Subdirectly irreducible modal algebras and initial frames

Giovanni Sambin
Dipartimento di Matematica Pura ed Applicata,
Università di Padova
Via Belzoni 7, I–35131 Padova, Italy
e-mail: sambin@math.unipd.it

This paper is centred around a nice conjecture, known to Wolfgang Rautenberg and myself since 1978[1]: a modal algebra $\mathcal{A}$ is subdirectly irreducible if and only if its dual frame $\mathcal{A}_*$ is initial, or generated (see definitions in the text). I here show that in full generality the conjecture is false, but that it becomes true under some mild additional assumptions. Unfortunately, some counterexamples seem to suggest that there is no room for just one general result containing all cases in which the conjecture can be proved. Still, some of the theorems here proved are general enough to include such interesting cases as the finite case, the case of Kripke frames and the case of modal logic $\mathsf{K4}$.

The literature on the subject consists, to my knowledge, exclusively of pages 154-156 of Rautenberg's book [Rautenberg 1979][2]. They contain what I call Rautenberg's criterion for a modal algebra to be subdirectly irreducible (see 1.11 below) and a reformulation of the conjecture in the special case of logics K^m, which however seems to be false (see counterexample in Fig. 2 below). Thus the results given here are *all* new. They can be seen as an application of the ground work made in [Sambin-Vaccaro 1988]; in fact, topology seems to be necessary to be able to express the additional conditions under which the conjecture becomes true.

A version of the present paper was sent in 1985 as a letter to Johan van Benthem answering to his curiosity about the conjecture, and circulated privately since then. Due to my change of interests, I lost trace of it. Destiny has given me a second chance only eleven years later, by means of Marcus Kracht. He came across a copy of my manuscript in Berlin, he realised that some of its results covered a gap in his book [Kracht 1998] and thus got in contact with me; in this way the paper was rescued. With remarkable patience, he also saved it from a second oblivion: he convinced me to publish it and assisted me in many ways to revise it (in particular, he rectified the statements of 2.1 and 3.8 and

[1] Sic; my private notes are unmistakably dated Oct. 4, 1978!

[2] One of the referees has drawn my attention to [Goldblatt 1989], where one direction of our theorem 3.6 is proved.

greatly simplified many counterexamples). I am very grateful to him.

1 Notation and preliminaries

We adopt the notation of [Sambin-Vaccaro 1988], henceforth TD, and certainly follow its spirit. However, we telegraphically repeat here some of the definitions while proving some general results[3] which are needed here, but are not contained in TD.

1.1 Frames

A *frame* is a triple $\mathcal{F} = (X, r, \mathcal{T})$ where X is a set, r is a binary relation on X and $\mathcal{T}$ is a boolean subalgebra of $\mathcal{P}X$ such that for every $C \in \mathcal{T}$ we have $r^{-1}C \in \mathcal{T}$, where $r^{-1}C := \{x \in X : x\,r\,y \text{ for some } y \in C\}$. A subset of X is called *internal* if it is in $\mathcal{T}$. The set X is topologized by taking $\mathcal{T}$ as a base for open subsets.

For every $C \subseteq X$, we put:

$$rC := \{y \in X : x\,r\,y \text{ for some } x \in C\}$$

We write rx for $r\{x\}$; then $rC = \cup\{rx : x \in C\}$. We also put:

$$r^*C := \{x \in X : rx \subseteq C\}$$

Both operations, r and r^*, can be iterated, and we put:

$$r^0C := C,\ r^1C := rC, \ldots, r^{n+1}C := r(r^nC)$$

$$r^{*0}C := C,\ r^{*1}C := r^*C, \ldots, r^{*n+1}C := r^*(r^{*n}C)$$

We write r^nx for $r^n\{x\}$; note that r^nx is the same as the image of x under the relation r^n. The operations r and r^* are tied together as follows:

Lemma 1.1 *For every $C, D \subseteq X$, $rC \subseteq D$ iff $C \subseteq r^*D$.*

Proof. Immediate, cf. TD.II.1.5.

Lemma 1.2 *For every $C \subseteq X$ and n, $(r^n)^*C = r^{*n}C$.*

Proof. A formal proof is by induction. Informally, it is immediate: $x \in (r^n)^*C$ iff $r^nx \subseteq C$ iff $r^{n-1}x \subseteq r^*C$ iff ... iff $x \in r^{*n}C$.

For every $C \subseteq X$, we put:

$$r^\infty C := \bigcup_{n \in N} r^nC$$

[3] Such results are so natural that one would expect them to be known; in any case, a novelty should be the fact that they are here proved using only minimal logic, namely intuitionistic logic deprived of the rule for absurdity.

$r_\infty C := \bigcap_{n\in N} r^{*n} C$

We write $r^\infty x$ for $r^\infty\{x\}$. Then $r^\infty C = \bigcup\{r^\infty x : x \in C\}$ and $r_\infty C = \{x : r^\infty x \subseteq C\}$.

If C is clopen, we see that $r^\infty C$ is open and $r_\infty C$ is closed. r^∞ and r_∞ are tied by the same relation as between r and r^* (Lemma 1.1 above):

Lemma 1.3 *For every $C, D \subseteq X$, $r^\infty C \subseteq D$ iff $C \subseteq r_\infty D$.*

Proof. For every $C, D \subseteq X$, $r^\infty C \subseteq D$ iff $\bigcup_{n\in N} r^n C \subseteq D$ iff $\forall n(r^n C \subseteq D)$ iff $\forall n(C \subseteq (r^n)^* D)$ iff (by Lemma 1.2) $\forall n(C \subseteq r^{*n} D)$ iff $C \subseteq r_\infty D$.

A subset C of X is called r-*hereditary*, or just *hereditary*, if for every $x \in X$, $x \in C$ and $x\,r\,y$ imply $y \in C$. That is, if every path beginning from C remains inside C.

Lemma 1.4 *For every $C \subseteq X$, the following are equivalent:*

1. *C is hereditary, i.e. $\forall x(x \in C \to rx \subseteq C)$*
2. *$rC \subseteq C$*
3. *$C \subseteq r^* C$*
4. *$r^\infty C = C$*
5. *$C = r_\infty C$*
6. *$\forall x(x \in C \to r^\infty x \subseteq C)$*

Proof. (1) $\Rightarrow$ (2). Clearly, $rC = \bigcup_{x\in C} rx$. Since $rx \subseteq C$ for any $x \in C$, $rC \subseteq C$ follows.
(2) $\Leftrightarrow$ (3) Is a particular case of Lemma 1.1.
(2) $\Rightarrow$ (4). $C \subseteq r^\infty C$ holds trivially by definition. By (2) we also have $rC \subseteq C$. Then, by monotonicity of the operator r, $r^{n+1} C \subseteq r^n C$ for all n. It follows that $r^n C \subseteq C$ for all C, and so $r^\infty C \subseteq C$. This shows (4).
(4) $\Leftrightarrow$ (5). By Lemma 1.3, $r^\infty C \subseteq C$ iff $C \subseteq r_\infty C$, and the rest is trivial.
(5) $\Rightarrow$ (6). If $x \in C$, then $x \in r_\infty C$, hence $r^\infty x \subseteq C$.
(6) $\Rightarrow$ (1). If $x \in C$, then $r^\infty x \subseteq C$ by (6), and hence a fortiori $rx \subseteq C$.

Lemma 1.5 *For every $x, y \in X$,*

$$r^\infty y \subseteq r^\infty x \text{ iff } y \in r^\infty x \text{ iff } \exists n(x r^n y).$$

Proof. Routine.

Lemma 1.6 *For every C, $r_\infty C$ is the greatest hereditary subset of C.*

Proof. $r_\infty C$ is hereditary: if $x \in r_\infty C$, then $r^\infty x \subseteq C$. So, if $x\,r\,y$ then $r^\infty y \subseteq r^\infty x$ by Lemma 1.5, and hence $r^\infty y \subseteq C$, that is, $y \in r_\infty C$.

$r_\infty C$ is the greatest hereditary subset: if $D \subseteq C$ and D is hereditary, then $D = r^\infty D \subseteq C$ and hence $D \subseteq r_\infty C$ by Lemma 1.3.

Lemma 1.7 *For every C, $r^\infty C$ is the minimal hereditary subset containing C.*

Proof. Very similar to that of Lemma 1.6.

Note that a corollary of Lemma 1.7 is: $r^\infty x$ is the smallest hereditary subset containing x.

1.2 Modal algebras

A *modal algebra* $\mathcal{A}$ is a pair (A, τ) where A is a boolean algebra and τ is a unary operation s.t. $\tau 1 = 1$ and $\tau(a \cdot b) = \tau a \cdot \tau b$. A filter F on A is said to be a *τ-filter* if it is closed under τ, i.e. if $a \in F$ implies $\tau a \in F$. The trivial τ-filter is $\{1\}$.

For every n, the n-th iteration of τ is defined by $\tau^0 a = a$, $\tau^1 a = \tau a$, ..., $\tau^{n+1} a = \tau(\tau^n a)$. For every n, we put $\tau_n a := a \cdot \tau a \cdot \ldots \cdot \tau^n a$, which allows to describe how τ-filters are generated.

Lemma 1.8 *For any subset $B \subseteq A$,*

$$\{a \in A : \textit{ for some } b_1, \ldots, b_k \in B \textit{ and } m_1, \ldots, m_k \in \mathbb{N}, a \geq \tau_{m_1} b_1 \cdot \ldots \cdot \tau_{m_k} b_k\}$$

is the least τ-filter containing B, and is denoted by $F[B]$.

Proof. Routine.

In particular, $F[b] = \{a \in A : a \geq \tau_m b \text{ for some } m\}$.

An element $c \in A$ is said to be *essential* if $F[c] \neq A$ and $F[c]$ is contained in every τ-filter distinct from $\{1\}$. We put $E_\mathcal{A} := \{c \in A : c \text{ is essential }\}$. Of course, $\mathcal{A}$ may have no essential elements. Actually, $E_\mathcal{A}$ is not empty iff $\mathcal{A}$ is subdirectly irreducible. In fact, according to the the standard definition of universal algebra [Grätzer 1968], a modal algebra $\mathcal{A}$ is subdirectly irreducible (s.i. from now on) if it has a least non-trivial congruence. So by the correspondence between τ-filters and congruences, see TD.II.5, $\mathcal{A}$ is s.i. iff it has a least non-trivial τ-filter. Moreover, we have:

Lemma 1.9 *$E_\mathcal{A} \cup \{1\}$ coincides with the least τ-filter. So $E_\mathcal{A}$ is not empty iff $\mathcal{A}$ has a least non-trivial τ-filter.*

Proof. If $E_\mathcal{A}$ is empty, the claim is trivial. If $E_\mathcal{A}$ is not empty, every element of $E_\mathcal{A}$ generates the least τ-filter. Conversely, every element different from 1 in the least τ-filter generates it, and hence is in $E_\mathcal{A}$.

We can characterise essential elements:

Lemma 1.10 *For every $\mathcal{A}$, $c \in A$ is essential iff $c \neq 1$ and $(\forall b \neq 1)\exists m(\tau_m b \leq c)$.*

Proof. Assume c is essential. Then $F[c]$ is the least non-trivial τ-filter. So for every $b \neq 1$, $F[c] \subseteq F[b]$ and hence $c \in F[b]$, that is $\tau_m b \leq c$ for some m.

Conversely, assume that the condition holds. Let F be any non-trivial τ-filter. For every $b \in F$, either $b = 1$ and hence trivially $b \in F[c]$, or $b \neq 1$ and hence $\tau_m b \leq c$ for some m. Since F is a τ-filter, $\tau_m b \in F$ and hence $c \in F$. So the inclusion $F[c] \subseteq F$ holds, and $F[c] = E_{\mathcal{A}} \cup \{1\}$, i.e. c is essential.

We have thus reached (see [Rautenberg 1979], page 155):

Theorem 1.11 *(Rautenberg's criterion) $\mathcal{A}$ is s.i. iff there exists $c \neq 1$ in A s.t. $(\forall b \neq 1)\exists m(\tau_m b \leq c)$*

1.3 Beginning duality

As in TD, * is the functor towards algebras, and $_*$ the functor towards frames. If $\mathcal{A} = (A, \tau)$ is a modal algebra, then $\mathcal{A}_* := (U(A), \tau_*, \beta A)$ where $U(A)$ is the set of ultrafilters on A, $S\,\tau_*\,T$ iff $\forall a(\tau a \in S \to a \in T)$ for any $S, T \in U(A)$, $\beta a := \{S \in U(A) : a \in S\}$ and $\beta A := \{\beta a : a \in A\}$. And if $\mathcal{F} = (X, r, \mathcal{T})$ is a frame, then $\mathcal{F}^* := (\mathcal{T}, r^*)$.

In TD.II.5 there is an indirect proof of the following theorem, but here a direct proof gives a better feeling:

Theorem 1.12 *The lattice of τ-filters of $\mathcal{A}$ and the lattice of τ_*-hereditary closed subsets of $\mathcal{A}_*$ are anti-isomorphic.*

Proof. From boolean duality, we know that the assignment

$$F \mapsto C_F := \{S \in U(A) : F \subseteq S\} = \bigcap\{\beta a : a \in F\}$$

gives an isomorphism between filters and closed subsets (help: it is better seen if we think of the copy ${\mathcal{A}_*}^*$ of $\mathcal{A}$; then a proof is almost immediate). So it is enough to show that F is closed under τ iff C_F is τ_*-hereditary. We have:

	$\forall a(a \in F \to \tau a \in F)$	
iff	$\forall a(C_F \subseteq \beta a \to C_F \subseteq \beta\tau a)$	because $a \in F$ iff $C_F \subseteq \beta a$
iff	$\forall a(C_F \subseteq \beta a \to C_F \subseteq {\tau_*}^*\beta a)$	because $\beta\tau a = {\tau_*}^*\beta a$
iff	$\forall a(C_F \subseteq \beta a \to \tau_* C_F \subseteq \beta a)$	because of Lemma 1.1 above
iff	$\tau_* C_F \subseteq C_F$	because C_F, and hence also $\tau_* C_F$, is closed
iff	C_F is τ_*-hereditary,	by Lemma 1.4 above.

Applying Theorem 1.12 to the definition of s.i. algebras, we have:

Corollary 1.13 *$\mathcal{A}$ is s.i. iff $\mathcal{A}_*$ has a greatest non-trivial τ_*-hereditary closed subset.*

Up to here, though some results were new, with respect to our topic it has been only warming up: now the point is to characterise those frames with a greatest non-trivial τ_*-hereditary closed subset.

2 The main arguments

The idea on which the connection between initial frames and s.i. modal algebras is based can be reduced to a few lines.

For every frame $\mathcal{F} = (X, r, \mathcal{T})$, we put $I_{\mathcal{F}} := \{x \in X : r^{\infty}x = X\}$. Every point in $I_{\mathcal{F}}$ is said to be *initial*, and $\mathcal{F}$ itself is said to be *initial*, or *generated* if $I_{\mathcal{F}} \neq \emptyset$. We will often deal with $X - I_{\mathcal{F}}$, and we thus give it a name, putting $H_{\mathcal{F}} := X - I_{\mathcal{F}}$. Keeping in mind Corollary 1.13, we see that the following observation connects initial frames with s.i. algebras:

Lemma 2.1 *For every frame $\mathcal{F}$, $H_{\mathcal{F}}$ is hereditary. Moreover, if X has a greatest non-trivial hereditary subset C then $H_{\mathcal{F}} = C$; otherwise $H_{\mathcal{F}} = X$. So $\mathcal{F}$ is initial iff $\mathcal{F}$ admits a greatest non-trivial hereditary subset.*

Proof. The last part follows from the first two.

$H_{\mathcal{F}}$ is hereditary: assume that $x \in H_{\mathcal{F}}$. Then $r^{\infty}x \neq X$. So for every $y \in r^{\infty}x$, we also have $r^{\infty}y \neq X$ (because $r^{\infty}y \subseteq r^{\infty}x$) and hence $y \in H_{\mathcal{F}}$. That is, $r^{\infty}x \subseteq H_{\mathcal{F}}$.

$H_{\mathcal{F}}$ is the greatest hereditary subset: if $D \neq X$ and D is hereditary, then $x \in D$ implies $r^{\infty}x \subseteq D$, that is $r^{\infty}x \neq X$ and $x \in H_{\mathcal{F}}$.

Unfortunately, we can not apply 1.13 to $H_{\mathcal{F}}$, since $H_{\mathcal{F}}$ is not necessarily closed (see the counterexample in Fig. 2 below).

Lemma 2.2 *For every frame $\mathcal{F}$ and every subset C, if $H_{\mathcal{F}} \subseteq C$ and $C \neq X$, then $H_{\mathcal{F}} = r_{\infty}C$.*

Proof. If $H_{\mathcal{F}} \subseteq C \neq X$, then $r_{\infty}C \subseteq H_{\mathcal{F}}$ because $r_{\infty}C$ is hereditary and non trivial, and hence $H_{\mathcal{F}}$ is the greatest such set. But also $H_{\mathcal{F}} \subseteq r_{\infty}C$, because $H_{\mathcal{F}} \subseteq C$ implies $r_{\infty}H_{\mathcal{F}} \subseteq r_{\infty}C$ and $H_{\mathcal{F}} = r_{\infty}H_{\mathcal{F}}$.

Lemma 2.3 *For every $\mathcal{F}$, either $H_{\mathcal{F}}$ is a dense subset of X or $H_{\mathcal{F}}$ is closed.*

Proof. If $H_{\mathcal{F}}$ is not dense, there is a clopen subset C with $H_{\mathcal{F}} \subseteq C \neq X$. But then $H_{\mathcal{F}} = r_{\infty}C$ by Lemma 2.2, and $r_{\infty}C$ is closed.

All this is enough to show that, in one direction, the situation is clear and not unsatisfying; as usual, a subset is said to be of measure zero if it does not contain a nonempty open subset:

Theorem 2.4 *For every algebra $\mathcal{A}$, if $I_{\mathcal{A}_*}$ is not of measure zero, then $\mathcal{A}$ is s.i.*

Proof. If $I_{\mathcal{A}_*}$ is not of measure zero, then $H_{\mathcal{A}_*}$ is not dense, and hence is closed by Lemma 2.3. So $H_{\mathcal{A}_*}$ is the greatest non trivial closed hereditary subset, and $\mathcal{A}$ is s.i. by Corollary 1.13.

Example 2.5 *Showing that in general*

$$\mathcal{A}_* \text{ initial} \not\Rightarrow \mathcal{A} \text{ s.i.}$$

Figure 1: The frame Ω

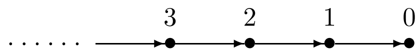
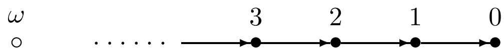


Figure 2: The frame $\Omega^*{}_*$



We find a descriptive initial frame $\mathcal{F}$ with $\mathcal{F}^*$ not s.i.; the point is that $H_{\mathcal{F}}$ will not be closed. Let $\Omega = (\mathbb{N}, >, \mathcal{T})$ be the frame consisting of the set $\mathbb{N}$ of natural numbers, the accessibility relation $>$, and the internal sets being the finite and cofinite subsets of $\mathbb{N}$. The corresponding Kripke frame is shown in Fig. 1. (We employ the following convention. $\bullet$ represents an irreflexive point, $\circ$ a reflexive point. Usually, all other relations are indicated by arrows, except in transitive frames, where we show only the immediate successor relation.) It is easily checked that the bidual of Ω has one more point, consisting of the ultrafilter of cofinite sets. Let us denote this point by ω. Furthermore, ω sees all other points. Hence $\Omega^*{}_* = (\mathbb{N} \cup \{\omega\}, >, \mathcal{U})$, where $\mathcal{U}$ consists of the finite sets not containing ω, and their complements. The corresponding Kripke frame is shown in Fig. 2. Now, put $\mathcal{A} := \Omega^*$. $\mathcal{A}_*$ is clearly initial, and $I_{\mathcal{A}_*} = \{\omega\}$. However, $\mathcal{A}$ is not s.i. (note that Theorem 2.4 does not apply, since $I_{\mathcal{A}_*} = \{\omega\}$ and $\{\omega\}$ is closed but not open, hence $H_{\mathcal{A}_*}$ is not closed). In fact, take the set $F_k := \{a : a \supseteq [k)\}$ of subsets a of $\mathbb{N}$ containing all numbers $\geq k$. This is a r^*-filter in $\mathcal{A}$. Moreover, $\bigcap_k F_k = \{\mathbb{N}\}$. This shows that $\mathcal{A}$ has no non-trivial smallest r^*-filter, and so $\mathcal{A}$ is not s.i.

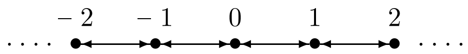
Incidentally, $\Omega^*{}_*$ is nothing but the universal frame on zero generators for GL, that is the dual of the free Magari algebra on the empty set (GL is the modal logic of provability, see [Boolos 1993] or [Smoryński 1985], Magari algebras were introduced in [Magari 1975], and called *diagonalizable algebras* there).

In the other direction, the situation is much more complex, and it seems that there is no room for just one general result. What I can prove, is based on the following observation:

Lemma 2.6 *Let $\mathcal{F}$ be a frame s.t. $r^\infty x$ is closed for every $x \in X$. Then*

$$C \text{ is essential in } \mathcal{F}^* \Rightarrow H_{\mathcal{F}} \subseteq C.$$

Figure 3: The frame $\mathcal{Z}$



Proof. Assume C is essential, and let $x \in H_{\mathcal{F}}$. Then $r^{\infty}x \neq X$ and since $r^{\infty}x$ is closed, there is a clopen $D \neq X$ with $r^{\infty}x \subseteq D$. From $r^{\infty}x \subseteq D$ we have $x \in r_{\infty}D$, but since C is essential, $r_{\infty}D \subseteq C$ and hence $x \in C$.

Lemma 2.7 *For every algebra $\mathcal{A}$, if $\mathcal{A}$ is s.i. and ${\tau_*}^{\infty}S$ is closed for every $S \in U(A)$, then $\mathcal{A}_*$ is initial, with $I_{\mathcal{A}_*}$ not of measure zero.*

Proof. If $\mathcal{A}$ is s.i., there exists $c \in A$ with c essential. Then βc is essential in ${\mathcal{A}_*}^*$, and hence Lemma 2.6 applies to yield $H_{\mathcal{A}_*} \subseteq \beta c$, that is $I_{\mathcal{A}_*} \subseteq \beta \nu c$ (where νc is the complement of c), and $\beta \nu c \neq \emptyset$ since $c \neq 1$.

Certainly the condition "$r^{\infty}x$ is closed" is elegant in theory, but difficult to verify in practice. Still, it includes some useful particular cases (see below). Moreover, I don't know whether it is really unavoidable. However, some kind of condition is necessary, as the following example shows.

Example 2.8 *Showing that in general*

$$\mathcal{A} \text{ s.i. } \not\Rightarrow \mathcal{A}_* \text{ initial}$$

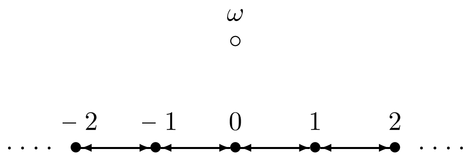
We find an example of a descriptive frame $\mathcal{F}$ which is not initial, but $\mathcal{F}^*$ is s.i. Namely, let $\mathcal{Z} := (\mathbb{Z}, r, \mathcal{T})$, where $\mathbb{Z}$ is the set of integers, and $x \; r \; y$ iff $x = y \pm 1$, and let $\mathcal{T}$ be the set of finite and cofinite subsets of $\mathbb{Z}$. This is a frame. $\mathcal{A} := \mathcal{Z}^*$ is the desired algebra. We can deduce already that $\mathcal{A}$ is s.i., because the only proper r^*-filter is that of cofinite subsets. In fact, assume a r^*-filter contains any subset C which is different from $\mathbb{Z}$ and let $z' \notin C$; then for any $z \in \mathbb{Z}$, $z \notin {r^*}^k C$, where $k := |z - z'|$, so that (by closure under supersets and intersection) all cofinite subsets are also contained. $\mathcal{A}_*$ is the bidual of $\mathcal{Z}$, ${\mathcal{Z}^*}_*$. This frame has one more point than $\mathbb{Z}$, based on the ultrafilter of cofinite subsets of $\mathbb{Z}$. We denote this point by ω. Put $x \; s \; y$ iff $x, y \in \mathbb{Z}$ and $x \, r \, y$ or $x = y = \omega$ and let $\mathcal{U}$ be the set of finite subsets of $\mathbb{Z} \cup \{\omega\}$ not containing ω and their complements. Then ${\mathcal{Z}^*}_* = (\mathbb{Z} \cup \{\omega\}, s, \mathcal{U})$. ${\mathcal{Z}^*}_*$ is not initial.

3 Adding some hypotheses

In at least two particular cases we can put Theorem 2.4 and Lemma 2.7 together:

Theorem 3.1 *For every finite algebra $\mathcal{A}$, $\mathcal{A}$ is s.i. iff $\mathcal{A}_*$ is initial.*

Figure 4: The frame $\mathcal{Z}^*{}_*$



Proof. The assumptions of Theorems 2.4 and Lemma 2.7 are trivially satisfied: ${\tau_*}^\infty S$ is always closed, and $I_{\mathcal{A}_*}$ is always open.

Theorem 3.2 *If $\mathcal{A}$ satisfies* $\mathsf{K4}$*, then $\mathcal{A}$ is s.i. iff $I_{\mathcal{A}_*}$ is not of measure zero.*

Proof. When $\mathcal{A}$ satisfies $\mathsf{K4}$, τ_* is transitive and for every transitive r the equation $r^\infty x = \{x\} \cup rx$ holds. So ${\tau_*}^\infty S$ is closed, because $\{S\}$ is closed ($\mathcal{A}_*$ is Hausdorff) and $\tau_* S$ is closed (τ_* is point-closed, see TD.II.2.9). Now apply 2.4 and 2.7.

Note that the proof of Theorem 3.2 does not use the fact that $\mathcal{A}_*$ is compact. In fact, we can prove a little more than Theorem 3.2, at the cost of proving another little Lemma:

Lemma 3.3 *If $\mathcal{F}$ is transitive and refined, then for every $C \in \mathcal{T}$, C is essential in $\mathcal{F}^*$ iff $H_\mathcal{F} \subseteq C$ and $C \neq X$.*

Proof. ($\Rightarrow$) If $\mathcal{F}$ is transitive and refined, $r^\infty x$ is closed for every x. Then apply Lemma 2.6.
($\Leftarrow$) Let D be clopen, with $D \neq X$. Then $r_\infty D = H_\mathcal{F}$ by Lemma 2.2. But $r_\infty D = D \cap r^* D$ because r is transitive, and $H_\mathcal{F} \subseteq C$. So $D \cap r^* D \subseteq C$ and C is essential.

Note that the above lemma does not hold if we substitute "transitive refined" with "Kripke":

Example 3.4 *Showing that in general for Kripke frames $\mathcal{F}$*

$$H_\mathcal{F} \subseteq C \text{ and } C \neq X \not\Rightarrow C \text{ essential}$$

Take the Kripke frame $(\mathbb{Z}, r)$ as in Fig. 3. We have $H_\mathcal{F} = \emptyset$. Now let D be the union of the intervals $[2^k - k, 2^k + k]$ for all k. Then $(r^*)^n D \neq \emptyset$ for all n, since $2^n \in (r^*)^n D$. Hence, $\emptyset$ is not essential. Nevertheless, there exist essential elements, for example $\mathbb{Z} - \{0\}$.

Theorem 3.5 *For every transitive refined frame $\mathcal{F}$, $\mathcal{F}$ is initial with $H_\mathcal{F}$ not dense iff $\mathcal{F}^*$ is s.i.*

Proof. $\mathcal{F}$ is initial with $H_{\mathcal{F}}$ not dense iff there exists $C \in \mathcal{T}$ with $H_{\mathcal{F}} \subseteq C$ and $C \neq X$ iff (by Lemma 3.3) there exists $C \in \mathcal{T}$ which is essential in $\mathcal{F}^*$ iff $\mathcal{F}^*$ is s.i.

Note that the theorem above has Theorem 3.2 as a particular case (when $\mathcal{F}$ is descriptive). The advantage is that it applies also to Kripke frames.

The assumption of transitivity in Theorem 3.5 can be substituted with that of $\mathcal{F}$ being a Kripke frame, but unfortunately the proof has to be changed:

Theorem 3.6 *For every Kripke frame $\mathcal{F}$, $\mathcal{F}$ is initial iff $\mathcal{F}^*$ is s.i.*

Proof. ($\Leftarrow$) Since $\mathcal{F}^*$ is s.i., there exists an essential $C \in \mathcal{T}$. And since $r^{\infty}x$ is trivially closed, we can apply Lemma 2.6 and 2.2 and obtain $H_{\mathcal{F}} = r_{\infty}C$. In particular, $H_{\mathcal{F}} \neq X$ and hence $I_{\mathcal{F}} \neq \emptyset$.
($\Rightarrow$) Assume $I_{\mathcal{F}} \neq \emptyset$, and let $x \in I_{\mathcal{F}}$. We want to show that $E := X - \{x\}$ is essential. To this aim, let $C \neq X$. Then there is a $y \notin C$. Since x is initial, $xr^n y$ for some n, and therefore $x \notin r^{*n}C$ (note that this holds trivially if $n = 0$, i.e. if $x = y$). A fortiori, $x \notin C \cap \ldots \cap r^{*n}C$ and hence $C \cap \ldots \cap r^{*n}C \subseteq E$, which is the claim.

3.1 More on K4

We first introduce a further definition (similar to that of [Rautenberg 1979], page 149). An element a of $\mathcal{A}$ is said to be *open* if the principal filter generated by a is closed under τ. That is, if $F[a] = \{b \in A : a \leq b\}$. If $\mathcal{A}$ satisfies K4, then a is open iff $a = \tau a \cdot a$ iff $a \leq \tau a$.

Lemma 3.7 *For every $\mathcal{A}$ and $a \in A$, a is open iff βa is τ^*-hereditary.*

Proof. The principal filter F generated by a corresponds to the clopen subset βa. And F is closed under τ, i.e. a is open, iff βa is τ^*-hereditary, because of Theorem 1.12.

Call $E_{\mathcal{A}}$ *principal* if $E_{\mathcal{A}} \cup \{1\}$ is a principal filter distinct from $\{1\}$.

Theorem 3.8 *For every $\mathcal{A}$ satisfying K4, the following are equivalent:*

1. *$\mathcal{A}$ is s.i.*
2. *$E_{\mathcal{A}}$ is principal*
3. *$\mathcal{A}$ has an element which is open and essential*
4. *$\mathcal{A}$ has a greatest open element $\neq 1$*
5. *$\mathcal{A}_*$ has a greatest non trivial clopen hereditary subset*
6. *$I_{\mathcal{A}^*}$ is a nonempty clopen subset*
7. *$I_{\mathcal{A}^*}$ is open and not empty*

Proof. (1) $\Rightarrow$ (7). By Lemma 2.3, $I_{\mathcal{A}^*}$ is either open or of measure zero. By Theoreom 2.4, if $\mathcal{A}$ is s.i., $I_{\mathcal{A}^*}$ is not of measure zero. Hence it is open and nonempty.

(7) $\Rightarrow$ (6). We show that $H_{\mathcal{A}^*}$ closed implies $H_{\mathcal{A}^*}$ clopen. In fact, if $H_{\mathcal{A}^*}$ is closed and not trivial, there is a clopen C with $H_{\mathcal{A}^*} \subseteq C \neq X$. Then $H_{\mathcal{A}^*} = r_\infty C = C \cap r^* C$, which is clopen.

(6) $\Rightarrow$ (5). If $H_{\mathcal{A}^*}$ is clopen, it is the greatest hereditary clopen subset.

(5) $\Rightarrow$ (4). By Lemma 3.7 above.

(4) $\Rightarrow$ (3). Let c be the greatest open element. Then for every $a \neq 1$, $a \cdot \tau a$ is open and hence $a \cdot \tau a \leq c$, that is, c is essential.

(3) $\Rightarrow$ (2). Let c be open and essential. Then $\{b : c \leq b\}$ is a τ-filter because c is open, and is equal to $E_{\mathcal{A}} \cup \{1\}$ because c is essential.

(2) $\Rightarrow$ (1). If $E_{\mathcal{A}}$ is principal, $E_{\mathcal{A}}$ is nonempty.

3.2 About logics K^m

Following [Rautenberg 1979], for every m we put

$$\mathsf{K}^m \ := \ \mathsf{K} \oplus p \wedge \Box p \ldots \wedge \Box^m p \rightarrow \Box^{m+1} p$$

that is, the least normal extension containing the formula $p \wedge \Box p \ldots \wedge \Box^m p \rightarrow \Box^{m+1} p$. Of course, this formula is expressed algebraically by $\tau_m a \leq \tau^{m+1} a$. Note that K^1 is not the same thing as $\mathsf{K4}$.

Lemma 3.9 *Let $\mathcal{F}$ be any refined frame with r closed and satisfying K^m for some m. Then for every $x \in X$,*

$$r^\infty x = \{x\} \cup rx \cup \ldots \cup r^m x$$

Proof. Let us write $r^*{}_m C$ for $C \cap r^* C \cap \ldots \cap r^{*m} C$. Then

$$\begin{aligned}
&\mathcal{F} \models p \wedge \Box p \ldots \wedge \Box^m p \rightarrow \Box^{m+1} p \\
&\text{iff } (\forall C \in \mathcal{T})(r^*{}_m C \subseteq r^{*m+1} C) \\
&\text{iff } (\forall C \in \mathcal{T})(\forall x \in X)(x \in r^*{}_m C \rightarrow x \in r^{*m+1} C) \\
&\text{iff } (\forall C \in \mathcal{T})(\forall x \in X)(\{x\} \cup rx \cup \ldots \cup r^m x \subseteq C \rightarrow r^{m+1} x \subseteq C) \\
&\text{iff } (\forall x \in X)(r^{m+1} x \subseteq \{x\} \cup rx \cup \ldots \cup r^m x)
\end{aligned}$$

The last equivalence is essentially based on the fact that both $\{x\} \cup rx \cup \ldots \cup r^m x$ and r^{m+1} are closed. Now $r^\infty x = \bigcup_{n \in \mathbb{N}} r^n x$, but for every n it holds that $r^{m+1+n} x \subseteq r^n(\{x\} \cup rx \cup \ldots \cup r^m x) = r^n x \cup \ldots \cup r^{m+n} x$ and hence $r^\infty = \{x\} \cup rx \cup \ldots \cup r^m x$.

In other words, a Kripke or descriptive frame $\mathcal{F}$ satisfies K^m iff every path in $\mathcal{F}$ can be reduced to a path of length at most m.

We immediately obtain that, when K^m is satisfied, any set C is hereditary iff $C = r_\infty C = C \cap r^* C \cap \ldots \cap r^{*m} C$. In particular, $r_\infty C$ is clopen whenever C is clopen. Moreover, a is open in $\mathcal{A} \models \mathsf{K}^m$ iff $a = \tau_m a$ iff $a \leq \tau a \cdot \ldots \cdot \tau^m a$. Thus *all the results we proved for* K4 *hold also for* K^m*, except Lemma 3.3 and Theorem 3.5* (we should restrict them to frames in which r is closed, to obtain that $r^\infty x$ is closed). I omit all the proofs, which however are just a variation on those for K4.

3.3 Half a theorem

When $\mathcal{A}$ is s.i. and satisfies K4 or K^m, $E_\mathcal{A}$ is principal and $I_{\mathcal{A}_*}$ is clopen. Thus one would hope to prove something like: for every $\mathcal{A}$, $\mathcal{A}$ is s.i. with $E_\mathcal{A}$ principal iff $\mathcal{A}_*$ is initial with $I_{\mathcal{A}_*}$ clopen. We will give the proof of only one half of it; my conjecture was that also the other half could be proved, but Marcus Kracht found a counterexample [Kracht 1998b].

We need a lemma with an interesting proof: compactness of $\mathcal{A}_*$ is used for the first time, and shows why $r_\infty C$ is difficult to handle.

Lemma 3.10 *For every descriptive frame* $\mathcal{F}$ *and every* $C \in \mathcal{T}$*:*

$$H_\mathcal{F} \textit{ is closed, } H_\mathcal{F} \neq X \textit{ and } C \textit{ is essential iff } H_\mathcal{F} \subseteq C \textit{ and } C \neq X$$

Proof. ($\Rightarrow$) Since $H_\mathcal{F}$ is closed and non trivial, $\mathcal{F}^*$ is s.i. and $H_\mathcal{F}$ corresponds to the filter $E_{\mathcal{F}^*}$. Thus any clopen C different form X and containing $H_\mathcal{F}$ is in $E_{\mathcal{F}^*}$, that is, is essential.
($\Leftarrow$) Let $D \in \mathcal{T}$ with $D \neq X$. Since $r_\infty D$ is hereditary, $r_\infty D \subseteq H_\mathcal{F} \subseteq C$. But C is open and $r_\infty D$ is an infinite intersection of closed subsets. By the compactness of $\mathcal{F}$, a finite intersection suffices, i.e. $D \cap \ldots \cap r^{*n} D \subseteq C$ for some n. So C is essential.

Theorem 3.11 *If* $I_{\mathcal{A}_*}$ *is clopen and nonempty,* $\mathcal{A}$ *is s.i. and* $E_\mathcal{A}$ *is principal.*

Proof. Since $I_{\mathcal{A}_*}$ is open and nonempty, $\mathcal{A}$ is s.i. and $E_\mathcal{A}$ non empty. If $I_{\mathcal{A}_*}$ is clopen, then $H_{\mathcal{A}_*}$ is clopen and a fortiori closed. We thus apply Lemma 3.10 to obtain that $H_{\mathcal{A}_*}$ is itself essential, and the least such. So, if $H_{\mathcal{A}_*} = \beta c$, $E_\mathcal{A}$ is principal and generated by c.

References

[Boolos 1993] G. Boolos, *The logic of provability*, Cambridge University Press, 1993.

[Goldblatt 1989] R. Goldblatt, *Varieties of complex algebras*, Ann. Pure Applied Logic 44 (1989), pp. 173–242.

[Grätzer 1968] *Universal algebra*, Van Nostrand, 1968.

[Kracht 1998] M. Kracht, *Tools and techniques in modal logic*, North - Holland, to appear.

[Kracht 1998b] M. Kracht, *On a conjecture by Giovanni Sambin*, typescript.

[Magari 1975] R. Magari, *The diagonalizable algebras*, Boll. Un. Mat. Ital. 4 (1975), pp. 321–331.

[Rautenberg 1979] W. Rautenberg, *Klassische und nichtklassische Aussagenlogik*, Vieweg Verlag, 1979.

[Smoryński 1985] C. Smoryński, *Self-reference and modal logic*, Springer Verlag, 1985.

[Sambin-Vaccaro 1988] G. Sambin & V. Vaccaro, *Topology and duality in modal logic*, Ann. Pure Applied Logic 37 (1988), pp. 249–296.