# The sequent calculus for the modal logic D

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**Summary**

We present a sequent calculus for the deontic logic **D** and prove its main syntactic and semantic properties, i.e. cut-elimination, interpolation, completeness with respect to serial frames, finite model property and decidability.

## 1. Introduction

The modal logic **D** (for deontic) is usually presented as the extension of the minimal normal modal logic **K** by the axiom schema $\Box A \to \neg\Box\neg A$ [Seg], i.e. **D** is the minimal modal logic obtained by adding to the classical propositional calculus the axioms

K-Ax: $\Box(A \to B) \to (\Box A \to \Box B)$

D-Ax: $\Box A \to \neg\Box\neg A$

and closing under

MP: $\dfrac{A \quad A \to B}{B}$ and Nec: $\dfrac{A}{\Box A}$

It is easy to see that **D** can equivalently be obtained by adding to a standard sequent calculus for the classical propositional logic, for instance **LK** in [Tak], the modal rules:

KR: $\dfrac{X \vdash A}{\Box X \vdash \Box A}$ and DR': $\dfrac{X \vdash A}{\Box X \vdash \neg\Box\neg A}$

where $\Box X$ stands for the set of formulas $\{\Box B : B \quad X\}$ if X is a set of formulas.

In fact the sequent $A_1,\ldots,A_n \vdash B_1,\ldots,B_m$ is provable in this sequent calculus if and only if the formula $A_1 \ \ldots \ A_n \to (B_1 \ \ldots \ B_m)$ is a theorem of **D** and in particular we have $\vdash B$ if and only if B is a theorem of **D**. Let us here show only the modal steps of the obvious proof by induction on the depth of the considered derivation since the non-modal ones are completely standard[1] .

On one hand we immediately have

[1]A more detailed proof of this theorem is shown in [Val] for the case of the modal logic **K** and the sequent calculus obtained by adding to **LK** only the rule **KR**.

$$\text{K-Ax: }\dfrac{\dfrac{\dfrac{A\to B, A\vdash B}{\Box(A\to B),\Box A\vdash \Box B}}{\Box(A\to B)\vdash \Box A\to\Box B}}{\vdash \Box(A\to B)\to(\Box A\to\Box B)}\qquad \text{D-Ax: }\dfrac{\dfrac{A\vdash A}{\Box A\vdash\neg\Box\neg A}}{\vdash\Box A\to\neg\Box\neg A}\qquad \text{Nec: }\dfrac{\vdash A}{\vdash\Box A}$$

and, on the other one, **D** is closed under KR and DR’ since from $C\to A$ using K-Ax and MP we obtain $\Box C\to\Box A$ and hence using D-Ax and MP we have $\Box C\to\neg\Box\neg A$ and it is well known that, in **K**, $\Box C_1 \;\dots\; \Box C_n\to\Box(C_1 \;\dots\; C_n)$.

Even if these rules are very natural they are not the simplest ones since the conclusion of DR’ can have more than one derivation, for instance by a ¬-introduction rule. This fact suggests the new rule

$$\text{DR: }\dfrac{X\vdash}{\Box X\vdash}$$

where, according with the intended meaning of a sequent, the empty set on the right hand side both in the premise and in the conclusion stands for falsum, i.e. the empty disjunction. In a language which contains also the symbol (to be interpreted in falsum) DR becomes $\dfrac{X\vdash}{\Box X\vdash}$ .

It is easy to see that DR’ and DR are equivalent over a calculus for **K**, i.e. which contains KR [Val]. In fact on one side we have

$$\dfrac{\dfrac{\dfrac{X\vdash A}{X,\neg A\vdash}}{\Box X,\Box\neg A\vdash}}{\Box X\vdash\neg\Box\neg A}\quad\text{and on the other}\quad \dfrac{\dfrac{\dfrac{X\vdash}{X\vdash}}{\Box X\vdash\neg\Box\neg}\qquad \dfrac{\dfrac{\dfrac{\vdash}{\vdash\neg}}{\vdash\Box\neg}}{\neg\Box\neg\quad\vdash}}{\Box X\vdash}\ .$$

In this way we have also proved that **D** is the modal logic obtained from **K** by adding only the axiom $\neg\Box$ , i.e. a particular instance of the characteristic axiom of **D** for A≡ because in **K** $\neg\Box\neg$ is logically equivalent to , since in this case DR is a consequence of KR and an occurrence of the cut-rule.

In the following we will refer to the modal system defined by KR and DR by **DS**.

## 2. Cut-elimination for DS

The theorem of cut-elimination can be easily proved for the sequent calculus **DS** by a standard double induction on the degree (principal induction) and the length of the thread (secondary induction) of the cut-formula. The steps to lower the thread and the non modal reductions are completely standard [Tak], while the modal reductions, characteristic of **DS**, are

$$\dfrac{\dfrac{X\vdash A}{\Box X\vdash\Box A}\qquad\dfrac{A,Y\vdash B}{\Box A,\Box Y\vdash\Box B}}{\Box X,\Box Y\vdash\Box B}\qquad\qquad \dfrac{\dfrac{X\vdash A\quad A,Y\vdash B}{X,Y\vdash B}}{\Box X,\Box Y\vdash\Box B}$$

and

$$\dfrac{\dfrac{X\vdash A}{\Box X\vdash\Box A}\qquad\dfrac{A,Y\vdash}{\Box A,\Box Y\vdash}}{\Box X,\Box Y\vdash}\qquad\qquad \dfrac{\dfrac{X\vdash A\quad A,Y\vdash}{X,Y\vdash}}{\Box X,\Box Y\vdash}$$

A standard consequence of cut-elimination is the interpolation theorem, which can be proved for **DS** by the well-known technique of Maehara-Takeuti [Tak]; i.e. we prove that if the sequent $X\vdash Y$ is derivable in **DS** then, for any partition $X_1, X_2$ of X and any partition $Y_1, Y_2$ of Y there is a formula C, the interpolant, which contains only the propositional variables common both to the formulas in $X_1$ $Y_1$ and $X_2$ $Y_2$ such that the sequents $X_1\vdash Y_1,C$ and $C,X_2\vdash Y_1$ are provable. Here we show only the modal steps of the usual proof by induction on the depth of a cut-free derivation of $X\vdash Y$ in **DS**:

(KR-1) Let us suppose that the sequents $X_1\vdash C$ and $C,X_2\vdash A$ are provable. Then obviously also the sequents $\Box X_1\vdash \Box C$ and $\Box C,\Box X_2\vdash \Box A$ are provable.

(KR-2) Let us suppose that the sequents $X_1\vdash A,C$ and $C,X_2\vdash$ are provable. Then it is not difficult to see that the sequents $\Box X_1\vdash \Box A,\neg\Box\neg C$ and $\neg\Box\neg C,\Box X_2\vdash$ are also provable.

(DR) let us suppose that the sequents $X_1\vdash C$ and $C,X_2\vdash$ are provable then the sequents $\Box X_1\vdash \Box C$ and $\Box C,\Box X_2\vdash$ are provable.

## 3. Semantics

Let us call *serial* [Seg] a Kripke frame <F,R> such that for any x F there is a y F such that xRy, i.e. a frame such that "there is always a future"; then any theorem of **D** is true in any serial frame. In fact, since any frame verifies K-Ax and is closed under MP and Nec [Seg], we must only show that $\Box A\rightarrow\neg\Box\neg A$ holds in any serial frame; this is obvious since, for any point w of a Kripke frame, $\Vdash_w \Box A,\Box\neg A$ if and only if w has no successor. Since KR is valid in any Kripke frame [Val], we can equivalently show that DR is valid in any serial frame; in fact if $\Vdash_w \Box X$ and wRy then $\Vdash_y X$, i.e. a **D**-countermodel for the sequent $\Box X\vdash$ is also a countermodel for the sequent $X\vdash$. We have hence shown the validity of **DS** with respect to the class of the serial frames. We give now a proof of the completeness theorem which shows at the same time also cut redundancy, decidability and the finite model property for **D**.

We can set up a proof procedure for **D** which looks for the provability of a sequent $X\vdash Y$ as follows. Let us write

$$\text{DRR:}\qquad O\,\frac{X\vdash C_1 \quad \dots \quad X\vdash C_n \quad X\vdash}{P,\Box X\vdash \Box C_1,\dots,\Box C_n,Q}\,O$$

where P and Q are sets of propositional variables and X is a set of formulas, to mean that the conclusion is derivable if at least one of the premises is derivable (DRR stands for **D** Ramification Rule).

**D** is obviously closed under DRR (use weakening) and hence it is a valid rule, but much more interesting is that DRR is sufficient to derive any theorem of **D**. In fact, consider a sequent calculus whose rules are the standard propositional rules and DRR and whose axioms are the sequents $X\vdash Y$ such that $X\cap Y\neq$ , then the procedure we look for is simply "apply any applicable rule (cut excluded!!) and stop on the axioms". First note that this procedure stops on any sequent since the premises of every rule contain only proper subformulas of the formulas present in the conclusion. Hence, provided the procedure is correct, we have proved that the logic

**D** is decidable. Moreover since DRR is a valid rule any sequent the procedure declares to be provable is really provable without using any cut. On the other hand if the procedure states a sequent not to be provable then we can inductively construct a **D**-countermodel for that sequent following the "proof-tentative" produced by the procedure starting from the leafs and going toward the root.

In fact any leaf in the proof-tentative of a non provable sequent is obviously a sequent $P \vdash Q$, where P and Q are sets of propositional variables such that no $p_i$ P is a $q_j$ Q; hence it can be falsified at a reflexive point with a valuation which forces any $p_i$ P and no $q_j$ Q.

If any premise of a propositional rule is falsified by one point then also the conclusion of that rule is falsified by the same point.

Finally if we have an occurrence of DRR then, by inductive hypothesis, we already have constructed n+1 **D**-countermodels for the n+1 sequents $X \vdash C_1 \ldots X \vdash C_n, X \vdash$, and hence we can obtain a **D**-countermodel for $P, \Box X \vdash \Box C_1, \ldots, \Box C_n, Q$ simply by adding a new irreflexive point such that it forces any $p_i$ P and no $q_j$ Q and is linked by an intransitive relation to all the n+1 countermodels.

Hence we have proved that a sequent that the procedure states not to be provable can really be falsified in a **D**-countermodel, which is obviously finite since the procedure always stops in a finite number of steps, constructed using only irreflexive points except for the top-most ones. Finally we observe that the cut-rule is redundant since the proof of a provable sequent that can be easily extracted by a successful proof tentative produced by the procedure uses no cut.

**References**

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