

# Pretopologies and completeness proofs

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Pretopologies were introduced in [S] and there shown to give a complete semantics for a propositional sequent calculus BL here called basic linear logic<sup>1</sup>, as well as for its extensions by structural rules, ex falso quodlibet or double negation. Immediately after the Logic Colloquium '88, conversation with Per Martin-Löf helped me to see how the pretopology semantics should be extended to predicate logic; the result now is a simple and fully constructive completeness proof for first order BL and virtually all its extensions, including usual, or structured, intuitionistic and classical logic. Such a proof clearly illustrates the fact that stronger set-theoretic principles and classical metalogic are necessary only when completeness is sought with respect to a special class of models, such as usual two-valued models.

To make the paper self-contained, I briefly review in section 1 the definition of pretopologies; section 2 deals with syntax and section 3 with semantics. The completeness proof in section 4, though similar in structure, is sensibly simpler than that in [S], and this is why it is given in detail. In section 5 it is shown how little is needed to obtain completeness for extensions of BL in the same language. Finally, in section 6 connections with proofs with respect to more traditional semantics are shortly investigated, and some open problems are put forward.

The content of this paper, except the last section, was already contained in a lecture given in March 1989 at the Department of Mathematics of the University of Stockholm; I thank Prof. P. Martin-Löf for his kind invitation. Soon after, a first draft of this paper was read by Prof. H. Ono, whose answers [O1] and [O2] in turn influenced the chapter on algebraic semantics in Prof. A. S. Troelstra's lectures [T]. So by now the completeness proof for BL has partly lost its originality; I will thus stress on the peculiarity of the approach via pretopologies.

The main advantage of pretopologies seems to be that of having a middle position: so on one hand little effort is needed to show the completeness of the semantics of pretopologies, as usual with algebraic semantics to which it is closely connected, but

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<sup>1</sup>In [S] the same calculus BL was baptized ML, M for minimal; I now prefer to avoid using the word minimal since its meaning is debatable in linear context. It is also debatable whether it is correct to call basic a logic in which  $\otimes$  is commutative; however, though commutativity is often not essential to obtain many of the results to follow, I believe that assuming it makes basic ideas more perspicuous. The reader can consult [T] for a recent survey and references on linear logic, keeping aware of some differences in notation. Also, see [S2] for a short survey on pretopologies.

on the other hand, contrary to algebraic semantics, pretopologies can be given an intuitive meaning which helps in forming a useful mental picture. A second advantage is that they allow fully constructive definitions and proofs; by this we mean, more specifically, that the set theory we need, unless otherwise stated, is a fragment of intuitionistic type theory [ITT], even if its notation and terminology is not adopted strictly here.

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## 1. Pretopologies.

A *base*  $\mathcal{S}$  is here simply a commutative monoid  $(S, \cdot, 1)$ . Elements of  $S$  are called objects and denoted by  $a, b, c, \dots$ ; the binary operation  $\cdot$  is called combination. A *precover* on  $\mathcal{S}$  is a relation  $\triangleleft$  between objects and subsets  $U, V, W, \dots$  of  $S$  which satisfies

$$\begin{array}{l} \text{reflexivity} \quad \frac{a \in U}{a \triangleleft U} \\ \text{transitivity} \quad \frac{a \triangleleft U \quad U \triangleleft V}{a \triangleleft V} \\ \text{stability} \quad \frac{a \triangleleft U \quad b \triangleleft V}{a \cdot b \triangleleft U \cdot V} \end{array}$$

where<sup>2</sup>  $U \triangleleft V \equiv (\forall b \in U)(b \triangleleft V)$  and  $U \cdot V \equiv \{a \cdot b : a \in U, b \in V\}$ . A *pretopology*  $\mathcal{F}$  is a base  $\mathcal{S}_{\mathcal{F}}$  provided with a precover  $\triangleleft_{\mathcal{F}}$  (subscripts are almost always omitted).

The meaning of the definition of pretopology can be grasped from several different points of view. First, the topological flavour of terminology is due to the fact that the notion of pretopology was born from that of formal topology<sup>3</sup>, on which an intuitionistic approach to pointfree topology is based (cf. [IFS], [S]); this is why we continue to read  $a \triangleleft U$  as “ $a$  is (pre)covered by  $U$ ”.

However, pretopologies can be given an independent intuitive interpretation. We think of  $\mathcal{S}_{\mathcal{F}}$  as a universe of concretely produced objects, or occurrences of pieces of information, which can be always combined by means of  $\cdot$ ; then 1 is the object which costs nothing to be produced. The “logical” structure of the universe is determined by the infinitary relation  $\triangleleft_{\mathcal{F}}$ , which is thought of as a generalized membership; then  $a \triangleleft U$  is read as “ $a$  is forced by  $\triangleleft_{\mathcal{F}}$  to be an element of  $U$ ” or “ $a$  is an  $\mathcal{F}$ -element of  $U$ ”.

It is then natural to consider the operator on subsets which associates with each subset  $U$  the subset  $\mathcal{F}U$  of all its  $\mathcal{F}$ -elements; formally,

$$\mathcal{F}U \equiv \{a \in S : a \triangleleft_{\mathcal{F}} U\}.$$

<sup>2</sup>Here and in the whole paper  $\equiv$  is the sign for definitional equality; when a definition is first given, the definiendum will always be at the left and the definiens at the right.

<sup>3</sup>Cf. section 6 below for a definition.

Intuitively,  $\mathcal{F}U$  is the least property which “behaves well” with respect to the logic given by  $\triangleleft$  and which contains  $U$ .

A subset of the form  $\mathcal{F}U$  for some  $U$  is called  $\mathcal{F}$ -saturated. Any condition involving  $\triangleleft$  can equivalently be expressed in terms of membership  $\in$  and  $\mathcal{F}$ -saturated subsets; let us call “rewriting” such a translation. So, by the definition,  $a \triangleleft U$  is rewritten as  $a \in \mathcal{F}U$ , and hence  $U \triangleleft V$  is rewritten as  $U \subseteq \mathcal{F}V$ . If we apply this to the definition of precover, we see that reflexivity is rewritten as  $U \subseteq \mathcal{F}U$  and transitivity as  $U \subseteq \mathcal{F}U \Rightarrow \mathcal{F}U \subseteq \mathcal{F}V$ ; it is easily seen that such two properties together are equivalent to  $U \subseteq \mathcal{F}U$ ,  $U \subseteq V \Rightarrow \mathcal{F}U \subseteq \mathcal{F}V$  and  $\mathcal{F}\mathcal{F}U \subseteq \mathcal{F}U$ , which are usually taken as the definition of closure operator on  $\mathcal{P}S$ . Stability is rewritten as  $\mathcal{F}U \cdot \mathcal{F}V \subseteq \mathcal{F}(U \cdot V)$ , namely a form of compatibility with combination.

Conversely, given any closure operator  $\mathcal{F}$  compatible with combination, by putting

$$a \triangleleft U \equiv a \in \mathcal{F}U$$

we immediately obtain a precover  $\triangleleft$ . The result is a biunivocal correspondence between precovers and closure operators compatible with combination; we are thus free to use either approach, according to convenience.

Obviously, the identity operator  $\mathfrak{S}U \equiv U$  trivially satisfies all the required conditions; the corresponding infinitary relation is crude membership  $\in$ , which then is the trivial precover (note that stability is just the definition of  $U \cdot V$ ); this confirms the idea that  $\triangleleft$  is a sort of generalized membership.

The third point of view is purely mathematical (for details, see [BS]). Given a base  $\mathcal{S}$ , the structure  $(\mathcal{P}S, \cdot, 1, \cup)$ , where  $\mathcal{P}S$  is the powerset of  $S$ ,  $\cdot$  is defined as above on subsets and  $\cup$  is set-theoretic union, is a (unital commutative) quantale (cf. e.g. [T] or [R] for definitions); then pretopologies with base  $\mathcal{S}$  can be seen as the shortest description of its quotients. In fact, a precover  $\triangleleft$  induces an equivalence between subsets

$$U =_{\mathcal{F}} V \equiv (U \triangleleft V \ \& \ V \triangleleft U)$$

which moreover respects the operations  $\cdot$  and  $\cup$ , i.e. is a quantale congruence (or a quantic nucleus, see [R], p. 29 and 32). Conversely, any quantale congruence  $\theta$  on  $(\mathcal{P}S, \cdot, 1, \cup)$  is induced in this way by a precover, namely the precover  $\triangleleft_{\theta}$  obtained by putting

$$a \triangleleft_{\theta} U \equiv U \cup \{a\} \theta U,$$

and the correspondence can be shown to be biunivocal. So, for any pretopology  $\mathcal{F}$ , the quotient  $\mathcal{P}S/_{=_{\mathcal{F}}}$  can be given the structure of a quantale (in the expected way: if  $[U]$  denotes the  $=_{\mathcal{F}}$ -equivalence class of  $U$ , put  $[U] \cdot [V] \equiv [U \cdot V]$  and  $\bigvee_{i \in I} [U_i] \equiv [\bigcup_{i \in I} U_i]$ ); moreover, any quantale can be presented in this way (and thus one can forget about the abstract algebraic definition).

An even simpler description is obtained in terms of  $\mathcal{F}$ -saturated subsets. In fact, since  $U =_{\mathcal{F}} V$  iff  $\mathcal{F}U = \mathcal{F}V$  and since  $\mathcal{F}\mathcal{F}U = \mathcal{F}U$ , the assignment  $[U] \mapsto \mathcal{F}U$  defines a bijection between  $\mathcal{P}S/_{=_{\mathcal{F}}}$  and the collection of all  $\mathcal{F}$ -saturated subsets, which

we call  $Sat(\mathcal{F})$ . Now, since  $\mathcal{F}$  is a closure operator,  $Sat(\mathcal{F})$  is automatically provided with a complete lattice structure (as usual, put  $\bigvee_{i \in I} \mathcal{F}U_i \equiv \mathcal{F}(\bigcup_{i \in I} \mathcal{F}U_i)$  and  $\bigwedge_{i \in I} \mathcal{F}U_i \equiv \mathcal{F}(\bigcap_{i \in I} \mathcal{F}U_i)$ ) and since moreover  $\mathcal{F}$  is compatible with combination, after providing  $Sat(\mathcal{F})$  with the operation  $\cdot_{\mathcal{F}}$  defined by  $\mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V \equiv \mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V)$ , the assignment  $[U] \mapsto \mathcal{F}U$  becomes an isomorphism of quantales.

We can now come back to the point of view of logic. Since  $\mathcal{F}$ -saturated subsets are thought of as “good” properties over  $S$ , it is quite natural to adopt them for the interpretation of formulas. Consequence between  $\mathcal{F}$ -saturated subsets is just inclusion, but since  $\mathcal{F}U \subseteq \mathcal{F}V$  iff  $U \triangleleft V$ , the precover relation  $\triangleleft$  lends itself to be the interpretation of consequence  $\vdash$  between formulas. The structure  $Sat(\mathcal{F})$  is rich enough with finitary and infinitary operations to interpret all logical constants of first-order basic linear logic (formally introduced in section 2 below). In fact, the complete lattice structure gives an interpretation to (additive) conjunction  $\&$  and disjunction  $\oplus$ , and to quantifiers, while the operation  $\cdot_{\mathcal{F}}$  will be the interpretation of multiplicative conjunction  $\otimes$ . In any pretopology, there are also three distinguished  $\mathcal{F}$ -saturated subsets, the top  $S$  and the bottom  $\mathcal{F}\emptyset$  of  $Sat(\mathcal{F})$  and the  $\mathcal{F}$ -saturation of the subset  $\{1\}$ : they will be the interpretation of the constant atomic formulas  $\top$ ,  $0$  and  $1$  respectively. What about implication? By stability of  $\triangleleft$  (which corresponds to infinite distributivity of  $\vee$  with respect to  $\cdot_{\mathcal{F}}$  in the quantale  $Sat(\mathcal{F})$ ), we can define it by putting, for any  $U, V \subseteq S$  (and writing  $a \cdot U$  as an abbreviation for  $\{a\} \cdot U$ ):

$$U \rightarrow_{\mathcal{F}} V \equiv \{a \in S : a \cdot U \triangleleft V\}.$$

Equivalently,  $a \in U \rightarrow_{\mathcal{F}} V$  iff  $a$  combined with any element of  $U$  gives an  $\mathcal{F}$ -element of  $V$ ; that is, the definition of  $\rightarrow_{\mathcal{F}}$  is the closest one can get, within the universe of objects with combination, to the usual intuitionistic semantic explanation of implication. The definition of negation too, strictly follows Brouwer’s spirit: once we know the extension  $\perp_{\mathcal{F}}$  of the property of being an “impossible” object in  $\mathcal{F}$ , we define  $-_{\mathcal{F}}U$  as  $U \rightarrow_{\mathcal{F}} \perp_{\mathcal{F}}$ , that is  $a \in -_{\mathcal{F}}U$  iff the combination of  $a$  with any object in  $U$  is “impossible”.

Note that the definition of  $\rightarrow_{\mathcal{F}}$  contains Girard’s linear implication  $\multimap$  between subsets as a particular case: in fact,  $\multimap$  can be defined ([G], 1.10, p. 20) as

$$U \multimap V \equiv \{a \in S : a \cdot U \subseteq V\}$$

which is the same thing as  $U \rightarrow_{\mathfrak{S}} V$ , where  $\mathfrak{S}$  is the trivial operator. It is shown below (lemma 3) that  $\rightarrow_{\mathcal{F}}$  is indeed a well-defined operation on  $Sat(\mathcal{F})$  and that, like all other operations, it can be described only in terms of the operator  $\mathcal{F}$  and set-theoretic notions (including  $\multimap$ ).

After the introduction of the sequent calculus BL and a precise formulation of the interpretation of formulas, we will rigourously see that the above semantics of logical constants is complete, which is the last good reason for the introduction of pretopologies. To avoid fragmentation of proofs, all the immediate properties of pretopologies needed in the paper are summed up in the following three lemmas.

LEMMA 1 (PROPERTIES OF  $\triangleleft$  ON SUBSETS). *In any pretopology, the following hold for any  $U, V, W, Z$ :*

- (1)  $U \triangleleft U, \quad \frac{U \subseteq V}{U \triangleleft V}$
- (2)  $\frac{U' \subseteq U \quad U \triangleleft V}{U' \triangleleft V}, \quad \frac{U \triangleleft V \quad V \subseteq V'}{U \triangleleft V'}$
- (3)  $\frac{W \triangleleft U \quad Z \triangleleft V}{W \cdot Z \triangleleft U \cdot V}$
- (4)  $U \cdot (U \rightarrow V) \triangleleft V$
- (5)  $\frac{Z \triangleleft U \quad W \triangleleft U \rightarrow V}{Z \cdot W \triangleleft V}$
- (6)  $\frac{W \cdot U \triangleleft V}{W \triangleleft U \rightarrow V}$

PROOF: Recalling the definition  $U \triangleleft V \equiv (\forall a \in U)(a \triangleleft V)$ , (1) and (2) follow immediately from reflexivity and transitivity, and (3) by stability. (4) is a reformulation of the definition of  $\rightarrow_{\mathcal{F}}$ , from which (5) follows by (3). Finally, (6) holds by the definition of  $\rightarrow_{\mathcal{F}}$  and reflexivity.

LEMMA 2 (EQUIVALENTS OF STABILITY). *The following are all equivalent to stability, and hence hold in any pretopology:*

- (1)  $\frac{Z \triangleleft U \quad W \cdot U \triangleleft V}{Z \cdot W \triangleleft V}$  (cut)
- (2)  $W \cdot \mathcal{F}U \triangleleft W \cdot U$
- (3)  $\mathcal{F}U \cdot \mathcal{F}V \subseteq \mathcal{F}(U \cdot V), \mathcal{F}(\mathcal{F}U \cdot \mathcal{F}V) = \mathcal{F}(U \cdot V)$
- (4)  $\frac{a \triangleleft U}{a \cdot b \triangleleft U \cdot b}$  (localization)

PROOF: (1) follows from (6) and (5) of lemma 1; applying (1) to  $\mathcal{F}U \triangleleft U$  we obtain (2), from which (3) is obvious; now note that (3) is the “rewriting” of stability. Equivalence of (4) with stability is obtained directly.

LEMMA 3 (PROPERTIES OF  $\rightarrow_{\mathcal{F}}$ ). *The following hold in any pretopology, for any  $U, V, W, Z$ :*

- (1)  $W \triangleleft U \rightarrow V$  iff  $W \cdot U \triangleleft V$ , hence  $1 \triangleleft U \rightarrow V$  iff  $U \triangleleft V$  ( $\rightarrow_{\mathcal{F}}$  is adjoint to  $\cdot$ )
- (2)  $U \rightarrow_{\mathcal{F}} V$  is  $\mathcal{F}$ -saturated
- (3)  $U =_{\mathcal{F}} U', V =_{\mathcal{F}} V' \Rightarrow U \rightarrow_{\mathcal{F}} V = U' \rightarrow_{\mathcal{F}} V'$  ( $\rightarrow_{\mathcal{F}}$  respects  $=_{\mathcal{F}}$ )
- (4)  $U \rightarrow_{\mathcal{F}} V = U \multimap \mathcal{F}V$

PROOF: From  $W \triangleleft U \rightarrow V$  and (4) of lemma 1, it follows that  $W \cdot U \triangleleft V$  by cut; the other direction of (1) is (6) of lemma 1. Then from  $a \triangleleft U \rightarrow V$  one has  $a \cdot U \triangleleft V$ , that is  $a \in U \rightarrow V$ , so (2) holds. (3) and (4) are immediate consequences of the fact that, by (2) of lemma 2,  $a \cdot U \triangleleft V$  iff  $a \cdot \mathcal{F}U \triangleleft V$  and that  $a \cdot U \triangleleft V$  iff  $a \cdot U \triangleleft \mathcal{F}V$  iff  $a \cdot U \subseteq \mathcal{F}V$  hold for any  $a, U, V$ .

## 2. Basic linear logic and its extensions.

The language of predicate basic linear logic contains two symbols<sup>4</sup>  $\otimes$  and  $\&$  for conjunction,  $\oplus$  for disjunction,  $\multimap$  for implication, and four constants for atomic formulas  $\top$ ,  $1$ ,  $0$  and  $\perp$ , beside quantifiers  $\exists$ ,  $\forall$ , individual variables  $x, y, \dots$ , signs for functions  $f, \dots$  and for predicates  $R, \dots$  as usual. Terms and formulas are defined as usual, and denoted by  $s, t, t_1 \dots$  and  $A, B, C, \dots$  respectively.

The behaviour of logical constants is governed by the following Gentzen-style sequent calculus, defining what we here call basic linear logic BL. Note the absence of structural rules of weakening and contraction, which compels to pay more attention than usual to the context of auxiliary formulas  $\Gamma, \Delta, \dots$

Axioms	$A \vdash A$		
Exchange	$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$	Cut	$\frac{\Gamma \vdash A \quad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}$
$\otimes$ L	$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$	$\otimes$ R	$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$
$\&$ L	$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \& B \vdash C}$	$\&$ R	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$
$\oplus$ L	$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}$	$\oplus$ R	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \oplus B}$
$\multimap$ L	$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$	$\multimap$ R	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$
1L	$\frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}$	1R	$\vdash 1$
0	$\Gamma, 0 \vdash C$	$\top$	$\Gamma \vdash \top$
$\forall$ L	$\frac{\Gamma, At \vdash C}{\Gamma, \forall x Ax \vdash C}$	$\forall$ R	$\frac{\Gamma \vdash Ax}{\Gamma \vdash \forall x Ax}$
$\exists$ L	$\frac{\Gamma, Ax \vdash C}{\Gamma, \exists x Ax \vdash C}$	$\exists$ R	$\frac{\Gamma \vdash At}{\Gamma \vdash \exists x Ax}$

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<sup>4</sup>To avoid adding further disorder, I refrain from proposing a variation in notation and rather adhere strictly to Girard's notation and terminology in [G].

(where as usual  $t$  is any term free for  $x$  in  $A$ , and in  $\forall R$  and  $\exists L$ ,  $x$  is not free in  $\Gamma$  and  $C$ ). Note that  $\perp$  is like any other formula, since no rules on it are assumed; it is there solely to define negation by  $\neg A \equiv A \rightarrow \perp$ .

By adding suitable rules or axioms to the above calculus, we can obtain (equivalent formulations of) several other logics. We will consider here the following possible additions:

weakening, or <b>w</b>	$\frac{\Gamma \vdash C}{\Gamma, A \vdash C}$
contraction, or <b>c</b>	$\frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$
ex falso quodlibet, or $\perp$ -rule	$\perp \vdash A$
double negation, or <b>dn</b>	$\neg\neg A \vdash A$

We will prove completeness of BL and of all its 11 extensions obtainable by adding any combination of the above four assumptions<sup>5</sup>; such extensions include several logics which already have a name, either traditional or recently introduced, as listed below:

- (1) BL +  $\perp$ -rule is here called IL, for Intuitionistic Linear logic (cf. [GL], [A], [S]);
- (2) IL + **w** is called intuitionistic logic without contraction (cf. [OK]) and its implicational fragment is equivalent to BCK-logic;
- (3) IL + **w** + **c** is equivalent to intuitionistic logic I;
- (4) BL + **dn** is equivalent to the exponential-free fragment of Girard's linear logic (cf. [G]), and is here called CL, for Classical Linear logic;
- (5) CL + **w** is equivalent to direct logic (cf. [KW]);
- (6) CL + **w** + **c** is equivalent to classical logic C.

All the proofs of such equivalences are syntactic routine, and are left to the reader (with one warning: note that also in the extensions of BL all provable sequents have exactly one formula on the right side).

Finally, we will also consider (end of section 5) the extension of each such system obtained by introducing an equality = which satisfies the standard axioms.

### 3. The semantics of pretopologies.

The basic idea, as in section 1 and in [S], is that  $\mathcal{F}$ -saturated subsets, for an arbitrary pretopology  $\mathcal{F}$ , act as generalized truth values of formulas. The interpretation of terms is standard: the domain of interpretation for individual variables is a set  $D$ , while the interpretation of a function sign is then an operation on  $D$ . The link between  $D$  and  $\mathcal{F}$  is given by the interpretation of atomic formulas  $Rt_1 \dots t_n$ , since they must be given a truth value, i.e. an  $\mathcal{F}$ -saturated subset, which depends on the interpretation of  $t_1, \dots, t_n$ . So, given an arbitrary pretopology  $\mathcal{F}$ , we call  $\mathcal{D} = (D, f^{\mathcal{D}}, \dots, R^{\mathcal{D}}, \dots)$  an  $\mathcal{F}$ -structure for the given language when:

- (1)  $D$  is an arbitrary set, called the domain;

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<sup>5</sup>Although the number of subsets of  $\{\mathbf{w}, \mathbf{c}, \perp\text{-rule}, \mathbf{dn}\}$  is  $16 = 2^4$ , the number of different extensions reduces to 11 since it is easy to see that in presence of **dn**, **w** and  $\perp$ -rule are equivalent.

- (2) for any n-ary function sign  $f$  in the language,  $f^{\mathcal{D}}$  is a function  $D^n \rightarrow D$ ;  
(3) for any n-ary predicate sign  $R$  in the language,  $R^{\mathcal{D}}$  is a function  $D^n \rightarrow \text{Sat}(\mathcal{F})$ .

The interpretation  $t^{\mathcal{D}}$  of a term  $t$  is defined inductively as usual; given an assignment  $\sigma$  to individual variables in  $D$ , and a term  $t$  with variables  $x_1, \dots, x_n$ , we write  $t^\sigma$  for the element of  $D$  obtained by applying  $t^{\mathcal{D}}$  to  $\sigma(x_1), \dots, \sigma(x_n)$ . Then, given  $\sigma$  and an arbitrary saturated subset  $\perp_{\mathcal{F}}$ , the truth value of a formula  $A$  is the  $\mathcal{F}$ -saturated subset  $V^\sigma(A)$  defined by induction as follows:

$$\begin{array}{ll}
V^\sigma(Rt_1 \dots t_n) \equiv R^{\mathcal{D}}t_1^\sigma \dots t_n^\sigma & \text{for any relation sign } R \\
V^\sigma(\top) \equiv S, V^\sigma(\perp) \equiv \perp_{\mathcal{F}} & V^\sigma(1) \equiv \mathcal{F}\{1\}, V^\sigma(0) \equiv \mathcal{F}\emptyset \\
V^\sigma(A \otimes B) \equiv \mathcal{F}(V^\sigma(A) \cdot V^\sigma(B)) & V^\sigma(A \& B) \equiv V^\sigma(A) \cap V^\sigma(B) \\
V^\sigma(A \oplus B) \equiv \mathcal{F}(V^\sigma(A) \cup V^\sigma(B)) & V^\sigma(A \multimap B) \equiv V^\sigma(A) \rightarrow_{\mathcal{F}} V^\sigma(B) \\
V^\sigma(\exists x Ax) \equiv \mathcal{F}(\bigcup_{d \in D} V^\sigma(Ad)) & V^\sigma(\forall x Ax) \equiv \bigcap_{d \in D} V^\sigma(Ad)
\end{array}$$

where  $V^\sigma(Ad)$  is an abbreviation for  $V^{\sigma_d}(Ax)$ ,  $\sigma_d$  differing from  $\sigma$  only in the value of  $x$ , which is  $d$ .

Note that the above definitions include the standard (classical) ones as a particular case, namely when  $\mathcal{F}$  is the trivial pretopology on the trivial monoid on  $\{1\}$ ; in fact, in this case the saturated subsets are only  $\emptyset$  and  $\{1\}$ , hence  $R^{\mathcal{D}}$  can be identified with a relation on  $D$  in the usual classical sense, and finally, choosing  $\perp_{\mathcal{F}} = \emptyset$ ,  $V^\sigma(A) = \{1\}$  iff  $\mathcal{D} \models_{\sigma} A$  holds in the usual tarskian sense. Note also that, in this case,  $A \otimes B$  and  $A \& B$  are given the same truth value.

For any  $\mathcal{F}$ -structure  $\mathcal{D}$ , a formula  $A$  is said to be valid in  $\mathcal{D}$  for  $V^\sigma$  if  $1 \in V^\sigma(A)$ , and valid in  $\mathcal{D}$  if it is valid for any  $V^\sigma$ , that is for any choice of  $\sigma$  and  $\perp_{\mathcal{F}}$ . The valuation  $V^\sigma$  is extended to sequents by saying that  $\Gamma \vdash A$  is valid iff, for  $\Gamma = B_1, \dots, B_n$ ,  $B_1 \otimes \dots \otimes B_n \multimap A$  is valid; by the definition of  $V^\sigma$  and lemmas 3.1 and 2.3, this means that  $\Gamma \vdash A$  is valid if  $V^\sigma(B_1) \cdot \dots \cdot V^\sigma(B_n) \triangleleft V^\sigma(A)$  or equivalently, writing  $V^\sigma(\Gamma)$  for  $V^\sigma(B_1 \otimes \dots \otimes B_n)$ , if  $V^\sigma(\Gamma) \triangleleft V^\sigma(A)$ .

**THEOREM 4 (BASIC VALIDITY).** *Every theorem of BL is valid in every  $\mathcal{F}$ -structure.*

**PROOF:** It is a potentially routine matter to prove that axioms are valid, and that rules are valid, in the sense that they preserve validity; actually, all rules except  $\forall R$  and  $\exists L$ , preserve validity under a fixed  $V^\sigma$ . However, here is an actual proof for the convenience of the reader. To improve readability, let  $U \equiv V^\sigma(A)$ ,  $V \equiv V^\sigma(B)$ ,  $W \equiv V^\sigma(\Gamma)$  and  $Z \equiv V^\sigma(C)$ ; then validity is shown as follows:

axioms: because  $U \triangleleft U$  by lemma 1.1;

exchange: by commutativity of  $\cdot$ ;

cut: by lemma 2.1;

$\otimes R$ : by lemma 1.3, namely stability for subsets;

$\otimes L$ : built in in the definition of  $V^\sigma$ ;

$\& R$ : from  $W \triangleleft U$  and  $W \triangleleft V$ , when  $U$  and  $V$  are  $\mathcal{F}$ -saturated we have  $W \subseteq U$  and  $W \subseteq V$ , hence  $W \subseteq U \cap V$  and a fortiori  $W \triangleleft U \cap V$ ;



&L and  $\oplus$ R: by lemma 1.2, since obviously  $W \cdot (U \cap V) \subseteq W \cdot U$  and  $U \subseteq \mathcal{F}(U \cup V)$  respectively;

$\oplus$ L: from  $W \cdot U \triangleleft Z$  and  $W \cdot V \triangleleft Z$  we have  $W \cdot U \cup W \cdot V \triangleleft Z$ , and hence, since  $W \cdot U \cup W \cdot Z = W \cdot (U \cup V)$  by definition, also  $W \cdot (U \cup V) \triangleleft Z$ , from which the claim by lemma 2.2;

— $\circ$ R: by lemma 3.1;

— $\circ$ L: because it is derivable from cut and  $A, A \rightarrow B \vdash B$ , which is valid by lemma 1.5;

1R and 1L: because trivially  $1 \in \mathcal{F}\{1\}$  and  $W \cdot 1 = W$ ;

$\top$ -rule and 0-rule: because trivially  $\mathcal{F}\emptyset \triangleleft W$  and  $W \triangleleft S$  for any  $W$ ;

$\forall$ L and  $\exists$ R: like for &L and  $\oplus$ R respectively, since by definition  $V^\sigma(\forall x Ax) \subseteq V^\sigma(At)$  and  $V^\sigma(At) \subseteq V^\sigma(\exists x Ax)$  for any term  $t$ .

To see that  $\forall$ R is valid, assume that  $\Gamma \vdash Ax$  is valid in the given  $\mathcal{F}$ -structure, that is  $V^\sigma(\Gamma) \triangleleft V^\sigma(Ax)$  under any assignment  $\sigma$ . Since  $\Gamma$  does not contain  $x$  free,  $W \equiv V^\sigma(\Gamma)$  does not depend on the value given to  $x$  by  $\sigma$ , and hence  $W \triangleleft V^\sigma(Ad)$  for any  $d \in D$ . Now proceed as for &R: since  $V^\sigma(Ad)$  is saturated, we have  $W \subseteq V^\sigma(Ad)$  and hence  $W \subseteq \bigcap_{d \in D} V^\sigma(Ad)$ , from which the claim by lemma 1.1.

Similarly, if  $\Gamma, Ax \vdash C$  is valid, then  $W \cdot V^\sigma(Ad) \triangleleft Z$  for any  $d \in D$ , from which  $W \cdot \mathcal{F}(\bigcup_{d \in D} V^\sigma(Ad)) \triangleleft Z$  by the same argument as in  $\oplus$ L, which shows validity of  $\exists$ L.

#### 4. The basic completeness proof.

The proof of completeness is based on the construction of a canonical  $\mathcal{F}$ -structure. Let  $Frm$  be the set of formulas; like in the construction of Lindenbaum algebras, we put

$$(A =_{BL} B) \equiv (\vdash A \circ \text{—} \circ B),$$

where  $A \circ \text{—} \circ B \equiv (A \text{—} \circ B) \& (B \text{—} \circ A)$ . Since  $A =_{BL} B$  iff  $A \vdash B$  and  $B \vdash A$ , it is immediate to check that  $=_{BL}$  is an equivalence relation on  $Frm$  which moreover, by  $\otimes$ -rules, respects the operation  $\otimes$ , that is,  $A =_{BL} A'$  and  $B =_{BL} B'$  imply  $A \otimes B =_{BL} A' \otimes B'$ . Therefore, if  $[A] \equiv \{B \in Frm : A =_{BL} B\}$ , we can define  $[A] \otimes [B] \equiv [A \otimes B]$ . For any  $A, B, C$ , it is easily seen that  $A \otimes (B \otimes C) =_{BL} (A \otimes B) \otimes C$ ,  $A \otimes B =_{BL} B \otimes A$ , and, by 1-rules,  $1 \otimes A =_{BL} A$ ; hence  $Frm_{BL} \equiv (Frm / =_{BL}, \otimes, [1])$  is a commutative monoid, here called the *Lindenbaum base* on BL.

We now want to define on the Lindenbaum base a precover  $\triangleleft_{Pr}$  which is to be connected with the derivability relation  $\vdash_{BL}$ . The simplest way is to require that  $A \vdash C$  iff  $[A] \triangleleft_{Pr} [C]$ , or equivalently to put

$$Pr[C] \equiv \{[A] \in Frm / =_{BL} : A \vdash_{BL} C\}.$$

Since when  $C =_{BL} B$  obviously  $A \vdash C$  iff  $A \vdash B$ , this is a sound definition, and hence omitting square brackets is harmless. We thus instead of  $[A] \in Pr[C]$  always write  $A \vdash C$  or  $A \in PrC$ ; note that then  $A \vdash C$ ,  $A \in PrC$ ,  $\{A\} \subseteq PrC$ ,  $PrA \subseteq PrC$  are all equivalent.

To obtain a closure operator, it is enough to close the family  $\{\text{Pr}C : C \in \text{Frm}\}$  under arbitrary intersections, or equivalently to extend  $\text{Pr}$  to any subset  $\Sigma$  of  $F_{BL}$  by putting

$$\text{Pr}\Sigma \equiv \bigcap \{\text{Pr}C : \Sigma \subseteq \text{Pr}C\}.$$

The equivalent formulation

$$B \triangleleft_{\text{Pr}} \Sigma \equiv (\forall C)(\Sigma \subseteq \text{Pr}C \Rightarrow B \vdash C).$$

will also be frequently used. Observe that notation is consistent, that is  $\text{Pr}A = \text{Pr}\{A\}$ , because for any  $B$ ,  $(\forall C)(A \vdash C \Rightarrow B \vdash C)$  iff  $B \vdash A$ . Also, note that for any  $\Sigma_1$  and  $\Sigma_2$ ,  $\text{Pr}\Sigma_1 = \text{Pr}\Sigma_2$  is, by definition of  $\text{Pr}$ , equivalent to: for any formula  $C$ ,  $\Sigma_1 \subseteq \text{Pr}C$  iff  $\Sigma_2 \subseteq \text{Pr}C$ .

LEMMA 5. *The structure  $\text{Pr}_{BL} = (\text{Frm}_{BL}, \text{Pr})$  is a pretopology, called the Lindenbaum pretopology of  $BL$ .*

PROOF: By its definition,  $\text{Pr}$  is a closure operator. Moreover, it is automatically compatible with combination, or equivalently, by lemma 2.4, it satisfies localization, which in this case amounts to

$$\frac{(\forall C)(\Sigma \subseteq \text{Pr}C \Rightarrow A \vdash C)}{(\forall C)(\Sigma \otimes B \subseteq \text{Pr}C \Rightarrow A \otimes B \vdash C)}$$

In fact, let  $C$  be arbitrary. From the premiss applied to  $B \multimap C$ , one has  $\Sigma \subseteq \text{Pr}(B \multimap C) \Rightarrow A \vdash (B \multimap C)$ ; but by  $\multimap$ -rules and cut,  $\Sigma \otimes B \subseteq \text{Pr}C$  iff  $\Sigma \subseteq \text{Pr}(B \multimap C)$ , and hence the conclusion.

Now we can easily define a  $\text{Pr}$ -structure  $\mathcal{T}$  with domain  $T$ , the set of all terms. As usual, the trick is that the interpretation  $f^{\mathcal{T}}$  of any function sign  $f$  is  $f$  itself. The new trick is to interpret a predicate sign  $R$  in the function  $R^{\mathcal{T}} : T^n \rightarrow \text{Sat}(\text{Pr})$  defined by

$$R^{\mathcal{T}}(t_1, \dots, t_n) \equiv \text{Pr}(R(t_1, \dots, t_n)).$$

The resulting  $\text{Pr}$ -structure is called the *BL-canonical structure*.

LEMMA 6 (ON THE CANONICAL VALUATION). *If  $\iota$  is the identity assignment on  $T$  (i.e.  $\iota x \equiv x$  for any variable  $x$ ) and  $V^\iota(\perp) \equiv \text{Pr}\perp$ , then for any formula  $A$ ,*

$$V^\iota(A) = \text{Pr}A.$$

*$V^\iota$  is called the canonical valuation.*

PROOF: By induction on formulas. For atomic formulas  $R(t_1, \dots, t_n)$ , the claim holds by the definition of  $V^\sigma$  and  $R^{\mathcal{T}}$ , and the fact that  $t^\iota = t$  for any  $t \in T$ . By the definition of valuation, the claim for  $\top$  and  $1$  amounts to  $\text{Pr}\top = \text{Frm}/_{=BL}$ , which holds by the  $\top$ -rule, and  $\text{Pr}\{1\} = \text{Pr}1$  respectively; the claim for  $0$  is  $\text{Pr}0 = \text{Pr}\emptyset$ , which by definition of  $\text{Pr}$  means that  $0 \in \text{Pr}C$  iff  $\emptyset \subseteq \text{Pr}C$  for any  $C$ , which is trivial by  $0$ -rule. Finally, recall that  $V^\iota(\perp) = \text{Pr}\perp$  is assumed.

Now assume, by inductive hypothesis, that  $V'(A) = \text{Pr}A$  and  $V'(B) = \text{Pr}B$ . Then, by the definition of valuation, the claim for connectives becomes:

$\text{Pr}(A \otimes B) = \text{Pr}(\text{Pr}A \otimes \text{Pr}B)$ , which, recalling that  $\text{Pr}C = \text{Pr}\{C\}$  for any  $C$  and that  $\{A \otimes B\} = \{A\} \otimes \{B\}$ , appears to be an equivalent formulation of stability, by lemma 2.3;

$\text{Pr}(A \& B) = \text{Pr}A \cap \text{Pr}B$ , which means  $C \in \text{Pr}(A \& B)$  iff  $C \in \text{Pr}A \cap \text{Pr}B$ , that is

for any formula  $C$ ,  $C \vdash A \& B$  iff  $C \vdash A$  and  $C \vdash B$ ,

which is immediate by  $\&$ -rules;

$\text{Pr}(A \oplus B) = \text{Pr}(\text{Pr}A \cup \text{Pr}B)$ , which, by the definition of  $\text{Pr}$ , means that for any  $C$ ,  $A \oplus B \in \text{Pr}C$  iff  $\text{Pr}A \cup \text{Pr}B \subseteq \text{Pr}C$ , which, since obviously  $\text{Pr}A \cup \text{Pr}B \subseteq \text{Pr}C$  iff  $\text{Pr}A \subseteq \text{Pr}C$  and  $\text{Pr}B \subseteq \text{Pr}C$ , is equivalent to

for any formula  $C$ ,  $A \oplus B \vdash C$  iff  $A \vdash C$  and  $B \vdash C$ ,

which is immediate by  $\oplus$ -rules;

$\text{Pr}(A \multimap B) = \text{Pr}A \rightarrow_{\text{Pr}} \text{Pr}B$  which, since  $\text{Pr}A \rightarrow_{\text{Pr}} \text{Pr}B = \{A\} \rightarrow_{\text{Pr}} \text{Pr}B$  by lemma 1.3, is equivalent to

for any formula  $C$ ,  $C \vdash A \multimap B$  iff  $C \otimes A \vdash B$ ,

which is immediate by rules for  $\multimap$  and  $\otimes$ .

We now turn to quantifiers, assuming by inductive hypothesis that  $V'(At) = \text{Pr}(At)$  for every term  $t$ . Since by definition  $V'(\forall xAx) \equiv \bigcap_{t \in T} V'(At)$ , the claim for  $\forall$  then reduces to  $\text{Pr}(\forall xAx) = \bigcap_{t \in T} \text{Pr}(At)$ , that is

for any formula  $C$ ,  $C \vdash \forall xAx$  iff  $C \vdash At$  for every term  $t$

which is obvious by  $\forall$ -rules (from right to left, if  $x$  is free in  $C$ , choose  $z$  not occurring in  $C$ , so that from  $C \vdash Az$  we can derive  $C \vdash \forall zAz$  and hence the claim since  $\forall zAz \vdash \forall xAx$ ).

Similarly, since  $V'(\exists xAx) \equiv \text{Pr}(\bigcup_{t \in T} V'(At))$ , the claim for  $\exists$  becomes  $\text{Pr}(\exists xAx) = \text{Pr}(\bigcup_{t \in T} \text{Pr}(At))$ , which by definition of  $\text{Pr}$  means that for every  $C$ ,  $\exists xAx \in \text{Pr}C$  iff  $\bigcup_{t \in T} \text{Pr}(At) \subseteq \text{Pr}C$ , that is:

for any formula  $C$ ,  $\exists xAx \vdash C$  iff  $At \vdash C$  for every term  $t$

which is obvious by  $\exists$ -rules (again, changing names to variables if necessary).

We now can easily conclude:

**THEOREM 7 (BASIC COMPLETENESS).** *A formula is provable in basic linear logic iff it is valid.*

**PROOF:** Validity was shown in section 3. Conversely, assume  $A$  is valid. Then in particular  $A$  is valid under the canonical valuation, that is  $[1] \in V'(A)$ . By lemma 6 on the canonical valuation,  $[1] \in \text{Pr}A$ , that is  $1 \vdash A$  and hence  $\vdash A$  by 1R and cut.

## 5. Completeness for extensions of BL.

Note that all the proofs in section 4 are completely constructive, and actually use only positive arguments (that is, roughly speaking, the metalogic is minimal). At least in this case, this is not just an ideological remark: it means that all arguments continue to hold whatever information we add. In particular, whatever extension  $L$  obtained by adding axioms or rules to the sequent calculus BL we consider, if we define an  $L$ -canonical structure in the obvious way (simply consider  $\vdash_L$  in place of  $\vdash_{BL}$ ), lemmas 5 and 6 continue to hold for  $L$ . Therefore, to prove completeness for such  $L$ , it is enough to:

1. characterize the class of models in which  $L$  is valid;
2. show that the  $L$ -canonical structure is in that class.

It is straightforward to work out task 1. for the extensions by weakening  $\mathbf{w}$ , contraction  $\mathbf{c}$  and  $\perp$ -rule:

LEMMA 8. *a. The rule of weakening is valid in a pretopology  $\mathcal{F}$  iff  $\mathcal{F}$  satisfies one of the following equivalent conditions:*

- (1)  $\frac{W \triangleleft Z}{W \cdot U \triangleleft Z}$
- (2)  $W \cdot U \triangleleft W$
- (3)  $a \cdot b \triangleleft a$
- (4)  $b \triangleleft 1$

*b. The rule of contraction is valid in  $\mathcal{F}$  iff  $\mathcal{F}$  satisfies one of the following equivalent conditions:*

- (1)  $\frac{W \cdot U \cdot U \triangleleft Z}{W \cdot U \triangleleft Z}$
- (2)  $U \triangleleft U \cdot U$
- (3)  $a \triangleleft a \cdot a$

*c. The  $\perp$ -rule is valid under a valuation  $V$  on  $\mathcal{F}$  iff  $V(\perp) = \mathcal{F}\emptyset$ ; any such  $V$  is called normal.*

PROOF: a. Condition (1) is a pedantic translation of weakening in terms of subsets. Each of the remaining conditions is obtained as a particular case of the preceding one (in detail, take in succession  $W = Z$ ,  $W = \{a\}$  and  $U = \{b\}$ ,  $a = 1$ ). Finally, (1) comes from (4) by stability.

b. Condition (1) is the translation of contraction, (2) and (3) are particular cases (first take  $W = 1$  and  $Z = U \cdot U$ , then  $U = \{a\}$ ). Conversely, (2) follows from (3) since for any  $a \in U$ ,  $\{a \cdot a\} \subseteq U \cdot U$ , and (1) comes from (2) by cut.

c. Obvious, since  $\mathcal{F}\emptyset$  is the least saturated subset.

Dealing now with double negation  $\mathbf{dn}$ , that is with a system in which, contrary to BL,  $\perp$  is no longer an arbitrary formula, it is more convenient to fix the interpretation of  $\perp$ , that is enrich a pretopology with an  $\mathcal{F}$ -saturated subset  $\perp_{\mathcal{F}}$  and say that a formula is valid in  $(\mathcal{F}, \perp_{\mathcal{F}})$  if it is valid for all valuations  $V$  such that  $V(\perp) = \perp_{\mathcal{F}}$ . But then working out task 1. for  $\mathbf{dn}$  brings us directly, and exactly, to Girard's

semantics for CL, namely phase spaces (cf. [G], p. 18). Given a base  $\mathcal{S} = (S, \cdot, 1)$  and an arbitrary subset  $\perp$ , following [G] we write  $U^\perp$  for  $U \multimap \perp \equiv \{a \in S : a \cdot U \subseteq \perp\}$ . The claim is that the operator  $(-)^{\perp\perp} : U \mapsto U^{\perp\perp}$  is a closure operator compatible with  $\cdot$ , that is that  $U \subseteq U^{\perp\perp}$ ,  $U \subseteq V^{\perp\perp} \Rightarrow U^{\perp\perp} \subseteq V^{\perp\perp}$  and  $U^{\perp\perp} \cdot V^{\perp\perp} \subseteq (U \cdot V)^{\perp\perp}$  hold for any  $U$  and  $V$ ; this could be verified by direct calculations (cf. [G], p. 18, 21 and 20 resp.), here we can obtain it as a corollary. In fact, if we think of the base  $\mathcal{S}$  as provided with the trivial precover  $\mathfrak{S}$ , then  $U \rightarrow_{\mathfrak{S}} \perp$  is exactly the same thing as  $U \multimap \perp$ , or  $U^\perp$ , so that for any valuation  $V$  such that  $V(\perp) = \perp$ , we have  $V(\neg\neg A) = V(A)^{\perp\perp}$  for any formula  $A$ . The claim then follows immediately by validity of BL, simply by checking that  $A \vdash \neg\neg A$ ,  $A \vdash \neg\neg B \Rightarrow \neg\neg A \vdash \neg\neg B$  and  $\neg\neg A \otimes \neg\neg B \vdash \neg\neg(A \otimes B)$  respectively are derivable in BL (hint: first show that if  $\Gamma, A \vdash \perp$  is derivable in BL, then also  $\Gamma, \neg\neg A \vdash \perp$ ).

So any subset  $\perp$  determines a pretopology on  $\mathcal{S}$ , which is called the *phase space* on  $\mathcal{S}$  with *dualizer*  $\perp$  and is denoted by  $\mathcal{S}^\perp$ ; its saturated subsets, i.e. subsets  $U$  which satisfy  $U^{\perp\perp} = U$ , are called *facts* in [G]. Note that the dualizer is a fact (because  $\neg\neg\perp \vdash \perp$  is derivable in BL).

It is easy to see that double negation **dn** is valid in every phase space  $\mathcal{S}^\perp$ . In fact, if  $V$  is any valuation on  $\mathcal{S}^\perp$  with  $V(\perp) = \perp$ , then for any formula  $A$  the definition  $V(\neg A) \equiv V(A) \rightarrow_{\mathcal{S}^\perp} V(\perp)$  gives  $V(\neg A) = V(A)^\perp$ , because  $V(A) \rightarrow_{\mathcal{S}^\perp} \perp = V(A) \multimap \perp^{\perp\perp}$  by lemma 3.4 and because  $\perp^{\perp\perp} = \perp$ ; so  $V(\neg\neg A) = V(A)^{\perp\perp}$ , but  $V(A)^{\perp\perp} = V(A)$  since  $V(A)$  is a fact.

Moreover, we can immediately add that no other pretopology can make **dn** valid (and this explains why, restricting to CL, the general notion of pretopology is inessential). In fact, the assumption that **dn** is valid in  $(\mathcal{F}, \perp)$  means that  $\mathcal{F}U = (\mathcal{F}U \rightarrow_{\mathcal{F}} \perp) \rightarrow_{\mathcal{F}} \perp$  for any  $U$ ; but  $\mathcal{F}U \rightarrow_{\mathcal{F}} \perp = U \rightarrow_{\mathcal{F}} \perp$  by properties of  $\rightarrow_{\mathcal{F}}$  and  $U \rightarrow_{\mathcal{F}} \perp = U \multimap \perp$  by lemma 3.4 because  $\perp$  is  $\mathcal{F}$ -saturated. So  $\mathcal{F}U = U^{\perp\perp}$  and hence, if  $\mathcal{S}$  is the base of  $\mathcal{F}$ ,  $\mathcal{F}$  indeed *is* the phase space  $\mathcal{S}^\perp$ . Summing up:

LEMMA 8. *d. For any base  $\mathcal{S}$  and any subset  $\perp$ , **dn** is valid in the phase space  $\mathcal{S}^\perp$ . Moreover, **dn** is valid in a pretopology  $(\mathcal{F}, \perp)$  with base  $\mathcal{S}$  iff  $\mathcal{F}$  is the phase space  $\mathcal{S}^\perp$ .*

It is now easy to work out task 2., that is to check that canonical pretopologies satisfy the respective conditions required by lemma 8:

LEMMA 9. *a.  $Pr_{BL+\mathbf{w}}$  satisfies  $A \triangleleft 1$ ;*

*b.  $Pr_{BL+\mathbf{c}}$  satisfies  $A \triangleleft A \otimes A$ ;*

*c. the canonical valuation on  $Pr_{BL+\perp rule}$  is normal;*

*d.  $Pr_{BL+\mathbf{dn}}$  is the phase space on the Lindenbaum base on CL with dualizer  $Pr\perp$ .*

PROOF: a. The claim is equivalent to  $A \in Pr_{BL+\mathbf{w}}1$ , i. e.  $A \vdash_{BL+\mathbf{w}} 1$ , which is immediate from 1R by weakening.

b. The claim is equivalent to  $(\forall C)(A \otimes A \vdash C \Rightarrow A \vdash C)$ , which is exactly what contraction tells.

c. By definition  $V^\perp(\perp) = Pr\perp$ , and  $Pr\perp = Pr\emptyset$  because  $Pr\perp = Pr0$  by the  $\perp$ -rule.

d. It must be shown that  $Pr\Sigma = (\Sigma \multimap Pr\perp) \multimap Pr\perp$  for an arbitrary set of formulas  $\Sigma$ . First note that, for any formula  $B$ ,  $B \in \Sigma \multimap Pr\perp$  iff  $(\forall D \in \Sigma)(B \otimes D \vdash \perp)$  iff

$(\forall D \in \Sigma)(D \in \text{Pr}(\neg B)) \equiv \Sigma \subseteq \text{Pr}(\neg B)$ . Then  $A \in (\Sigma \multimap \text{Pr}\perp) \multimap \text{Pr}\perp$ , which by definition is  $A \otimes (\Sigma \multimap \text{Pr}\perp) \subseteq \text{Pr}\perp$ , that is  $(\forall B)(B \in \Sigma \multimap \text{Pr}\perp \Rightarrow A \otimes B \vdash \perp)$ , becomes equivalent to  $(\forall B)(\Sigma \subseteq \text{Pr}(\neg B) \Rightarrow A \vdash \neg B)$  which, since any formula  $C$  is equivalent to a formula of the form  $\neg B$ , is equivalent to  $(\forall C)(\Sigma \subseteq \text{Pr}C \Rightarrow A \vdash C)$ , which is the definition of  $A \in \text{Pr}\Sigma$ .

The above lemmas 8 and 9 tell that BL extended by **w**, **c**,  $\perp$ -rule or **dn** is complete with respect to the corresponding class of models, defined by the condition in lemma 8 a., b., c. and d. respectively. Now note that again all arguments are positive, and hence we can repeat them in any combination; so

**THEOREM 10 (COMPLETENESS OF EXTENSIONS).** *All logics obtained from BL by adding one or more of the four assumptions **w**, **c**,  $\perp$ -rule and **dn**, are complete with respect to the class of models which satisfy all the corresponding conditions (in lemma 8 a., b., c. and d. resp.).*

We have thus achieved a completely constructive and uniform proof of completeness, with respect to the semantics of pretopologies and its specifications, of all logics mentioned in section 2.

Adding equality to the language and assuming the standard equality axioms, that is

$$\begin{aligned} & \vdash t = t \\ & s_1 = t_1, \dots, s_n = t_n \vdash f s_1, \dots, s_n = f t_1, \dots, t_n \\ & s_1 = t_1, \dots, s_n = t_n, R t_1, \dots, t_n \vdash R s_1, \dots, s_n \end{aligned}$$

for any  $s, t, f, R$ , it is natural to modify the definition of  $\mathcal{F}$ -structure by adding a function  $=^{\mathcal{D}}: D^2 \rightarrow \text{Sat}(\mathcal{F})$  and by requiring that it makes equality axioms valid, that is, writing  $\|c = d\|$  for  $=^{\mathcal{D}}(c, d)$ :

$$\begin{aligned} & 1 \triangleleft \|c = c\| \\ & \|c_1 = d_1\| \cdot \dots \cdot \|c_n = d_n\| \triangleleft \|f^{\mathcal{D}} c_1, \dots, c_n = f^{\mathcal{D}} d_1, \dots, d_n\| \\ & \|c_1 = d_1\| \cdot \dots \cdot \|c_n = d_n\| \cdot R^{\mathcal{D}}(d_1, \dots, d_n) \triangleleft R^{\mathcal{D}}(c_1, \dots, c_n). \end{aligned}$$

The trouble is that the addition of equality axioms to BL is not sufficient to derive general substitutivity of equal terms, namely

$$s_1 = t_1, \dots, s_n = t_n, A t_1, \dots, t_n \vdash A s_1, \dots, s_n$$

for every formula  $A$ . It is easily seen, however, that the obvious induction on formulas is in fact possible if we add also

$$\begin{array}{l} \text{weakening for equality} \quad \frac{\Gamma \vdash C}{\Gamma, s = t \vdash C} \\ \\ \text{contraction for equality} \quad \frac{\Gamma, s = t, s = t \vdash C}{\Gamma, s = t \vdash C} \end{array}$$

This further extension is certainly syntactically consistent, and does not destroy the peculiarities of BL; actually, to justify structural rules for equality one could argue

that a proposition asserting identity of terms, that is of operations, can have no other proof than an abstract verification, and hence reproducible and disposable *ad libitum*. However, the next trouble is validity in the semantics of pretopologies. A simple way out (as suggested by Silvio Valentini) is to require the valuation of formulas  $s = t$  to be an  $\mathcal{F}$ -saturated subset  $W$  which can be “weakened” and “contracted”, that is (by lemma 8.a and b) which satisfies  $W \triangleleft 1$  and  $W \triangleleft W \cdot W$ . Then we put

$$Center(\mathcal{F}) \equiv \{W \in Sat(\mathcal{F}) : W \triangleleft 1 \text{ and } W \triangleleft W \cdot W\}$$

and say that  $\mathcal{D}$  is an  $\mathcal{F}$ -structure with equality if, beside being an  $\mathcal{F}$ -structure, it is equipped with a function  $=^{\mathcal{D}}: D^2 \rightarrow Center(\mathcal{F})$  which makes equality axioms valid. Note that, since  $Center(\mathcal{F})$  always contains  $\mathcal{F}\emptyset$  and  $\mathcal{F}1$ , it becomes trivial only if the valuations of 0 and 1 coincide in  $\mathcal{F}$ .

With such setting, it is straightforward to apply the above method to prove completeness also to systems with equality (being careful to recall, in the proof of lemma 9, that  $\text{Pr}(s = t)$  must be shown to be in  $Center(\text{Pr})$ , which is obvious by structural rules for equality). We thus have:

**THEOREM 11 (COMPLETENESS FOR LOGICS WITH EQUALITY).** *Let  $L$  be BL or any of the extensions considered in theorem 10, and let  $L_e$  be obtained from  $L$  by adding the sign  $=$ , equality axioms and structural rules for equality. Then  $L_e$  is complete with respect to the class of models of  $L$  which are based on  $\mathcal{F}$ -structures with equality.*

## 6. Some connections with traditional completeness proofs.

Having obtained a uniform method to prove completeness of several different logics, it is quite natural to investigate on its connections with well established proofs. In this final section, a few results and problems in this direction are given.

The closest connection is with standard algebraic semantics. In fact, it is immediate to check that for any pretopology  $\mathcal{F}$ ,  $Sat(\mathcal{F})$  is a quantale (the only non-trivial property is infinite distributivity, which, recalling the definition of operations in  $Sat(\mathcal{F})$ , follows from  $U \cdot (\cup_{i \in I} V_i) = \cup_{i \in I} (U \cdot V_i)$  by several applications of lemma 2.3). In particular,  $Sat(\text{Pr}_{BL})$  is a quantale. A *locale*, or complete Heyting algebra, can be defined as a quantale in which  $a \cdot b = a \wedge b$  holds for any  $a, b$ ; it is immediate to check that  $a \cdot b = a \wedge b$  holds for any  $a, b$  iff  $a \cdot b \leq a$  and  $a \leq a \cdot a$  hold for any  $a, b$ . So, by lemma 8.a and 8.b,  $Sat(\mathcal{F})$  is a locale iff **w** and **c** are valid in  $\mathcal{F}$ . Then  $Sat(\text{Pr}_I)$  is a locale. Finally, a *complete Boolean algebra*, cBa for short, is a locale in which every element is regular, that is  $--a = a$  for any  $a$  (where  $-a \equiv a \rightarrow 0$ ). So, by lemma 8.c and 8.d,  $Sat(\mathcal{F})$  is a cBa iff **w**, **c** and **dn** are valid in  $\mathcal{F}$  for all valuations  $V$  with  $V(\perp) = V(0) = \mathcal{F}\emptyset$ , that is iff classical logic C is valid in  $(\mathcal{F}, \mathcal{F}\emptyset)$ . Then, by lemma 9.c and 9.d,  $Sat(\text{Pr}_C)$  is a cBa.

Now note that the usual definition of valuation of formulas in an algebraic structure (cf. e.g. [TvD], [T]) when applied to  $Sat(\mathcal{F})$  coincides with our definition. Thus the theorems of section 5 give:

**COROLLARY 12 (CONNECTION WITH ALGEBRAIC SEMANTICS).** *Basic linear (intuitionistic, classical) logic is complete with respect to the semantics given by structures*

with truth values in a quantale (locale,  $cBa$ )<sup>6</sup>.

The above corollary obviously applies also to logics with equality. It is worthwhile to note explicitly, however, that when  $\mathbf{w}$  and  $\mathbf{c}$  are valid in a pretopology  $\mathcal{F}$ , then  $Sat(\mathcal{F})$  is a locale, which of course coincides with  $Center(\mathcal{F})$ ; so  $\mathcal{F}$ -structures with equality include (also in view of theorem 13 below) the usual notion of  $\Omega$ -structure and  $\Omega$ -set (cf. [TvD]).

The connection with algebraic semantics is actually much deeper than it appears from the above corollary; in fact, not only  $Sat(\mathcal{F})$  is a quantale for any pretopology  $\mathcal{F}$ , but also, conversely, any quantale  $\mathcal{Q} = (Q, \bigvee, \cdot, 1)$  is isomorphic to  $Sat(\mathcal{F})$  for some pretopology  $\mathcal{F}$  (see [BS] or work out the hint: for any  $U \subseteq Q$ , put  $\mathcal{F}U \equiv \{a \in Q : a \leq \bigvee U\}$ ). Consequently, any locale is isomorphic to  $Sat(\mathcal{F})$  for a pretopology  $\mathcal{F}$  satisfying  $\mathcal{F}U \cdot_{\mathcal{F}} \mathcal{F}V = \mathcal{F}U \cap \mathcal{F}V$  for any  $U, V$ , a condition which is easily seen (cf. [S], p. 265) to be equivalent to:  $U \cdot V \triangleleft U$  and  $U \triangleleft U \cdot U$  for any  $U, V$ . Any such pretopology is here called a formal topology<sup>7</sup>, and the letter  $\mathcal{A}$  instead of  $\mathcal{F}$  is used to denote it. We thus have:

**THEOREM 13 (PRESENTATION OF QUANTALES AND LOCALES).** *Any quantale (locale) can be isomorphically presented as  $Sat(\mathcal{F})$  for some pretopology  $\mathcal{F}$  ( $Sat(\mathcal{A})$  for some formal topology  $\mathcal{A}$ ).*

By a similar argument and by lemma 9.d, we could also add: any  $cBa$  is isomorphic to  $Sat(\mathcal{A})$  for some formal topology satisfying  $\mathcal{A}U = U^{\perp\perp}$  for some subset  $\perp$ . With a little further work, however, one can obtain a more perspicuous characterization of  $cBa$ 's, which seems to be new and of independent mathematical interest. For any semilattice  $\mathcal{L} = (L, \cdot, 1)$  and any  $X \subseteq L$ , let  $\mathcal{L}_X \equiv \mathcal{L}^{\emptyset^{XX}}$ ; that is,  $\mathcal{L}_X$  is defined to be the phase space with base  $\mathcal{L}$  and with dualizer  $\emptyset^{XX}$  (recall that  $Y^{XX} \equiv (Y \multimap X) \multimap X$  for arbitrary subsets  $X$  and  $Y$ ), which in turn is the least fact in the phase space with dualizer  $X$ . Then:

**THEOREM 14 (REPRESENTATION OF COMPLETE BOOLEAN ALGEBRAS).** *For any semilattice  $\mathcal{L} = (L, \cdot, 1)$  and any subset  $X$ , if  $\mathcal{L}_X$  is defined as above, then  $Sat(\mathcal{L}_X)$  is a  $cBa$ ; conversely, for any  $cBa$   $\mathcal{B}$ , there is a semilattice  $\mathcal{L}$  with a subset  $X$  such that  $\mathcal{B}$  is isomorphic to  $Sat(\mathcal{L}_X)$ .*

**PROOF:** Putting together the arguments above, we can deduce that a quantale is in fact a  $cBa$  iff it is presented as  $Sat(\mathcal{F})$  for a pretopology  $\mathcal{F}$  in which classical logic  $C$  is valid for all normal valuations. So it is enough to prove that:

- (1)  $C$  is valid in  $\mathcal{L}_X$  for all normal valuations;
- (2) if  $C$  is valid in  $\mathcal{F}$  for all normal valuations, then  $Sat(\mathcal{F}) \cong Sat(\mathcal{L}_X)$  for some semilattice  $\mathcal{L}$  and some  $X \subseteq L$ .

The idea of the proof of (2) is quite simple, though details would be tedious. If  $\mathbf{w}$  and  $\mathbf{c}$  are valid in  $\mathcal{F}$ , then  $a \cdot a =_{\mathcal{F}} a$ , and hence  $\mathcal{S}/=_{\mathcal{F}}$  is a semilattice; the precover of  $\mathcal{F}$

<sup>6</sup>Of course, we could state a similar corollary for all other extensions of BL, if we had names for the corresponding algebraic structures.

<sup>7</sup>Note that in [IFS] a formal topology is defined as provided also with a positivity predicate.



can easily be lifted to  $\mathcal{S}/_{=\mathcal{F}}$ , but the structure of saturated subsets remains exactly the same. So we may assume, without loss of generality, that  $\mathcal{F}$  is based on a semilattice  $\mathcal{L}$ . Now, by lemma 9.d, assuming **dn** to be valid in  $\mathcal{F}$  for all normal valuations, that is **dn** valid in  $(\mathcal{F}, \mathcal{F}\emptyset)$ , means that  $\mathcal{F}$  is the phase space  $\mathcal{L}^\perp$  for  $\perp \equiv \mathcal{F}\emptyset$ ; since by definition  $\mathcal{F}\emptyset = \emptyset^{\perp\perp}$ , choosing  $X = \perp$  we have  $\perp = \emptyset^{XX}$ , so that  $\mathcal{L}^\perp$  is  $\mathcal{L}_X$ .

To prove (1), note that **c** is valid because  $a \triangleleft a \cdot a$  follows from  $a = a \cdot a$ , and **w** because it is derivable from **dn** by  $\perp$ -rule. So it remains to be proved that **dn** is valid for all normal valuations; since  $\mathcal{L}_X \equiv \mathcal{L}^{\emptyset^{XX}}$ , by lemma 9.d, **dn** is valid for all  $V$  with  $V(\perp) = \emptyset^{XX}$ . Then the proof is completed by showing that  $\emptyset^{XX}$  is the least fact in  $\mathcal{L}_X$ , i.e. that  $\emptyset^{XX} = \emptyset^{(\emptyset^{XX})(\emptyset^{XX})}$  holds. To this aim, note that

$$Y^{Y^{XX}} = (Y^{XX})^{Y^{XX}}$$

holds for any  $X, Y$ , because (by lemma 3, and writing  $\rightarrow_{XX}$  for implication in the phase space with dualizer  $X$ ) it is  $Y^{Y^{XX}} \equiv Y \multimap Y^{XX} = Y \rightarrow_{XX} Y^{XX} = Y^{XX} \rightarrow_{XX} Y^{XX} = Y^{XX} \multimap Y^{XX} \equiv (Y^{XX})^{Y^{XX}}$ . Such equation gives in particular  $\emptyset^{\emptyset^{XX}} = (\emptyset^{XX})^{\emptyset^{XX}}$ , from which, applying the same ‘‘exponent’’  $\emptyset^{XX}$ , also  $\emptyset^{(\emptyset^{XX})(\emptyset^{XX})} = (\emptyset^{XX})^{(\emptyset^{XX})(\emptyset^{XX})}$ ; now the left member of this is equal to  $\emptyset^{XX}$  (because  $Y^{YY} = Y$  for any  $Y$ ), so the claim is proved.

Also the construction itself of the L-canonical pretopology  $\text{Pr}_L$  is closely connected to a well-known technique (which is used e.g. in [O2] and [T] to prove completeness). In fact, simply by isolating the relevant assumptions, we can extract from the proofs of lemma 5 and 6 a proof of the following more general result:

**THEOREM 15 (DEDEKIND–MACNEILLE COMPLETION).** *Let  $\mathcal{S}$  be a base equipped with a partial ordering  $\leq$  and with a binary operation  $\rightarrow$  such that  $a \cdot b \leq c$  iff  $a \leq b \rightarrow c$ , for any  $a, b, c$ . Then, putting*

$$a \triangleleft U \equiv (\forall c \in S)(U \leq c \rightarrow a \leq c)$$

(where  $U \leq c$  stands for  $(\forall b \in U)(b \leq c)$ ) defines a precover on  $\mathcal{S}$ , and  $a \mapsto \{b : b \leq a\}$  is an embedding of  $\mathcal{S}$  into the quantale of its saturated subsets, and such embedding respects all existing infima and suprema.

In particular, if  $L$  is any of the logics considered above, this means that the Lindenbaum algebra of  $L$  is embedded in  $\text{Sat}(\text{Pr}_L)$ . The lemma on the canonical valuation can then be seen as a corollary of this embedding and of the fact that  $[\exists xAx] = \bigvee_{t \in T} [At]$  and  $[\forall xAx] = \bigwedge_{t \in T} [At]$  hold for any  $A$  in the Lindenbaum algebra of  $L$ .

This remark is relevant to connect our method with the usual Henkin style proof of completeness of classical logic  $C$  with respect to two-valued models. As usual, we say that a set of formulas  $H$  is a *Henkin set* if it is maximal consistent and if for any formula  $A$ , there exists a term  $t$  such that  $\exists xAx \rightarrow At \in H$ . The idea is to apply directly to  $H$  the method developed in section 4. We thus define  $\Gamma \vdash_H A$  to mean that  $H_0, \Gamma \vdash_C A$  for some finite  $H_0 \subseteq H$ ; since  $H$  is closed under modus ponens,

$\vdash_H$  is closed under all propositional rules, and this is enough to put into action the machinery of section 4. Since  $H$  is maximal consistent,  $Frm/\equiv_H$  has exactly two elements, namely  $[\perp]$  and  $[\top]$ . So, putting  $\text{Pr}_H C \equiv \{[A] : A \vdash_H C\}$  as usual, it follows that for any formula  $C$  either  $\text{Pr}_H C = \text{Pr}_H \perp$  or  $\text{Pr}_H C = \text{Pr}_H \top$ . Hence for any  $\Sigma$ , reasoning classically, either  $\text{Pr}_H \Sigma = \text{Pr}_H \perp$  (when  $\Sigma \cap H = \emptyset$ ) or  $\text{Pr}_H \Sigma = \text{Pr}_H \top$  (when  $\Sigma \cap H \neq \emptyset$ ); in other words,  $\text{Sat}(\text{Pr}_H)$  is isomorphic to  $\{0, 1\}$ . So the  $H$ -canonical structure, together with the identity assignment  $\iota$ , is just the well-known and usual two-valued interpretation on terms, call it  $\mathcal{M}_H$ . The usual proof proceeds by showing that for any  $A$ ,

$$\mathcal{M}_H \models A \text{ iff } A \in H,$$

but this is exactly what our lemma on the canonical valuation tells in this case, since  $V^\iota(A) = 1$  iff  $\mathcal{M}_H \models A$  (cf. a remark in section 3) and  $\text{Pr}_H A = 1$  iff  $A \in H$ . So it remains to be proved that  $V^\iota(A) = \text{Pr}_H A$  actually holds. Inductive steps for connectives follow from the fact that  $\vdash_H$  is closed under all propositional rules. For quantifiers, the assumption that  $H$  is Henkin is crucial. In fact, even if  $\vdash_H$  may fail to be closed under  $\exists L$  or equivalently  $\forall R$ , the property that  $H$  is Henkin, or equivalently, that  $\exists xAx \in H$  iff  $At \in H$  for some term  $t$ , that is  $[\exists xAx] = \bigvee_{t \in T} [At]$ , is exactly what is needed to prove the inductive step for  $\exists$ . Similarly for  $\forall$ , because since  $H$  is Henkin,  $\forall xAx \in H$  iff  $At \in H$  for all terms  $t$ , that is  $[\forall xAx] = \bigwedge_{t \in T} [At]$ .

This gives a proof of the usual completeness theorem with respect to two-valued semantics which shows clearly that non-constructivity is concentrated in the proof of existence of Henkin sets. So, abandoning the classical conception of truth with only two truth values, that is, technically, relaxing the requirement of maximality of  $H$ , one is free to see that the above argument applies to any set of sentences, and thus obtain a constructive proof of completeness not only for pure logics, but also for any theory based on them, i.e. the so-called strong completeness.

In fact, let us say that a theory over a logic  $L$  (BL or any of the extensions considered above) is just a set of *sentences*  $\Sigma$ ; then there is no obstacle to define  $\Gamma \vdash_\Sigma A$  to mean that  $\Gamma \vdash A$  is derivable by *all* the rules of  $L$  from axioms  $\vdash B$  for any  $B \in \Sigma$  (one must be a bit careful here, since if  $L$  does not include structural rules, then  $\Gamma \vdash_\Sigma A$  is not the same thing as  $\Sigma_0, \Gamma \vdash_L A$  for some finite  $\Sigma_0 \subseteq \Sigma$ , cf. [T]). If  $\vdash_\Sigma A$ , we say that  $A$  is  $\Sigma$ -*derivable over*  $L$ . Since by definition  $\vdash_\Sigma$  is closed under all rules of  $L$ , all the results of sections 4 and 5 apply to it. Hence we obtain an  $\mathcal{F}$ -structure  $\text{Pr}_\Sigma$  which is the canonical structure for  $\Sigma$  over  $L$ , that is, which satisfies: for any formula  $A$ ,  $\vdash_\Sigma A$  iff  $A$  is valid in  $\text{Pr}_\Sigma$  under the canonical valuation. Note that all formulas of  $\Sigma$  are trivially valid in the  $\Sigma$ -canonical structure. So, defining  $A$  to be  $\Sigma$ -*valid over*  $L$  when, in any  $\mathcal{F}$ -structure which is a model for  $L$ ,  $A$  is valid whenever all formulas of  $\Sigma$  are valid, we immediately have:

**THEOREM 16 (STRONG COMPLETENESS).** *For any of the pure logics  $L$  considered above, any set of sentences  $\Sigma$  and any formula  $A$ ,  $A$  is  $\Sigma$ -derivable over  $L$  iff  $A$  is  $\Sigma$ -valid over  $L$ .*

Clearly, in the case of structured logics the above definitions of  $\Sigma$ -derivability and  $\Sigma$ -validity coincide with the usual ones; we thus have a constructive proof of strong

completeness, even for classical logic! This makes clearer the fact that it is the use of two-valued models which, in a completeness proof, is responsible for the need both of set-theoretic principles and of classical metalogic; I believe that this shows how, more generally, the relation between a logic and the metalogic needed to prove its completeness is a topic open to research.

I conclude with a couple of technical problems which arise naturally from the work done. First of all, one problem is whether  $\text{Pr}_L$  is the *minimal* completion of the Lindenbaum algebra, or equivalently, of the Lindenbaum base for L. I have good reasons to believe that an affirmative answer would bring also an affirmative answer to a second problem, relative to classical logic, namely that of finding an intrinsic mathematical characterization of the notion of Henkin set, in the following terms. Given a completely prime filter  $F$  over the cBa  $\text{Sat}(\text{Pr}_C)$ , it is easy to see how to obtain a Henkin set  $H_F$ : simply put  $A \in H_F$  iff  $\text{Pr}_C A \in F$ . So the problem is whether the converse holds, that is whether there is a biunivocal correspondence between Henkin sets and completely prime filters, which in pointfree topology are defined to be the (formal) points, of  $\text{Sat}(\text{Pr}_C)$ .

Finally, I also leave open the problem of finding a satisfactory connection between the semantics of pretopologies and more traditional semantics for intuitionistic logic, based on topological spaces and Kripke models.

#### REFERENCES

- [A] V. M. Abrusci, *A sequent calculus for intuitionistic linear propositional logic*, in “Mathematical Logic,” ed. by P.P. Petkov Plenum, 1990, pp. 223-242.
- [BS] G. Battilotti – G. Sambin, *Presentation of locales and quantales by means of formal topologies and pretopologies*, to appear.
- [G] J-Y. Girard, *Linear logic*, Theor. Computer Sc. **50** (1987), pp. 1–102.
- [GL] J-Y. Girard – Y. Lafont, *Linear logic and lazy computation*, in “TAPSOFT ’87,” Springer Lecture Notes Computer Science 250, 1987, pp. 52–66.
- [KW] J. Ketonen – R. Weyrauch, *A decidable fragment of predicate calculus*, Theor. Computer Sc. **32** (1984), pp. 293–307.
- [ITT] P. Martin-Löf, “Intuitionistic type theory,” Bibliopolis, Naples, 1984.
- [O1] H. Ono, *Algebraic aspects of logics without structural rules*, in “Proceedings of Malcev Conference on algebra, Novosibirsk 1989,” to appear.
- [O2] H. Ono, *Phase structures and quantales, a semantical study of logics without structural rules*, in “Substructural Logics,” K. Došen, P. Schroeder-Heister eds., Oxford U. P., to appear.
- [OK] H. Ono - Y. Komori, *Logics without the contraction rule*, J. Symbolic Logic **50** (1985), pp. 169–201.
- [R] K. Rosenthal, “Quantales and their applications,” Longman, 1990.
- [IFS] G. Sambin, *Intuitionistic formal spaces – a first communication*, in “Mathematical logic and its applications,” D. Skordev ed., Plenum, 1987, pp. 187–204.
- [S] G. Sambin, *Intuitionistic formal spaces and their neighbourhood*, in “Logic Colloquium ’88,” R. Ferro et al. (eds.), North Holland, 1989, pp. 261–285.
- [S2] G. Sambin, *The semantics of pretopologies*, in “Substructural Logics,” K. Došen, P. Schroeder-Heister eds., Oxford U. P., to appear.
- [T] A. S. Troelstra, “Lectures on linear logic,” CSLI Lecture Notes no. 29, Center for the Study of Language and Information, Stanford, 1992.