REPRESENTATION THEOREMS FOR QUANTALES

Silvio Valentini

Dip. di Matematica - Università di Padova

Abstract. In this paper we prove that any quantale $\mathcal{Q}$ is (isomorphic to) a quantale of suitable relations on $\mathcal{Q}$. As a consequence two isomorphism theorems are also shown with suitable sets of functions of $\mathcal{Q}$ into $\mathcal{Q}$. These theorems are the mathematical background one needs in order to give natural and complete semantics for (non-commutative) Linear Logic using relations.

1991 *Mathematics Subject Classification.* 06D05, 06D10, 06D20..
Key words and phrases. Quantale, Locale, Order relation.

Introduction

The interest for quantales has grown up in the communities of the logicians and of the theoretical computer scientist after the works by Rosenthal [Rosenthal 90] and Yetter [Yetter 90] which show the relevance of these structures in giving complete semantics of Girard's Linear Logic [Girard 87].

Let us recall the definition of quantale [Rosenthal 90] and state the basic results we are going to use. The structure

$$\mathcal{Q} \equiv \langle Q, \bullet, 1, \vee \rangle$$

is a *(unital) quantale* if

$$\langle Q, \bullet, 1 \rangle$$

is a monoid with unity 1,

$$\langle Q, \vee \rangle$$

is a complete semilattice and $\bullet$ distributes over $\vee$, i.e. $(\vee_{i \in I} x_i) \bullet t = \vee_{i \in I}(x_i \bullet t)$ and $t \bullet (\vee_{i \in I} x_i) = \vee_{i \in I}(t \bullet x_i)$.

As usual, $\vee$ induces an order relation $\leq$ on Q by putting

$$x \leq t \equiv x \vee t = t.$$

It is well known that $\leq$ is reflexive, transitive and antisymmetric; moreover, because of distributivity, the following lemma holds.

Lemma: (stability). *For every quantale $\mathcal{Q}$, $\leq$ is compatible with $\bullet$, i.e. for every $x, t, u, v \in Q$ if $x \leq t$ and $u \leq v$ then $x \bullet u \leq t \bullet v$.*

Proof. First observe that if $w \leq z$ then, for any $s \in Q$, $s \bullet w \leq s \bullet z$ and $w \bullet s \leq z \bullet s$ because $z = w \vee z$ implies $s \bullet z = s \bullet (w \vee z) = (s \bullet w) \vee (s \bullet z)$ and $z \bullet s = (w \vee z) \bullet s = (w \bullet s) \vee (z \bullet s)$; now suppose $x \leq t$ and $u \leq v$ then $x \bullet u \leq t \bullet u$ and $t \bullet u \leq t \bullet v$ hence $x \bullet u \leq t \bullet v$ by transitivity.

In addition, $\vee\emptyset$ and $\vee Q$ are respectively the minimum and the maximum element of the quantale $\mathcal{Q}$. Hence

$$\wedge S \equiv \vee\{x \in Q : x \leq y, \text{ for each } y \in S\}$$

defines an operation of infimum on Q.

The representation theorem

Now we want to show that any quantale $\mathcal{Q}$ is isomorphic to a sub-quantale of a quantale of suitable relations on $\mathcal{Q}$. To this purpose we introduce the following definition.

Definition: (ordered relation). Let $\langle Q, \bullet, 1, \vee \rangle$ be a quantale; a binary relation R on Q is called *ordered* if

- for any $u, x, y, z \in Q$, if $u \leq x$, xRy and $y \leq z$ then uRz, i.e. R is compatible with $\leq$;

- for any $t, x_1, x_2, \dots \in Q$, if, for every $i \in I$, $x_i Rt$ then $\vee_{i\in I} x_i Rt$, i.e. R is compatible with $\vee$;

- for any $x, t_1, t_2, \dots \in Q$, if, for every $i \in I$, xRt_i then $xR \wedge_{i\in I} t_i$, i.e. R is compatible with $\wedge$.

Note that if R is an ordered relation on $\mathcal{Q}$ then, for each $x \in Q$, $\vee\emptyset Rx$ and $xR\wedge\emptyset = \vee Q$ hold. Moreover observe that the order relation $\leq$ is a very simple example of ordered relation which will be called I in the following. In fact the ordered relations get their name from the fact that they can be thought as an abstract kind of order relation where reflexivity and antisimmetry are not concerned, while working in a framework where arbitrary suprema and infima exist.

The conditions we require on ordered relations are suitably chosen in order to give them the strutture of a quantale; in particular compatibility with respect to $\vee$ and $\wedge$ are needed in order to show distributivity. Anyhow the first step is to define the quantale operations. The most natural monoid operation is the relation composition, i.e.

$$R \circ S \equiv \{(x, z) \in Q \times Q : (\exists y \in Q)\ xRy \text{ and } ySz\}.$$

In fact the basic intuition in this work was that the standard theorem *a la* Cayley for monoids could be extended to quantales.

Lemma: (closure under composition). *Let $\langle Q, \bullet, 1, \vee \rangle$ be a quantale and R and S be ordered relations on $\mathcal{Q}$; then also $R \circ S$ is an ordered relation on $\mathcal{Q}$.*

Proof. Compatibility of $R \circ S$ with respect to $\leq$ is a direct consequence of the compatibility of R and S with respect to $\leq$; whereas to prove compatibility of $R \circ S$ with respect to $\vee$ one uses compatibility of R with respect to $\vee$ and to prove compatibility of $R \circ S$ with respect to $\wedge$ one uses compatibility of S with respect to $\wedge$, beside compatibility of R and S with respect to $\leq$ and the fact that $w_i \leq \vee_{i\in I} w_i$ and $\wedge_{i\in I} w_i \leq w_i$ hold for each $i \in I$.

Now we can explain why we call the ordered relation $\leq$ "I"; as one can easily check I is the *identity* for composition among ordered relations, i.e. $R \circ I = R = I \circ R$ holds for each ordered relation R. Hence $\langle R_Q, \circ, I \rangle$ is a monoid, where R_Q is the set of the ordered relations on $\mathcal{Q}$.

The next step is to define the supremum of an arbitrary set of ordered relations $\{R_i : i \in I\}$. The natural choice is the union but in general the union of ordered relations is not an ordered relation. For this reason we put

$$\vee_{i\in I} R_i \equiv \cap_{i\in I} R_i$$

which is obviously a semilattice operation provided that $\cap_{i\in I} R_i$ is an ordered relation.

Lemma: (closure under arbitrary intersection). *Let $\langle Q, \bullet, 1, \vee\rangle$ be a quantale and $\{R_i : i \in I\}$ a set of ordered relations on $\mathcal{Q}$; then $\vee_{i\in I} R_i$ is an ordered relation on $\mathcal{Q}$.*

Proof. The result is almost obvious since $x \vee_{i\in I} R_i y$ holds if and only if, for any $i \in I$, xR_iy and each R_i satisfies the compatibility conditions.

By means of the previous definitions we have constructed the structure $\langle R_Q, \circ, I, \vee_R\rangle$. In order to prove that it is a quantale we must only show that $\circ$ distributes over $\vee_R$. One inclusion is obvious: in fact let S and R_i, $i \in I$, be ordered relations then we obviously have $\cap_{i\in I} R_i \subseteq R_i$, for each $i \in I$, and hence $(\cap_{i\in I} R_i) \circ S \subseteq R_i \circ S$ and $S \circ (\cap_{i\in I} R_i) \subseteq S \circ R_i$ and finally $(\cap_{i\in I} R_i) \circ S \subseteq \cap_{i\in I}(R_i \circ S)$ and $S \circ (\cap_{i\in I} R_i) \subseteq \cap_{i\in I}(S \circ R_i)$. In order to prove the other inclusion we put the compatibility conditions at work: in fact, to prove that $\cap_{i\in I}(R_i \circ S) \subseteq (\cap_{i\in I} R_i) \circ S$, suppose $y \cap_{i\in I} (R_i \circ S) z$, then, for each $i \in I$, $yR_i \circ Sz$ and hence there is an element u_i such that yR_iu_i and u_iSz; thus $yR_i \vee_{i\in I} u_i$, since R_i is compatible with $\leq$, so $y \cap_{i\in I} R_i \vee_{i\in I} u_i$; moreover $\vee_{i\in I} u_i Sz$, because S is compatible with $\vee$, and hence $y(\cap_{i\in I} R_i) \circ Sz$; on the other hand to prove that $\cap_{i\in I}(S \circ R_i) \subseteq S \circ (\cap_{i\in I} R_i)$, suppose $y \cap_{i\in I} (S \circ R_i) z$, then, for each $i \in I, yS \circ R_i z$ and hence there is an element u_i such that ySu_i and u_iR_iz; thus $yS \wedge_{i\in I} u_i$, because S is compatible with $\wedge$ and $\wedge_{i\in I} u_i R_i z$, since R_i is compatible with $\leq$, hence $\wedge_{i\in I} u_i \cap_{i\in I} R_i z$ and so $yS \circ (\cap_{i\in I} R_i)z$.

Hence we have proved the following theorem.

Theorem. *Let $\langle Q, \bullet, 1, \vee\rangle$ be a quantale and R_Q be the set of the ordered relations on $\mathcal{Q}$; then $\langle R_Q, \circ, I, \vee_R\rangle$ is a quantale.*

Now we can prove the representation theorem.

Theorem: (representation theorem). *Any quantale $\langle Q, \bullet, 1, \vee\rangle$ is isomorphic to a sub-quantale of the quantale $\langle R_Q, \circ, I, \vee_R\rangle$.*

In fact, supposing x as being an element of Q, let us consider the relation

$$R_x \equiv \{(y, z) : x \bullet y \leq z\}$$

which will be called the *relation generated* by x. It is easy to prove that R_x is an ordered relation since compatibility with respect to $\leq$ is a consequence of stability, compatibility with respect to $\vee$ is a consequence of distributivity of $\bullet$ over $\vee$ whereas compatibility with respect to $\wedge$ is straightforward.

Now, an immersion of the quantale $\mathcal{Q}$ into R_Q is obtained by the morphism ϕ which maps the element $x \in Q$ into the relation R_x generated by x. In fact ϕ respects the operations of the quantale $\mathcal{Q}$ since $R_{t\bullet x} = R_x \circ R_t$ because if $yR_{t\bullet x}z$, i.e. $t \bullet x \bullet y \leq z$, then $x \bullet yR_tz$ and hence $yR_x \circ R_tz$, since $yR_xx \bullet y$ and if $yR_x \circ R_tz$ then there is an element u such that yR_xu and uR_tz, i.e. $x \bullet y \leq u$ and $t \bullet u \leq z$, hence $t \bullet x \bullet y \leq z$, i.e. $yR_{t\bullet x}z$; moreover $R_1 = I$, because xR_1y if and only if $x = 1 \bullet x \leq y$; finally $R_{\vee x_i} = \vee_{i\in I} R_{x_i}$, because $yR_{\vee x_i}z$ iff $(\vee_{i\in I} x_i) \bullet y = \vee_{i\in I}(x_i \bullet y) \leq z$ iff for each $i \in I$, $x_i \bullet y \leq z$ iff for each $i \in I$, $yR_{x_i}z$ iff $y \cap_{i\in I} R_{x_i} z$.

Finally ϕ is injective, and hence it is an isomorphism between $\mathcal{Q}$ and the sub-quantale of the generated relations $\{R_x : x \in Q\}$. In fact $x \leq t$ if and only if $R_t \subseteq R_x$ since if yR_tz, i.e. $t \bullet y \leq z$, then $x \bullet y \leq z$, i.e. yR_xz, because $x \leq t$ implies $x \bullet y \leq t \bullet y$ and so we have proved that $R_t \subseteq R_x$; on the other hand $1R_tt$ thus $R_t \subseteq R_x$ implies $1R_xt$ i.e. $x = x \bullet 1 \leq t$. So if $R_x = R_t$, i.e. $R_t \subseteq R_x$ and $R_x \subseteq R_t$, then $x \leq t$ and $t \leq x$ and hence $x = t$.

The isomorphism theorem

In the previous paragraph we have proved that the set of ordered relations $\{R_x : x \in Q\}$, as a sub-quantale of the quantale of the ordered relations on $\mathcal{Q}$, is isomorphic to $\mathcal{Q}$. Here we give the conditions that characterize such a set of relations. In order to obtain this result let us introduce the definition of the operation of right implication between elements of a quantale.

Definition: (right implication). Let $\mathcal{Q}$ be a quantale and $x, y \in Q$; then the *right implication* $y \leftarrow x$, to be read y *is implied by* x, is defined by putting

$$y \leftarrow x \equiv \vee\{w \in Q : w \bullet x \leq y\}$$

The right implication is already considered in [Rosenthal 90] and its definition is standard in a non-commutative linear logic framework [Abrusci 91]. The following lemma justifies its name.

Lemma. *Let $\mathcal{Q}$ be a quantale and $x, y, w, t \in Q$; then the following properties hold*

1) $(y \leftarrow x) \bullet x \leq y$;
2) $w \bullet x \leq y$ if and only if $w \leq y \leftarrow x$;
3) if $x \leq y$ and $w \leq t$ then $(w \leftarrow y) \bullet x \leq t$;
4) if $x \leq y$ and $w \leq t$ then $w \leftarrow y \leq t \leftarrow x$.

Proof.
1) $(y \leftarrow x) \bullet x = \vee\{w : w \bullet x \leq y\} \bullet x = \vee\{w \bullet x : w \bullet x \leq y\} \leq y$;
2) If $w \bullet x \leq y$ then obviously $w \leq \vee\{w \in Q : w \bullet x \leq y\} = y \leftarrow x$; on the other hand, if $w \leq y \leftarrow x$ then $w \bullet x \leq (y \leftarrow x) \bullet x \leq y$, by point 1;
3) if $x \leq y$ then $(w \leftarrow y) \bullet x \leq (w \leftarrow y) \bullet y \leq w \leq t$, using point 1;
4) straightforward consequence of point 3. and 2.

Now we can define the relations on $\mathcal{Q}$ we are interested in in order to state the isomorphism theorem.

Definition: (right ordered relation). Let $\mathcal{Q}$ be a quantale; then R is a right ordered relation on $\mathcal{Q}$ if
- R is an ordered relation on $\mathcal{Q}$, i.e. R is compatible with $\leq, \vee$ and $\wedge$,
- for any $x, y, u \in Q$, if xRy then $x \bullet uRy \bullet u$, i.e. R is right compatible with $\bullet$,
- for any $x, y, u \in Q$, if xRy then $x \leftarrow uRy \leftarrow u$, i.e. R is right compatible with $\leftarrow$.

Theorem: (isomorphism theorem). *Let $\mathcal{Q}$ be a quantale; then $\mathcal{Q}$ is isomorphic to the quantale of the right ordered relations on $\mathcal{Q}$.*

In order to prove the isomorphism theorem, because of the results in the previous section, it is sufficient to show that all the generated relations are also right ordered relations and that any right ordered relation is a generated relation.

The left-to-right implication is straightforward. In fact we have already proved that any generated relation is an ordered relation; moreover if xR_wy, i.e. $w \bullet x \leq y$, then $w \bullet x \bullet u \leq y \bullet u$, i.e. $x \bullet uR_wy \bullet u$, by stability and hence R_w is right compatible with $\bullet$; finally if xR_wy, i.e. $w \bullet x \leq y$, then $w \bullet x \leftarrow u \leq y \leftarrow u$ and so $w \bullet (x \leftarrow u) \leq y \leftarrow u$, i.e. $x \leftarrow uR_wy \leftarrow u$, since $w \bullet (x \leftarrow u) \leq w \bullet x \leftarrow u$, because $w \bullet (x \leftarrow u) \bullet u \leq w \bullet x$.

In order to prove the other implication we need a preliminary lemma.

Lemma. *Let R be a right ordered relation; then $R_w \subseteq R$ if and only if $1Rw$.*

Proof. One implication is straightforward because $R_w \subseteq R$ implies $1Rw$ since $1R_ww$; to prove the other suppose xR_wy, i.e. $w \bullet x \leq y$, then xRy since $1Rw$ implies $x = 1 \bullet xRw \bullet x$.

The next lemma proves that any right ordered relation is a generated relation. In fact it shows that any right ordered relation R is determined by its *1-image*, i.e. by the set $\{y : 1Ry\}$.

Lemma. *Let R be a right ordered relation; then $R \subseteq \cup_{R_w \subseteq R} R_w \subseteq \cup_{1Rw} R_w \subseteq R_{\wedge\{w:1Rw\}} \subseteq R$.*

Proof. We will prove the inclusions following the order from left to right. To prove the first let us suppose xRy then $x \leftarrow xRy \leftarrow x$, hence $1Ry \leftarrow x$ because $1 \leq x \leftarrow x$, and so $R_{y \leftarrow x} \subseteq R$ but $xR_{y\leftarrow x}y$ because $(y \leftarrow x) \bullet x \leq y$, and so $R \subseteq \cup_{R_w \subseteq R} R_w$. The second inclusion is straightforward since $R_w \subseteq R$ implies $1Rw$. The third inclusion holds since, for each w such that $1Rw$, $\wedge\{w : 1Rw\} \leq w$ hence $R_w \subseteq R_{\wedge\{w:1Rw\}}$ and so $\cup_{1Rw} R_w \subseteq R_{\wedge\{w:1Rw\}}$; finally in order to prove the last inclusion observe that $1R \wedge \{w : 1Rw\}$ and hence $x = 1 \bullet xR \wedge \{w : 1Rw\} \bullet x$ and so if $xR_{\wedge\{w:1Rw\}}y$, i.e. $\wedge\{w : 1Rw\} \bullet x \leq y$, then xRy.

Hence we have proved that a right ordered relation R is a relation generated by $\wedge\{w : 1Rw\}$ and this concludes the proof of the isomorphism theorem.

Representing quantales by means of functions

In this section we use the results so far obtained to show some new forms of the representation and the isomorphism theorems using suitable functions instead of relations.

Let us begin by considering the functions of the quantale $\mathcal{Q}$ into itself which respect arbitrary infima ($\wedge$-*ordered functions* in the following), i.e. the functions $f : Q \to Q$ such that $\wedge_{i\in I} f(y_i) = f(\wedge_{i\in I} y_i)$ or equivalently the monotonic functions, i.e. $x \le y$ implies $f(x) \le f(y)$, such that $\wedge_{i\in I} f(y_i) \le f(\wedge_{i\in I} y_i)$.

Now we state the link between the ordered relations and the $\wedge$-ordered functions. Let $f : Q \to Q$ be an $\wedge$-ordered function then the relation R_f defined by putting

$$xR_f y \equiv x \le f(y)$$

will be called the *relation associated with* f.

Lemma. *Let $f : Q \to Q$ be an $\wedge$-ordered function and R_f its associated relation; then R_f is an ordered relation.*

Proof. While compatibility with respect to $\vee$ is straightforward, those with respect to $\le$ and $\wedge$ are immediate consequences respectively of monotonicity of f and of the fact that it respects $\wedge$.

Note that to prove this lemma it is merely necessary that $\wedge_{i\in I} f(y_i) \le f(\wedge_{i\in I} y_i)$ but the other inequality is an immediate consequence of monotonicity.

Also the other implication can be proved. Let us say that the function $f_R : Q \to Q$ *is associated with* the relation R if

$$f_R(y) = \vee\{z : zRy\}.$$

Lemma. *Let R be an ordered relation on the quantale $\mathcal{Q}$; then f_R is a monotonic function which respects $\wedge$.*

Proof. Monotonicity of f_R is immediate since $x \le y$ implies that if zRx then zRy and hence $\vee\{z : zRx\} \le \vee\{z : zRy\}$. In order to show that $\wedge_{i\in I} f_R(y_i) \le f_R(\wedge_{i\in I} y_i)$ observe that $\vee\{z : zRy_i\}Ry_i$ holds, hence $\wedge_{i\in I}(\vee\{z : zRy_i\})Ry_i$, since R is compatible with $\le$, so $\wedge_{i\in I}(\vee\{z : zRy_i\})R \wedge_{i\in I} y_i$, since R is compatible with $\wedge$, and thus $\wedge_{i\in I}(\vee\{z : zRy_i\}) \le \vee\{z : zR \wedge_{i\in I} y_i\}$.

Moreover the two constructions are inverse to one another, i.e. the following lemma holds.

Lemma. *Let R be an ordered relation on the quantale $\mathcal{Q}$ and $f : Q \to Q$ be an $\wedge$-ordered function; then*

1) $R_{f_R} = R$

2) $f_{R_f} = f$

Proof. Both points are straightforward consequences of the above definitions:

1) xRy iff $x \le \vee\{z : zRy\}$ iff $x \le f_R(y)$ iff $xR_{f_R}y$;

2) $f_{R_f}(y) = \vee\{z : zR_f y\} = \vee\{z : z \le f(y)\} = f(y)$

By the previous lemmas we have proved that there is a bijection between the set of the ordered relation R_Q on the quantale $\mathcal{Q}$ and the set F_Q of the

$\wedge$-ordered functions of $\mathcal{Q}$ into $\mathcal{Q}$. Then we can define a quantale structure on F_Q by transporting the operations from the quantale R_Q as follows:

$$f_R \circ_F f_S \equiv f_{R \circ S}$$
$$I_F \equiv f_I$$
$$\vee_F f_{R_i} \equiv f_{\vee R_i}.$$

The following lemma makes these definitions more explicit.

Lemma. *Let $R, S, R_1, R_2, \ldots$ be ordered relations on the quantale $\mathcal{Q}$; then*
1) $f_{R \circ S} = f_R \circ f_S$, i.e. $f_{R \circ S}(y) = f_R(f_S(y))$
2) f_I = identity
3) $f_{\vee R_i} = \wedge f_{R_i}$, i.e. $f_{\vee R_i}(y) = \wedge_{i \in I}\{f_{R_i}(y)\}$.

Proof. To prove the first point it is sufficient to observe that $tR \circ Sy$ if and only if $tR \vee \{z : zSy\}$ since this implies that $f_{R \circ S}(y) = \vee\{t : tR \circ Sy\} = \vee\{t : tR \vee \{z : zSy\}\} = f_R \circ f_S(y)$. The second point is straightforward since $f_I(y) = \vee\{t : tIy\} = \vee\{t : t \leq y\} = y$. In order to prove the last point we must show that $\vee\{z : \text{ for each } i, zR_iy\} = \wedge_{i \in I}\{\vee\{z : zR_iy\}\}$. First we will show that $\wedge_{i \in I}\{\vee\{z : zR_iy\}\} \leq \vee\{z : \text{ for each } i, zR_iy\}$; in fact, for each i, $\vee\{z : zR_iy\}R_iy$ and hence $\wedge_{i \in I}\{\vee\{z : zR_iy\}\}R_iy$, since R_i is compatible with $\leq$. Now, in order to prove the other inequality, observe that $\wedge_{i \in I}\{\vee\{z : zR_iy\}\} = \vee\{w : \text{ for each } i, w \leq \vee\{z : zR_iy\}\} \leq \vee\{z : \text{ for each } i, zR_iy\}$ since $\vee\{z : zR_iy\}R_iy$.

In order to obtain a sub-quantale of F_Q which is isomorphic to $\mathcal{Q}$ we have to characterize the $\wedge$-ordered functions which correspond to the right ordered relations or, equivalently, to the generated relations. To this purpose let us consider the $\wedge$-ordered functions f such that also the following conditions hold:
- $f(y) \bullet w \leq f(y \bullet w)$
- $f(y) \leftarrow w \leq f(y \leftarrow w)$

which will be called *implicative functions*.

Lemma. *Let $f : Q \to Q$ be an implicative function; then R_f is a right ordered relation.*

Proof. We already know that R_f is an ordered relation. Moreover the two conditions we add on $\wedge$-ordered functions to have an implicative function are suitably chosen to prove the validity of the two conditions we need to satisfy in order to show that R_f is also a right ordered relation.

Also the other implication can be proved, i.e. the function associated to a right ordered relation is an implicative function. Instead of giving a direct proof we will give an indirect one which will also provide an explicit characterization of the implicative functions and explain their name. To this purpose we will introduce the following definition.

Definition: (left implication). Let $\mathcal{Q}$ be a quantale and $x, y \in Q$; then the *left implication* $x \to y$, to be read x *implies* y, is defined by putting

$$x \to y \equiv \vee\{w \in Q : x \bullet w \leq y\}.$$

This definition is very similar to the one of the right implication but one must consider that in the case of a non-commutative quantale the left and right implications are different operations. The following lemma on the left implication, whose proof is completely similar to the one for the case of the right implication, justifies its name.

Lemma. *Let $\mathcal{Q}$ be a quantale and $x, y, w, t \in Q$; then the following properties hold:*
1) $x \bullet (x \to y) \leq y$;
2) $x \bullet w \leq y$ if and only if $w \leq x \to y$;
3) if $x \leq y$ and $w \leq t$ then $x \bullet (y \to w) \leq t$;
4) if $x \leq y$ and $w \leq t$ then $y \to w \leq x \to t$.

Now, suppose $x \in Q$ and let us consider the function

$$f_x(y) = x \to y$$

which will be called the *implicative function generated by* x.

Lemma. *Let $x \in Q$; then the implicative function generated by x is an implicative function.*

Proof. Monotonicity is an immediate consequence of the point 4 of the previous lemma and arbitrary infima are respected since one can prove that $x \bullet \wedge_{i \in I}\{x \to y_i\} \leq \wedge_{i \in I}\{x \bullet (x \to y_i)\}$. Moreover it is not difficult to prove that $(x \to y) \bullet w \leq x \to (y \bullet w)$ and $(x \to y) \leftarrow w \leq x \to (y \leftarrow w)$, because of associativity of $\bullet$, and hence $f_x(y) \bullet w \leq f_x(y \bullet w)$ and $f_x(y) \leftarrow w \leq f_x(y \leftarrow w)$.

Much more interesting is to observe that any implicative function is an implicative generated function. In fact we have already proved in the previous section that any right ordered relation is determined by its 1-image. We can state this result also in the following way: let R be a right ordered relation then xRy iff $xR_{\wedge\{w:1Rw\}}y$ iff $\wedge\{w : 1Rw\} \bullet x \leq y$ iff $x \leq \wedge\{w : 1Rw\} \to y$.

Applying this fact to the case of a function associated to a right ordered relation R we obtain that $f_R(y) = \vee\{z : zRy\} = \vee\{z : z \leq \wedge\{w : 1Rw\} \to y\} = \wedge\{w : 1Rw\} \to y$ and hence we have proved the following lemma.

Lemma. *Let R be a right ordered relation on the quantale $\mathcal{Q}$; then the function f_R associated to R is the implicative function generated by $\wedge\{w : 1Rw\}$.*

Now, suppose that f is an implicative function then we have already proved that the relation R_f associated to it is a right ordered relation. Hence we have that $f(y) = f_{R_f}(y) = \wedge\{w : 1R_f w\} \to y = \wedge\{w : 1 \leq f(w)\} \to y$ and so the following theorem holds.

Theorem. *Let $f : Q \to Q$ be an implicative function; then f is the generated implicative function $f_{\wedge\{w:1\leq f(w)\}}$.*

So we have closed the circle since we have proved that there is a bijective correspondence between right ordered relations and implicative functions and hence that any quantale $\mathcal{Q}$ is isomorphic to the quantale of its implicative functions.

In a way completely similar to the case of the $\wedge$-ordered functions, we can introduce the $\vee$-*ordered functions*, i.e. the functions $f : Q \to Q$ such that $f(\vee_{i\in I} x_i) = \vee_{i\in I} f(x_i)$, or equivalently the monotonic functions such that $f(\vee_{i\in I} x_i) \leq \vee_{i\in I} f(x_i)$. As in the case of the $\wedge$-ordered functions, it is easy to associate a relation R^f with a $\vee$-ordered function f by the position

$$xR^f y \equiv f(x) \leq y$$

and to prove that R^f is an ordered relation. Moreover the function

$$f^R(x) = \wedge\{z : xRz\}$$

associated with the relation R is a $\vee$-ordered function such that the two constructions are inverse to one another, i.e. $R = R^{f^R}$ and $f = f^{R^f}$. Hence also the $\vee$-ordered functions define a quantale F^Q by transporting the operations from the quantale R_Q and it is easy to prove that the following holds:

$$f^R \circ^F f^S(x) = f^R(f^S(x)) \quad I^F(x) = x \quad \vee^F f^{R_i}(x) = \vee_{i\in I}\{f^{R_i}(x)\}$$

Also in the case of F^Q, we can characterize the $\vee$-ordered functions which correspond to the right ordered relations and hence have an isomorphism theorem with the quantale $\mathcal{Q}$. To this purpose let us consider the $\vee$-ordered functions f which satisfy also the conditions:

- $f(x \bullet w) \leq f(x) \bullet w$
- $f(x \leftarrow w) \leq f(y) \leftarrow w$

and which will be called *traslations*. It is immediate to prove that the relation associated with a traslation is a right ordered relation. As for the implicative functions, the most interesting way to prove the converse is to give an explicit definition of the traslations. Supposing $x \in Q$, let us consider the function

$$f^x(y) = x \bullet y$$

which will be called the *traslation generated by* x. It is immediate to verify that any generated traslation is a traslation. Moreover every traslation is generated. In fact, supposing R as being a right ordered relation, xRy iff $1Ry \leftarrow x$ iff $\wedge\{z : 1Rz\} \leq y \leftarrow x$ iff $f^R(1) \bullet x \leq y$ and hence $f^R(x) = \wedge\{z : xRz\} = \wedge\{z : f^R(1) \bullet x \leq z\} = f^R(1) \bullet x$, i.e. if R is a right ordered relation then f^R is the traslation generated by $f^R(1)$. Finally let f be a traslation then R^f is a right ordered relation and hence $f(x) = f^{R^f}(x) = f^{R^f}(1) \bullet x = f(1) \bullet x$.

Applications

It is easy to apply the results of the previous sections to particular kinds of quantales. For instance, let us analyze the case of *locales*, i.e. quantales where the operations $\bullet$ and $\wedge$ coincide, which turn out to be the complete Heyting algebras. Then we have that any complete Heyting algebra $\mathcal{H}$ is (isomorphic to) a Heyting algebra of generated relations over $\mathcal{H}$ where the operations are defined by $R_x \wedge R_y \equiv \cap \{S : R_x \cup R_y \subseteq S\}$, which is equal to $R_{x \wedge y}$ but also to $R_x \circ R_y$ in this particular case; $R_x \vee R_y \equiv R_x \cap R_y$, which is equal to $R_{x \vee y}$; $R_x \to R_y \equiv \cap\{S : R_y \subseteq S \circ R_x\}$, which is equal to $R_{x \to y}$; $0 \equiv H \times H$, which is equal to $\{(y, z) : 0 \wedge y \leq z\} = R_0$ and $1 = \leq$, which is equal to $\{(y, z) : 1 \wedge y \leq z\} = R_1$. In a completely similar way, using the results of section 4 instead of those of section 3, we have that any complete Heyting algebra $\mathcal{H}$ is isomorphic to the Heyting algebra F_H of the implicative functions $f_x(y) = x \to y$ of $\mathcal{H}$ into itself and also to the Heyting algebra F^H of the traslations $f^x(y) = x \wedge y$ of $\mathcal{H}$ into itself with the obvious definitions of the operations.

We have a much more interesting application of the representation theorems in the case of Linear Logic [Girard 87]. In fact they are the mathematical background one needs in order to obtain *natural* and complete semantics of the (non-commutative) Linear Logic such that any formula is interpreted in a generated relation and the (non-commutative) connective $\otimes$ (*times*) is interpreted in the composition of relations. The underlying idea is that a formula specifies an action on a domain while the generated relation where it is interpreted specifies the result of applying such an action on a given element of the domain. The interpretation is mainly similar to the previous one for the complete Heyting algebras but some problems arise since the algebra *a la* Lindembaum of linear logic is not a quantale, because of the lack of arbitrary suprema. For this reason the whole completeness proof, which turns out to be a representation theorem again, is not as simple as the proofs we have presented here for the quantales and hence the reader is invited to look it in [Valentini 92].

References

[Abrusci 91] V.M. Abrusci, *Phase semantics and sequent calculus for pure non commutative classical propositional linear logic*, Journal of Symbolic Logic **56-4** (1991), 1403-1452.

[Girard 87] J.I. Girard, *Linear Logic*, Theoretical Computer Science **50** (1987), 1-102.

[Rosenthal 90] K.I. Rosenthal, *Quantales and their applications*, Longman Scientific, Technical, New York, 1990.

[Valentini 92] S.Valentini, *A simple interpretation of the non-commutative linear logic: the semantics of the ordered relations*, to appear.

[Yetter 90] D.N. Yetter, *Quantales and (noncommutative) linear logic*, Journal of Symbolic Logic **55** (1990), 41-64.

Dip. di Matematica - Università di Padova, via G. Belzoni n.7, I-35131 Padova, Italy

E-mail address: valentini@pdmat1.unipd.it