A preview of the basic picture

Giovanni Sambin - Silvia Gebellato
Dipartimento di Matematica Pura ed Applicata,
Università di Padova
via Belzoni, 7 - 35131 Padova
sambin,silvia@math.unipd.it

Introduction. If the aim is to develop mathematics within a constructive set theory, topology seems to be a good test since it is a field in which foundational problems are particularly evident. This is a fortiori true if constructivity is meant in a stricter sense to include predicativity, like in Martin-Löf's constructive type theory [5]. In fact, the usual definition of topological space involves a quantification over subsets, which has to be justified predicatively. Moreover, in many well known topological spaces the definition of points requires an infinite amount of information (one example is given by real numbers) and thus it is not a priori granted that the collection of points form a set.

Such problems are solved in formal topology (see [8] and [10]) which is strictly constructive since it is developed fully within Martin-Löf's type theory (henceforth simply type theory) equipped with a notation for subsets to support intuition (as introduced and justified in [12]).

For our present purposes, the definition of formal topology can be motivated as follows. Assume that a topology ΩX on a set of points X is given by means of a base; this is expressed in type theory as a family of subsets of X indexed on a set S, that is a function $ext : S \to \mathcal{P}X$, which is the same (cf. [12]) as a binary relation $x \Vdash a$, for $x \in X$ and $a \in S$. The main idea is then to transfer the structure of ΩX over the set S, and to this aim S is equipped with some new primitives. A natural choice is to add a binary operation $\cdot$ satisfying $x \Vdash a \cdot b$ iff $x \Vdash a$ and $x \Vdash b$ and thus called formal intersection, a distinguished element $1 \in S$ satisfying $x \Vdash 1$ for any $x \in X$, and an infinitary relation $a \lhd U$ for $a \in S$ and $U \subseteq S$, satisfying $a \lhd U$ iff $(\forall x \in X)(x \Vdash a \to (\exists b \,\epsilon\, U)(x \Vdash b))$ and thus called formal cover. A unary predicate $Pos(a)$ *prop* $(a \in S)$ is also added, satisfying $Pos(a)$ iff $(\exists x \in X)(x \Vdash a)$ and called the positivity predicate.

The definition of formal topology is then obtained by expressing the above situation in pointfree terms, that is by requiring the structure $\mathcal{A} = (S, \cdot, 1, \lhd, Pos)$ to satisfy all the properties of the new primitives $\cdot, 1, \lhd, Pos$ which can be formulated without any mention of points of X. This leads to (cf. [8] and [9]): $\mathcal{A} = (S, \cdot, 1, \lhd, Pos)$ is a formal topology if:

$(S, \cdot, 1)$ is a commutative monoid;

$\lhd$ satisfies:

$$\text{reflexivity}\ \frac{a\,\epsilon\,U}{a \lhd U} \qquad \text{transitivity}\ \frac{a \lhd U \quad (\forall b\,\epsilon\,U)(b \lhd V)}{a \lhd V}$$

$$\cdot\text{-left}\ \frac{a \lhd U}{a \cdot b \lhd U} \qquad \cdot\text{-right}\ \frac{a \lhd U \quad a \lhd V}{a \lhd \{b \cdot c : b\,\epsilon\,U, c\,\epsilon\,V\}}$$

Pos satisfies

$$\text{monotonicity}\ \frac{Pos(a) \quad a \lhd U}{(\exists b\,\epsilon\,U)Pos(b)} \qquad \text{positivity}\ \frac{Pos(a) \to a \lhd U}{a \lhd U}$$

(for an analytic explanation of such conditions see [10]).

Any infinitary relation $\lhd$ is equivalently represented as an operator on subsets $\mathcal{A}U \equiv \{a \in S : a \lhd U\}$; then $\lhd$ is a cover if and only if $\mathcal{A}$ is a closure operator which moreover satisfies distributivity in the form $\mathcal{A}(U \cdot V) = \mathcal{A}(U) \cap \mathcal{A}(V)$ (where $U \cdot V \equiv \{b \cdot c : b\,\epsilon\,U, c\,\epsilon\,V\}$). Then a *formal open* can be defined as a subset U of S such that $U = \mathcal{A}U$.

The presence of the positivity predicate Pos has been felt by some scholars as a redundancy; from the above considerations, we see that $Pos(a)$ is the only primitive corresponding to an existential quantification over points, and it thus becomes a positive pointfree way to express that $ext(a)$ is inhabited. Its presence was due (apart from the convenience in the definition of formal points and in the treatment of Scott domains [13]) to the expectation of obtaining a good definition of formal closed subsets. As we will see here, to obtain this not only Pos must be kept, but the way it expresses existential quantification over points must be strengthened, reaching a binary predicate which is as relevant as the formal cover and dual to it.

What is the point of the move to pointfree terms? An ideological rejection of points altogether is not a far reaching justification in our opinion; to the contrary, we believe that when points form a set, the information given by them should by no means be thrown away (two examples: rational numbers and all finite sets). The trouble is that in the most interesting examples there is no simple way to generate inductively all the points one would like to have. In the case of real numbers, this problem was solved by Dedekind with the introduction of Dedekind cuts and by Brouwer with choice sequences. Formal topology allows to solve the same problem in more general terms by introducing the abstract notion of formal point as follows. Using the notation with $\Vdash$ adopted above, when a point x is concretely given it satisfies the conditions:

$$x \Vdash 1 \qquad\qquad x \Vdash a \cdot b \text{ iff } x \Vdash a \text{ and } x \Vdash b$$

$$\frac{x \Vdash a \quad a \lhd U}{(\exists b\,\epsilon\,U)(x \Vdash b)} \qquad \frac{x \Vdash a}{Pos(a)}$$

Like the definition of formal topology $\mathcal{A}$ is obtained by requiring all the properties which can be expressed in the pointfree language with $\cdot, 1, \lhd$ and Pos, formal points of $\mathcal{A}$ are now *defined* to be those subsets of S which cannot be distinguished, in the language of $\mathcal{A}$, from the subset $\alpha_x \equiv \{a \in S : x \Vdash a\}$ traced by an hypothetical generic concrete point x. Note that this idea is exactly the same which led Dedekind from rational numbers to cuts, and to the definition of real numbers as cuts (cf. [2]). Formally, a subset α of S is said to be a formal point if, after writing $\alpha \Vdash a$ in place of $\alpha \ni a$ (i.e. $a \epsilon \alpha$), all the above conditions are satisfied with α replacing x; we reach in this way the same definition as that given in [8].

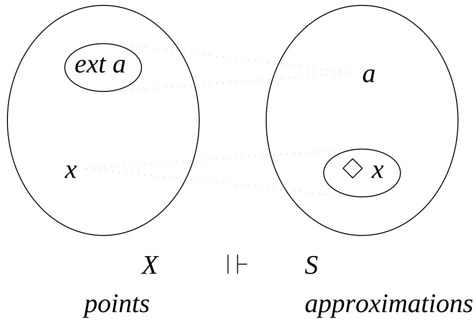
The collection of formal points over a formal topology $\mathcal{A}$ is denoted by $Pt(\mathcal{A})$. The structure $(Pt(\mathcal{A}), \mathcal{A}, \ni)$ is called a formal space, since it is similar to the structure $(X, \mathcal{A}, \Vdash)$ which was used above to describe a topological space. It is type theory which gives a precise foundational meaning to the distinction between topological spaces and formal spaces, since it refrains from considering $Pt(\mathcal{A})$ a set like any other, and in this sense it has favoured the emergence of formal topology itself.

In a similar way, we now see how a new quite rich structure emerges after a radical rejection of the law of excluded third, and in particular after rejection of the identification of $\exists\neg$ with $\neg\forall$, which in topology takes the form of the identification of closed subsets with complements of open subsets. To this aim, we have to go back to the simple structure $(X, S, \Vdash)$ introduced above and take it as our main object of study. The aim of this paper is to give a sort of preview of the fact that this will be rewarding, mainly from a conceptual point of view[1].

Basic pairs. A structure $\mathcal{X} \equiv (X, S, \Vdash)$, where X and S are arbitrary sets and $\Vdash$ is an arbitrary binary relation between them, is here called a *basic pair*. To help the intuition, we may (as in the introduction above) think of X as a set of concrete points and S as a set of basic formal opens (or observables); $x \Vdash a$ can be read as "a is a formal neighborhood of x" or more neutrally as "x forces a" and then the relation $\Vdash$ itself is called forcing. This way of reading introduces a distinction between the left side, which is called concrete side, and the right one, which is called formal side. The relation $\Vdash$ is the way to pass from the formal to the concrete side, and conversely. For any $a \in S$, the *extension* of a is the subset of X of all concrete points forcing a, that is $ext\, a \equiv \{x \in X : x \Vdash a\}$. In topological terms, the family of subsets $(ext\, a)_{a \in S}$ is of course a sub-base for a topology on X, like any family of subsets of X.

In the other direction, any element $x \in X$ on the concrete side determines a subset $\Diamond x \equiv \{a \in S : x \Vdash a\}$ on the formal side, which is called the system of neighborhoods (or approximations) of x. The picture we have in mind is something like:

[1] We have at least two debts of gratitude to Per Martin-Löf. One is his interest in formal closed subsets, which is almost as old as formal topology itself. It indirectly stimulated the discovery (by Sambin) of binary *Pos* and hence much of the basic picture in December '95; morphisms and a correct appreciation of symmetry came later (and are due to both authors). The other is more recent discussions, in particular on the topics in the last paragraph here. It is a pleasure for us to thank him.



The definition of $\Diamond$ is immediately extended to any subset $A \subseteq X$ by defining as usual $\Diamond A \equiv \bigcup_{x \epsilon A} \Diamond x$. Spelling out the definition of union of subsets (see [12]), we see that $\Diamond A$ is just the image of A along $\Vdash$ through an existential quantification:

$$\Diamond A \equiv \{a \in S : (\exists x \in X)(x \Vdash a \;\&\; x \,\epsilon\, A)\}.$$

Because of the option for intuitionism, the image of A obtained through a universal quantification is not definable in terms of $\Diamond$ and we thus are lead to put

$$\Box A \equiv \{a \in S : (\forall x \in X)(x \Vdash a \to x \,\epsilon\, A)\}.$$

So both $\Diamond$ and $\Box$ are operators on subsets, i.e. functions from $P(X)$ to $P(S)$. The fact that they are given by an existential and universal quantification respectively is immediately visible by adopting a notation for quantification relativized to subsets (as justified in [12]):

$$\begin{array}{rcl} \Diamond A & \equiv & \{a \in S : (\exists x \,\epsilon\, \mathit{ext}\, a)(x \,\epsilon\, A)\} \\ \Box A & \equiv & \{a \in S : (\forall x \,\epsilon\, \mathit{ext}\, a)(x \,\epsilon\, A)\} \end{array}$$

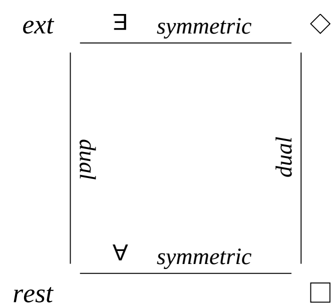
In the other direction, also *ext* is extended to any subset $U \subseteq S$ by putting $\mathit{ext} U \equiv \bigcup_{a \epsilon U} \mathit{ext}\, a$; $\mathit{ext} U$ is called the *extension* of U. As above, we consider also the universal image $\mathit{rest}\, U$ and call it the *restriction* of U. Using quantifiers relativized to $\Diamond x$, the formal definitions are:

$$\begin{array}{rcl} \mathit{ext} U & = & \{x \in X : (\exists a \,\epsilon\, \Diamond x)(a \,\epsilon\, U)\} \\ \mathit{rest}\, U & = & \{x \in X : (\forall a \,\epsilon\, \Diamond x)(a \,\epsilon\, U)\} \end{array}$$

A glance at the definitions shows that the definition of the operators *ext* and *rest* could be obtained from that of $\Diamond$ and $\Box$, respectively, just by switching the role of the sets X and S. In fact, writing as usual $\Vdash^-$ for the inverse of the relation $\Vdash$, we see that $\mathcal{X}^- \equiv (S, X, \Vdash^-)$ is still a basic pair, perfectly as good as $\mathcal{X} \equiv (X, S, \Vdash)$; we call $\mathcal{X}^-$ the symmetric of $\mathcal{X}$. So the operators *ext* and *rest* in $\mathcal{X}$ are just the same thing as $\Diamond$ and $\Box$, respectively, but in its symmetric $\mathcal{X}^-$.

In purely mathematical terms, $\Diamond A$ and $\Box A$ just give what is sometimes called the weak and strong image, respectively, of the subset A along a relation, which in this case is $\Vdash$. For a relation denoted by r, the notation rA and $r^\circ A$, respectively, is sometimes used. Symmetrically, $extU$ and $rest\, U$ are just the weak and strong anti-image, respectively, of U along $\Vdash$. They are denoted by $r^- U$ and $r^* U$, respectively, if the relation is r. Notice that $r^- U$ and $r^* U$ are the same thing as weak and strong image along the relation r^- . Even if the mathematical content is exactly the same, to help the intuition we here have preferred to adopt a specific terminology and notation, namely $\Diamond$, $\Box$, ext, $rest$, for weak and strong (anti-)images along the forcing relation $\Vdash$, which according to a uniform notation should have been called $\Vdash$, $\Vdash^\circ$, $\Vdash^-$, $\Vdash^*$ respectively.

Beside the geometrical symmetry between the left side X and the right side S, there is also a logical duality clearly present: the definition of $\Diamond$ and $\Box$ are obtained one from the other by interchanging the roles of $\forall$ with $\exists$ and $\rightarrow$ with & . The same of course holds for ext and $rest$. So a picture could be:



What is the use of all such structure? We begin by seeing that the topological notions of interior and closure are immediately obtained by combinations of the four operators $\Diamond$, $\Box$, ext, $rest$. The symmetry of the picture will then produce also their pointfree, or formal, versions.

Interior and closure. The interior of a subset A of X is usually defined as the set of points of X with a neighborhood all contained in A (see for instance [4], pp. 42, 44). In our notation, such definition becomes

$$int A \equiv \{x \in X : (\exists a \,\epsilon\, \Diamond x)(\forall y \in X)(y \Vdash a \rightarrow y \,\epsilon\, A)\},$$

and then it is clear that such combination of quantifiers is just the composition of ext after $\Box$, that is $int A \equiv ext \Box A$.

As usual we say that A is *open* if $A = int A$; but of course, we cannot expect int so defined to be a topological interior operator, since nowhere it has been assumed anything telling that the intersection of two open subsets is open. However, it is easy to prove that int is an interior operator, that is

i. $int A \subseteq A$,

ii. if $A \subseteq B$ then $int A \subseteq int B$,

iii. $int A \subseteq int\, int A$,

for any $A, B \subseteq X$. Condition i. follows immediately from the adjunction

$$extU \subseteq A \text{ iff } U \subseteq \Box A, \text{ for any } U \subseteq S \text{ and } A \subseteq X \tag{1}$$

by taking U to be $\Box A$, condition ii. follows from the fact that the operators *ext* and $\Box$ preserve inclusion of subsets and iii. is a consequence of

$$\Box\, ext \Box A = \Box A, \quad \text{for any} \quad A \subseteq X$$

which follows easily from (1) above.

Quite similarly, the usual definition of the closure clA of a subset A of X says that $x \,\epsilon\, clA$ if any neighborhood of x intersects A. In our notation,

$$clA \equiv \{x \in X : (\forall a \,\epsilon \diamond x)(\exists y \,\epsilon\, ext\, a)(y \,\epsilon\, A)\}$$

that is $clA \equiv rest \diamond A$. It is now easy to prove that cl is a closure operator, that is

i. $A \subseteq clA$,

ii. if $A \subseteq B$ then $clA \subseteq clB$,

iii. $cl\,clA \subseteq clA$,

for any $A, B \subseteq X$. Like above, the proof is based on the adjunction

$$\diamond A \subseteq U \quad \text{iff} \quad A \subseteq rest\, U, \quad \text{for any } A \subseteq X \text{ and } U \subseteq S \tag{2}$$

and the fact that $\diamond$ and *rest* preserve inclusion.

Like we did above with open subsets, we say that A is *closed* if $A = clA$, even if cl is not a closure operator in the sense of topology (since the union of two closed subsets is not necessarily closed).

Formal open and formal closed subsets. Because of the symmetry between the left and the right side of a basic pair $X \xrightarrow{\Vdash} S$, the above definitions of $int \equiv ext\Box$ and $cl \equiv rest \diamond$ also have symmetric definitions, obtained by replacing each operator with its symmetric: $\mathcal{C} \equiv \diamond\, rest$ and $\mathcal{A} \equiv \Box\, ext$. By symmetry, it is immediate that $\mathcal{C}$ is an interior operator and $\mathcal{A}$ is a closure operator. We now see that actually $\mathcal{A}$ is something already known, while $\mathcal{C}$ is in a sense what we were looking for. In fact, spelling out the definition of $\mathcal{A}$, we see that

$$a \,\epsilon\, \mathcal{A}U \equiv (\forall x \in X)(x \Vdash a \to (\exists b \,\epsilon\, U)(x \Vdash b))$$

that is, $a \,\epsilon\, \mathcal{A}U$ if all concrete points forcing a also force U, which is the relation between a and U which was meant to be expressed by the formal cover $a \lhd U$. So we extend the previous definition and say that U is formal open if $U = \mathcal{A}U$; note however that in the wider generality of any basic pair, the closure operator $\mathcal{A}$ does not satisfy distributivity $\mathcal{A}(U \cdot V) = \mathcal{A}U \cap \mathcal{A}V$ since X has no topology in the traditional sense. Such generality, however, allows to see that $\mathcal{A}$ is symmetric to cl, which means that the notion of "a being covered by U", i.e. $a \,\epsilon\, \mathcal{A}U$, is just the symmetric of $x \,\epsilon\, cl A$, that is "x is an adherence point of A"; to our knowledge, the simple (but unexpected!) fact that the formula defining one notion can be obtained from the other by interchanging points and opens had not been noticed before.

More interesting is the second operator $\mathcal{C}$, which is the novelty emerged, by symmetry with int or equivalently by duality with $\mathcal{A}$, from the general study of basic pairs. Spelling out its definition, we have

$$a \,\epsilon\, \mathcal{C} F \equiv (\exists x \in X)(x \Vdash a \;\&\; (\forall b \in S)(x \Vdash b \rightarrow b \,\epsilon\, F))$$

which we can now recognize as a strengthening of the intuitive pointwise interpretation of the positivity predicate. In fact, writing $\Diamond x \subseteq F$ in place of $(\forall b \in S)(x \Vdash b \rightarrow b \,\epsilon\, F)$ as usual, we see that $a \,\epsilon\, \mathcal{C} F$ means not only that a is inhabited by a concrete point x, but also that $\Diamond x \subseteq F$, i.e. all neighborhoods of such point x are elements of F. As we write $a \lhd U$ for $a \,\epsilon\, \mathcal{A}U$, we also will write $Pos(a, F)$ for $a \,\epsilon\, \mathcal{C} F$ and call $Pos(a, F)$, for any $a \in S$ and $F \subseteq S$, the binary positivity predicate. The previous unary positivity predicate is now obtained as a special case, by putting $Pos(a) \equiv Pos(a, S)$.

The relevance of binary Pos is that it allows to define by symmetry the notion of formal closed: we say that a subset F of S is *formal closed* if $F = \mathcal{C} F$, i.e. $a \,\epsilon\, F$ iff $Pos(a, F)$.

In this way we see that the notions of concrete and formal, open and closed subsets are all defined by means of a couple of relativized quantifiers of the form $\exists\forall$ or $\forall\exists$ (see the picture below). The logical structure is so evident that one could even reverse the perspective and conceive such topological notions as a conceptual tool to treat combinations of quantifiers in a more synthetic way.

The isomorphism theorem. By definition of int, any concrete open subset A is of the form $ext U$ for some $U \subseteq S$. Conversely, any subset of X of the form $ext U$, for any $U \subseteq S$, is concrete open, because $ext U = ext \Box ext U = int\, ext U$. Therefore:

$A \subseteq X$ is concrete open iff $A = ext U$ for some $U \subseteq S$

Quite similarly, one can prove that:

$A \subseteq X$ is concrete closed iff $A = rest\, U$ for some $U \subseteq S$

$U \subseteq S$ is formal open iff $U = \Box A$ for some $A \subseteq X$

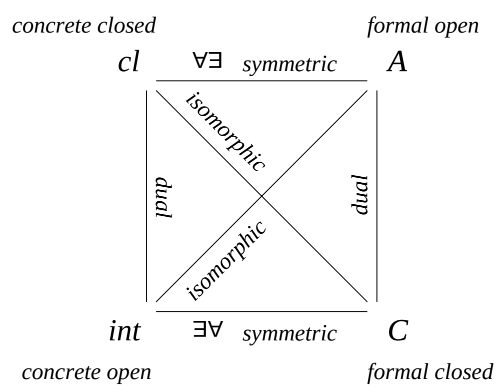
$F \subseteq S$ is formal closed iff $F = \Diamond A$ for some $A \subseteq X$.

It is then easy to see that, when restricted to open subsets (either concrete or formal), the operators ext and $\Box$ are bijective, and one inverse of the other. Similarly for closed subsets, with $rest$ and $\Diamond$.

Just from the fact that *int* is an interior operator, it follows that an arbitrary union of concrete open subsets is concrete open. Symmetrically, an arbitrary union of formal closed subsets is formal closed. Dually, an arbitrary intersection of concrete closed (formal open) subsets is concrete closed (formal open). We can as usual define the meet of an arbitrary family of concrete open subsets $int\,A_i$ as the interior of the intersection, i.e. $\bigwedge_{i\in I} int\,A_i \equiv int(\bigcap_{i\in I} int\,A_i)$; equivalently, but less constructively, one could define $\bigwedge_{i\in I} int\,A_i$ as the join of all concrete open subsets contained in $\bigcap_{i\in I} int\,A_i$. Dually, the join of an arbitrary family of formal open subsets is defined by $\bigvee_{i\in I} \mathcal{A}\,U_i \equiv \mathcal{A}(\bigcup_{i\in I} \mathcal{A}\,U_i)$. So concrete and formal open subsets form two complete lattices. Quite similarly for closed subsets. The following theorem gives then further evidence of the correctness of our definitions:

Theorem. The operator *ext* is an isomorphism between the lattice of formal open and that of concrete open subsets. Dually, the operator *rest* is an isomorphism between the lattices of formal closed and of concrete closed subsets.

The following picture summarizes most of the information about open and closed subsets:



Of course, the vertical line at the right refers to the formal side, and at the left to the concrete side. Also, the top horizontal line refers to closure operators, and the bottom one to interior operators. One diagonal refers to open subsets, the other to closed subsets.

Continuity. What we have seen so far could be summarized by saying that topology begins with basic pairs. They are the simplest extension of the notion of set, that is two sets linked in the weakest possible way, namely by a relation. We are now going to see that continuity begins with the weakest possible way to link two basic pairs, namely a pair of relations giving rise to a commutative square.

Given two basic pairs $\mathcal{X} \equiv X \xrightarrow{\Vdash} S$ and $\mathcal{Y} \equiv Y \xrightarrow{\Vdash} T$, we say that a pair of relations $r : X \to Y$ and $s : S \to T$ is a *morphism* from $\mathcal{X}$ to $\mathcal{Y}$ if the diagram

$$\begin{tikzcd} X \arrow[r, "\Vdash"] \arrow[d, "r"] & S \arrow[d, "s"] \\ Y \arrow[r, "\Vdash"] & T \end{tikzcd}$$

is commutative. Here we assume that composition of relations is defined as usual; then, writing rx for $r\{x\}$, commutativity of the above diagram is expressed by the equation

$$\Diamond rx = s\Diamond x \quad \text{for any } x \in X. \tag{3}$$

Several motivations lead to consider relations rather than functions and then to adopt the above definition of morphisms between basic pairs. First of all, relations are more general than functions and they allow to grasp better the essence of continuity. Secondly, on one hand we obtain the usual definition of continuity for functions as a particular case, but on the other hand we will also be able to give a natural constructive definition of topological Kripke structures. A third good reason for considering relations is that the inherent symmetry of basic pairs is somehow preserved: if $(r, s) : \mathcal{X} \to \mathcal{Y}$ is a morphism, also the inverse (s^-, r^-) is a morphism, from $\mathcal{Y}^-$ into $\mathcal{X}^-$. This statement would be impossible with functions.

Given a relation $r : X \to Y$, a simple minded extension of the usual definition of continuity for functions is to require that r^- is open. Since any open subset of Y is of the form $ext U$ for some $U \subseteq T$, this amounts to

$$r^-(ext U) = int(r^- ext U) \quad \text{for any } U \subseteq T.$$

Since *ext* distributes over union, it is enough to require that

$$r^-(ext b) = int(r^- ext b) \quad \text{for any } b \in T. \tag{4}$$

One can see that, putting $a\, s\, b \equiv ext a \subseteq r^-(ext b)$, such requirement is equivalent to (3) above. So, (3) is satisfied when r^- is open, for a suitable choice of s. On the other hand, it can easily be proved that if (r, s) is a morphism, then r^- is open and s is essentially uniquely determined by r; in fact, if (r, s') is any other morphism, then s and s' coincide "topologically", that is $\mathcal{A}\, s'^- b = \mathcal{A}\, s^- b$ for any $b \in T$. So in a certain sense (3) is equivalent to r^- being open; we prefer the former for aesthetic reasons.

An equivalent characterization is reached through a different path. Assume we express the fact that r^- is open as:

$$\text{for any } U \subseteq T, \text{ there is } V \subseteq S \text{ such that } r^-(ext U) = ext V.$$

More constructively, this can be expressed by requiring the existence of a family of subsets $V_b \subseteq S$ for $b \in T$ such that

$$r^-(ext b) = ext V_b \qquad \text{for any } b \in T. \tag{5}$$

But as we have seen already, the family of subsets $(V_b)_{b\in T}$ is equivalently represented as a relation $a\,s\,b \equiv a \,\epsilon\, V_b$, and then (5) becomes

$$r^- \,(ext b) \;=\; ext(s^- \,b) \qquad \text{for any } b \in T. \tag{6}$$

It is a matter of fact that (6) is equivalent to (3). Actually, one can prove that also

$$r^* \,(rest\ F) \;=\; rest\,(s^*\,F) \text{ for any } F \subseteq T,$$

$$\Box\,(r^\circ\,A) \;=\; s^\circ\,(\Box\,A) \text{ for any } A \subseteq X$$

are equivalent formulations of morphisms.

Basic pairs and morphisms, as defined above, form a category which we call **BP**. From a strictly technical point of view, **BP** is nothing but the category $\mathbf{Rel}^2$, that is the category whose objects are arrows in **Rel**, the category of sets and relations, and morphisms are indeed defined as commutative squares. We have preferred to adopt a new name to recall both that topology is now involved, which makes **BP** conceptually different from $\mathbf{Rel}^2$, and that the underlying set theory is constructive type theory. Nevertheless, the usual nice tricks with diagrams are possible in **BP** as they were in $\mathbf{Rel}^2$. For instance, the commutative square of the definition of a morphism (r, s) can be read also as a morphism $(\Vdash, \Vdash)$ from the basic pair $X \xrightarrow{r} Y$ into the basic pair $S \xrightarrow{s} T$.

A basic pair is technically also the same thing as a boolean Chu space (see [7]); we have chosen to adopt a new name for the same reasons as above. A fortiori in this case since the category of Chu spaces is quite different from **BP**, since morphisms of Chu spaces are defined as pairs of functions, and in opposite directions. Thus **BP** strictly generalizes the category of boolean Chu spaces and provides it with a topological taste which was so far neglected. One can therefore expect from **BP** an even wider range of applications than those developed and foreseen by V. Pratt for Chu spaces (see his www page [6]).

The notion of continuity for relations (sometimes euphemistically called "many-valued functions") has been considered by various authors, particularly in the past; a textbook is [1]. Two more recent references are [15], especially section 4.4 where some bibliographic references can also be found, and [16], which generalizes[2] the notion of continuous relation as introduced in [11].

When X and Y are topological spaces, a relation $r : X \to Y$ is said to be lower semi-continuous if r^* is closed, i.e. $r^*\,A$ is closed in X whenever A is closed in Y, and upper semi-continuous if r^* is open (see [15]). Lower semi-continuity is classically equivalent to r^- open, and hence to our (4), which does not need any free variable on subsets, while the free variable on subsets to express upper semi-continuity is not eliminable. This is why we have adopted the former as our definition (while continuous relations of [1] are required to satisfy both).

Note that our definition is still sufficient to give the usual definition of continuity for functions as a special case when the relation r is actually a function.

[2] Note that the generalization of [11] given by M. B. Smyth in [16] is in the opposite direction of that presented here. We will bring more arguments for our present choice in [3].

Topological Kripke structures. A Kripke structure is usually nothing but a set X together with a relation $r : X \to X$. It clearly is a special case of basic pair (in which $S = X$). We are more interested however in the fact that basic pairs allow to introduce a constructive definition of topological Kripke structure in a natural way. In fact, we say that $(\mathcal{X}, r)$ is a topological Kripke structure if $X \xrightarrow{\Vdash} S$ is a basic pair, so that X is topologized by S through $\Vdash$, and $r : X \to X$ is a relation whose inverse r^- is open. In other terms, a topological Kripke structure is essentially nothing but a morphism from a basic pair into itself. Then also the notion of contraction (see e.g. [11], also called p-morphism, etc.) can now be generalized, and described simply as a commutative cube, of which one face is $(\mathcal{X}, r)$ and the opposite face is $(\mathcal{Y}, s)$.

A new trend in formal topology. To conclude this preview, we can repeat the process described in the introduction as a motivation for the definition of formal topology, but now starting from a more general situation, given by a basic pair $(X, S, \Vdash)$. The unfolding of the basic picture in the previous pages has shown that to describe in the best possible way the concrete topological structure of X by means only of the formal side, we have to adopt two primitive relations $\lhd$ and Pos or equivalently two operators $\mathcal{A}$ and $\mathcal{C}$, which will be assumed to be a closure operator and an interior operator respectively. When $\mathcal{A}$ and $\mathcal{C}$ are defined by means of the relation $\Vdash$ in a basic pair, the link between them is automatically given by the fact that $\mathcal{A} \equiv \Box\, ext$ and $\mathcal{C} \equiv \Diamond\, rest$ with respect to *the same* forcing relation. We now have to add a condition expressing this with no mention of X, and hence of $\Vdash$. We thus arrive at

$$\text{compatibility} \quad \frac{a \,\epsilon\, \mathcal{A}U \quad a \,\epsilon\, \mathcal{C}V}{(\exists b \,\epsilon\, U)(b \,\epsilon\, \mathcal{C}V)}$$

which is easily seen to hold in any basic pair. In the equivalent notation with $\lhd$ and Pos, we thus reach the:

Definition of basic formal topology. A triple $\mathcal{S} \equiv (S, \lhd, Pos)$ is called a basic formal topology if S is a set, $\lhd$ and Pos are infinitary relations satisfying:

$$\text{reflexivity} \quad \frac{a \,\epsilon\, U}{a \lhd U} \qquad \text{transitivity} \quad \frac{a \lhd U \quad (\forall b \,\epsilon\, U)(b \lhd V)}{a \lhd V}$$

$$\text{antirefl.} \quad \frac{Pos(a, F)}{a \,\epsilon\, F} \quad \text{trans.} \quad \frac{Pos(a, F) \quad (\forall b \in S)(Pos(b, F) \to b \,\epsilon\, G)}{Pos(a, G)}$$

$$\text{compatibility} \quad \frac{a \lhd U \quad Pos(a, F)}{(\exists b \,\epsilon\, U)(Pos(b, F))}$$

The difference with the previous definition of formal topology is twofold. On one hand, no condition expressing that $(ext\, a)_{a \in S}$ is a base is now present, or equivalently, no condition guaranteeing that formal opens form a frame. Since they *do* form a complete lattice in any case, the difference is distributivity. This was previously expressed by the requirement $\mathcal{A}(U \cdot V) = \mathcal{A}U \cap \mathcal{A}V$, or equivalently $\cdot$-left and $\cdot$-right, but it can be expressed even in absence of the primitive operation $\cdot$ taking up an idea in [14]. So in this sense basic formal topologies become a more general definition. On the other hand, however, now a binary Pos is required, which means that formal topologies are *not* obtained as a special case. Rather, this leads to the definition of formal topologies in which a binary Pos is assumed. We expect that the study of such structures will highlight the stronger expressiveness of binary Pos, in particular with regards to negative notions.

The definition of basic formal topology has in any case some interest in itself. A notion of formal point can still be given, but one can see that in absence of distributivity it collapses with that of formal closed subset. Also, a good notion of formal continuous map between basic formal topologies can be obtained by using both weak and strong anti-images, to deal with formal open and formal closed subsets respectively.

As a final remark, note that, due to the complete symmetry of a basic pair, we would reach exactly the same definition by transferring the structure of the formal-right side to the concrete-left side, rather than conversely as in formal topology. If we actually write down the result, we see that it is nothing but a structure (X, cl, int), where cl is a closure operator and int an interior operator, linked by the condition

$$\frac{x \in cl A \quad x \in int B}{(\exists y \in A)(y \in int B)}$$

which has an immediate intuitive content. This is a quite simple but rich structure, which never came to life before because the equalities of classical logic made it impossible to be conceived.

References

[1] C. Bergè, *Espace topologiques - functions multivoques*, Dunod, Pris, 1959.

[2] R. Dedekind, *Stetigkeit und irrationale Zahlen*, Vieweg, 1872. also in Gesammelte mathematische Werke, vol III, Vieveg 1932.

[3] S. Gebellato and G. Sambin, *The basic picture II: steps towards relational topology*, 1997. to appear.

[4] J. L. Kelley, *General topology*, D. van Nostrand company inc., Toronto, New York, London, 1955.

[5] P. Martin-Löf, *"Intuitionistic type theory", notes by Giovanni Sambin of a series of lectures given in Padua, June 1980*, Bibliopolis, Napoli, 1984.

[6] V. Pratt, *A guide to Chu spaces*. page of the World Wide Web with address: http://boole.stanford.edu/chuguide.html.

[7] ——, *Chu spaces and their interpretation as concurrent objects*, in Computer Science Today: Recent Trends and Developments, J. van Leeuwen, ed., vol. 1000 of Lecture Notes in Computer Science, Springer-Verlag, 1995, pp. 392–405.

[8] G. Sambin, *Intuitionistic formal spaces - a first communication*, in Mathematical Logic and its Applications, D. Skordev, ed., Plenum, New York - London, 1987, pp. 187–204.

[9] ——, *Intuitionistic formal spaces vs. Scott domains*, in Atti del Congresso "Temi e prospettive della logica e della filosofia della scienza contemporanee", Cesena, January 7-10 1987, C. Cellucci and G. Sambin, eds., vol. 1, Bologna, 1988, CLUEB, pp. 159–163.

[10] ——, *Formal topology - twelve years of development*, 1997. to appear.

[11] G. Sambin and V. Vaccaro, *Topology and duality in modal logic*, Annals of Pure and Applied Logic, 37 (1988), pp. 249–296.

[12] G. Sambin and S. Valentini, *Building up a toolbox for Martin-Löf's type theory: subset theory*, in Twenty-five years of constructive type theory", Venice, October 19-21, 1995", G. Sambin and J. Smith, eds., Oxford U. P., 1997. to appear.

[13] G. Sambin, S. Valentini, and P. Virgili, *Constructive domain theory as a branch of intuitionistic pointfree topology*, Theoretical Computer Science, 159 (1996), pp. 319–341. (former Preprint n. 13, Dipartimento di Matematica P. e A., Università di Padova, August 1992).

[14] I. Sigstam, *Formal spaces and their effective presentations*, Archive for Mathematical Logic, 34 (1995), pp. 211–246.

[15] M. B. Smyth, *Topology*, in Handbook of Logic in Computer Science, S. Abramsky, D. Gabbay, and T. Maibaum, eds., Oxford U. P., 1992.

[16] ——, *Semi-metrics, closure spaces and digital topology*, Theoretical computer science, 151 (1995), pp. 257–276.