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On the formal points of the formal topology of the binary tree

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Abstract Formal topology is today an established topic in the development of constructive mathematics and constructive proofs for many classical results of general topology have been obtained by using this approach. Here we analyze one of the main concepts in formal topology, namely, the notion of formal point. We will contrast two classically equivalent definitions of formal points and we will see that from a constructive point of view they are completely different. Indeed, according to the first definition the formal points of the formal topology of the real numbers can be indexed by a set whereas this is not possible according to the second one.

1 Basic definitions

In this section the basic definitions of formal topology will be quickly recalled. Anyhow, the reader interested in having more details on formal topology is invited to look at [CSSV].

1.1 Concrete topological spaces

The classical definition of topological space reads as follows: $(X, \Omega(X))$ is a topological space if X is a set and $\Omega(X)$ is a subset of $\mathcal{P}(X)$ which satisfies:

(Ω_1) $\emptyset, X \in \Omega(X)$;
(Ω_2) $\Omega(X)$ is closed under finite intersection;
(Ω_3) $\Omega(X)$ is closed under arbitrary union.

Usually, elements of X are called *points* and elements of $\Omega(X)$ are called *opens*.

The quantification implicitly used in (Ω_3) is of the third order, since it says

$$(\forall F \in \mathcal{P}(\mathcal{P}(X)))\ (F \subseteq \Omega(X) \rightarrow \bigcup F \in \Omega(X))$$

We can "go down" one step by thinking of $\Omega(X)$ as a family of subsets indexed by a set S through a map $\mathsf{ext} : S \rightarrow \mathcal{P}(X)$, that is, a binary relation between S and X. In fact, we can now quantify on S rather than on $\Omega(X)$. But we still have to say

$$(\forall U \in \mathcal{P}(S))(\exists c \in S)\ (\cup_{a \varepsilon U} \mathsf{ext}(a) = \mathsf{ext}(c))$$

which is still impredicative[1].

We can "go down" another step by defining opens to be of the form $\mathsf{Ext}(U) \equiv \cup_{a \varepsilon U} \mathsf{ext}(a)$ for an arbitrary subset U of S. In this way $\emptyset$ is open, because $\mathsf{Ext}(\emptyset) = \emptyset$, and closure under union is automatic, because obviously $\cup_{i \in I} \mathsf{Ext}(U_i) = \mathsf{Ext}(\cup_{i \in I} U_i)$. So, all we have to do is to require that $\mathsf{Ext}(S)$ be the whole X, that is,

$$(B_1) \quad X = \mathsf{Ext}(S)$$

and that closure under finite intersections holds, that is,

$$\begin{array}{ll}(B_2) & (\forall a, b \in S)(\forall x \in X)\ ((x \ \varepsilon\ \mathsf{ext}(a) \cap \mathsf{ext}(b)) \rightarrow \\ & \quad (\exists c \in S)\ (x \ \varepsilon\ \mathsf{ext}(c)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)))\end{array}$$

It is not difficult to realize that this amounts to the standard definition saying that $\{\mathsf{ext}(a) \subseteq X |\ a \in S\}$ is a base (see for instance [Eng77]). We can make (B_2) a bit shorter by introducing an abbreviation, that is

$$a \downarrow b \equiv \{c : S |\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)\}$$

so that it becomes $(\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{Ext}(a \downarrow b)$.

Note that $c\ \varepsilon\ a \downarrow b$ implies that $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$, and hence $\mathsf{Ext}(a \downarrow b) \equiv \cup_{c\ \varepsilon\ a \downarrow b} \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$. Then the definition of concrete topological space can be rewritten as follows:

Definition 1 *A* concrete topological space *is a triple* $\mathcal{X} \equiv (X, S, \mathsf{ext})$ *where* X *and* S *are sets and* ext *is a binary relation from* S *to* X *satisfying:*

$$(B_1)\ X = \mathsf{Ext}(S)$$

$$(B_2)\ (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{Ext}(a \downarrow b)$$

[1] All the set-theoretical notions that we use are conform to the subset theory for Martin-Löf's type theory as presented in [SV98]. In particular, we will use the symbol $\in$ for the membership relation between an element and a set or a collection and ε for the membership relation between an element and a subset, which is never a set but a propositional function.

1.2 Formal topologies

The notion of formal topology arises by describing as well as possible the structure induced by a concrete topological space on the *formal side*, that is the side of the set S of the names, and then by taking the result as an axiomatic definition. The reason for such a move is that the definition of concrete topological space is too restrictive, given that the most interesting cases of topological space do not have, from a predicative point of view[2], a *set* of points to start with, and in the definition of concrete topological space we have to require that X and S are sets in order to be able to give a constructive meaning to the quantifications in (B_1) and (B_2).

The problem how to identify the open sets on the formal side is easily solved. Since the elements in S are names for basic opens of the topology on X, then we can obtain their *extension*, that is the concrete basic open, by using the operator ext. Now, any open set is the union of basic opens and hence it can be specified on the formal side by using the subset of all the (names of the) basic opens which are used to form it. It is easy to check that, provided the conditions (B_1) and (B_2) are satisfied, in this way, we really obtain a topology on the set X.

From a topological point of view an open subset of X is characterized by the property of being the union of all the basic opens that it contains or, equivalently, to coincide with its interior $\mathsf{Int}(A)$, where, for any $A \subseteq X$,

$$\mathsf{Int}(A) \equiv \{x \in X \mid (\exists a \in S)\ x \,\varepsilon\, \mathsf{ext}(a) \;\&\; \mathsf{ext}(a) \subseteq A\}$$

Of course, for any $A \subseteq X$, $\mathsf{Int}(A) \subseteq A$ and thus a subset A is open if and only if $A \subseteq \mathsf{Int}(A)$.

Theorem 1 *Let $A \subseteq X$. Then A is an open subset if and only if there exists a subset U of S such that $A = \mathsf{Ext}(U)$.*

Proof. Let A be an open subset of X and consider the subset $U \equiv \{a \in S \mid \mathsf{ext}(a) \subseteq A\}$. Then $A = \mathsf{Ext}(U)$. On the other hand, let U be any subset of S and suppose that $x \,\varepsilon\, \mathsf{Ext}(U)$; then there exists $a \in S$ such that $a \,\varepsilon\, U$ and $x \,\varepsilon\, \mathsf{ext}(a)$; but the former yields $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and hence $x \,\varepsilon\, \mathsf{Int}(\mathsf{Ext}(U))$, that is, $\mathsf{Ext}(U)$ is open. □

The proof of the previous theorem shows how to find, for any given open subset A of X, a suitable subset U of S such that A and $\mathsf{Ext}(U)$ are extensionally equal; we chose the *biggest* among the possible subsets, that is, the one which contains *all* of the suitable basic opens. It is clear that in general this is not the only choice and that it is well possible that two different subsets of S have the same extension. Thus we don't have a one-to-one

[2] Here we commit ourselves to Martin-Löf's constructive set theory [ML84]; hence we distinguish between sets, which can be inductively generated, and collections.

correspondence between concrete opens and subsets of S and we need to introduce an equivalence relation on the formal side which identifies the subsets U and V when $\mathsf{Ext}(U) = \mathsf{Ext}(V)$. Of course, within a constructive set theory, we cannot introduce such a relation among subsets since the collection of the subsets of a set is not a set, but we can simplify a bit the problem if we realize that the following theorem holds.

Theorem 2 *Let U and V be subsets of S. Then* $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ *if and only if* $(\forall a \in S)\ \mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \leftrightarrow \mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$.

Thus, we need just to introduce, on the formal side, a new relation $\lhd$ between elements and subsets of S such that

$$a \lhd U \equiv \mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$$

In fact, after the previous theorem, we can define an equivalence relation between the subsets of S by setting

$$U =_{\lhd} V \equiv (\forall a \in S)\ a \lhd U \leftrightarrow a \lhd V$$

and it is immediately possible to prove the following theorem.

Theorem 3 *Let U and V be two subsets of S. Then $U =_{\lhd} V$ if and only if* $\mathsf{Ext}(U) = \mathsf{Ext}(V)$.

Now, in order to obtain a one-to-one correspondence between formal and concrete open subsets, we could simply state that a formal open is an equivalence class of the relation $=_{\lhd}$. But, to avoid dealing with collections of collections of subsets, we simply choose the "fullest" among the representative of an equivalence class by setting

$$\lhd(U) \equiv \{a \in S |\ a \lhd U\}$$

and say that a *formal open* is any subset $\lhd(U)$ for some subset U. Now, $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ if and only if $\lhd(U) = \lhd(V)$. Moreover it is possible to prove that $\lhd(U)$ is a good representative of the equivalence class of the subset U because $\lhd(U) =_{\lhd} U$, i.e. $\lhd(\lhd(U)) = \lhd(U)$. In fact, it is easy to check that the following two conditions on $\lhd$ are valid and hence we can assume them like axiomatic conditions on the formal side. They are, first,

$$\text{(reflexivity)} \qquad \frac{a\ \varepsilon\ U}{a \lhd U}$$

which holds since if $a\ \varepsilon\ U$, then $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$, and, secondly,

$$\text{(transitivity)} \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

where $U \lhd V$ is a shorthand for a derivation of $u \lhd V$ under the assumption that $u \,\varepsilon\, U$. The validity of *transitivity* is straightforward because the first assumption means that $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and the second yields that $\mathsf{Ext}(U) \subseteq \mathsf{Ext}(V)$.

We can re-write *reflexivity* and *transitivity* by using a set-theoretical notation

$$\text{(reflexivity)} \quad U \subseteq \lhd(U) \qquad \text{(transitivity)} \quad \frac{U \subseteq \lhd(V)}{\lhd(U) \subseteq \lhd(V)}$$

and hence we obtain both $\lhd(U) \subseteq \lhd(\lhd(U))$ by using *reflexivity* and also $\lhd(\lhd(U)) \subseteq \lhd(U)$ as a consequence of *transitivity*.

Thus we found a relation, that is, $\lhd$, and some conditions on it, that is, *reflexivity* and *transitivity*, which allow to deal on the formal sides with concrete open subsets. But these conditions are not sufficient to describe completely the concrete situation; for instance there are no conditions which formally describe the conditions (B_1) and (B_2).

To formulate (B_2) completely on the formal side let us first write $U \downarrow V$ to mean the subset $\{c \in S |\ (\exists u \,\varepsilon\, U)\ c \lhd u \,\&(\exists v \,\varepsilon\, V)\ c \lhd v\}$. Then, supposing $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$, we immediately obtain $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \cap \mathsf{Ext}(V)$ and hence $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U \downarrow V)$ since it is easy to prove that $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \mathsf{Ext}(U \downarrow V)$. Its formal counterpart is

$$(\downarrow\text{-right}) \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

We thus arrived at the main definition[3].

Definition 2 *A* formal topology *is a couple $\mathcal{A} \equiv (S, \lhd)$ where S is a set, $\lhd$ is an infinitary relation, called* cover *relation, between elements and subsets of S satisfying the following conditions:*

$$\text{(reflexivity)} \qquad \frac{a \,\varepsilon\, U}{a \lhd U}$$

$$\text{(transitivity)} \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$(\downarrow\text{-right}) \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

[3] The reader which knows formal topology will notice that we avoided to deal with the positivity predicate; we chose this approach for the sake of simplicity.

1.3 Inductive generation of formal topologies

One of the main tools in formal topology is inductive generation of the cover relation since this allows to develop proofs by induction. The problem of inductively generated formal topologies has been completely solved and the reader can look in [CSSV] and [Val99] for a detailed discussion of the problems that an inductive generation of formal topologies requires to solve and for their solutions. We will recall here, without any proofs, only the results that we will use in the next sections.

The conditions appearing in the definition of formal topology, though written in the shape of rules, must be understood as requirements of validity: if the premises hold, also the conclusion must hold. As they stand, they are by no means acceptable rules to generate inductively a cover relation.

An inductive definition of a cover will start from some axioms, which at the moment we assume to be given by means of any relation $R(a, U)$ for $a \in S$ and $U \subseteq S$. We thus want to generate the least cover $\lhd_R$ which satisfies the following condition:

$$\text{(axioms)} \qquad \frac{R(a, U)}{a \lhd_R U}$$

From an impredicative point of view, $\lhd_R$ is easily obtained "from above" simply as the intersection of the collection $\mathcal{C}_R$ of all the reflexive, transitive infinitary relations containing R. In fact, it is clear that the total relation is in $\mathcal{C}_R$ and that the intersection preserves all such conditions.

Predicatively the method of defining $\lhd_R$ as the intersection of $\mathcal{C}_R$ is not acceptable, since there is no way of producing $\mathcal{C}_R$ above as a set-indexed family and hence to define its intersection.

Therefore, we must obtain $\lhd_R$ "from below" by means of some introductory rules. The first naive idea is that of using axioms, *reflexivity* and *transitivity* for this purpose. But then a problem emerges: in the premises of *transitivity* there is a subset which does not appear in the conclusion. This means that the tree of possible premises to conclude that $a \lhd_R U$ has an unbounded branching: each subset V satisfying $a \lhd_R V$ and $V \lhd_R U$ would be enough to obtain $a \lhd_R U$, and there is no way to survey them all. Also, a dangerous vicious circle seems to be present: the subset V, whose existence would be enough to obtain $a \lhd_R U$, could be defined by means of the relation $\lhd_R$ itself which we are trying to construct.

This is the reason why we have to put some constraints on the infinitary relation $R(a, U)$. Thus, we are going to generate a cover relation only when we have an *axiom set*, that is a set-indexed family $I(a)$ set $[a : S]$ and an indexed family $C(a, i) \subseteq S$ $[a : S, i : I(a)]$ of subsets of S, whose intended meaning is to state that, for all $i \in I(a)$, $a \lhd C(a, i)$. Then, an infinitary relation $R(a, U)$ is safe if

$$R(a, U) \equiv (\exists i \in I(a))\ C(a, i) \subseteq U$$

In fact, in this case we can generate the cover relation which satisfies *reflexivity* and *transitivity* by using the following rules

$$\text{(reflexivity)}\ \frac{a\ \varepsilon\ U}{a \lhd U} \quad \text{and} \quad (\lhd\text{-infinity})\ \frac{i \in I(a) \qquad C(a, i) \lhd U}{a \lhd U}$$

In this way any reference to the subset V disappeared and the implicit use of an existential quantification on the collection $\mathcal{P}(S)$ is transformed into an existential quantification on the elements of the set $I(a)$.

We can now extend the previous rules into new ones which allow to generate a cover relation which satisfies also $\downarrow$*-right*. To this aim, we must add, to those of a formal topology, an extra primitive expressing what in the concrete case is $\mathsf{ext}(a) \subseteq \mathsf{ext}(b)$. We can obtain this by adding directly a pre-order relation $a \leq b$ among names. Thus we obtain the following definition.

Definition 3 *A* $\leq$-formal topology *is a triple* $(S, \leq, \lhd)$ *where* S *is a set,* $\leq$ *is a pre-order relation over* S*, that is* $\leq$ *is reflexive and transitive, and* $\lhd$ *is a relation between elements and subsets of* S *which satisfies* reflexivity, transitivity *and the two following conditions*

$$(\leq\text{-left})\quad \frac{a \leq b \qquad b \lhd U}{a \lhd U} \qquad (\leq\text{-right})\quad \frac{a \lhd U \qquad a \lhd V}{a \lhd \downarrow U \cap \downarrow V}$$

where $\downarrow U \equiv \{c : S |\ (\exists u\ \varepsilon\ U)\ c \leq u\}$.

It is straightforward to verify that the new conditions are valid in any concrete topological space under the intended interpretation. And only a little more work is required to prove that any $\leq$-formal topology is a formal topology. The converse is trivial: given any formal topology $(S, \lhd)$, we can define $a \leq b$ as $a \lhd \{b\}$ and we obtain a $\leq$-formal topology with the original one as a cover relation. Thus all we need is to be able to inductively generate a $\leq$-formal topology.

We will say that an axiom set I and C satisfies the *axiom condition* if, whenever $c \leq a$ and $a \lhd C(a, i)$ for some $i \in I(a)$, then there exists $j \in I(c)$ such that $(\forall x\ \varepsilon\ C(c, j))(\exists y\ \varepsilon\ C(a, i))\ x \leq c\ \&\ x \leq y$. The axiom condition states that, whenever $\mathsf{ext}(c) \subseteq \mathsf{ext}(a)$ and, for some axiom index $i \in I$, $\mathsf{ext}(a) \subseteq \mathsf{Ext}(C(a, i))$, then, by distributivity, a set, covering $\mathsf{ext}(c)$, can be built by collecting all the open subsets of $\mathsf{ext}(c) \cap \mathsf{ext}(y)$ for $y \in C(a, i)$.

In [CSSV] it is proved that, given any axiom set, we can always build a new axiom set which satisfies the axiom condition, and that, if the axiom condition is satisfied, a cover relation can be inductively generated by using *reflexivity*, $\leq$*-left* and $\lhd$*-infinity*.

1.4 Formal points

When working in formal topology one is in general interested in the properties of a topological space $(X, \Omega(X))$ which make no reference to the points. Thus one dispenses with the collection X and it is possible to work by using only the set of the names for the basic opens. But this does not mean that points are out of reach. In fact, we can identify a point with the filter of all the basic opens which, in the concrete case, contain it.

Definition 4 *Let* $(S, \leq, \lhd)$ *be a* $\leq$*-formal topology. Then a* formal point *is any non-empty subset* α *of* S *which, for any* $a, c \in S$ *and any* $U \subseteq S$*, satisfies the following conditions:*

$$\text{(up-closure)} \quad \frac{a\ \varepsilon\ \alpha \qquad a \leq c}{c\ \varepsilon\ \alpha}$$

$$\text{(completeness)} \quad \frac{a\ \varepsilon\ \alpha \qquad a \lhd U}{(\exists u\ \varepsilon\ U)\ u\ \varepsilon\ \alpha}$$

$$\text{(directness)} \quad \frac{a\ \varepsilon\ \alpha \qquad c\ \varepsilon\ \alpha}{(\exists b \in S)\ b \leq a\ \&\ b \leq c\ \&\ b\ \varepsilon\ \alpha}$$

It is easy to show that in the case of a generated formal topology one can simplify the *completeness* condition by requiring that it holds only for the axioms, that is:

$$\text{(axiom completeness)} \qquad \frac{a\ \varepsilon\ \alpha \qquad i \in I(a)}{(\exists x\ \varepsilon\ C(a,i))\ x\ \varepsilon\ \alpha}$$

2 The formal topology of the binary tree

In this section we are going to deal with the formal topology of the binary tree. This is a very simple generated formal topology; indeed, it can be generated by using, for any basic open, one single axiom. Thus we will write $a \lhd F(a)$ to mean the only axiom which states that the element a is covered by the subset $F(a)$.

To define the formal topology of the binary tree we will use the set $\mathbf{2}^*$ of the finite lists of elements of the set $\mathbf{2} \equiv \{0, 1\}$. We obtain a $\sqsubseteq$-formal topology from the set $\mathbf{2}^*$ by using the order relation $\sqsubseteq$ such that, for any $\sigma, \tau \in \mathbf{2}^*$, $\sigma \sqsubseteq \tau$ holds if and only if τ is an initial segment of σ. The intended meaning of this order relation can be understood if one thinks of a list as a *partial* information on an infinite sequence (see also [Vic89]); hence a longer list is a more precise information on the sequence and there are less infinite sequences containing it. Thus $\sigma \sqsubseteq \tau$ means that there are less infinite sequences which contain σ than infinite sequences which contain τ.

With the same intuitive reading, the cover relation $\sigma \lhd U$ means that the infinite sequences which contain σ are all among the sequences that contain at least one of the lists in U. But, in order to define this relation without any reference to the *collection* of the infinite sequences, we have to find suitable axioms for it. Here, we require that the list σ is covered by all of its one-step successors, i.e. the only form of axiom is $\sigma \lhd \{\sigma 1, \sigma 0\}$. Then, the cover relation can be inductively generated according to the previous section since the axiom condition can easily be proved.

It is convenient to summarize the generation rules that we use. Supposing $\sigma, \tau \in \mathbf{2}^*$ and $U \subseteq \mathbf{2}^*$, they are

$$\frac{\sigma \;\varepsilon\; U}{\sigma \lhd U} \qquad \frac{\sigma \sqsubseteq \tau \quad \tau \lhd U}{\sigma \lhd U} \qquad \frac{\sigma 1 \lhd U \quad \sigma 0 \lhd U}{\sigma \lhd U}$$

Let us now specialize the definition of formal point to the case of the formal topology on the binary tree. Since this is a generated topology we can use the following definition: a formal point is a non-empty subset α of $\mathbf{2}^*$ such that, for any $\sigma, \tau \in \mathbf{2}^*$, the following conditions hold:

$$\frac{\sigma \;\varepsilon\; \alpha \quad \sigma \sqsubseteq \tau}{\tau \;\varepsilon\; \alpha} \qquad \frac{\sigma \;\varepsilon\; \alpha}{(\exists k \in \mathbf{2})\; \sigma k \;\varepsilon\; \alpha} \qquad \frac{\sigma \;\varepsilon\; \alpha \qquad \tau \;\varepsilon\; \alpha}{(\exists \eta \in \mathbf{2}^*)\; \eta \sqsubseteq \sigma \;\&\; \eta \sqsubseteq \tau \;\&\; \eta \;\varepsilon\; \alpha}$$

In fact, the second condition used here is an immediate consequence of *axiom completeness* which requires that, supposing σ is an element of a point α, an element in $\{\sigma 1, \sigma 0\}$ is in α.

Note that the empty list nil is an element of any formal point. Moreover, formal points are subsets whose elements are lists which form an infinite path in the tree. To see this fact, let us begin by stating a technical lemma.

Lemma 1 *Suppose that η, σ, τ are elements of $\mathbf{2}^*$ such that $\eta \sqsubseteq \sigma$, $\eta \sqsubseteq \tau$ and $\mathsf{len}(\sigma) = \mathsf{len}(\tau)$, where len is the function which computes the length of a list. Then $\sigma = \tau$.*

Corollary 1 *Let α be a formal point of the formal topology of the binary tree. Then, for any natural number n, α contains at most one list of length n.*

Proof. Suppose σ and τ are two lists in α of length n; then *directness* yields that there exists a list η such that $\eta \sqsubseteq \sigma$ and $\eta \sqsubseteq \tau$ and hence, by the previous lemma 1, $\sigma = \tau$. □

We can also prove that in any point there is a list of any given length.

Lemma 2 *Let α be a formal point of the formal topology of the binary tree, then*

$$(\forall n \in \mathsf{N})(\exists \sigma \in \mathbf{2}^*)\; (\mathsf{len}(\sigma) =_{\mathsf{N}} n) \;\&\; (\sigma \;\varepsilon\; \alpha)$$

Proof. The proof is by induction on $n \in \mathsf{N}$. If $n = 0$, then nil has the required properties whereas if $n > 0$, then, by inductive hypothesis, there is a list σ such that $\mathsf{len}(\sigma) = n - 1$ and $\sigma \,\varepsilon\, \alpha$; then *axiom completeness* yields that there exists $k \in \mathbf{2}$ such that $\sigma k \,\varepsilon\, \alpha$, and $\mathsf{len}(\sigma k) = n$. □

An immediate consequence of the preceding lemma and the so-called axiom of choice, which holds in Martin-Löf's type theory, is that, for any formal point α, there exists a function $f_\alpha \in \mathsf{N} \rightarrow \mathbf{2}^*$ that, for any natural number n, picks the only list $f_\alpha(n)$ in α of length n.

3 Main results

This section is devoted to the proof that the collection of the formal points of the formal topology on the binary tree can be indexed by a set. This result is not new in locale theory (see for instance [Vic89]), but locale theory is usually developed within topos theory, where the power-set constructor is allowed, while we are here working within Martin-Löf's type theory, where the choice principle is justified, and it is well known that the two approaches are not compatible from a constructive point of view since the well known Diaconescu's argument would yield classical logic (see [MV99]). We will prove here, within type theory, the correspondence between the following collections:

1. The collection of the formal points of the formal topology of the binary tree;
2. The collection of the functions $f : \mathsf{N} \rightarrow \mathbf{2}^*$ such that, $\mathsf{len}(f(n)) = n$ and, for any $n \in \mathsf{N}$, $f(n+1) \sqsubseteq f(n)$;
3. The set of the functions from natural numbers into $\mathbf{2}$.

Indeed, the first collection is not a set since it cannot be inductively defined because formal points are subsets defined only by specifying their properties and the second collection is only the projection of a set.

After Lemma 2, we already know that, given any formal point α of the formal topology of the binary tree, there exists a function $f_\alpha \in \mathsf{N} \rightarrow \mathbf{2}^*$ which, for any natural number n, yields the only list $f_\alpha(n)$ in α of length n, so that $f_\alpha(n+1) \sqsubseteq f_\alpha(n)$.

Moreover, it is obvious that given any function $f : \mathsf{N} \rightarrow \mathbf{2}^*$ we can obtain a function $F_f : \mathsf{N} \rightarrow \mathbf{2}$ by setting

$$F_f(n) \equiv f(n+1)[n]$$

where $\sigma[n]$ is the n-th element of the list σ.

Consider now any function $G \in \mathsf{N} \rightarrow \mathbf{2}$. Then we obtain a formal point of the topology of the binary tree by setting

$$\alpha_G = \{\sigma \in \mathbf{2}^* |\ (\forall x \in \mathsf{N})\ (x < \mathsf{len}(\sigma)) \rightarrow \sigma[x] =_{\mathbf{2}} G(x)\}$$

In fact, the following lemma holds.

Lemma 3 *Let $G \in \mathsf{N} \to \mathbf{2}$. Then α_G is a formal point of the formal topology of the binary tree.*

Proof. We have to check that the four required conditions hold for α_G. The proof of most of them is straightforward. We show here only axiom completeness. Suppose $\sigma \; \varepsilon \; \alpha_G$, i.e. $(\forall x < \mathsf{len}(\sigma)) \; \sigma[x] =_{\mathbf{2}} G(x)$; then we obtain that $(\forall x < \mathsf{len}(\sigma \cdot G(\mathsf{len}(\sigma))) \; \sigma \cdot G(\mathsf{len}(\sigma))[x] =_{\mathbf{2}} G(x)$ and hence $\sigma \cdot G(\mathsf{len}(\sigma)) \; \varepsilon \; \alpha_G$; thus $(\exists k \in \mathbf{2}) \; \sigma k \; \varepsilon \; \alpha_G$. □

We can now prove that the relevant constructions are inverse to each other.

Lemma 4 *Let β be a formal point of the formal topology of the binary tree. Then $\alpha_{F_{f_\beta}} = \beta$.*

Proof. We know that $\sigma \; \varepsilon \; \alpha_{F_{f_\beta}}$ if and only if $(\forall x \in \mathsf{N}) \; (x < \mathsf{len}(\sigma)) \to \sigma[x] =_{\mathbf{2}} F_{f_\beta}(x)$, that is, $(\forall x \in \mathsf{N}) \; (x < \mathsf{len}(\sigma)) \to \sigma[x] =_{\mathbf{2}} f_\beta(x+1)[x]$, which means that $\sigma = f_\beta(\mathsf{len}(\sigma))$, since two lists are equal if and only if they coincide on all their components; then $\sigma \; \varepsilon \; \beta$, since $f_\beta(\mathsf{len}(\sigma)) \; \varepsilon \; \beta$. On the other hand, if $\sigma \; \varepsilon \; \beta$, then $\sigma = f_\beta(\mathsf{len}(\sigma))$ and hence $\sigma \; \varepsilon \; \alpha_{F_{f_\beta}}$. □

Lemma 5 *Let $G \in \mathsf{N} \to \mathbf{2}$. Then, for any $n \in \mathsf{N}$, $F_{f_{\alpha_G}}(n) = G(n)$.*

Proof. The proof consists just in unfolding the definitions: $F_{f_{\alpha_G}}(n) = f_{\alpha_G}(n+1)[n] = G(n)$ □

Note that this lemma is sufficient to state that $F_{f_{\alpha_G}} = G$ only if we are working within the extensional version of Martin-Löf type theory (see [NPS90]).

4 A formal topology for the real numbers

The results of the previous section can easily be generalized to the formal topology of any tree such that the immediate successors of any branch are determined only by the branch itself, that is, the formal topology of the *generic computable tree*. In fact, let A be any set and consider the set A^* of the finite list of elements in A. Then, it is immediate to generalize the order relation $\sqsubseteq$ that we introduced in section 2 to elements of A^*. Now, consider the formal topology inductively generated by using, for any list $\sigma \in A^*$, the axiom

$$\sigma \lhd F(\sigma)$$

where $F(\sigma) \equiv \{\sigma a_1, \ldots, \sigma a_n, \ldots\}$ for some subset $\{a_1, \ldots, a_n, \ldots\}$ of A which in general can depend on σ[4]. Then, given any formal point α of such a formal topology we can obtain as in section 3 a function $f_\alpha : \mathsf{N} \to A$ such that

$$\begin{array}{l} \mathsf{nil} \cdot f_\alpha(0) \;\varepsilon\; F(\mathsf{nil}) \\ f_\alpha(0) f_\alpha(1) \;\varepsilon\; F(\mathsf{nil} \cdot f_\alpha(0)) \\ f_\alpha(0) f_\alpha(1) f_\alpha(2) \;\varepsilon\; F(f_\alpha(0) f_\alpha(1)) \\ \ldots \end{array}$$

that is, a function f_α which *stays inside the axioms*. Moreover, the results analogous to Lemma 1 and Lemma 2 can be proved and hence a list σ is an element in the formal point α if and only if there exists a natural number n such that $\sigma = f_\alpha(0) \ldots f_\alpha(n-1) f_\alpha(n)$.

Also the converse direction can be proved, that is, given any *suitable* function from N into A it is possible to determine a formal point of the formal topology of the generic computable tree. The new problem is that we cannot use all of the functions in $\mathsf{N} \to A$ but only those functions f which stay inside the axioms of the formal topology. Note that in the case of the formal topology of the binary tree this condition was automatically satisfied since the subset considered for an axiom was always the full set $\mathbf{2}$.

Thus we define

$$\alpha_f \equiv \{f(0) \ldots f(n) \in A^* |\; n \in \mathsf{N}\}$$

and we obtain a formal point. It is clear that the collection of the formal points of the formal topology of the generic computable tree can be indexed by the elements of the set of the functions from N to A which stay inside the axioms[5].

We can apply this result to a formal topology whose formal points are the real numbers. Write $\mathcal{Q}$ for the set of the rational numbers. Then we can generate a formal topology on $\mathcal{Q} \times \mathcal{Q}$ by using the following axioms [Neg96]:

$$(1) \qquad (p, q) \lhd \{(p, r), (s, q)\} \text{ provided } s < r$$
$$(2) \qquad (p, q) \lhd \{(r, s) |\; p < r \text{ and } s < q\}$$

Of course, this is not a formal topology on a set of lists and hence we cannot directly apply our results on the possibility of indexing by means of a set to the collection of its formal points. Nonetheless, we can obtain an equivalent

[4] This definition can be easily formalized within type theory. Let $U(\sigma)$ be the subset of A that we are considering, that is, U is a propositional function with arguments in A^* and A, then the subset $F(\sigma)$ of A^* is the propositional function $(\eta : A^*)(\exists u \;\varepsilon\; U(\sigma))\; \eta = \sigma u$.

[5] Note however that the correspondence is only onto and not one-to-one since two elements of such a set can be different even if they are obtained from the same function from N to A but with different proofs that it satisfies the condition of being inside the axioms.

topology on a set of lists. Indeed, consider the set $\mathsf{List}(\mathcal{Q}\times\mathcal{Q})$; its elements are lists such that the list $(p_1, q_1)\dots(p_n, q_n)$ can be used to denote the interval of rational numbers $(\mathsf{max}\{p_i|\ 1 \leq i \leq n\}, \mathsf{min}\{q_j|\ 1 \leq j \leq n\})$ that is, the intersection of all the intervals $(p_1, q_1), \dots, (p_n, q_n)$.

The next problem is that given any list, that is, any "interval", the axioms that we need according to the previous definition of the topology of the real numbers are still too many. In fact, our result applies only when the generated topology has exactly one axiom for each element in $\mathsf{List}(\mathcal{Q}\times\mathcal{Q})$. But we can indeed reduce (1) and (2) to a single axiom by only requiring:

$$(3) \qquad (p, q) \lhd \{(r, s)|\ p < r \text{ and } s < q \text{ and } |s - r| < \tfrac{|q-p|}{2}\}$$

for any $(p, q) \in \mathcal{Q} \times \mathcal{Q}$. The idea standing behind (3) is that in order to cover the interval (p, q), we do not need all subinterval of (p, q) but only the considerably shorter ones. In this way, we force a formal point to choose where to stay within the interval (p, q), which was the purpose of axiom (1) above. This intuitive explanation is reflected from a technical point of view in the fact that it is now possible to show that (3) is valid in the previous formal topology of the real numbers and that in the formal topology over $\mathcal{Q} \times \mathcal{Q}$ generated by using only (3) the axioms (1) and (2) are provable.

It is easy to adapt axioms (3) to the case of the formal topology on the set $\mathsf{List}(\mathcal{Q} \times \mathcal{Q})$. Indeed they become

$$(p_1, q_1)\dots(p_n, q_n) \lhd \{(p_1, q_1)\dots(p_n, q_n)(p_{n+1}, q_{n+1}) \in \mathsf{List}(\mathcal{Q} \times \mathcal{Q})| \\ |\mathsf{min}_{1\leq i\leq n+1} q_i - \mathsf{max}_{1\leq i\leq n+1} p_i| < \tfrac{|\mathsf{min}_{1\leq i\leq n} q_i - \mathsf{max}_{1\leq i\leq n} p_i|}{2}\}$$

In this way we can generate a formal topology on a set of lists such that its formal points, that is, the real numbers, can be indexed by a set.

5 A formal topology for the recursively enumerable subsets of N.

To illustrate by another example how to apply the method that we developed in the previous sections, let us introduce the formal topology of the recursively enumerable subsets of the natural numbers and the formal topology of the recursive subsets of the natural numbers.

First we extend the set N of the natural numbers to a new set N_+ by adding the new element $*$. Then, we say that a subset U of N is *recursively enumerable* if there is a function $f : \mathsf{N} \to \mathsf{N}_+$ such that $U = \mathsf{Im}[f] \setminus \{*\}$, where $\mathsf{Im}[f]$ is the image of the function f. The need for the extension from N to N_+ should be clear: it is necessary in order to have the empty subset among the recursively enumerable subsets. In a similar way, we will say that a subset U of N is *recursive* if it is the image of an *increasing* function $f : \mathsf{N} \to \mathsf{N}_+$, that is, a function such that, for any $x, y \in \mathsf{N}$, if $x < y$, then

$f(x) <^{+} f(y)$, where $<^{+}$ is the extension of the usual order relation $<$ between natural numbers to the elements of N_{+} such that, for any $x \in \mathsf{N}_{+}$, $x <^{+} *$.

We can index the colletion of finite subsets of N_{+} by the elements of the set N_{+}^{*} of the lists of elements in N_{+}, neglecting the problem of identifying lists which differ by the order or the repetitions of their elements (see [Mai99] and [Val00] for an analysis of the problem of extending Martin-Löf type theory with extensional set constructors like quotient sets). Also the collection of finite subsets of N can be indexed by such lists since the occurrences of $*$ within a list are effectively recognizable. In this vein, we may set, for any $n \in \mathsf{N}$ and $\sigma \in \mathsf{N}_{+}^{*}$,

$$n \;\varepsilon\; \sigma \equiv (\exists x \in \mathsf{N})\; x < \mathsf{len}(\sigma) \;\&\; n =_{\mathsf{N}_{+}} \sigma[x]$$

Now, let us introduce the family of axioms that we will use to generate a formal topology over N_{+}^{*} whose points correspond to the recursively enumerable subsets of N. Our only axiom, for any $\sigma \in \mathsf{N}_{+}^{*}$, is

$$\sigma \lhd \{\tau \in \mathsf{N}_{+}^{*} |\; (\exists k \in \mathsf{N}_{+})\; \tau =_{\mathsf{N}_{+}^{*}} \sigma k\}$$

We can associate with any formal point α of this formal topology a function $f_{\alpha} : \mathsf{N} \rightarrow \mathsf{N}_{+}$ in a way completely analogous to what we did in section 3, that is, the function f_{α} is such that, if $\sigma \;\varepsilon\; \alpha$, then $\sigma \cdot f_{\alpha}(\mathsf{len}(\sigma))$ is the only list of length $\mathsf{len}(\sigma) + 1$ which is contained in α. Moreover, given any function $f : \mathsf{N} \rightarrow \mathsf{N}_{+}$, we obtain a formal point by setting

$$\alpha_{f} \equiv \{\sigma \in \mathsf{N}_{+}^{*} |\; (\forall k \in \mathsf{N})\; (k < \mathsf{len}(\sigma)) \rightarrow \sigma[k] =_{\mathsf{N}_{+}} f(k)\}$$

Finally, it is obvious that the correspondence between functions and formal points is one-to-one.

Now, given a formal point α of this formal topology, we can obtain a subset of N by setting:

$$U_{\alpha} \equiv \{n \in \mathsf{N} |\; (\exists \sigma \;\varepsilon\; \alpha)\; n \;\varepsilon\; \sigma\}$$

and U_{α} is recursively enumerable since we have that $U_{\alpha} = \mathsf{Im}[f_{\alpha}] \setminus \{*\}$.

On the other hand, for any recursively enumerable subset U of S, we can define a formal point by setting

$$\alpha_{U} \equiv \alpha_{f_{U}}$$

where f_{U} is a function from N into N_{+} which shows that U is a recursively enumerable subset of N.

In a similar way one can obtain a formal topology whose formal points correspond to the recursive subsets of the natural numbers if one uses the set of the increasing sequences of elements of N_{+} instead of N_{+}^{*}.

6 Formal points do not always form a set

After the previous examples it is possible to begin to wonder whether the collection of the formal points of a (generated) formal topology can always be indexed by a set. The answer is negative as this example by Thierry Coquand shows (see [CSSV]).

Let us consider the $\leq$-formal topology defined on the set $\mathbf{2}$ by using the usual order relation and by setting, for any $a \in \mathbf{2}$ and any $U \subseteq \mathbf{2}$,

$$a \lhd U \equiv (\exists u \,\varepsilon\, U)\; a \leq u$$

It is not difficult to check that a subset α of $\mathbf{2}$ is a formal point of this formal topology if and only if $1 \,\varepsilon\, \alpha$ and, whenever $a \,\varepsilon\, \alpha$ and $a \leq b$, then also $b \,\varepsilon\, \alpha$. Thus, for any closed proposition A, $\alpha_A \equiv \{x \in \mathbf{2} |\; A \vee (x =_{\mathbf{2}} 1)\}$ is a point.

Now, note that $\alpha_A = \alpha_B$ if and only if $A \leftrightarrow B$. Thus, there is a one-to-one correspondence between the collection of formal points of this formal topology and the collection of the closed propositions, because we can associate to any formal point α the closed proposition $A_\alpha \equiv 0 \,\varepsilon\, \alpha$ and in this way we obtain that $\alpha = \alpha_{A_\alpha}$. But then if we could index by a set the collection of the formal points of this formal topology we could as well index by a set the collection of all the closed propositions and it is not expected that such a collection can be indexed by any set.

7 Conclusion

The results of the previous sections show that the collection of the formal points of the formal topology on the binary tree can be indexed by a set. This seems not agree with the intuition which would refuse the idea that such a collection is a set since it roughly corresponds to the collection of the real numbers between 0 and 1. It is clear that this situation arises from the *axiom completeness* condition on formal points, i.e.

$$\frac{a \,\varepsilon\, \alpha}{(\exists c \,\varepsilon\, F(a))\; c \,\varepsilon\, \alpha}$$

because the constructive meaning of the existential quantifier in the conclusion allows (and forces) to obtain a function from N into $\mathbf{2}$.

A possible way out is then to change this condition into the weaker

$$\text{(weak axiom completeness)} \qquad \frac{a \,\varepsilon\, \alpha}{\neg\neg(\exists c \,\varepsilon\, F(a))\; c \,\varepsilon\, \alpha}$$

which suggests to change the *completeness* condition in the very definition of formal point by setting

$$\text{(weak completeness)} \qquad \frac{a \,\varepsilon\, \alpha \quad a \lhd U}{\neg\neg(\exists u \,\varepsilon\, U)\; u \,\varepsilon\, \alpha}$$

In this way one only knows that it is not the case that all the elements in U are not in the neighborhood filter of the formal point α, but one has no possibility to know which element is really in α and hence it becomes not possible to construct the function from N into $\mathbf{2}$ associated with α.

In the following we will call *weak point* any subset which satisfies *non-emptiness*, *up-closure*, *directness* and this weaker form of *completeness*.

We show now that the collection of weak points cannot be indexed by a set. In order to prove this result, let us first recall that a subset U of the set of the natural numbers is called *stable* if $\neg\neg U \subseteq U$, that is, for all $x \in \mathsf{N}$, $\neg\neg(x \varepsilon U) \rightarrow (x \varepsilon U)$. Then, it is possible to prove that the collection of the stable subsets of the set of the natural numbers is not a set. In fact, it is possible to adapt to such a collection the standard proof that if the collection of all the subsets of N is a set and the axiom of choice holds, then any proposition A is decidable, that is, $A \vee \neg A$ holds (see [MV99]). Indeed, the same proof can be adapted to any collection $\mathcal{S}$ of subsets such that the union of a subset in $\mathcal{S}$ with a decidable subset is an element of $\mathcal{S}$, where a subset U is decidable if for any $x \in \mathsf{N}$, $(x \varepsilon U) \vee \neg(x \varepsilon U)$ holds. Now, the union of a stable subset with a decidable one is stable and thus, supposing the collection of stable subsets to form a set, it would be possible to prove that every stable subset is decidable, which is not to be expected[6].

Now we can show that there are at least as many weak points as elements in the collection of the stable subsets of natural numbers. Hence if a set could be used to index the collection of weak points then one of its subsets could be used to index the collection of stable subsets.

Indeed, given any subset U of N, we can define a weak point by setting

$$\alpha_U \equiv \{\sigma \in \mathbf{2}^* |\ (\forall x < \mathsf{len}(\sigma))\ (\sigma[x] =_{\mathbf{2}} 1) \leftrightarrow (x \varepsilon U)\}$$

In fact, the following lemma holds.

Lemma 6 *For any subset $U \subseteq N$, α_U is a weak point.*

Proof. *Non-emptiness* and *up-closure* are immediate and *directness* holds because, supposing σ and τ are two lists in α_U, then one is an initial segment of the other.

To prove *weak axiom completeness*, assume that $\sigma \varepsilon \alpha_U$ and let us suppose that $(\forall k \in \mathbf{2})\ \neg(\sigma k \varepsilon \alpha_U)$. Then, by setting first $k = 1$ and then $k = 0$, we obtain both $\neg(\forall x < \mathsf{len}(\sigma) + 1)\ (\sigma 1[x] = 1) \leftrightarrow (x \varepsilon U)$ and $\neg(\forall x < \mathsf{len}(\sigma) + 1)\ (\sigma 0[x] = 1) \leftrightarrow (x \varepsilon U)$ which, together with $\sigma \varepsilon \alpha_U$, yield $\neg(1 = 1 \leftrightarrow \mathsf{len}(\sigma) + 1 \varepsilon U)$ and $\neg(0 = 1 \leftrightarrow \mathsf{len}(\sigma) + 1 \varepsilon U)$. Hence $\neg(\mathsf{len}(\sigma) + 1 \varepsilon U)$ and $\neg\neg(\mathsf{len}(\sigma) + 1 \varepsilon U)$ follow by intuitionistic logic,

[6] Suppose that A is any proposition such that its negation is not decidable, i.e. suppose $\neg A \vee \neg\neg A$ does not hold, and define $U_A \equiv \{x \in \mathsf{N} |\ \neg A\}$. Then $x \varepsilon U_A$ if and only if $\neg A$ holds and hence U_A is a stable subset which is not decidable.

that is, we found a contradiction. Thus we get $\neg(\forall k \in \mathbf{2})\ \neg(\sigma k\ \varepsilon\ \alpha_U)$, i.e. $\neg\neg(\exists k \in \mathbf{2})\ \sigma k\ \varepsilon\ \alpha_U$. □

Note that, if the subset U is decidable, then the definition of α_U above coincides with the one we gave in section 3. In fact, a subset U of N is decidable if and only if there exists a function $\phi_U \in \mathsf{N} \to \mathbf{2}$ such that $(\forall x \in N)\ (x\ \varepsilon\ U) \leftrightarrow (\phi(x) =_{\mathbf{2}} 1)$ (see [Val96]).

On the other hand, given any weak formal point α we can obtain a subset of N by setting

$$U_\alpha \equiv \{n \in \mathsf{N} |\ (\forall \sigma\ \varepsilon\ \alpha)\ (n < \mathsf{len}(\sigma)) \rightarrow \sigma[n] =_{\mathbf{2}} 1\}$$

Then, we can prove the following theorem.

Theorem 4 *For any subset $V \subseteq \mathsf{N}$, $V \subseteq U_{\alpha_V} \subseteq \neg\neg V$. Hence, for any stable subset $V \subseteq \mathsf{N}$, $V = U_{\alpha_V}$.*

Proof. The second statement is an immediate consequence of the first and of the definition of stable subset. Thus, let us prove the first. Let $n \in \mathsf{N}$, then $n\ \varepsilon\ U_{\alpha_V}$ if and only if $(\forall \sigma\ \varepsilon\ \alpha_V)\ (n < \mathsf{len}(\sigma)) \rightarrow \sigma[n] =_{\mathbf{2}} 1$ that is, if and only if

$$\begin{array}{l}(\forall \sigma \in \mathbf{2}^*)\\ \quad ((\forall x < \mathsf{len}(\sigma))\ (\sigma[x] =_{\mathbf{2}} 1) \leftrightarrow (x \varepsilon V))\ \&\ (n < \mathsf{len}(\sigma))) \rightarrow \sigma[n] =_{\mathbf{2}} 1\end{array}$$

Let us suppose now that $n\ \varepsilon\ V$ and consider any list $\sigma \in \mathbf{2}^*$ such that $(\forall x < \mathsf{len}(\sigma))\ (\sigma[x] =_{\mathbf{2}} 1) \leftrightarrow (x\ \varepsilon\ V)$ and $n < \mathsf{len}(\sigma)$; then $\sigma[n] =_{\mathbf{2}} 1$ and hence $n\ \varepsilon\ U_{\alpha_V}$. To prove the second inclusion, suppose that $n\ \varepsilon\ U_{\alpha_V}$ and consider all the lists σ whose length is $n+1$ and whose last component is 0. Then, by means of some intuitionistic logic, for each of them we can prove that

$$\neg(\forall x < \mathsf{len}(\sigma))\ (\sigma[x] =_{\mathbf{2}} 1) \leftrightarrow (x\ \varepsilon\ V)$$

since $\neg(0 = 1)$. Hence, by using again a little of intuitionistic logic, for any such list σ we can obtain a proof of the sentence $C_\sigma \equiv \neg(A_0\ \&\ \ldots\ \&\ A_n)$ where A_i is $i\ \varepsilon\ V$ or $\neg(i\ \varepsilon\ V)$ according to the fact that in the i-th position in the list σ there is 1 or 0. Thus, by using all together the sentences C_σ, we can finally prove that $\neg\neg(n\ \varepsilon\ V)$. □

Corollary 2 *Let V_1 and V_2 be stable subsets of N. Then, if $\alpha_{V_1} = \alpha_{V_2}$, then $V_1 = V_2$.*

Proof. If $\alpha_{V_1} = \alpha_{V_2}$, then $U_{\alpha_{V_1}} = U_{\alpha_{V_2}}$ and hence the previous theorem yields $V_1 = V_2$ since V_1 and V_2 are supposed to be stable subsets. □

Hence we proved that there are at least as many weak points as stable subsets.

At present it is not clear if the solution to use weak points instead of points is acceptable from the point of view of a formal development of topology or if it is too weak. For instance, we know that according to the standard definition of formal point it is not possible to obtain the traditional order completeness property for the real numbers, since we have too few points, while this result is a consequence of the use of the weak completeness condition (see [Neg96]). This fact seems to suggest to change the definition of formal point. On the other hand, as the referee of the first version of this paper pointed out, one can consider the lack of the classical order completeness more like a peculiarity of constructive analysis than like a defect. Hence, no definite choice should be made before a better comprehension of the situation will be achieved.

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