Points and Co-points in Formal Topology

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Dopo una breve introduzione delle principali idee della topologia formale [Sam87] vengono introdotte le nozioni di punto e co-punto. Si presentano quindi alcuni metodi per costruire punti e co-punti in opportune topologie formali e si forniscono alcune applicazioni logiche di tali costruzioni[1].

1. Introduction

In this paper we analyse the notion of point in the formal topology framework [Sam87] and introduce the new idea of co-point. Since formal topology is not commonly known, even if it is deeply related to the localic approach to topology [Joh82, Vic89], and the references are, at present, very few, we think that a short introduction to the subject can be helpful.

The main idea of the formal topology is describing, in a constructive framework, the properties of a topological space (X,ΩX), where ΩX is the family of the open subsets of X, which can be expressed without any reference to the points, that is to the elements of X. Here the philosophical motif is that an infinite amount of information is in general needed in order to discern one point while a finite one is required in order to distinguish a set of points (i.e. an open set). The idea is considering the properties instead of the concepts as primitive entities; a concept can eventually be recovered as a suitable "set of properties", i.e. the properties that the concept satisfies.

Since we know a topology as soon as one of its bases is specified and it is not restrictive looking just at the bases which are closed under intersection, the structure which we may consider turns out to be a semilattice <S,•,1>, where S is the set of the elements of the base, • is the operation of intersection between base elements and 1 is the identity of the operation •, i.e. the whole set X.

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[1] An extended version of this paper, with fully annotated proofs, can be found in [Val89].

Every open set of a topology can be obtained as a union of elements of the base, but the union does not make sense if we refuse any reference to the points; hence we are naturally led to identify a generic open set with the subset of S of the elements of the base whose union is the given set.

We have now to characterize the relation between opens and elements of the base. Following an hint of Grothendieck topology we can introduce a relation ≤ of coverage between the base elements and the open sets whose intended meaning is to explain when an element of the base is a subset of an open set, i.e. if *a*   S and V∏S we want that $a \le V \equiv a \prod \approx V$.

The conditions[2] we require on this relation are a straightforward rephrasing of the similar set-theoretic situation. Let *a,b*   S and U,V∏S:

Reflexivity: $$\frac{a\ \ U}{a \le U}$$

i.e. "if the element *a* of the base is used to construct the open U then it is a subset of U"

Transitivity: $$\frac{a \le U \quad (\ \ x\ \ U)\ x \le V}{a \le V}$$

i.e. "if the element *a* is a subset of the open U and each element of U is a subset of the open V then *a* must be a subset of V"

•-Left: $$\frac{a \le U}{a \bullet b \le U}$$

i.e. "if an element is a subset of the open U then every smaller element of the base is also a subset of U"

•-Right: $$\frac{a \le U \qquad a \le V}{a \le \{b \bullet c:\ b\ \ U,\ c\ \ V\}}$$

i.e. "if the element *a* is a subset of the open U and also of the open V then it is a subset of the open obtained by the union of the intersections of all the elements of the base that form U and V respectively".

In order to partly recover points we ask to know if an element of the base is inhabited, i.e. if there is a positive evidence that it is not empty; to distinguish such elements we introduce a predicate Pos on the elements of S and require that it satisfies the following conditions:

Monotonicity: $$\frac{\mathrm{Pos}(a) \qquad a \le U}{(\ \ x\ \ U)\ \mathrm{Pos}(x)}$$

i.e. "if the element of the base *a* contains some point and is a subset of the open U then there must be a not empty element in U"

Ex_falso quodlibet: $$\frac{\neg \mathrm{Pos}(a)}{a \le \emptyset}$$

i.e. "if the element of the base *a* is not inhabited then it can be covered by every open"

Openness: $$\frac{a \le U}{a \le \{b\ \ U:\ \mathrm{Pos}(b)\}}$$

i.e. "the not empty elements of the open set U are sufficient to cover everything U covers"

[2] We express the conditions in a form that recalls the rules of a standard deductive system just to make easier in the following to present the proofs of the properties of the formal topologies.

The following remarks can be easily proved.

- The converse of Ex_falso quodlibet holds.
- (a S) $a \le \{1\}$, i.e. every element of the base is covered by 1.
- Pos(1) if and only if (x S) Pos(x)

We call *proper* any formal topology such that Pos(1), namely such that "there is some point".

2. Points and co-points

Let us now try to recover points in a formal topology. As we already said, we can think of a point as a "sufficiently complete" set of informations, i.e. open sets, on a concept. Of course we can limit ourselves to consider only elements of the base to collect informations, since any open set is obtained as a collection of elements of the base. So we can define a point as a special set of elements of the base obeying suitable conditions.

Let α be the point we are going to define, the first condition is obviously

Not-degeneration: $1 \quad \alpha$

i.e. "every point belong to the whole space" or, in term of information, "the always true information is true also of the concept α".

Moreover

Intersection: $\dfrac{a \quad \alpha \qquad b \quad \alpha}{a \bullet b \quad \alpha}$

i.e. "if the point α belongs both to a and b then it belongs to their intersection" or "if the informations a and b concern the concept α then their conjunction does".

Point consistency: $\dfrac{a \quad \alpha}{\mathrm{Pos}(a)}$

i.e. "if the point α belongs to the base element a then a is not empty" or "if the information a concern some concept then it is consistent".

All the previous conditions turn each point into a consistent filter of the semilattice of elements of the base, but none of them makes it "sufficiently complete". To this aim we need to correlate the notion of point to that of coverage.

Point completeness: $\dfrac{a \le U \qquad a \quad \alpha}{(\ x \quad U) \ x \quad \alpha}$

i.e. "if the point α belongs to the base element a and a is is a subset of the open U then the point α must belong to one of the element of U" or "if the information a concerns the concept α and the set of information U is less sharp than a then there must be some information in U concerning α".

Hence a point is just a completely prime filter of the semilattice S.

Remarks

- if Pos is decidable, i.e. (a S) Pos(a) ¬Pos(a), then the point consistency condition is a consequence of point completeness.
- the converse of the intersection condition is a consequence of point completeness.
- the condition of not-degeneration can be substituted with the equivalent condition that α is not empty.

It is interesting to analyse also the properties of the complement of a point, which will be called *co-point* in the following, even if at present we have no geometrical intuition on it. Our interest in the co-points can be found in the applications that will follow.

In reading the conditions on co-points, in order to help the intuition, the reader should think of the operation • as the disjunction, and hence of 1 as , and of the relation a≤U as "a is a consequence of U"; it is easy to convince ourselves that all the conditions we gave on cover relation are still valid using this interpretation, even if the consequence relation is considered in intuitionistic logic. With this interpretation, the first property of a co-point $\alpha^{¾}$, complement of the point α, is consistency,

Co-point consistency: 1 $\alpha^{¾}$

that is "$\alpha^{¾}$ must not contain ".

Then we have deductive completeness, which can be expressed by the following condition

Co-point completeness: $$\frac{a\leq U \quad (\quad x \quad U)\ x \quad \alpha^{¾}}{a \quad \alpha^{¾}}$$

i.e. "if a is a consequence of U and each element in U belongs to the co-point $\alpha^{¾}$ then also a must be in $\alpha^{¾}$", whose derivation from point completeness requires classical logic.

The next property concerns the operation

Disjunction: if a•b $\alpha^{¾}$ then a $\alpha^{¾}$ or b $\alpha^{¾}$

that just expresses the intended meaning of the operation and whose proof is trivial.

Finally also point consistency has an interesting translation

Validity: $$\frac{\neg \mathrm{Pos}(a)}{a \quad \alpha^{¾}}$$

whose proof is immediate and whose meaning is clear if the intended meaning of Pos(a) is that a is not provable, i.e. it is not derivable from Ø.

Remarks

- the converse of the disjunction condition is a consequence of the co-point completeness condition.

- the previous conditions are sufficient to make deductively closed a co-point $\overline{\alpha}$, i.e. $\overline{\alpha}$ contains everything is deducible from it, because if $a\leq\overline{\alpha}$ then co-point completeness implies that a $\overline{\alpha}$ since (x $\overline{\alpha}$) x $\overline{\alpha}$ holds.
- the four properties we have given completely characterize a co-point, that is the complement of a set of base elements which satisfies such properties is a point[3].

3. Points construction

In this section we will discuss the issue of the existence of points in a formal topology. Two questions arise: is there always any point in a formal topology and, in the positive case, what methods can be used to construct (to find?) one of them?

It is immediate to see that the not_degeneration condition and the point consistency condition imply that the existence of points requires that Pos(1) holds i.e. the formal topology must be proper (not a surprise I suppose!). Anyhow this condition is not sufficient; for instance, it is not difficult to show that the formal topology based on <B, ,1>, where B is a complete atomless boolean algebra, with the coverage relation $a\leq U$ iff $a\leq$ U and the positivity predicate Pos(a) iff $a\neq 0$, has no completely prime filter and hence no point[4]. For this reason let us show two techniques to construct points in suitable proper formal topologies.

In the first case, let us suppose we are dealing with a formal topology <S,•,1,≤,Pos> with a denumerable and effectively listable family of open sets (later on we will show a meaningful example of such a formal topology); then for each positive element a we can construct a point α containing it.

In fact let $W_0,W_1,\ldots,W_n,\ldots$ be a list of the open sets, and let $L \equiv U_0,U_1,\ldots,U_m,\ldots$ be a new list, obtained by the previous one, in such a way that each W_i is denumerably repeated (for instance $U_0=W_0$, $U_1=W_0$, $U_2=W_1$, $U_3=W_0$, $U_4=W_1$, $U_5=W_2$, $U_6=W_0$, $U_7=W_1,\ldots$) and consider the following inductive construction for α.

$\alpha_0 = \{1,a\}$

Let $\alpha_n=\{d_1,\ldots,d_m\}$ be the (finite) set obtained at the step n of the inductive construction; as we will show, α_n is a set closed under •, which contains 1, whose elements are all positive and hence $d=d_1\bullet\ldots\bullet d_m$ is a positive element. Let U_n be the n-th open set in the list L, then define

$\alpha_{n+1}=\alpha_n$ if $d \not\leq U_n$

otherwise let x^* be the element of U_n whose existence is assured by the following proof

$$\dfrac{\mathrm{Pos}(d) \qquad \dfrac{d\leq U_n \qquad d\leq\{d\}}{d\leq\{d\bullet u:\ u\ \ U_n\}}}{(\ x\ \ U_n)\ \mathrm{Pos}(d\bullet x)}$$

and define

[3] Classical logic is needed to prove this result.
[4] This fact was observed by G.Sambin during a talk on points in formal topology.

$\alpha_{n+1}=\alpha_n \approx \{x^*\bullet d_i\colon d_i \ \ \alpha_n\}$

Finally define

$\alpha = {}_{i \ \ \omega}\, \alpha_i$

Let us show that the statements we made on each α_n in the inductive construction hold.

- each α_n is a finite set: α_0 is finite and ™α_{n+1}™≤ 2 ™α_n™.
- each α_n is closed under •: an easy proof by induction on the construction.
- for each n, 1 α_n: 1 α_0 and, for each i, $\alpha_i \sqcap \alpha_{i+1}$.
- for each n, each element of α_n is positive: the proof is again by induction on the construction; as regard to α_0 we supposed that *a* is positive and Pos(*a*) implies Pos(1); let us suppose, by inductive hypothesis, that each element of α_n is positive, then the result is obvious if $\alpha_{n+1}=\alpha_n$; otherwise $\alpha_{n+1}=\alpha_n \approx \{x^*\bullet d_i\colon d_i \ \ \alpha_n\}$, for a suitable x^*; in this case let $c \ \ \alpha_{n+1}$: if $c \ \ \alpha_n$ the result is straightforward otherwise $x^*\bullet d = x^*\bullet d_1\bullet\ldots\bullet d_m \leq \{x^*\bullet d_i\} = \{c\}$ and the result follows by monotonicity since Pos($x^*\bullet d$) holds by the choice of x^*.

<u>Theorem 3.1</u>: (Point construction: first method)

Let *a* be a positive element of a proper formal topology with a denumerable and effectively listable family of open sets, then there is a point α containing *a*.

Proof: In the previous construction 1 α and each element of α is positive. Moreover if $c,d \ \ \alpha$ then, for some n, $c,d \ \ \alpha_n$ hence $c\bullet d \ \ \alpha_n$ then $c\bullet d \ \ \alpha$. Finally we must show that if $c \ \ \alpha$ and $c\leq$U then (x U)$x \ \ \alpha$. In fact if $c \ \ \alpha$ then, for some n, $c \ \ \alpha_n$ and, since the open set U numerably appears in the list *L* of open sets, there will be a step k≥n such that U≡U_k; but $\alpha_k=\{c,d_1,\ldots,d_m\}$ then $c\leq$U implies $c\bullet d_1\bullet\ldots\bullet d_m\leq$U and hence $\alpha_{k+1}=\alpha_k\approx\{x^*\bullet d_i\colon d_i \ \ \alpha_k\}$ for a suitable x^* U; then $x^* \ \ \alpha_{k+1}$, since 1 α_k, and finally $x^* \ \ \alpha$. Moreover $a \ \ \alpha$ since $a \ \ \alpha_0$.

Note that, using classical logic, under the same assumptions on the considered formal topology we can obtain the following stronger result.

<u>Theorem 3.2</u>:

Let $a\not\leq$ W in a proper formal topology with a denumerable and effectively listable family of open sets, then there exists a point α such that $a \ \ \alpha$ and (x W) x / α.

The proof will be shown after some preliminary lemmas whose proofs are almost immediate.

<u>Lemma 3.3</u>[5]: If $c\not\leq$ W then Pos(c).

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[5] The proof requires classical logic.

Lemma 3.4[5]: If $c \leq$U and $c \not\leq$ W then there is x^* U such that $c \bullet x^* \not\leq$ W.

Lemma 3.5: If $c \leq \{f\}$ and $c \not\leq$ W then $f \not\leq$ W.

The construction of a point α which omits any element of the open set W is now a slight modification of the previous one.

$\alpha_0 = \{1, a\}$

Let $\alpha_n = \{d_1, \ldots, d_m\}$ be the (finite) set obtained at the step n of the inductive construction; as we will show, α_n is a set closed under • which contains 1 and elements none of which is covered by W, hence $d = d_1 \bullet \ldots \bullet d_m \not\leq$ W. Let U_n be the n-th open set in the list *L*, then define

$\alpha_{n+1} = \alpha_n$ if $d \not\leq U_n$

otherwise let x^* be the element of U_n such that $d \bullet x^* \not\leq$ W, whose existence is assured by lemma 3.4, and define

$\alpha_{n+1} = \alpha_n \approx \{x^* \bullet d_i : d_i \;\; \alpha_n\}$

Finally define

$\alpha = {}_{i \;\; \omega}\, \alpha_i$

The only novelty with respect to the proof of the theorem 3.1 is that we must show that, for each n, none of the elements of α_n is covered by W. The proof is by induction on the construction: as regard to α_0 we supposed that *a* is not covered by W and hence also 1 is not covered by W since $a \leq \{1\}$; suppose, by inductive hypothesis, that none of the elements of α_n is covered by W, hence the result is obvious if $\alpha_{n+1} = \alpha_n$; otherwise $\alpha_{n+1} = \alpha_n \approx \{x^* \bullet d_i : d_i \;\; \alpha_n\}$, for a suitable x^*; let $c \;\; \alpha_{n+1}$, then if $c \;\; \alpha_n$ the result is straightforward otherwise $x^* \bullet d = x^* \bullet d_1 \bullet \ldots \bullet d_m \leq \{x^* \bullet d_i\} = \{c\}$ and the result follows again by lemma 3.5 since $x^* \bullet d \not\leq$ W by the choice of x^*.

The proof that α is a point containing *a* is completely similar to the previous one and we just have to show that none of the elements of W belongs to α; this is almost obvious since if $x \;\; \alpha$ then, for some n, $x \;\; \alpha_n$ and hence $x \not\leq$ W while reflexivity implies that each element of W is covered by W.

Beside theorem 3.1, an obvious corollary of the previous theorem is

Corollary 3.6:

Let $a \not\leq$ W in a proper formal topology with a denumerable and effectively listable family of open sets, then there is a co-point α¾ such that a α¾ and (x W) x α¾.

It is interesting to note that theorem 3.2 allows to recover the original intuition about the cover relation and positivity predicate since it assures us that our topology has “enough points”. In fact, for any formal topology such that it holds, we also have that

$a \leq W$ iff for each point α, if a α then there exists w W such that w α

and

Pos(a) iff there is a point α such that a α

since it is easy to prove that $a \not\leq \emptyset$ is a consequence of Pos(a).

Another technique for constructing points can be given provided that, for any couple of base elements a and b, Pos(a) and Pos(b) imply Pos(a•b). Recalling the intended meaning of • and Pos this may seem a very strange condition, and it is surely not necessary in order to have points; anyhow we will show in the following an interesting example of such a formal topology. Moreover the reader should note that this condition is just a generalization of the 'finite intersection property' that assures the existence of filters, and hence ultrafilters, in a boolean algebra. On the other hand, the previous example about the atomless complete boolean algebra shows that the usual finite intersection property is not sufficient to have points.

In this hypothesis, let A={$a_1,\ldots,a_n,\ldots$} be a set of positive elements of S, then the following inductive construction yields a point α which contains all the elements of A.

$\alpha_0=\{1\}\approx A$

Let $\alpha_n=\{b_1,\ldots,b_m,\ldots\}$ be the set obtained at the step n of the inductive construction; as we will show α_n contains, beside 1, only positive elements; thus if one of its elements b_i is covered by an open set U, monotonicity assures that there exists a positive element x^* U. Then define

$\alpha_{n+1}= \alpha_n$

$\approx\{x^*\}$ for each U and b_i such that $b_i \leq U$

$\approx\{b_i \bullet b_j\}$ for all b_i, b_j α_n.

Finally define

$\alpha=_{h \ \omega} \alpha_h$

Let us show that, for any n, each element of α_n is positive. The proof is by induction on the construction. Obviously α_0 contains only positive elements. Let us now suppose that c α_{n+1} then three possibilities can arise: c α_n, and the result follows by the inductive hypothesis; c was added to α_n since an element of α_n was covered by an open set U, and the result is obvious since we added just positive elements; finally $c=b_i \bullet b_j$, where b_i, b_j α_n, and in this case the result follows by the inductive hypothesis and the assumption that Pos(b_i) and Pos(b_j) imply Pos($b_i \bullet b_j$).

Theorem 3.7: (Point construction: second method)

Let A be a set of positive elements in a proper formal topology such that, for any a and b, Pos(a) and Pos(b) imply Pos(a•b); then there is a point which contains each element of A.

Proof: Consider the previous construction: it is obvious that $1 \ \alpha$ and that each element of α is positive. Moreover if $b,c \ \alpha$ then, for some n, $b,c \ \alpha_n$ and hence $b\bullet c \ \alpha_{n+1}$ and finally $b\bullet c \ \alpha$ and if $b \ \alpha$ and $b \leq U$ then, for some n, $b \ \alpha_n$ hence there is $x^* \ U$ such that $x^* \ \alpha_{n+1}$ and hence $x^* \ \alpha$.

4. A representation theorem.

The theorems on the point construction of the previous section are the basis to prove a representation theorem for formal topologies. Consider a formal topology τ for which theorem 3.2 holds; then it can be represented by a concrete topology where each base element of the formal topology is identified with the set of points which contains it. In fact, let $C_a=\{\alpha \ \mathrm{Pt}(\tau): a \ \alpha\}$, where Pt(τ) are the points of the formal topology $\tau \equiv$ <S,•,1,≤,Pos>, and consider the structure $<\{C_a: a \ S\}, \leftrightarrow, C_1>$, with the coverage relation defined by

$C_a \leq^* \{C_{b1},\ldots,C_{bn},\ldots\}$ iff $C_a \sqcap \approx\{C_{b1},\ldots,C_{bn},\ldots\}$

and the positivity predicate

Pos*(C_a) iff $(\ \alpha \ \mathrm{Pt}(\tau))\ a \ \alpha$,

i.e. a constructive way to require that $C_a \neq \emptyset$

There is no difficult in checking that $<\{C_a: a \ S\}, \leftrightarrow, C_1>$ is a (concrete) formal topology. Let us show that it is "isomorphic" to τ. In fact consider the map: $a \ \ C_a$. It is obviously surjective and if $a \neq b$ (i.e. $a \not\leq \{b\}$ or $b \not\leq \{a\}$ holds) then theorem 3.2. assures us that there exists a point α such that $a \ \alpha$ and b / α or there exists a point β such that $b \ \beta$ and a / β, i.e. $C_a \neq C_b$. Moreover:

- $C_a \leftrightarrow C_b = C_{a\bullet b}$, and hence in particular $C_1 \leftrightarrow C_a = C_a$,

  since $\alpha \ C_a \leftrightarrow C_b$ iff $\alpha \ C_a$ and $\alpha \ C_b$ iff $a \ \alpha$ and $b \ \alpha$ iff $a\bullet b \ \alpha$ iff $\alpha \ C_{a\bullet b}$
- $a \leq \{b_1,\ldots,b_n,\ldots\}$ iff $C_a \sqcap \approx\{C_{b1},\ldots,C_{bn},\ldots\}$,

  in fact, let $\alpha \ C_a$, i.e. $a \ \alpha$, then $a \leq \{b_1,\ldots,b_n,\ldots\}$ implies that there is $b_i \ \alpha$ hence $\alpha \ C_{bi}$ so $\alpha \ \approx\{C_{b1},\ldots,C_{bn},\ldots\}$; on the other hand if $a \not\leq \{b_1,\ldots,b_n,\ldots\}$ theorem 3.2 assures us that there is a point α such that $a \ \alpha$ while no b_i belongs to α; so there is a point α such that $\alpha \ C_a$ while it does not belong to any of C_{bi} and hence does not belong to their union.
- Pos(a) iff Pos*(C_a),

  in fact theorem 3.1, which is a consequence of theorem 3.2, establishes that Pos(a) implies that there is a point α containing a; on the other hand if there is a point α containing a then a must be positive.

The following corollary will be used in the next section.

Corollary 4.1:

Consider a formal topology for which theorem 3.2 holds, then *a*≤Ø iff *a* $\alpha^{3/4}$ for each co-point $\alpha^{3/4}$.

Proof: *a*≤Ø iff C_a≤*Ø iff C_a⊓≈Ø=Ø iff a α for each point α iff a $\alpha^{3/4}$ for each co-point $\alpha^{3/4}$.

5. An application to logic.

After introducing the co-points we observed that a standard deductive system is a particular formal topology where the coverage relation is the relation of derivability between a (finite) set of formulas and a formula, the operation is the disjunction, the unity is and the positivity predicate is the non-provability assertion[6].

We can easily put to work the theorems on points, and hence on co-points, construction of section 3. on this formal topology.

First of all we consider a deductive system for which disjunction property holds (that is |—A B iff |—A or |—B) then Pos(A B) (i.e. |⁄— A B) iff Pos(A) (i.e. |⁄— A) and Pos(B) (i.e. |⁄— B). Hence, given a not provable formula A, i.e. if Pos(A) holds, theorem 3.7 shows the existence of a co-point $\alpha^{3/4}$ such that A $\alpha^{3/4}$. A suitable meaning of a co-point is now clear: it is a model for the derivability relation; in fact it is a consistent set of formulas (co-point consistency condition), closed under deduction (co-point completeness condition) and disjunction (co-point disjunction condition) which contains all the theorems (validity condition).

An analogous result can be achieved, using theorem 3.1 instead of 3.7, under the hypothesis that the formal topology has a denumerable and effectively listable family of open sets; note that this fact obviously holds for any deductive system in which the formulas are constructed on a denumerable language. In this case corollary 4.1 holds and its meaning is obviously a completeness theorem: A≤Ø, i.e. |—A, iff A belongs to all co-points, i.e. to all models.

It is interesting to note that the co-points provide an interpretation also for the other propositional connectives. Let us use the terminology of section 4.; we already proved that $C_{A\ B}=C_A \leftrightarrow C_B$ and that A B belongs to the co-point $\alpha^{3/4}$, i.e. A B is true in the model $\alpha^{3/4}$, iff A $\alpha^{3/4}$ or B $\alpha^{3/4}$. Let us now consider the conjunction: using the valid consequences A B|—A, A B|—B and A,B|—A B it is not difficult to prove that $C_{A\ B}=C_A \approx C_B$ and that, for each co-

[6] To be more precise one would consider the quotient of formulas with respect to the equi-derivability relation, but we think that the reader has no difficulty to supply such a standard part of the work.

point α¾, A B α¾ iff A α¾ and B α¾. Moreover if the derivability relation is classic, i.e. |—A ¬A holds, then, using the valid consequence A,¬A| — , we obtain that, for each point α, α $C_{\neg A}$ iff α C_A and, for each co-point α¾,¬A α¾ iff A α¾. Finally using A,A B|—B and the fact that C,A|—B implies C|—A B we obtain that $C_{A\ \ B}$= ↔ {C_C: C_B∏C_C≈C_A} and hence, for each co-point α¾, A B α¾ iff A α¾ implies B α¾.

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