b)^+ \wedge c = C(b)^+$) or by reflexivity (when $c < b, b \wedge c = c$ and $C(b)^+ \wedge c = \{c\}$). Note that the result is a real derivation of $a \lhd U^+$ in which only the three rules of transitivity, $\wedge$-left and reflexivity are applied.

Now it is easy to prove closure under monotonicity and weak openness by induction on such three rules (the assumption $\mathsf{Pos}(a)$ is used to show that $a \lhd U^+$ whenever $a \in U$). This completes the proof of the fact that $\mathcal{C}$ is a formal topology.

The definition of inductive covering is a precise formulation of the following informal but clear notion: U is an intuitive cover of a if starting from a we fall into U after a finite number of arbitrary choices. Certainly the axioms and rules for $\lhd$ fulfill such interpretation; the claim is that they embrace it completely. Incidentally, note that the above informal notion is equivalent to that of wellfounded tree, i.e. a cover of Δ.

Similarly, the definition of point is the precise counterpart of the informal notion of branch (or arrow, or infinitely proceeding sequence). By condition 2 of definition 4.1, a point α which meets a node a, in the sense that $\alpha(a)$ is true, also meets any cover U of a, in the sense that $\alpha(U)$. In particular, $\alpha(a) \to (\exists b \in C(a))\alpha(b)$ holds for any node a. Moreover, whenever $b, c \in C(a)$ and $\alpha(b), \alpha(c)$ are both true, then $\alpha(b \wedge c)$ and hence $\mathsf{Pos}(b \wedge c)$, which can only hold if $b = c$, because b, c have equal length. So α satisfies

$$\alpha(a) \to (\exists! b \in C(a))\alpha(b) \tag{1}$$

which, together with $\alpha(\Delta)$ and $\alpha(a) \to \mathsf{Pos}(a)$, tells that α is a branch. The converse holds if we agree that any branch satisfies (1) and contains only consistent nodes, beside the root. In fact, from $\alpha(a) \to (\exists b \in C(a))\alpha(b)$, which takes care of the axioms, condition 2 is proved by induction on the rules, while condition 1 follows by the uniqueness of the immediate successor granted by (1). Of course, the question remains open whether (1) is inherent our intuitive notion of branch, in particular since it asserts the constructive existence of the immediate successor (one could be satisfied with $\alpha(a) \to \neg\neg(\exists b \in C(a))(\alpha(b))$. This is why I have not used the word choice sequence.

Usually the order of definitions is the opposite (two good sources on such matters are [FIM] and [MG]): one assumes to know what a branch is so well to make sense of quantification over branches and then defines wellfoundedness, or equivalently covers, by saying that U is a bar of a if

$$(\forall \alpha)(\alpha(a) \to \alpha(U)) \tag{2}$$

holds, where α ranges over branches. But the notion of bar so defined is difficult to be used, unless one assumes also that, under some weak restrictions, bars can be defined inductively (principle of bar induction). Inductive bars are defined by the rules (read U i.b. a as U is an inductive bar of a):

$$\frac{a \in U}{U \text{ i.b. } a}\ \eta \qquad\qquad \frac{\begin{array}{c}(b \in C(a))\\ |\\ U \text{ i.b. } b\end{array}}{U \text{ i.b. } a}\ \mathcal{F}$$

A subset U is monotone if $a \in U \ \&\ b \leq a \to b \in U$. Then monotone bar induction is the statement

$\mathrm{BI_M}$ $\qquad U$ monotone $\&\ (\forall \alpha)(\alpha(a) \to \alpha(U)) \to U$ i.b. a

One of Brouwer's arguments for $\mathrm{BI_M}$ is based on the assumption (Brouwer's Dogma) that any fully analysed proof of (2) only makes use of η, $\mathcal{F}$, ζ-inferences, where a ζ-inference allows to obtain U bars b from $b \in C(a)$ and U bars a. It is easy to see that, for a consistent node a, a fully analysed proof of U bars a exists iff $a \lhd U$ (hint: first reduce the derivation of $a \lhd U$ to one in which transitivity is applied only with an axiom as left premiss, and note that then η, $\mathcal{F}$, ζ-rules correspond exactly to our reflexivity, $\wedge$-left and transitivity+axioms, respectively). Hence, assuming BD, $\mathrm{BI_M}$ is proved simply by lifting $\wedge$-left over transitivity (obvious) and then noting that any application of $\wedge$-left under reflexivity can be eliminated if U is monotone. Moreover, the fact that any derivation of $a \lhd U$ can be reduced to a fully analysed proof in Brouwer's sense, is a little argument in favour of BD.

Brouwer himself gives also another, shorter argument for $\mathrm{BI_M}$: thought through intuitionistically, a bar is nothing else than an inductive bar. We agree with him (and hence also with Kleene, see [FIM], end of p. 50). However, our foundational framework allows to go a little step further, and simply define the meaning of $(\forall\alpha)(\alpha(a) \rightarrow \alpha(U))$ to be $a \lhd U$. The content of $\mathrm{BI_M}$ then becomes part of the definition of universal quantification over points, and the above claim by Brouwer is substituted by: thought through intuitionistically, an intuitive cover is nothing else than an inductive cover.

Finally, it is now easy to keep a promise made in section 4. Conceiving of $Pt(\mathcal{C})$ in the classical way, the function ϕ from opens of $\mathcal{C}$ onto opens of $Pt(\mathcal{C})$ is an isomorphism iff for any monotone subset U and any consistent node a, $\phi(a) \subseteq \phi(U) \rightarrow a \lhd U$; but $\phi(a) \subseteq \phi(U)$ is just (2), and we have shown above that $a \lhd U$ is equivalent, when U is monotone and a is consistent, to U i.b. a. Thus ϕ is an isomorphism iff $\mathrm{BI_M}$ holds.

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